Consensus in Networked Multiagent Systems With Stochastic Sampling

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***Abstract*—This brief addresses the consensus problem in networks with stochastic sampling, assuming that the sampling intervals are independently and identically distributed. Necessary and sufficient conditions to guarantee mean square consensus of both first-order and second-order multiagent systems are derived in terms of statistical properties of the sampling intervals. Numerical examples are also given to verify the theoretical results, revealing that the conditions on the sampling intervals are significantly relaxed in contrast to the previous sufficient conditions. By applying the first-order model to opinion consensus problems with intermittent interactions, it is finally found that the temporal heterogeneity of human activities impedes the opinion consensus forming.**

***Index Terms*—Consensus, multiagent systems, opinion consensus, stochastic sampling, temporal heterogeneity.**

I. Introduction

Consensus of multiple agents, which is one of the most fundamental issues in the distributed control of multiagent systems, has been a research focus for decades. This is partly motivated by broad applications of multiagent systems in various areas, including flocking control [1], attitude alignment of spacecrafts [2], distributed sensor networks [3], and opinion forming in social networks [4], [5].

Due to the intermittent communication among agents or the prevalence of digital controllers in modern control systems, the majority of current literatures concerning consensus [6]–[10] were devoted to sampled-data consensus problems with nonuniform sampling intervals [11]–[16], focusing on deriving sufficient conditions on the bounds of the sampling intervals for guaranteeing consensus. All these sufficient conditions are more or less conservative in the sense that occasional appearances of large sampling intervals, which are out of bounds, may not drastically deteriorate the system consensus. Such conservatism is rooted in its framework of deterministic systems. In this regard, it is of both theoretical and practical significance to explore necessary and sufficient conditions from the perspective of stochastic sampling by assuming that the sampling intervals are random variables. Another more important motivation to address the consensus problem with stochastic sampling lies in the fact that both the clock and the sampling device may be subject to noises and randomly occurring uncertainties in a time-triggered sampled-data control system.

Note that the robust H_∞ control and synchronization of networks with stochastic sampling were investigated in [17] and [18], respectively, in which the sampling interval was assumed to switch between two or more different values in a random way. More recently, the node-to-node consensus problem with stochastically varied sampling intervals was studied in [19]. All these results are based on the assumption that the distribution of the sampling intervals is *a priori*, which is, however, not the case in many real scenarios. Therefore, this brief addresses the consensus problem with stochastic sampling, where the sampling intervals are assumed independently and identically distributed with statistical properties obtainable only.

We derive necessary and sufficient conditions in terms of the statistical properties of sampling intervals for guaranteeing both first-order and second-order consensus. We also present numerical examples to show that the previous sufficient conditions with respect to sampling intervals given in [6] and [14] are too conservative in contrast to the necessary and sufficient conditions proposed in this brief. Finally, we apply the first-order model to opinion consensus problems in social networks such that the sampling process can be interpreted as intermittent person-to-person interactions. Motivated by the fact that various human activities exhibit temporal heterogeneity [20], [21], we further investigate the impact of the interaction intervals' heterogeneity on the opinion consensus forming, which is completely neglected in current studies [4], [5] on opinion consensus problems.

This brief is organized as follows. Mathematical notations, definitions, and lemmas to be used are presented in Section II. The first-order and second-order consensus problems in networks with stochastic sampling are addressed in Section III. The simulation studies with an application to opinion consensus problems is presented in Section IV. Section V concludes this brief.

II. Preliminaries

We first present some mathematical notations and definitions in graph theories. Let $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the set of $n \times 1$-dimensional real vectors and the set of $n \times m$-dimensional real matrices, respectively.

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$I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. $\mathbf{1} = [1, 1, \ldots, 1]^T$, and $\mathbf{0}$ denotes a null matrix, both of which have appropriate dimensions if no confusion arises. $\rho(\cdot)$ denotes the spectral radius of a matrix. $\otimes$ and $\oplus$ denote the Kronecker product and sum, respectively. For a random variable, $\mathbf{E}[\cdot]$ is the expectation, and $\mathbf{Var}[\cdot]$ is the variance.

In a multiagent system, information exchanges between agents can be modeled by directed graphs (digraphs). A weighted digraph is usually denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where the vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, the edge set $\mathcal{E} \subseteq \{(v_i, v_j) : v_i, v_j \in \mathcal{V}\}$, and the weighted adjacency matrix $A = [a_{ij}]$ with non-negative adjacency elements a_{ij}. Edge (v_i, v_j) is an arrow, where v_i denotes the tail and v_j the head of the arrow. The adjacency elements associated with the edges are positive, i.e., $(v_j, v_i) \in \mathcal{E} \Leftrightarrow a_{ij} > 0$, and assume that $a_{ii} = 0$ for any $i \in \mathcal{V}$. The neighbors of agent v_i are denoted by $N_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. Let $\deg(i) = \sum_{j=1}^{n} a_{ij}$, $D = \mathrm{diag}(\deg(1), \deg(2), \ldots, \deg(n))$, and the Laplacian matrix of $\mathcal{G}$ is defined as $L = D - A$.

Hereinafter, we introduce several definitions and lemmas in stochastic linear systems to be used in this brief. Consider the discrete-time stochastic linear system consisting of n agents

$$x(k+1) = W_k x(k) \tag{1}$$

where $x = [x_1, x_2, \ldots, x_n]^T$ with $x_i \in \mathbb{R}$ denoting the state of agent i, $\{W_k\}$ are independently and identically distributed matrices in $\mathbb{R}^{n\times n}$ with common random variable W, and W has the property that $W\mathbf{1} = \mathbf{1}$. There have been emerging investigations [22]–[24] on consensus over random networks based on system (1), and three types of consensus, i.e., mean square consensus, almost sure consensus, and consensus with probability 1, are proved equivalent in [22]. For the sake of consistence, we call it mean square consensus throughout this brief and give its definition as below.

Definition 1: System (1) is said to reach mean square consensus if $\mathbf{E}[|x_i(k)|^2] < \infty$ for all $k \geq 0$ and any i, and there exists a random variable x^* such that

$$\lim_{k\to\infty} \mathbf{E}\left[\left|x_i(k) - x^*\right|^2\right] = 0$$

for any i.

We then present the lemmas that will be used in drawing the main conclusion of this brief.

Lemma 1 [23]: System (1) reaches mean square consensus if and only if the system

$$\xi(k+1) = \tilde{W}_k \xi(k) \tag{2}$$

where $\xi(k) = Q^T x(k) \in \mathbb{R}^{n-1}$ is called the error vector with $Q \in \mathbb{R}^{n\times(n-1)}$ satisfying $Q^T\mathbf{1} = \mathbf{0}$ and $Q^T Q = I_{n-1}$, and $\tilde{W}_k = Q^T W_k Q$, is mean square stable which means $\lim_{k\to\infty} \mathbf{E}[\|\xi(k)\|^2] = 0$ for any $\xi(0) \in \mathbb{R}^{n-1}$.

Hence, we characterize the convergence speed of mean square consensus by computing the convergence speed of the error vector $\xi(k)$. Then the asymptotic convergence factor for system (1) is defined as

$$r_a = \sup_{\|\xi(0)\|\neq 0} \lim_{k\to\infty} \left(\frac{\mathbf{E}\left[\|\xi(k)\|^2\right]}{\|\xi(0)\|^2} \right)^{1/k}$$

and it can be computed from the statistical properties of $\tilde{W}$ by the following lemma.

Lemma 2 [23]: System (2) is mean square stable if and only if $r_a = \rho(\mathbf{E}[\tilde{W} \otimes \tilde{W}]) < 1$.

III. Consensus With Stochastic Sampling

In this section, we will address both the first-order and second-order consensus problems of a multiagent system with stochastic sampling, focusing on deriving necessary and sufficient conditions on the sampling intervals for guaranteeing consensus.

A. First-Order Consensus With Stochastic Sampling

We consider a continuous-time multiagent system consisting of n identical agents with dynamics represented by

$$\dot{x}_i(t) = u_i(t), u_i(t) = \sum_{j\in N_i} a_{ij}\big(x_j(t) - x_i(t)\big),\ i \in \mathcal{I}_n \tag{3}$$

where $\mathcal{I}_n = \{1, 2, \ldots, n\}$, $x_i \in \mathbb{R}$ denotes the state of agent i.

Assume $\{t_k\}_{k=0}^{+\infty}$ are the discrete sampling instants, satisfying $0 = t_0 < t_1 < \cdots < t_k < \ldots$, and the sampling intervals $t_{k+1} - t_k = h_k, k = 0, 1, \ldots$ are independently and identically distributed with common random variable h. Then the dynamics for agent $i \in \mathcal{I}_n$ is

$$\dot{x}_i(t) = \sum_{j\in N_i} a_{ij}\big(x_j(t_k) - x_i(t_k)\big),\ \ t \in \big[t_k, t_{k+1}\big). \tag{4}$$

In the following theorem, we present a necessary and sufficient condition in terms of the statistical properties of h for guaranteeing mean square consensus of system (4).

Theorem 1: Assume that h has finite second-order moment. Then system (4) reaches mean square consensus if and only if the asymptotic convergence factor $\rho((\mathbf{E}[h]RL-R)\otimes(\mathbf{E}[h]RL-R) + \mathbf{Var}[h](RL \otimes RL)) < 1$ with $R = I_n - (1/n)\mathbf{1}\mathbf{1}^T$.

Proof: We rewrite system (4) into the compact discrete-time form

$$x(k+1) = P_k x(k) \tag{5}$$

with k standing for t_k and $P_k = I_n - h_k L$. Since $\{h_k\}$ are independently and identically distributed with common random variable h, $\{P_k\}$ are independently and identically distributed matrices, and we refer to $P = I_n - hL$ as their common random variable satisfying

$$P\mathbf{1} = \mathbf{1}. \tag{6}$$

Lemma 1 implies that mean square consensus of system (5) is equal to the mean square stability of system

$$\xi(k+1) = \tilde{P}_k \xi(k) \tag{7}$$

with $\xi(k) = Q^T x(k)$ and $\tilde{P}_k = Q^T P_k Q$, where $Q \in \mathbb{R}^{n\times(n-1)}$ is defined with the properties $Q^T\mathbf{1} = \mathbf{0}$ and $Q^T Q = I_{n-1}$. Following Lemma 2, it further gives that system (5) reaches mean square consensus if and only if the asymptotic convergence factor

$$r_a = \rho\big(\mathbf{E}\big[Q^T P Q \otimes Q^T P Q\big]\big) < 1.$$

From the orthogonality of $[Q \quad (1/\sqrt{n})\mathbf{1}]$, we have $QQ^T = R$ with $R = I_n - (1/n)\mathbf{1}\mathbf{1}^T$. Then we obtain

$$\begin{aligned}&(Q \otimes Q)\mathbf{E}\big[Q^T PQ \otimes Q^T PQ\big]\big(Q^T \otimes Q^T\big)\\&\quad = (Q \otimes Q)\mathbf{E}\big[Q^T PR \otimes Q^T PR\big]\\&\quad = (Q \otimes Q)\mathbf{E}\big[Q^T\big(P - (1/n)\mathbf{1}\mathbf{1}^T\big) \otimes Q^T\big(P - (1/n)\mathbf{1}\mathbf{1}^T\big)\big]\\&\quad = (R \otimes R)\mathbf{E}[P \otimes P] \end{aligned} \tag{8}$$

the second equality of which follows from (6) and the third is due to $Q^T\mathbf{1} = \mathbf{0}$. By defining the orthogonal matrix $[Q \otimes Q \quad V]$ where $V \in \mathbb{R}^{n^2 \times (2n-1)}$ such that $(Q \otimes Q)^T V = 0$ and $V^T V = I_{2n-1}$, we have

$$\begin{aligned}&\begin{bmatrix} Q \otimes Q & V \end{bmatrix}\begin{bmatrix} \mathbf{E}\big[Q^T PQ \otimes Q^T PQ\big] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} Q^T \otimes Q^T \\ V^T \end{bmatrix}\\&\qquad = (Q \otimes Q)\mathbf{E}\big[Q^T PQ \otimes Q^T PQ\big]\big(Q^T \otimes Q^T\big)\end{aligned}$$

which implies that the first line of (8) is similar to

$$\begin{bmatrix} \mathbf{E}\big[Q^T PQ \otimes Q^T PQ\big] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

and so is the last line of (8). Therefore

$$\rho\big(\mathbf{E}\big[Q^T PQ \otimes Q^T PQ\big]\big) = \rho((R \otimes R)\mathbf{E}[P \otimes P]).$$

Furthermore, we have

$$\begin{aligned} r_a &= \rho((R \otimes R)\mathbf{E}[P \otimes P])\\ &= \rho((R \otimes R)\mathbf{E}[(I_n - hL) \otimes (I_n - hL)])\\ &= \rho\Big((R \otimes R)\Big(I_{n^2} - \mathbf{E}[h](L \oplus L) + \mathbf{E}[h^2](L \otimes L)\Big)\Big)\\ &= \rho((\mathbf{E}[h]RL - R) \otimes (\mathbf{E}[h]RL - R) + \mathbf{Var}[h](RL \otimes RL)).\end{aligned}$$

Till now, it is straightforward to obtain the conclusion in Theorem 1. ■

B. Second-Order Consensus With Stochastic Sampling

Motivated by a broad class of vehicles modeled by double-integrators, we consider a continuous-time multiagent system consisting of n identical agents with double-integrator dynamics

$$\begin{cases} \dot{r}_i(t) = v_i(t) \\ \dot{v}_i(t) = u_i(t) \end{cases}, i \in \mathcal{I}_n \tag{9}$$

where $\mathcal{I}_n = \{1, 2, \ldots, n\}$, $r_i \in \mathbb{R}$ and $v_i \in \mathbb{R}$ are the position and velocity of agent i, respectively, and $u_i \in \mathbb{R}$ is the distributed control input for agent i based on the state information of its own and its neighbors. A classic consensus protocol for system (9) studied in [7] and [14] is given as

$$u_i(t) = \sum_{j \in N_i} a_{ij}\big(r_j(t) - r_i(t)\big) - \alpha v_i(t) \tag{10}$$

where α is a positive control gain introducing the absolute velocity damping. Applying the stochastic sampling as described in the first-order consensus case, system (9) with consensus protocol (10) is represented by

$$\begin{cases} \dot{r}_i(t) = v_i(t) \\ \dot{v}_i(t) = \sum\limits_{j \in N_i} a_{ij}\big(r_j(t_k) - r_i(t_k)\big) - \alpha v_i(t_k) \end{cases} \tag{11}$$

where $i \in \mathcal{I}_n$, $t \in [t_k, t_{k+1})$.

Definition 2: System (11) is said to reach mean square consensus if $\mathbf{E}[|r_i(t)|^2] < \infty$ for all $t \geq 0$ and any i, and there exists a random variable r^* such that $\lim_{t\to\infty} \mathbf{E}[|r_i(t) - r^*|^2] = 0$ and $\lim_{t\to\infty} \mathbf{E}[|v_i(t)|^2] = 0$ for any i.

Denote $r = [r_1, r_2, \ldots, r_n]^T$, and $v = [v_1, v_2, \ldots, v_n]^T$. Let k stand for t_k, and then system (11) can be rewritten in the following discrete time compact matrix form:

$$\begin{bmatrix} r(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} I_n - \frac{h_k^2}{2}L & \Big(h_k - \frac{\alpha h_k^2}{2}\Big)I_n \\ -h_k L & (1 - \alpha h_k)I_n \end{bmatrix}\begin{bmatrix} r(k) \\ v(k) \end{bmatrix}. \tag{12}$$

Let $\tilde{r} = r + (1/\alpha)v$, and $x = [r^T, \tilde{r}^T]^T$. Then rewrite system (12) into

$$x(k+1) = \hat{P}_k x(k) \tag{13}$$

with

$$\hat{P}_k = \begin{bmatrix} \Big(1 - \alpha h_k + \frac{\alpha^2 h_k^2}{2}\Big)I_n - \frac{h_k^2}{2}L & \Big(\alpha h_k - \frac{\alpha^2 h_k^2}{2}\Big)I_n \\ \frac{\alpha^2 h_k^2}{2}I_n - \Big(\frac{h_k}{\alpha} + \frac{h_k^2}{2}\Big)L & \Big(1 - \frac{\alpha^2 h_k^2}{2}\Big)I_n \end{bmatrix}.$$

Lemma 3: System (11) reaches mean square consensus if and only if system (13) reaches mean square consensus.

Proof: We conduct the proof from both necessity and sufficiency.

Necessity: Assume system (11) reaches mean square consensus, i.e., there exists a random variable r^* such that $\lim_{t\to\infty} \mathbf{E}[|r_i(t) - r^*|^2] = 0$ and $\lim_{t\to\infty} \mathbf{E}[|v_i(t)|^2] = 0$ for any i. Due to $\tilde{r} = r + (1/\alpha)v$ and $x = [r^T, \tilde{r}^T]^T$, it gives that $\lim_{k\to\infty} \mathbf{E}[|x_j(k) - r^*|^2] = 0$ for any j, which immediately implies that system (13) reaches mean square consensus.

Sufficiency: Assume system (13) reaches mean square consensus, i.e., there exists a random variable r^* such that $\lim_{k\to\infty} \mathbf{E}[|x_j(k) - r^*|^2] = 0$ for any j. For any $t > 0$, there exists $s \in \mathbb{N}$ such that $t_s \leq t < t_{s+1}$. Let $t - t_s = \tau$, and then we have

$$x(t) = Z(\tau)x(t_s) \tag{14}$$

with

$$Z(\tau) = \begin{bmatrix} \Big(1 - \alpha\tau + \frac{\alpha^2\tau^2}{2}\Big)I_n - \frac{\tau^2}{2}L & \Big(\alpha\tau - \frac{\alpha^2\tau^2}{2}\Big)I_n \\ \frac{\alpha^2\tau^2}{2}I_n - \Big(\frac{\tau}{\alpha} + \frac{\tau^2}{2}\Big)L & \Big(1 - \frac{\alpha^2\tau^2}{2}\Big)I_n \end{bmatrix}.$$

When $t_s \to \infty$, $\mathbf{E}[|x_j(t_s) - r^*|^2] = 0$ for any j. Since $Z(\tau)$ is row stochastic, it follows from (14) that $\lim_{t\to\infty} \mathbf{E}[|x_j(t) - r^*|^2] = 0$ for any j. Till now, it is straightforward that $\lim_{t\to\infty} \mathbf{E}[|r_i(t) - r^*|^2] = 0$ and $\lim_{t\to\infty} \mathbf{E}[|v_i(t)|^2] = 0$ for any i, i.e., system (11) reaches mean square consensus. ■

Redefine $R = I_{2n} - (1/2n)\mathbf{1}\mathbf{1}^T$, and let

$$\hat{P}_I = \begin{bmatrix} -\alpha I_n & \alpha I_n \\ -\frac{1}{\alpha}L & \mathbf{0} \end{bmatrix}, \hat{P}_{II} = \begin{bmatrix} \frac{\alpha^2}{2}I_n - \frac{1}{2}L & -\frac{\alpha^2}{2}I_n \\ \frac{\alpha^2}{2}I_n - \frac{1}{2}L & -\frac{\alpha^2}{2}I_n \end{bmatrix}.$$

Based on these notations, we present a necessary and sufficient condition on the sampling intervals for guaranteeing second-order consensus in the following theorem.

Theorem 2: Assume that h has finite fourth-order moment. Then system (11) reaches mean square consensus if and only if the asymptotic convergence factor $\rho(R \otimes R + \mathbf{E}[h](R\hat{P}_I \otimes R + R \otimes R\hat{P}_I) + \mathbf{E}[h^2](R\hat{P}_I \otimes R\hat{P}_I + R\hat{P}_{II} \otimes R + R \otimes R\hat{P}_{II}) + \mathbf{E}[h^3](R\hat{P}_I \otimes R\hat{P}_{II} + R\hat{P}_{II} \otimes R\hat{P}_I) + \mathbf{E}[h^4](R\hat{P}_{II} \otimes R\hat{P}_{II})) < 1$.

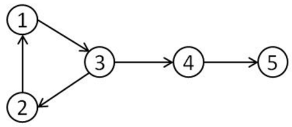


Fig. 1. Illustrative example of the communication topology.

Proof: Induced by $\{h_k\}$ and $\{\hat{P}_k\}$ are independently and identically distributed matrices with common random variable $\hat{P}$ and $\hat{P}$ satisfies

$$\hat{P}\mathbf{1} = \mathbf{1}. \tag{15}$$

Following Lemma 3 and the arguments in the proof of Theorem 1, we easily obtain that system (11) reaches mean square consensus if and only if

$$r_a = \rho\Big((R \otimes R)\mathbf{E}\Big[\hat{P} \otimes \hat{P}\Big]\Big) < 1.$$

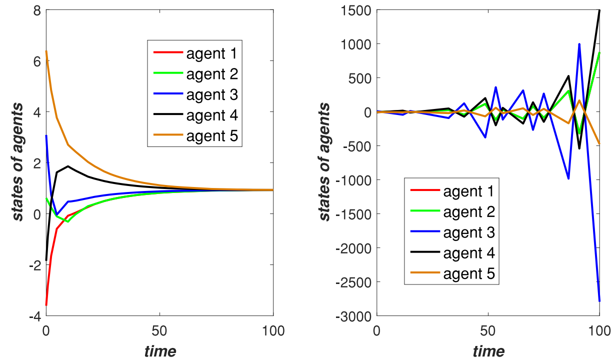
Moreover, $\hat{P} = I_{2n} + h\hat{P}_I + h^2\hat{P}_{II}$, based on which, we get the conclusion in terms of the statistical properties of sampling intervals as stated in the theorem. ■

IV. Simulation Studies With Application to Opinion Consensus

In this section, numerical examples are given to demonstrate the theoretical results concerning first-order and second-order consensus. Consider a network of $n = 5$ agents with the communication topology as shown in Fig. 1 throughout the following examples, in which the adjacency elements associated with the edges are all 0.2.

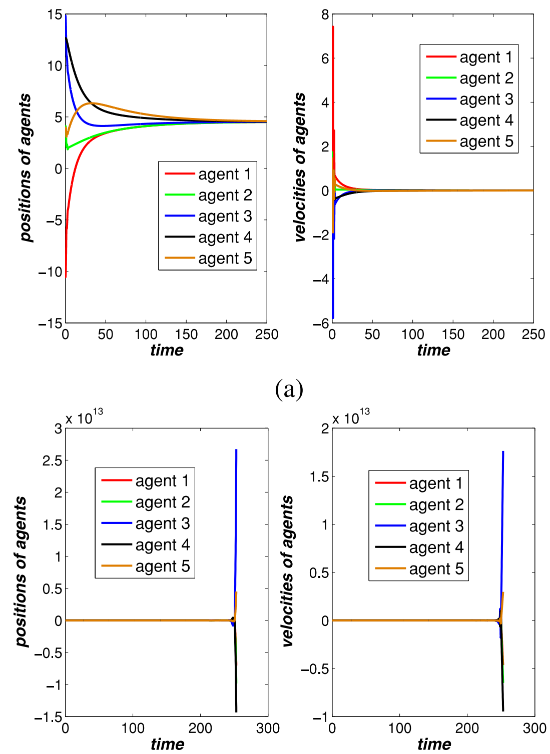
We first assume the agents follow the first-order dynamics (4) with h randomly chosen from exponential distribution with $\mathbf{E}[h] = 2$ and $\mathbf{Var}[h] = 4$. It can be easily computed that the asymptotic convergence factor defined in Theorem 1 is $0.76 < 1$. The agents' state trajectories are shown in Fig. 2(a), indicating that the system converges and reaches consensus. According to the result in [6], $h < 1/\max_i l_{ii} = 5$ is the sufficient condition for guaranteeing consensus. In contrast, the maximum of h in this example reaches 13.12, and it is thus evident that the previous sufficient condition with respect to h is conservative. To validate the necessary part of Theorem 1, we let h be randomly chosen from exponential distribution with $\mathbf{E}[h] = 4$ and $\mathbf{Var}[h] = 16$ such that the asymptotic convergence factor is $2.44 > 1$. Then the agents' state trajectories are shown in Fig. 2(b), revealing that the system diverges.

We further assume the agents follow the second-order dynamics (11) with $\alpha = 3$, in which h is randomly chosen from exponential distribution with $\mathbf{E}[h] = 0.3$. It can be easily obtained $\mathbf{E}[h^2] = 2\times 0.3^2$, $\mathbf{E}[h^3] = 6\times 0.3^3$, and $\mathbf{E}[h^4] = 24\times 0.3^4$ such that the asymptotic convergence factor defined in Theorem 2 is $0.9607 < 1$. Fig. 3(a) shows the agents' position and velocity trajectories, which reveals that the system converges and reaches consensus. As reported in [14], h should be in the region $(2\max_i l_{ii}/(\alpha^3 - \alpha \max_i l_{ii}), \sqrt{2}/\alpha)$, which means that $0.0152 \leq h \leq 0.4714$ is sufficient for ensuring consensus in this example. In contrast, the minimum and maximum of h in this example are 0.0012 and 2.1976, respectively, indicating that the previous sufficient condition is rather conservative. To validate the necessary part of Theorem 2, we let h be randomly chosen from exponential distribution with $\mathbf{E}[h] = 0.8$. Then $\mathbf{E}[h^2] = 2 \times 0.8^2$, $\mathbf{E}[h^3] = 6 \times 0.8^3$, $\mathbf{E}[h^4] = 24 \times 0.8^4$ such that the asymptotic convergence factor is $9.7332 > 1$, and the agents' position and velocity trajectories shown in Fig. 3(b) indicate that the system diverges.



(a) (b)

Fig. 2. State trajectories of the agents with the sampling intervals randomly chosen from exponential distribution with (a) $\mathbf{E}[h] = 2$ and (b) $\mathbf{E}[h] = 4$.



(b)

Fig. 3. Position and velocity trajectories of the agents with the sampling intervals randomly chosen from exponential distribution with (a) $\mathbf{E}[h] = 0.3$ and (b) $\mathbf{E}[h] = 0.8$.

Next, we apply the first-order model to opinion consensus problems in social networks, where person-to-person interactions are usually intermittent with independently and identically distributed random intervals, such as Web page visits, messages in online forums, and e-mail activity. It has also been commonly observed that person-to-person interaction intervals present power-law distributions [20], [21] due to the temporal heterogeneity of human activities. Therefore, we let the interaction intervals be randomly chosen from power-law distributions to study the impact of the temporal heterogeneity of human activities on the opinion consensus forming.

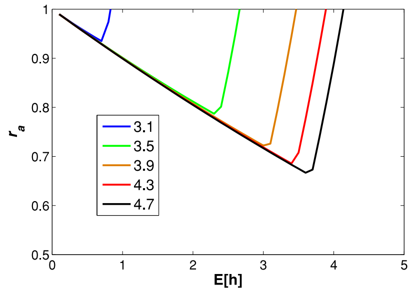


Fig. 4. Asymptotic convergence factor for the power-law distributions of the interaction intervals with the changing $\mathbf{E}[h]$ and different γ.

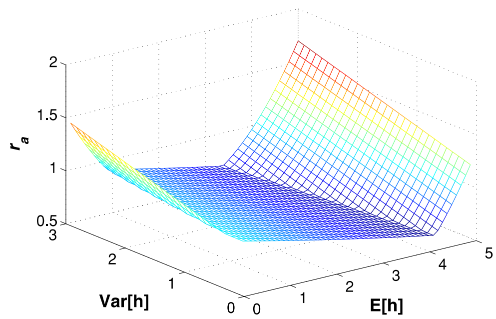


Fig. 5. Change of the asymptotic convergence factor with respect to $\mathbf{E}[h]$ and $\mathbf{Var}[h]$.

According to Theorem 1, we refer to the range of $\mathbf{E}[h]$ satisfying $r_a < 1$ as the feasible range. Fig. 4 shows the feasible range of $\mathbf{E}[h]$ for power-law distributions of h with different exponents denoted by γ. It can be obviously observed that the feasible range of $\mathbf{E}[h]$ shrinks and the asymptotic convergence factor increases with the decrease of γ. Since a larger asymptotic convergence factor implies a slower convergence speed, and a smaller γ implies a more heterogeneous distribution of h, we conclude that the heterogeneity of h slows down the convergence speed or even destroys the system convergence. In other words, the heterogeneity of h impedes the opinion consensus forming. To uncover the impact of the heterogeneity of h on the opinion consensus forming more explicitly, we show in Fig. 5 the change of r_a with respect to $\mathbf{E}[h]$ and $\mathbf{Var}[h]$, revealing that larger $\mathbf{Var}[h]$ tends to achieve slower convergence speed for an arbitrary $\mathbf{E}[h]$.

V. Conclusion

In this brief, we have analyzed the sampled-data consensus problem for a group of single-integrator or double-integrator agents based on the assumption that the sampling intervals are independently and identically distributed random variables. We have derived necessary and sufficient conditions in terms of the statistical properties of sampling intervals for guaranteeing mean square consensus. In some sense, the conditions with respect to sampling intervals are substantially relaxed in contrast to the previous sufficient conditions, all of which were too conservative and derived in the framework of deterministic systems. We have also provided numerical examples to demonstrate the theoretical results. Finally, we have applied the first-order model to opinion consensus problems in social networks, finding that the temporal heterogeneity of human activities impedes the opinion consensus forming.

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