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1 Inverse Transpose

I will show that

$$\frac{\partial ln|A|}{\partial A} = (A^{-1})^T$$

We'll name $|x|$ the function $g(x)$, and $ln(x)$ the function $f(x)$, and declaring $F(x) = f(g(x))$, the chain rule says that $\frac{F'(x)}{x'} = \frac{f'(g(x))}{g'}\frac{g'(x)}{x'}$.

Thus the term on the left side of the equation, $\frac{\partial(f(g(A)))}{\partial A}$ derives to:

$$\frac{\partial f(g(A))}{\partial g(A)}\frac{\partial g(A)}{\partial A}$$

And back in the original terms:

$$\frac{\partial ln|A|}{\partial |A|}\frac{\partial |A|}{\partial A}$$

Now since $\frac{d}{dx}lnx = \frac{1}{x}$, and since $\partial[constant] = 1$, and $|A|$ is just a constant, we just get:

$$\frac{1}{|A|}\frac{\partial |A|}{\partial A}$$

The fraction on the right hand side can alternatively be notated: $\nabla_A|A|$, or written out as:

$$\begin{bmatrix} \frac{\partial |A|}{\partial A_{11}} & \frac{\partial |A|}{\partial A_{12}} & \cdots \\ \frac{\partial |A|}{\partial A_{21}} & \ddots & \cdots \\ \vdots & \vdots & \frac{\partial |A|}{\partial A_{mn}} \end{bmatrix}$$

As in Andrew Ng's course notes, we define the adjoint as $(\text{adjoint}(A))_{ij} = (-1)^{i+j}|A_{\backslash j,\backslash i}|$.

And keep that definition of the determinant: $|A| = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}|A_{\backslash i,\backslash j}|$

Let's transpose the determinant in the definition of the adjoint: $(\text{adjoint}(A))_{ij} = (-1)^{i+j}|(A^T)_{\backslash i,\backslash j}|$

Which of course we can rewrite as: $(\text{adjoint}(A^T))_{ij} = (-1)^{i+j}|(A)_{\backslash i,\backslash j}|$.

And so we can redefine the determinant as: $|A| = \sum_{i=1}^{n} a_{ij}\text{adjoint}(A^T)$

And now it's relatively clear why $\nabla_A|A| = \text{adjoint}(A)$

By definition of the determinant, $\text{adjoint}(A)A = |A|I$. This can be reordered: $\text{adjoint}(A)A = |A|I \rightarrow \frac{\text{adjoint}(A)A}{|A|} = I$ and since $|A|$ is a scalar, it doesn't really matter where it goes. Then we get $\frac{\text{adjoint}(A)AA^{-1}}{|A|} = IA^{-1} \rightarrow \frac{\text{adjoint}(A)}{|A|} = A^{-1}$.

Now since $\nabla_A|A| = \text{adjoint}(A)$, we get $\frac{\nabla_A|A|}{|A|} = A^{-1}$.

2 Positive eigenvalues

For a matrix A and any eigenvector x and eigenvalue λ:

$$Ax = \lambda x$$

A (square) positive definite matrix A is defined as: $x^T Ax > 0$, where x is any vector of the same width as A (and x is non-zero). We wish to show that if x is an eigenvector of A, which is positive definite, then the corresponding eigenvalue λ is positive.

First, multiply both sides of the eigenvector equation by x^T:

$$x^T Ax = x^T \lambda x$$

Because we presume that A is positive definite, and we are dealing with non-zero eigenvectors (x), we see that the left side of this equation must be greater than zero, and so must be the right side:

$$x^T \lambda x > 0$$

Now, since λ is just a scalar number, we can move it over:

$$x^T x\lambda > 0$$

A vector multiplied by its transpose is a sum of squares:

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1x_1 + x_2x_2 + x_3x_3 + x_4x_4$$

We're only dealing with real numbers, so our squares must be non-negative. But we've also asserted that our vector x is non-zero, so the sum of our squares must be positive (non-zero). We declare that $c = x^T x$,

$$c\lambda > 0$$

And since we've shown that $c > 0$, divide both sides by c, and we're done:

$$\lambda > 0$$

Of course, this works for any eigenvalue λ, since it works for the general case.

3 Matrix Algebra

We will show:

$$(A + BD^{-1}C)^{-1} = A^{-1} - A^{-1}B(D + CA^{-1}B)^{-1}CA^{-1}$$

Begin by multiplying both sides by $(A + BD^{-1}C)$

$$(A + BD^{-1}C)^{-1}(A + BD^{-1}C) \stackrel{?}{=} (A^{-1} - A^{-1}B(D + CA^{-1}B)^{-1}CA^{-1})(A + BD^{-1}C)$$

The left side reduces to unity:

$$I \stackrel{?}{=} (A^{-1} - A^{-1}B(D + CA^{-1}B)^{-1}CA^{-1})(A + BD^{-1}C)$$

Factor out the A^{-1} on the right side:

$$I \stackrel{?}{=} (A^{-1})(1 - B(D + CA^{-1}B)^{-1}CA^{-1})(A + BD^{-1}C)$$

Now distribute the $(A + BD^{-1}C)$ term.

$$I \stackrel{?}{=} (A^{-1})(A + BD^{-1}C - B(D + CA^{-1}B)^{-1}CA^{-1}A - B(D + CA^{-1}B)^{-1}CA^{-1}BD^{-1}C)$$

Simplify out $A^{-1}A$.

$$I \stackrel{?}{=} (A^{-1})(A + BD^{-1}C - B(D + CA^{-1}B)^{-1}C - B(D + CA^{-1}B)^{-1}CA^{-1}BD^{-1}C)$$

Factor out B to the left, and C to the right.

$$I \stackrel{?}{=} (A^{-1})(A + B(D^{-1} - (D + CA^{-1}B)^{-1} - (D + CA^{-1}B)^{-1}CA^{-1}BD^{-1})C)$$

Factor out $(D + CA^{-1}B)^{-1}$:

$$I \stackrel{?}{=} (A^{-1})(A + B(D^{-1} - (D + CA^{-1}B)^{-1}(1 + CA^{-1}BD^{-1}))C)$$

And inject a DD^{-1}:

$$I \stackrel{?}{=} (A^{-1})(A + B(D^{-1} - (D + CA^{-1}B)^{-1}(1 + CA^{-1}BD^{-1})DD^{-1})C)$$

And distribute towards the left:

$$I \stackrel{?}{=} (A^{-1})(A + B(D^{-1} - (D + CA^{-1}B)^{-1}(D + CA^{-1}BD^{-1}D)D^{-1})C)$$

The $D^{-1}D$ cancels to I:

$$I \stackrel{?}{=} (A^{-1})(A + B(D^{-1} - (D + CA^{-1}B)^{-1}(D + CA^{-1}B)D^{-1})C)$$

And $(D + CA^{-1}B)^{-1}(D + CA^{-1}B)$ also cancels to I:

$$I \stackrel{?}{=} (A^{-1})(A + B(D^{-1} - D^{-1})C$$

I'll multiply it out, just to be clear. First, A^{-1}:

$$I \stackrel{?}{=} A^{-1}A + A^{-1}B(D^{-1} - D^{-1})C$$

Simplify the left term, distribute the right:

$$I \stackrel{?}{=} I + A^{-1}BD^{-1}C - A^{-1}BD^{-1}C$$

Subtract I from both sides:

$$0 \stackrel{?}{=} A^{-1}BD^{-1}C - A^{-1}BD^{-1}C$$

Add $A^{-1}BD^{-1}C$ to both sides, and the equality is obvious.

$$A^{-1}BD^{-1}C = A^{-1}BD^{-1}C$$

4 Cost + Reward function optimization

We begin with the dynamic equation (acceleration is a function of gas applied at some time):

$$\ddot{x} = u(t)$$

and initial conditions (which just say we start at the beginning, with a speed of zero):

$$x = 0$$

$$\dot{x} = 0$$

and the reward - cost function:

$$J = x(T) - \int_0^T u^2(t)dt$$

In that equation, $x(T)$ represents the distance traveled at time T (the bigger the better). And $\frac{1}{2}u^2(t)$ represents the acceleration (amount of gas) applied at point t, and the integration of that function from time $= 0$ to time $= T$ is the total amount of power ($=$ gas) used between the starting point and the end time, but squared in order to encourage a gradual application of power.

$\dot{x}$ is a vector, which consists of two scalars, which correspond to the derivatives of position and velocity:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \begin{pmatrix} x_2 \\ \dot{x}_2 \end{pmatrix}$$