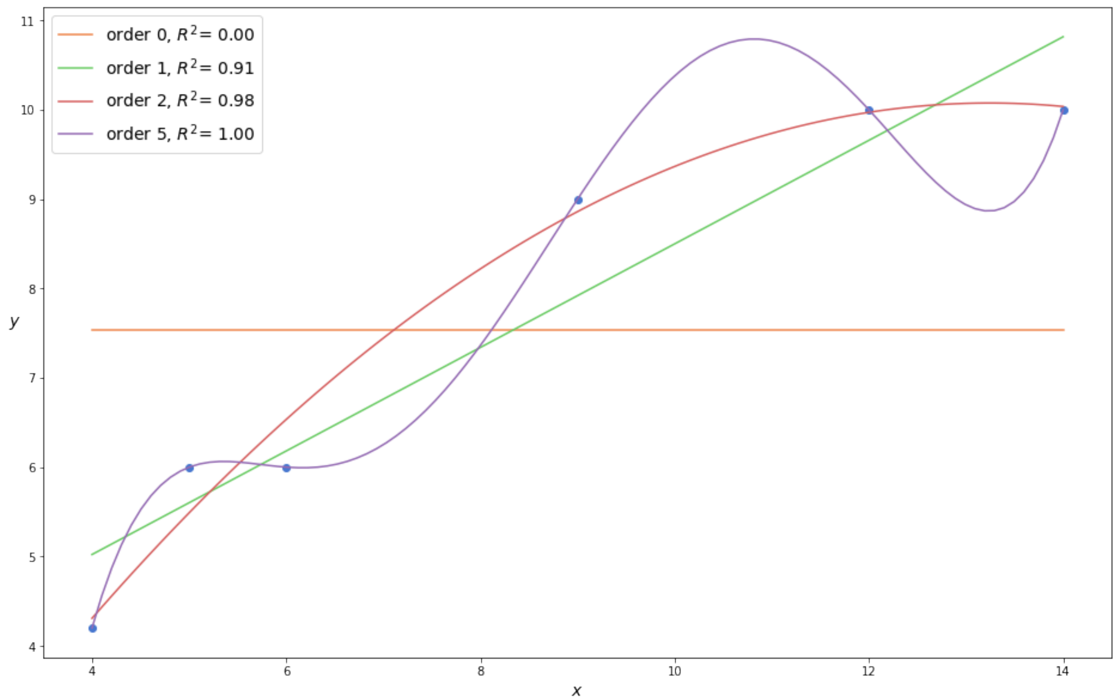
Bayesian Data Analysis Chapter 6

January 20, 2019

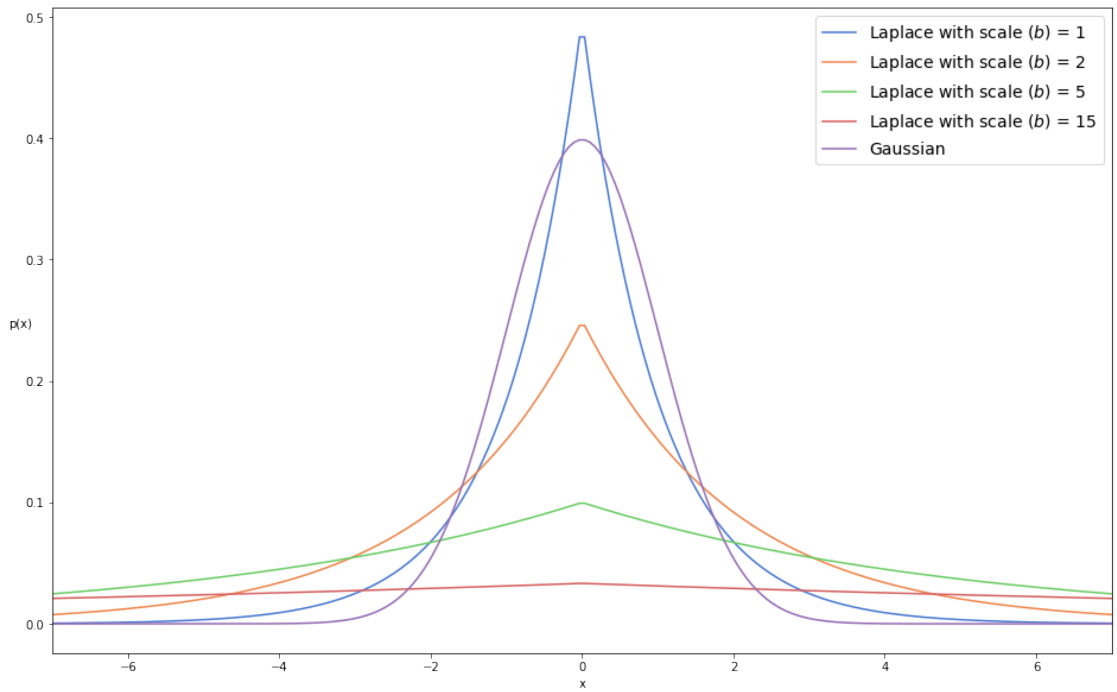
```
In [1]: %matplotlib inline
        import pymc3 as pm
        import numpy as np
        import scipy.stats as stats
        import matplotlib.pyplot as plt
        import seaborn as sns
        import matplotlib
        palette = 'muted'
        sns.set_palette(palette); sns.set_color_codes(palette)
```

1 Overfitting vs underfitting

```
In [2]: x = np.array([4.,5.,6.,9.,12, 14.])
        y = np.array([4.2, 6., 6., 9., 10, 10.])

        order = [0, 1, 2, 5]
        plt.plot(x, y, 'o')
        for i in order:
            x_n = np.linspace(x.min(), x.max(), 100)
            coeffs = np.polyfit(x, y, deg=i)
            ffit = np.polyval(coeffs, x_n)

            p = np.poly1d(coeffs)
            yhat = p(x)
            ybar = np.mean(y)
            ssreg = np.sum((yhat-ybar)**2)
            sstot = np.sum((y - ybar)**2)
            r2 = ssreg / sstot

            plt.plot(x_n, ffit, label='order {}, $R^2$= {:.2f}'.format(i, r2))

        plt.legend(loc=2, fontsize=14);
        plt.xlabel('$x$', fontsize=14);
        plt.ylabel('$y$', fontsize=14, rotation=0);
        fig = matplotlib.pyplot.gcf()
        fig.set_size_inches(16,10)
```



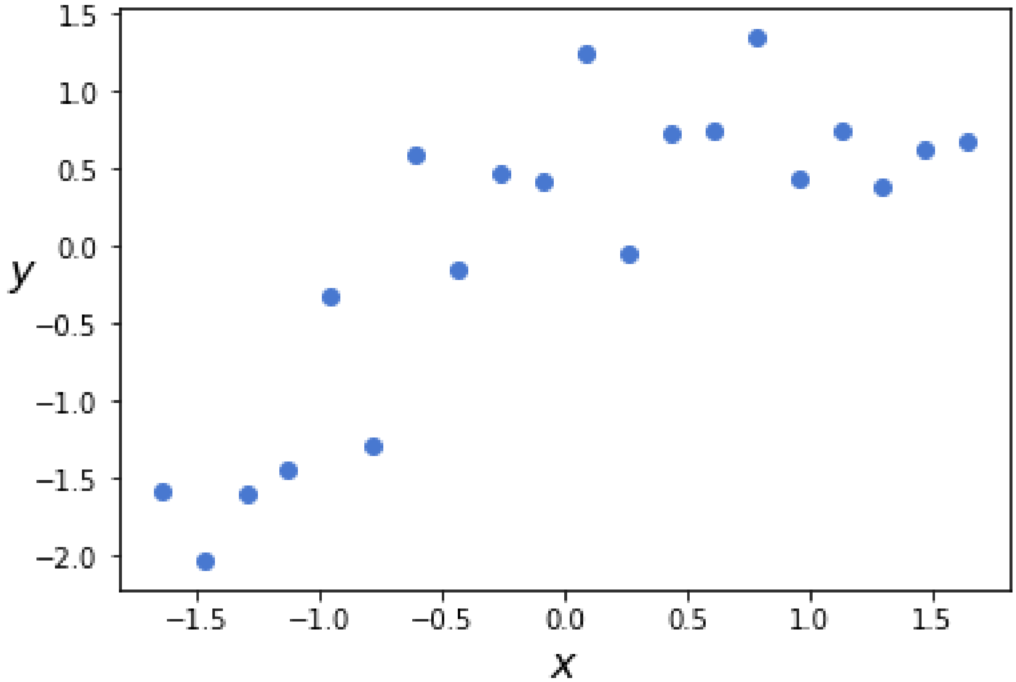
In [3]:
```
plt.figure(figsize=(8, 6))
x_values = np.linspace(-10, 10, 300)
for df in [1, 2, 5, 15]:
    distri = stats.laplace(scale=df)
    x_pdf = distri.pdf(x_values)
    plt.plot(x_values, x_pdf, label='Laplace with scale ($b$) = {}'.format(df))

x_pdf = stats.norm.pdf(x_values)
plt.plot(x_values, x_pdf, label='Gaussian')
plt.xlabel('x')
plt.ylabel('p(x)', rotation=0)
plt.legend(loc=0, fontsize=14)
plt.xlim(-7, 7);
fig = matplotlib.pyplot.gcf()
fig.set_size_inches(16,10)
```




```
In [4]: x_1 = np.array([ 10.,   8.,  13.,   9.,  11.,  14.,   6.,   4.,  12.,   7.,   5.])
        y_1 = np.array([  8.04,   6.95,   7.58,   8.81,   8.33,   9.96,   7.24,   4.26,
                10.84,   4.82,   5.68])
```

```
In [5]: np.random.seed(1)
        real_alpha = 4.25
        real_beta = [8.7, -1.2]
        data_size = 20
        noise = np.random.normal(0, 2, size=data_size)
        x_1 = np.linspace(0, 5, data_size)
        y_1 = real_alpha + real_beta[0] * x_1 + real_beta[1] * x_1**2 + noise
```

```
In [6]: order = 2#5
        x_1p = np.vstack([x_1**i for i in range(1, order+1)])
        x_1s = (x_1p - x_1p.mean(axis=1, keepdims=True))/x_1p.std(axis=1, keepdims=True)
        y_1s = (y_1 - y_1.mean())/y_1.std()
        plt.scatter(x_1s[0], y_1s);
        plt.xlabel('$x$', fontsize=16);
        plt.ylabel('$y$', fontsize=16, rotation=0);
```

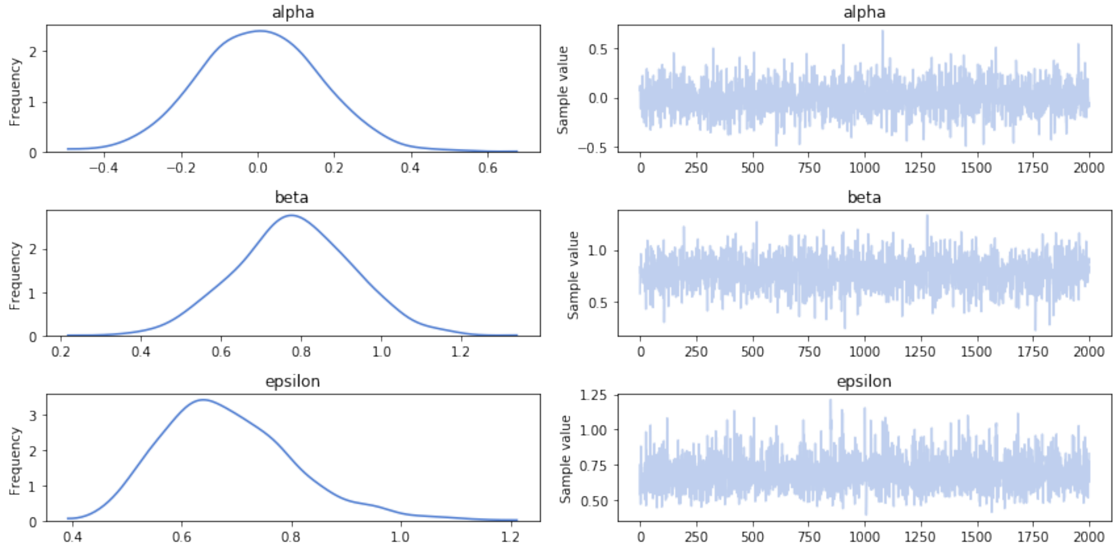


model_l: Not including polynomial term

```
In [7]: with pm.Model() as model_l:
            alpha = pm.Normal('alpha', mu=0, sd=10)
            beta = pm.Normal('beta', mu=0, sd=1)
            epsilon = pm.HalfCauchy('epsilon', 5)

            mu = alpha + beta * x_1s[0]

            y_l = pm.Normal('y_l', mu=mu, sd=epsilon, observed=y_1s)

            trace_l = pm.sample(2100,chains=1,njobs=1)
```

```
Auto-assigning NUTS sampler...
Initializing NUTS using jitter+adapt_diag...
Sequential sampling (1 chains in 1 job)
NUTS: [epsilon, beta, alpha]
100%|| 2600/2600 [00:02<00:00, 1189.52it/s]
Only one chain was sampled, this makes it impossible to run some convergence checks
```

```
In [8]: chain_l = trace_l[100:]
        pm.traceplot(chain_l);
```




```
In [9]: pm.summary(chain_l)
```

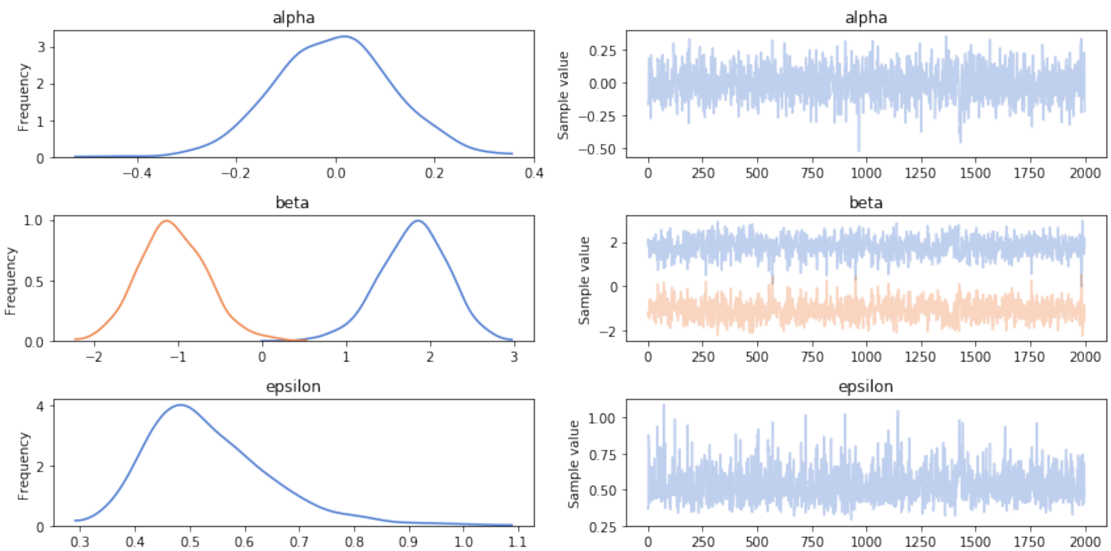
```
Out[9]:              mean        sd  mc_error   hpd_2.5  hpd_97.5
        alpha    0.002707  0.157234  0.004058 -0.297762  0.309386
        beta     0.779145  0.148678  0.003046  0.482230  1.051304
        epsilon  0.683799  0.120173  0.002663  0.459332  0.920043
```

model_p: Iincluding polynomial term

```
In [10]: with pm.Model() as model_p:
             alpha = pm.Normal('alpha', mu=0, sd=10)
             beta = pm.Normal('beta', mu=0, sd=1, shape=x_1s.shape[0])
             epsilon = pm.HalfCauchy('epsilon', 5)

             mu = alpha + pm.math.dot(beta, x_1s)

             y_l = pm.Normal('y_l', mu=mu, sd=epsilon, observed=y_1s)

             trace_p = pm.sample(2100,chains=1,njobs=1)
```

```
Auto-assigning NUTS sampler...
Initializing NUTS using jitter+adapt_diag...
Sequential sampling (1 chains in 1 job)
NUTS: [epsilon, beta, alpha]
100%|| 2600/2600 [00:04<00:00, 575.17it/s]
The acceptance probability does not match the target. It is 0.8964686391665999, but should be c
Only one chain was sampled, this makes it impossible to run some convergence checks
```

In [11]:
```
chain_p = trace_p[100:]
pm.traceplot(chain_p);
```

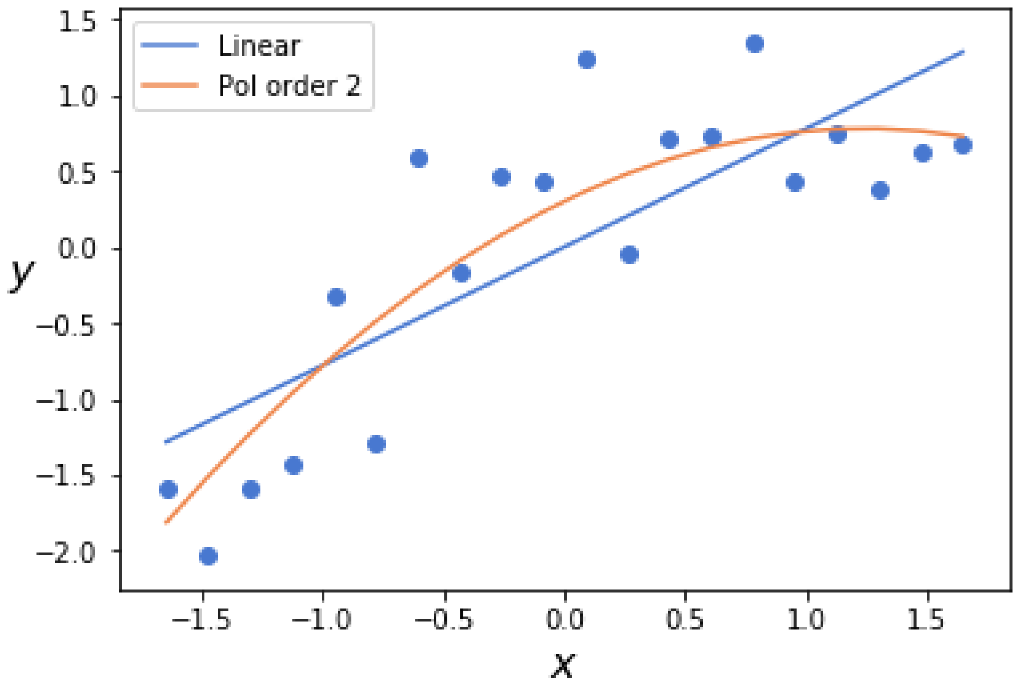


In [12]:
```
pm.summary(chain_p)
```

Out[12]:

	mean	sd	mc_error	hpd_2.5	hpd_97.5
alpha	-0.006345	0.118836	0.003146	-0.220809	0.234278
beta__0	1.796237	0.404589	0.014456	0.996769	2.578225
beta__1	-1.061327	0.403376	0.014966	-1.888992	-0.312826
epsilon	0.535186	0.113336	0.003481	0.339629	0.762821

In [13]:
```
alpha_l_post = chain_l['alpha'].mean()
betas_l_post = chain_l['beta'].mean(axis=0)
idx = np.argsort(x_1s[0])
y_l_post = alpha_l_post + betas_l_post * x_1s[0]

plt.plot(x_1s[0][idx], y_l_post[idx], label='Linear')


alpha_p_post = chain_p['alpha'].mean()
betas_p_post = chain_p['beta'].mean(axis=0)
y_p_post = alpha_p_post + np.dot(betas_p_post, x_1s)

plt.plot(x_1s[0][idx], y_p_post[idx], label='Pol order {}'.format(order))

plt.scatter(x_1s[0], y_1s)
plt.xlabel('$x$', fontsize=16)
plt.ylabel('$y$', fontsize=16, rotation=0);
plt.legend();
```



2 Information criteria

```
In [14]: model_p.name='Polynomial'
         model_l.name='Linear'
         dic_l = pm.compare({model_p: trace_p, model_l: trace_l})
         dic_l

/home/damianos/miniconda3/envs/pymc3/lib/python3.7/site-packages/pymc3/stats.py:167: FutureWarn
  return np.stack(logp)
/home/damianos/miniconda3/envs/pymc3/lib/python3.7/site-packages/pymc3/stats.py:167: FutureWarn
  return np.stack(logp)


Out[14]:              WAIC pWAIC dWAIC weight    SE   dSE var_warn
         Polynomial  32.67  2.64     0      1  4.61     0        0
         Linear       42.5  2.14  9.83      0  4.16  2.71        0

In [15]: waic_l = pm.waic(trace=trace_l, model=model_l)
         waic_l

Out[15]: WAIC_r(WAIC=42.50361731898587, WAIC_se=4.157641613844861, p_WAIC=2.1412246201899223, 

In [16]: waic_p = pm.waic(trace=trace_p, model=model_p)
         waic_p
```

Out[16]: WAIC_r(WAIC=32.67190882656928, WAIC_se=4.610510226210436, p_WAIC=2.6390976063746674,

In [17]: loo_l = pm.loo(trace=trace_l, model=model_l)
loo_l

```
loo_l = pm.loo(trace=trace_l, model=model_l)
loo_l
```

Out[17]: LOO_r(LOO=42.62390890178155, LOO_se=4.1873713733381175, p_LOO=2.2013704115877637, sha

In [18]:

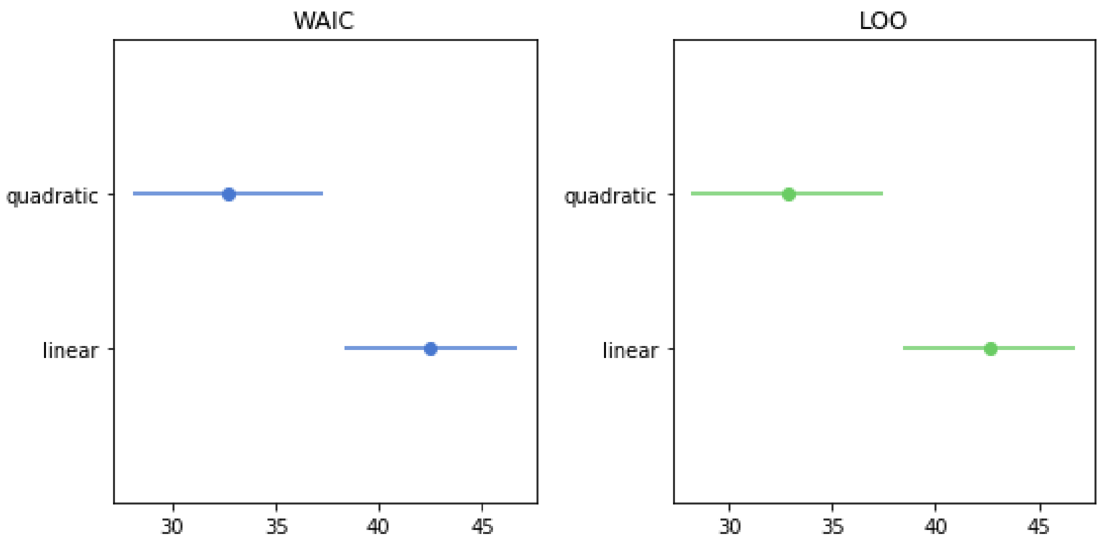
```
loo_p = pm.loo(trace=trace_p, model=model_p)
loo_p
```

Out[18]: LOO_r(LOO=32.83617091380626, LOO_se=4.654341727757128, p_LOO=2.721228649993158, shape

2.0.1 Lower is better

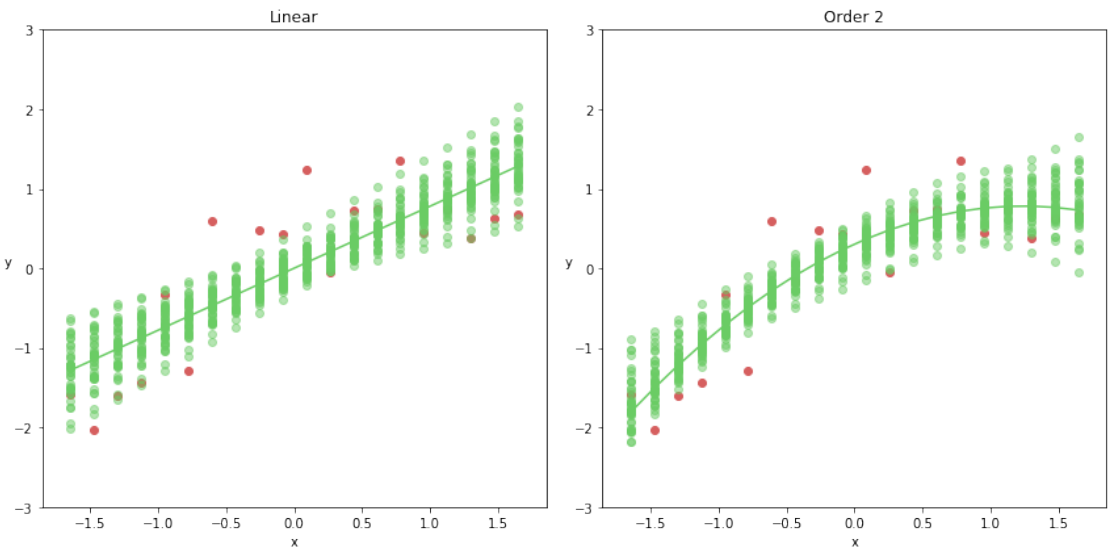
In [19]:

```
plt.figure(figsize=(8, 4))
plt.subplot(121)
for idx, ic in enumerate((waic_l, waic_p)):
    plt.errorbar(ic[0], idx, xerr=ic[1], fmt='bo')
plt.title('WAIC')
plt.yticks([0, 1], ['linear', 'quadratic'])
plt.ylim(-1, 2)

plt.subplot(122)
for idx, ic in enumerate((loo_l, loo_p)):
    plt.errorbar(ic[0], idx, xerr=ic[1], fmt='go')
plt.title('LOO')
plt.yticks([0, 1], ['linear', 'quadratic'])
plt.ylim(-1, 2)

plt.tight_layout()
```



2.1 Posterior predictive checks

In [20]:
```
plt.figure(figsize=(12,6))
plt.subplot(121)
plt.scatter(x_1s[0], y_1s, c='r');
plt.ylim(-3, 3)
plt.xlabel('x')
plt.ylabel('y', rotation=0)
plt.title('Linear')


for i in range(0, len(chain_l['alpha']), 50):
    plt.scatter(x_1s[0], chain_l['alpha'][i] + chain_l['beta'][i]*x_1s[0], c='g',
                edgecolors='g', alpha=0.5);
plt.plot(x_1s[0], chain_l['alpha'].mean() + chain_l['beta'].mean()*x_1s[0], c='g', al


plt.subplot(122)
plt.scatter(x_1s[0], y_1s, c='r');
plt.ylim(-3, 3)
plt.xlabel('x')
plt.ylabel('y', rotation=0)
plt.title('Order {}'.format(order))
for i in range(0, len(chain_p['alpha']), 50):
    plt.scatter(x_1s[0], chain_p['alpha'][i] + np.dot(chain_p['beta'][i], x_1s), c='g
                edgecolors='g', alpha=0.5)
idx = np.argsort(x_1)
plt.plot(x_1s[0][idx], alpha_p_post + np.dot(betas_p_post, x_1s)[idx], c='g', alpha=1)

plt.tight_layout()
```



3 Bayes factors

In [21]:
```
coins = 30 # 300
heads = 9 # 90
y = np.repeat([0, 1], [coins-heads, heads])
print('These are the coin tosses we are modelling:\n {}'.format(y))
```

```
These are the coin tosses we are modelling:
 [0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1]
```

In [22]:
```
with pm.Model() as model_BF:
    p = np.array([0.5, 0.5])
    # model_index is a stochastic variable governed by the Categorical distribution
    # returning 0 or 1 for each model respectively
    model_index = pm.Categorical('model_index', p=p)
    # there are two models with different priors
    # one alpha=4,beta=8 and another alpha=8,beta=4
    m_0 = (4, 8)
    m_1 = (8, 4)
    # m returns the alpha,betas based on whether model index ==0
    # or not (`pm.math.eq(model_index, 0)`)
    m = pm.math.switch(pm.math.eq(model_index, 0), m_0, m_1)
    # prior on theta of the Bernouli
    theta = pm.Beta('theta', m[0], m[1])
    # likelihood, y-> 1 heads, 0 tails
    y_pred = pm.Bernoulli('y', theta, observed=y)

    trace_BF = pm.sample(5000,chains=1,njobs=1)
```

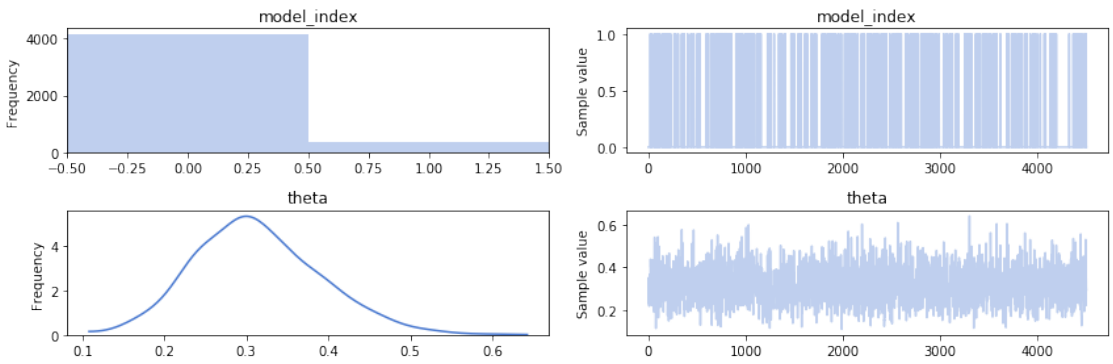
```
Sequential sampling (1 chains in 1 job)
CompoundStep
>BinaryGibbsMetropolis: [model_index]
>NUTS: [theta]
100%|| 5500/5500 [00:03<00:00, 1774.92it/s]
Only one chain was sampled, this makes it impossible to run some convergence checks
```

4 $BayesFactor = \frac{p(y|M_0)}{p(y|M_1)} = \frac{p(M_0|y) \times p(M_1)}{p(M_1|y) \times p(M_0)}$

In [23]:
```
chain_BF = trace_BF[500:]
pm.traceplot(chain_BF)
pM1 = chain_BF['model_index'].mean()
pM0 = 1 - pM1
print('Prior of selecting model 0 = {:.2f}'.format(p[0]))
```

```
print('Prior of selecting model 1 = {:.2f}'.format(p[1]))
print('Posterior mean of selecting model 0 = {:.2f}'.format(pM1))
print('Posterior mean of selecting model 1 = {:.2f}'.format(pM0))
print('Bayes factor = {:.2f}, thus model 0 is more likely'.format((pM0/pM1)*(p[1]/p[0
```

```
Prior of selecting model 0 = 0.50
Prior of selecting model 1 = 0.50
Posterior mean of selecting model 0 = 0.08
Posterior mean of selecting model 1 = 0.92
Bayes factor = 11.43, thus model 0 is more likely
```



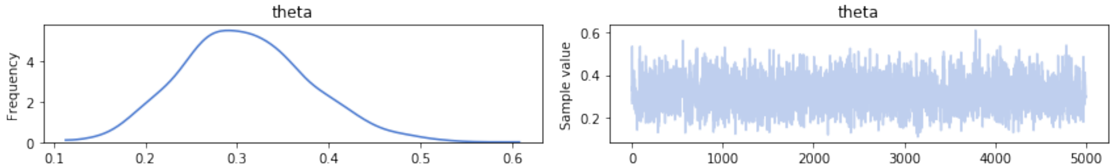
4.0.1 Also model 0 $\alpha = 4, \beta = 8$ is more compatible with the observations of mostly tails and $\theta < 0.5$

4.1 Comparison of models using Information Criteria

```
In [24]: with pm.Model() as model_BF_0:
             theta = pm.Beta('theta', 4, 8)
             y_pred = pm.Bernoulli('y', theta, observed=y)

             trace_BF_0 = pm.sample(5000,chains=1,njobs=1)
```

```
Auto-assigning NUTS sampler...
Initializing NUTS using jitter+adapt_diag...
Sequential sampling (1 chains in 1 job)
NUTS: [theta]
100%|| 5500/5500 [00:02<00:00, 2580.73it/s]
Only one chain was sampled, this makes it impossible to run some convergence checks
```

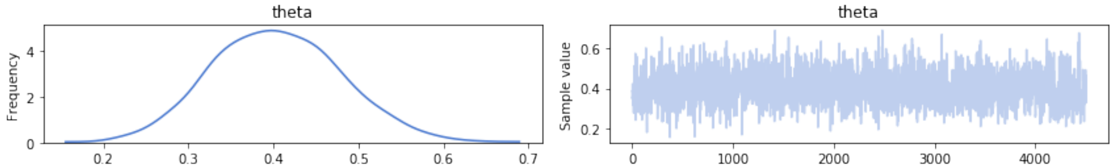
```
In [25]: chain_BF_0 = trace_BF_0[500:]
         pm.traceplot(trace_BF_0);
```




```
In [26]: with pm.Model() as model_BF_1:
             theta = pm.Beta('theta', 8, 4)
             y_pred = pm.Bernoulli('y', theta, observed=y)

             trace_BF_1 = pm.sample(5000,chains=1,njobs=1)
```

```
Auto-assigning NUTS sampler...
Initializing NUTS using jitter+adapt_diag...
Sequential sampling (1 chains in 1 job)
NUTS: [theta]
100%|| 5500/5500 [00:02<00:00, 2729.76it/s]
Only one chain was sampled, this makes it impossible to run some convergence checks
```

```
In [27]: chain_BF_1 = trace_BF_1[500:]
         pm.traceplot(chain_BF_1);
```




```
In [28]: waic_0 = pm.waic(chain_BF_0, model_BF_0)
         waic_0
```

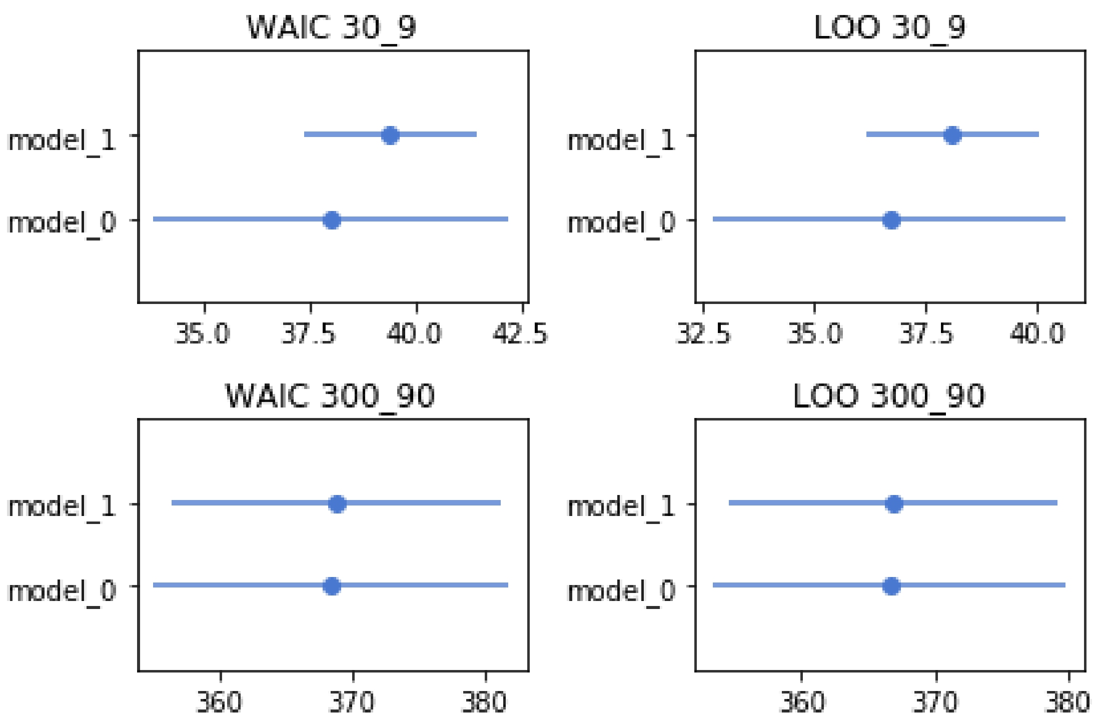
```
/home/damianos/miniconda3/envs/pymc3/lib/python3.7/site-packages/pymc3/stats.py:167: FutureWarn
  return np.stack(logp)
```

```
Out[28]: WAIC_r(WAIC=38.09485595294723, WAIC_se=4.250210218039174, p_WAIC=0.7153551590167374,
```

```
In [29]: waic_1 = pm.waic(chain_BF_1, model_BF_1)
         waic_1
```

```
Out[29]: WAIC_r(WAIC=39.45664158585026, WAIC_se=2.0465734755680027, p_WAIC=0.6998503578974625,
```

```
In [30]: loo_0 = pm.loo(chain_BF_0, model_BF_0)
         loo_0

Out[30]: LOO_r(LOO=38.096185450169244, LOO_se=4.250485469459661, p_LOO=0.7160199076277465, sha

In [31]: loo_1 = pm.loo(chain_BF_1, model_BF_1)
         loo_1

Out[31]: LOO_r(LOO=39.45818767404322, LOO_se=2.0467180085626864, p_LOO=0.7006234019939441, sha

In [32]: est = [((38.02, 4.17), (39.41, 2.04)), ((36.69, 3.96), (38.09, 1.94)),
                ((368.41, 13.40), (368.76, 12.48)) , ((366.61, 13.31), (366.87, 12.34))]

         title = ['WAIC 30_9', 'LOO 30_9', 'WAIC 300_90', 'LOO 300_90']

         for i in range(4):
             plt.subplot(2,2,i+1)
             for idx, ic in enumerate(est[i]):
                 plt.errorbar(ic[0], idx, xerr=ic[1], fmt='bo')
             plt.title(title[i])
             plt.yticks([0, 1], ['model_0', 'model_1'])
             plt.ylim(-1, 2)

         plt.tight_layout()
```



4.1.1 Bayes factors are sensitive to the selection of priors and this selection becomes less relevant as we increase the data. As you can see from using the Information Criteria there isn't much of a difference between the models, and these differences are going to become less and less as we increase the number of data. This sensitivity of Bayes factors to the selection of priors makes people not want to use them as much.

```
In [33]: import sys, IPython, scipy, matplotlib, platform
         print("This notebook was created on a %s computer running %s and using:\nPython %s\nIP
```

```
This notebook was created on a x86_64 computer running debian buster/sid and using:
Python 3.7.2
IPython 7.2.0
PyMC3 3.6
NumPy 1.16.0
SciPy 1.2.0
Matplotlib 3.0.2
Seaborn 0.9.0


/home/damianos/miniconda3/envs/pymc3/lib/python3.7/site-packages/ipykernel_launcher.py:2: Depre
```