Quantum Computing with Very Noisy Gates

Produced with pdflatex and xfig

- The C_4/C_6 architecture.
- Performance data from simulation.
- Resource requirements.

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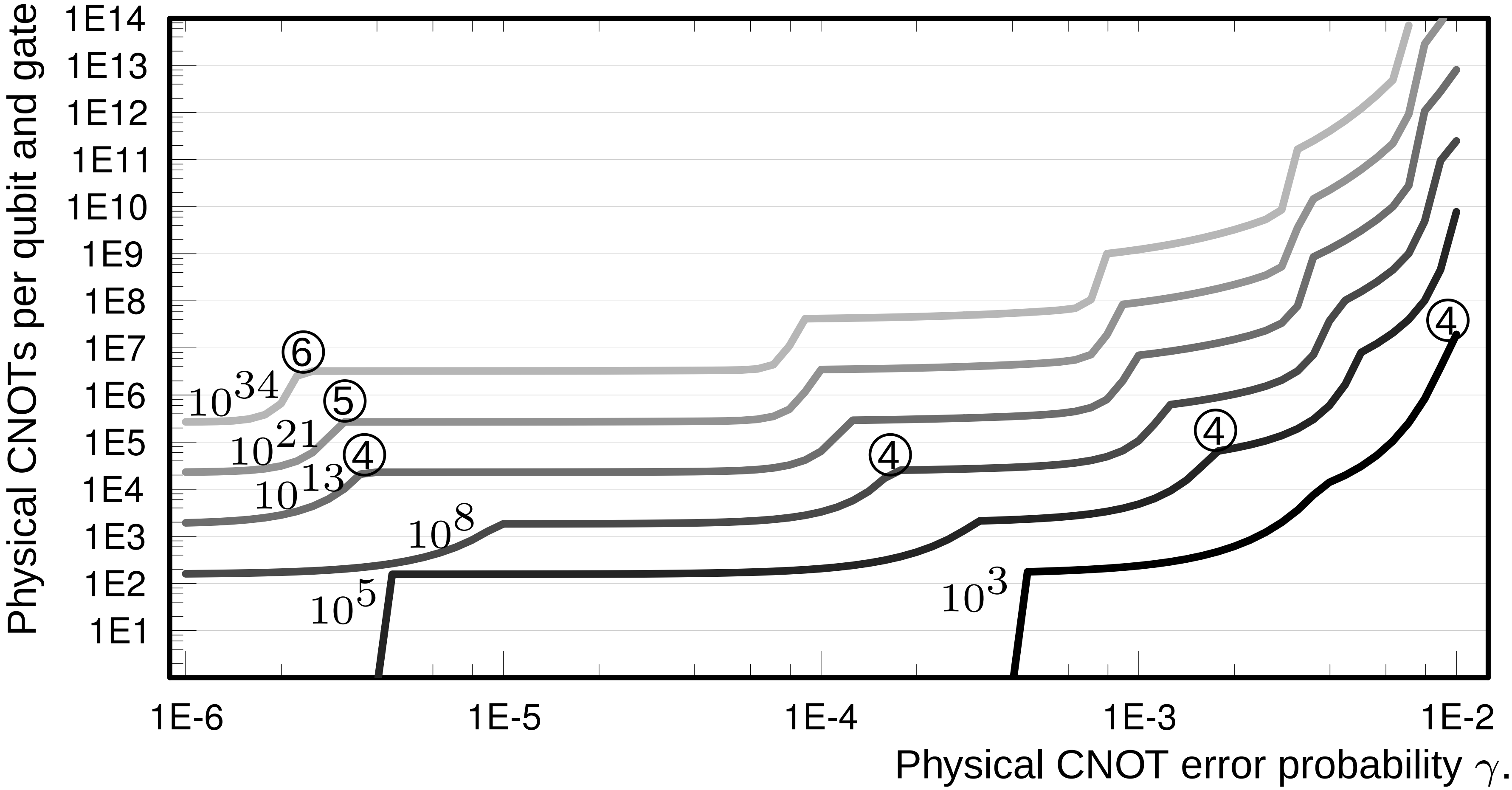
Typical Resource Requirements



- Resource requirements for the C_4/C_6-architecture and different computation sizes (by simulation and modelling).

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- Partial decoding for state preparation.
- Fault-tolerant implementation of Clifford gates $+ \delta$ suffices.
- Evidence that depolarizing errors $> 3\%$ per gate are ok.

The [[4,2,2]]-Code C_4

- A $[[4,2,2]]$ code: Check ops: $[XXXX], [ZZZZ]$. See: Pauli products.

  Logical ops: $X_L = [XXII], Z_L = [ZIZI]$

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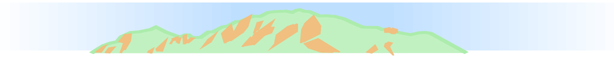

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- State space representation: Logical states

$$\underbrace{|a\rangle_{\mathsf{L}}}_{\text{logical qubit L}} \quad \overbrace{|b\rangle_{\mathsf{S}}}^{\text{logical qubit S}} \quad \underbrace{|s_1 s_2\rangle_{\mathsf{Y}}}_{\text{syndrome bits}}$$

a, b, s_1, s_2 are eigenvalues labels.
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- Some encoded states:

$$\begin{aligned}
|0000\rangle_{\text{LSY}} &= \frac{1}{\sqrt{2}}\left(|0000\rangle_{1234} + |1111\rangle_{1234}\right) \\
|{+}{+}00\rangle_{\text{LSY}} &= \frac{1}{\sqrt{2}}\left(|{+}{+}{+}{+}\rangle_{1234} + |{-}{-}{-}{-}\rangle_{1234}\right) \\
|{+}000\rangle_{\text{LSY}} &= \frac{1}{2}\left(|00\rangle_{12} + |11\rangle_{12}\right)\left(|00\rangle_{34} + |11\rangle_{34}\right)
\end{aligned}$$

The [[3,1,2]]$_4$-Code C_6

- C_4 encodes *qubit pairs*.
- C_6 encodes one qubit pair in three.
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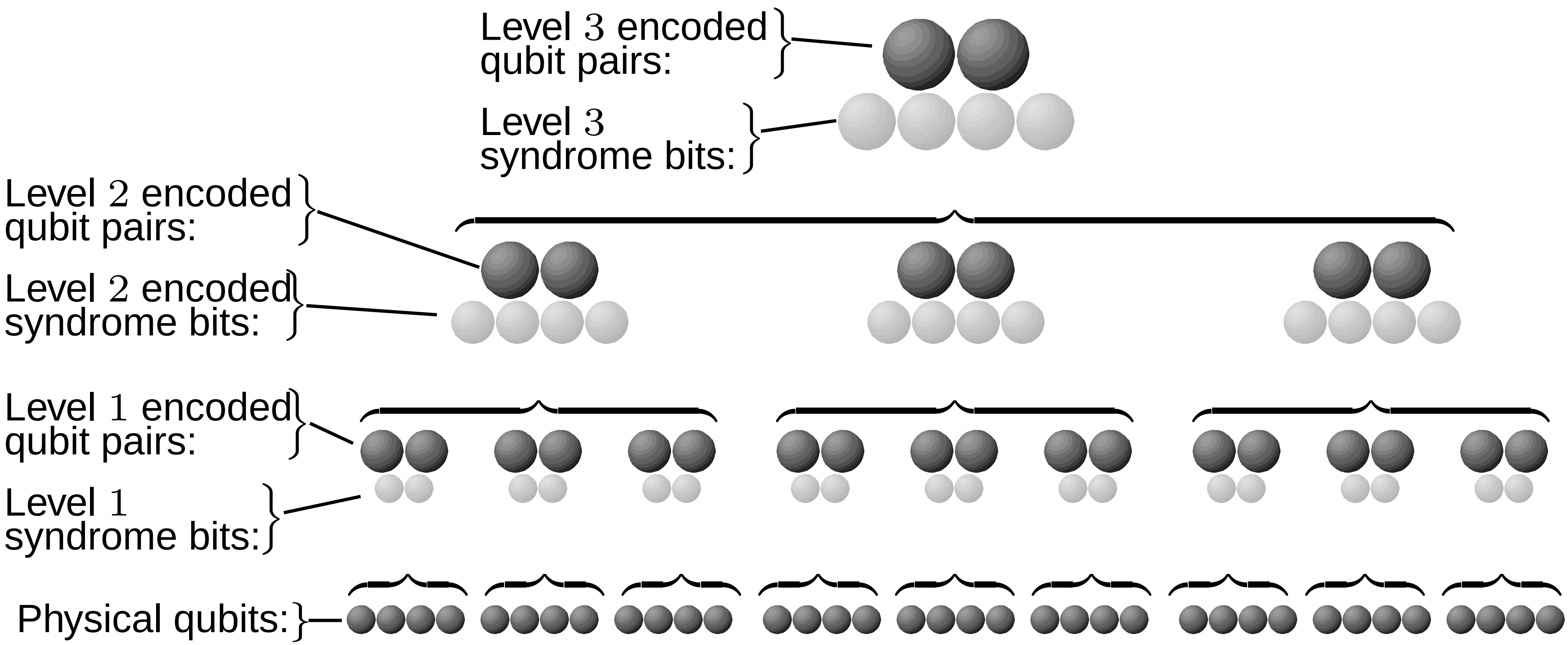
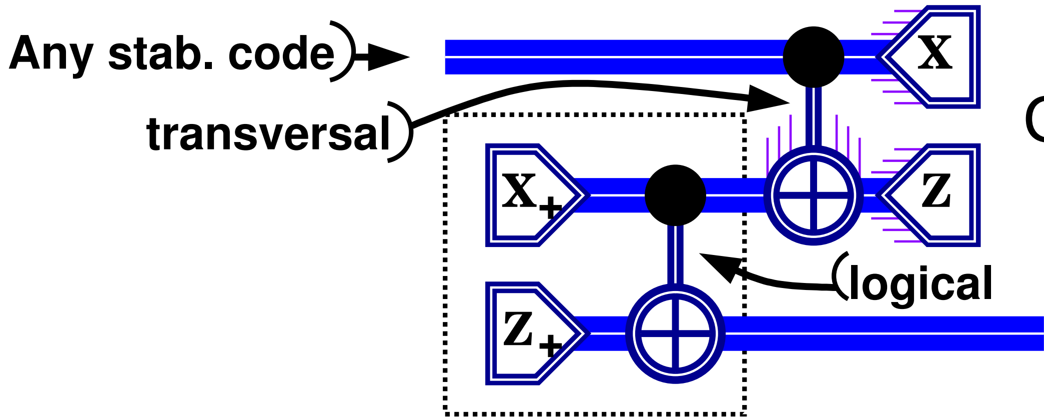
- Encoded states include:
  Local modifications of generalized GHZ states.

C_4/C_6 concatenation hierarchy.



Error-correcting Teleportation

- Syndrome measurement from qubit-wise teleportation.



Complete syndrome.

Fresh qubits.

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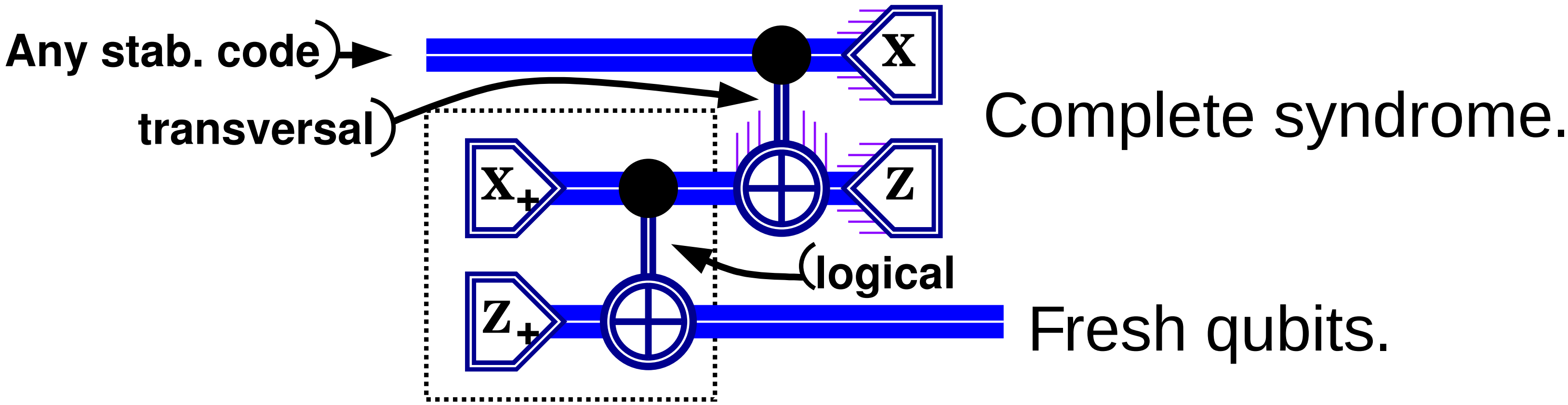

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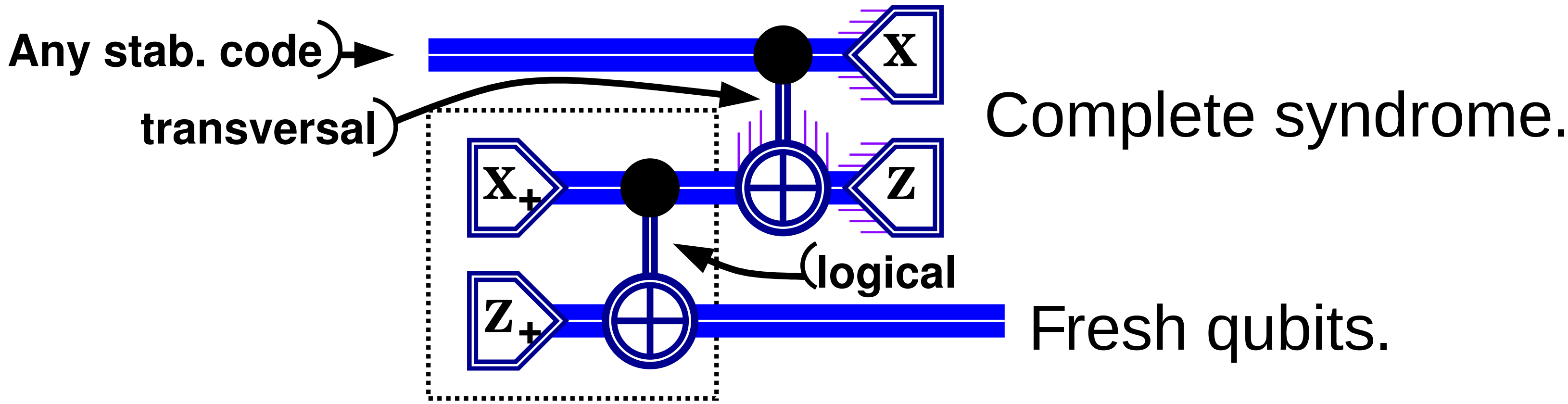

- Syndrome → error detection, correction, tracking.
- Use *Pauli frame* to avoid explicit correction gates.

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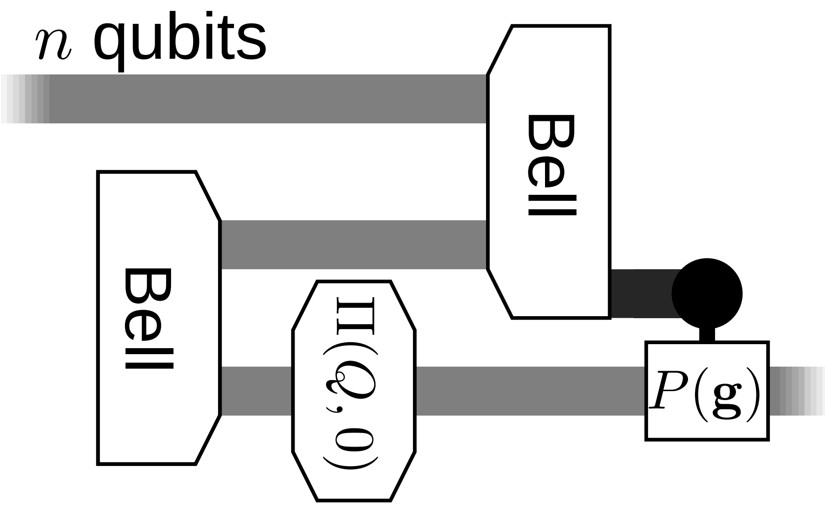

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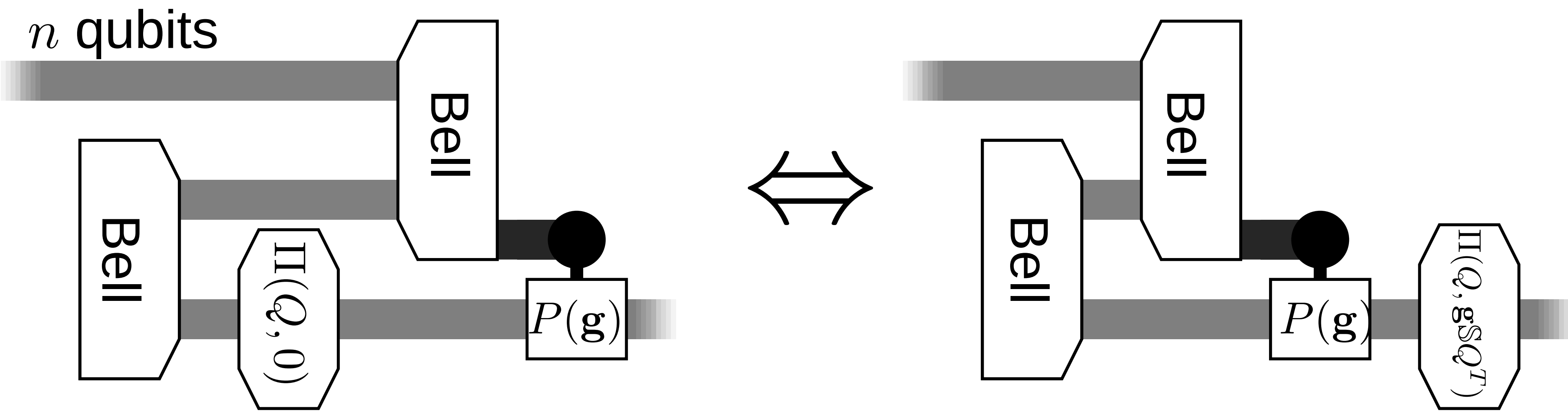

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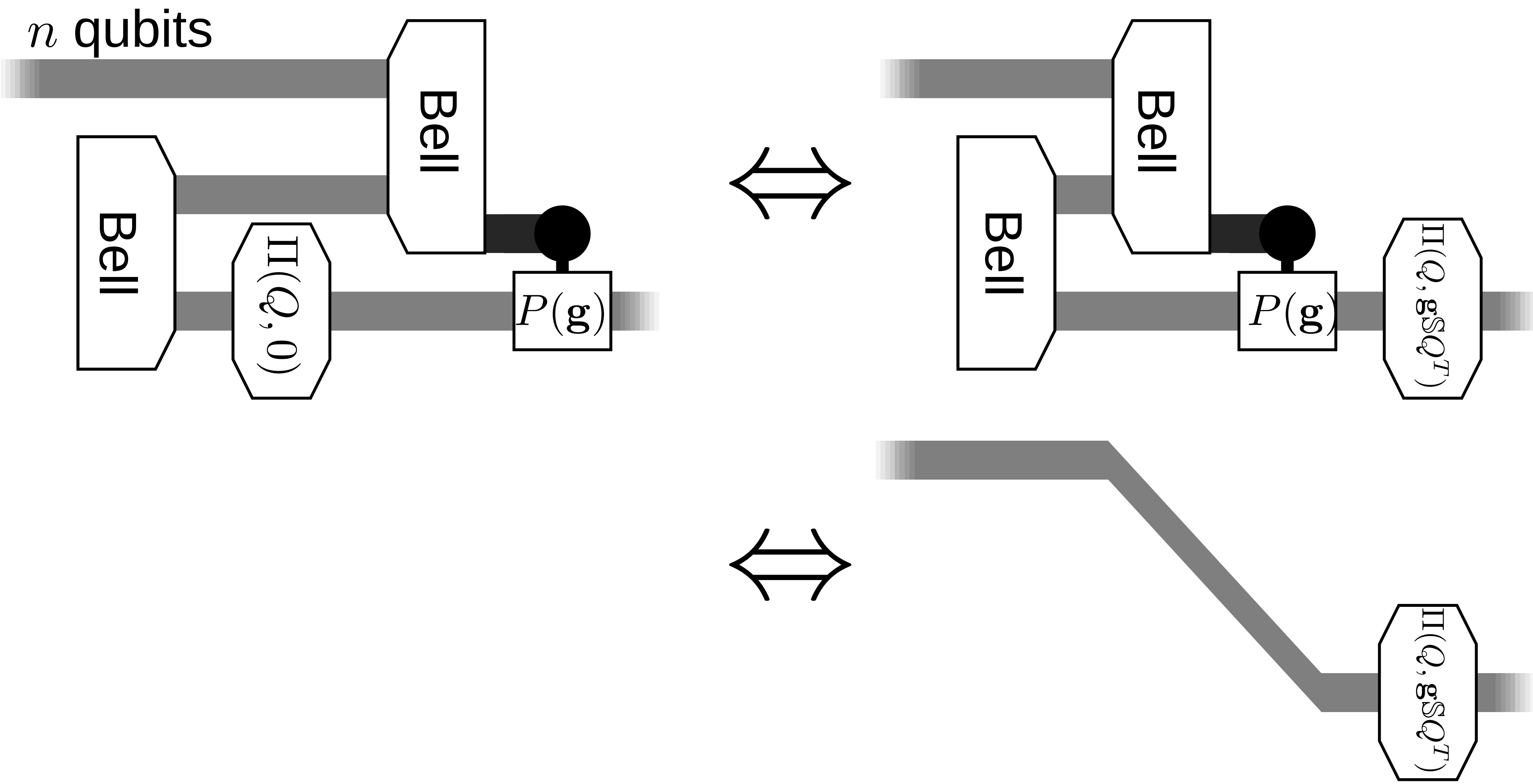
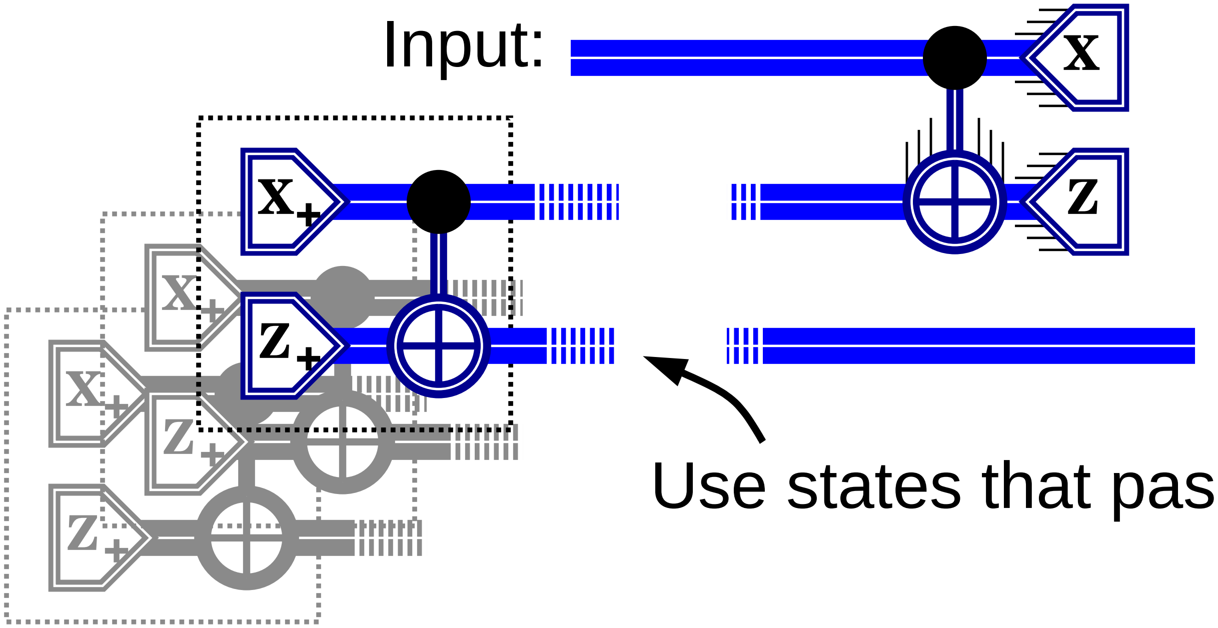


Post-selected State Preparation



Use states that pass all error checks.

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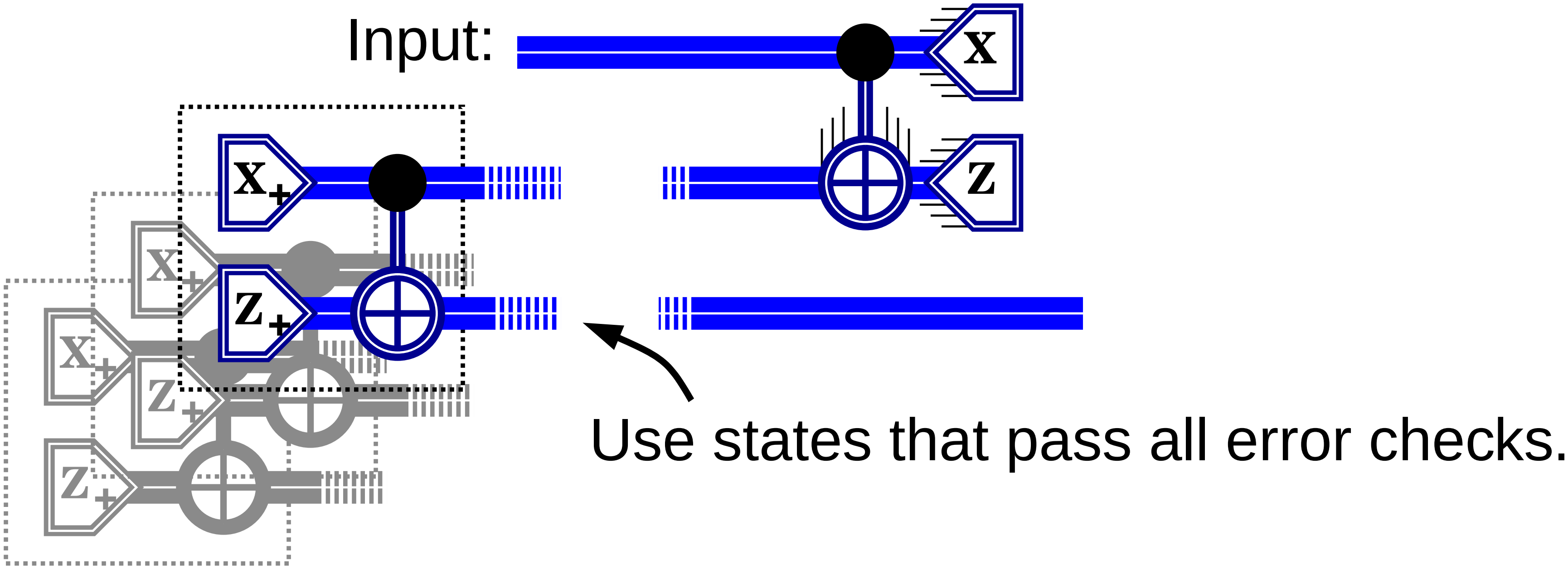

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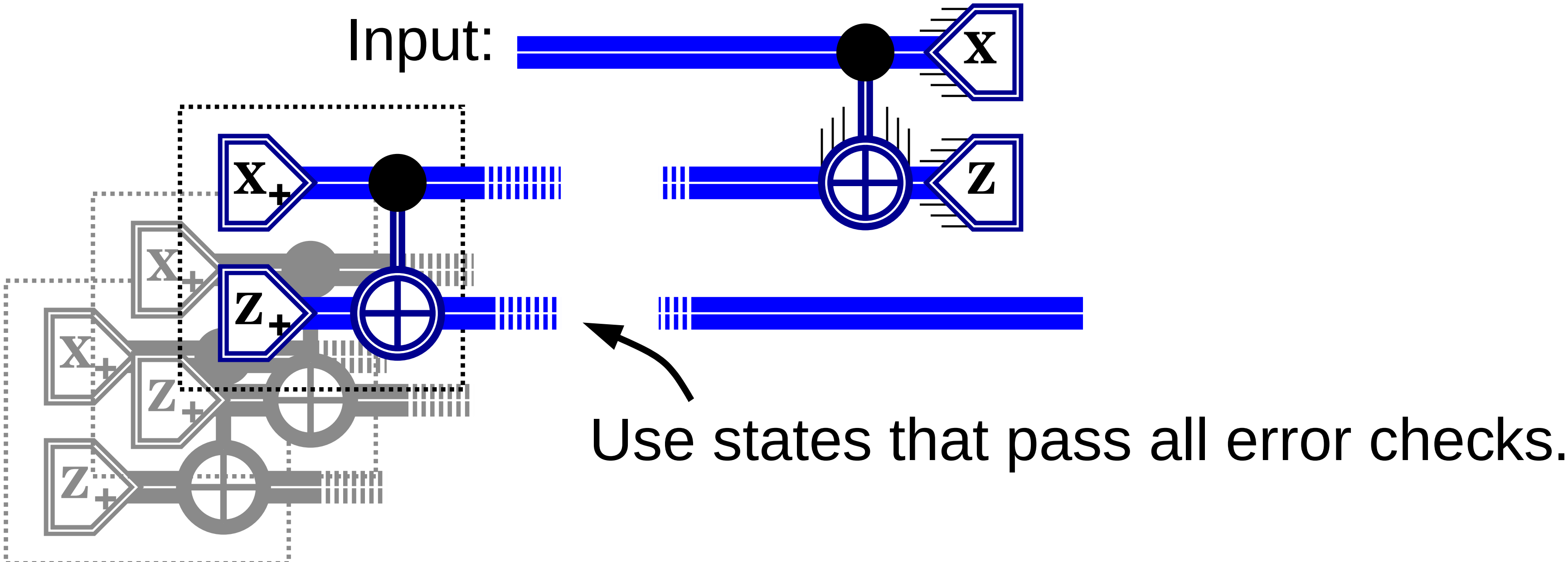

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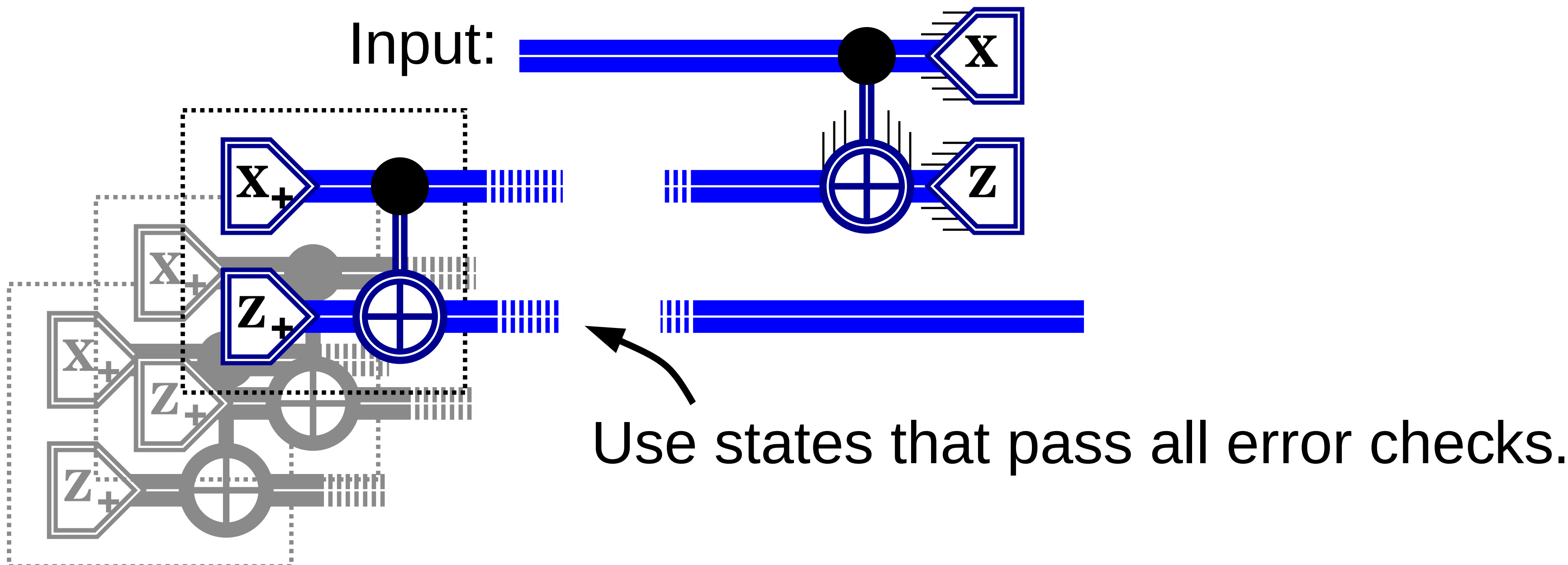

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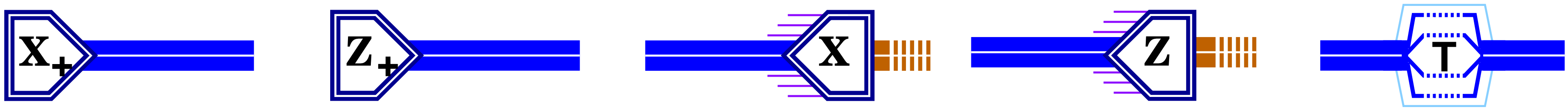
- Can use parallel state preparation factories.

C_4 Bell-State Preparation



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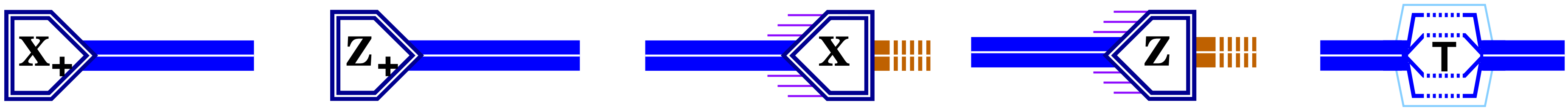

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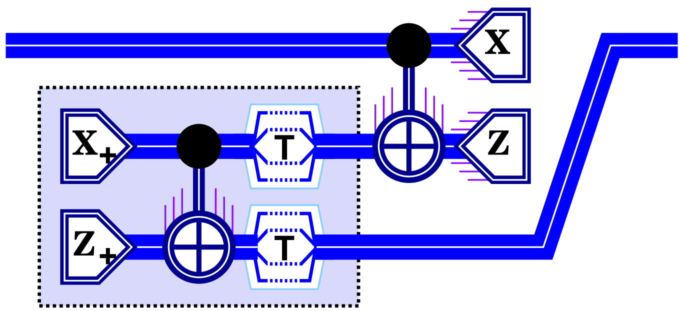

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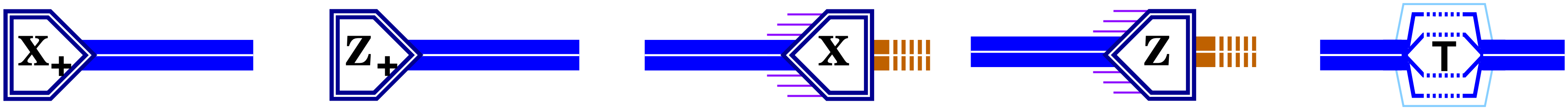

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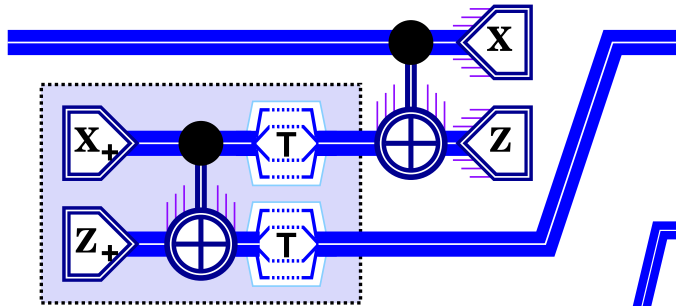

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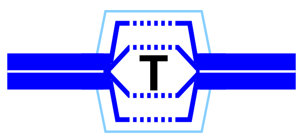

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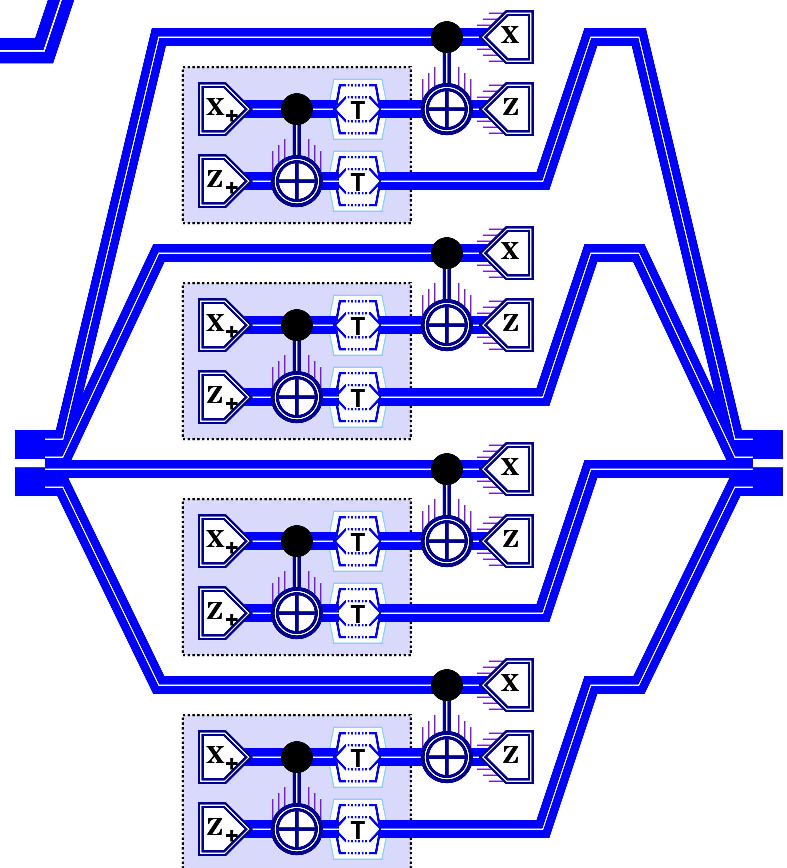

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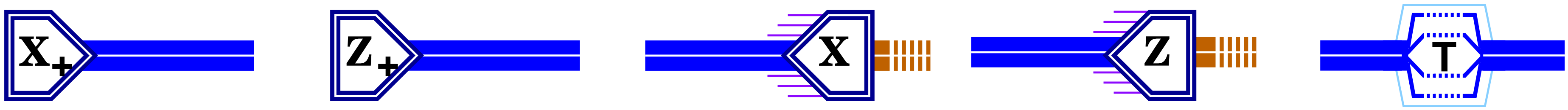

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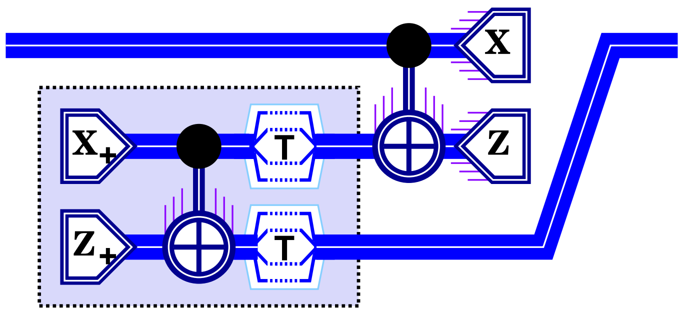

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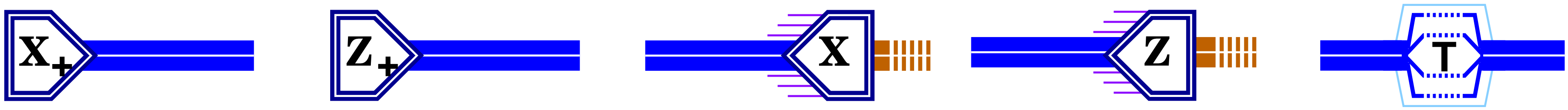

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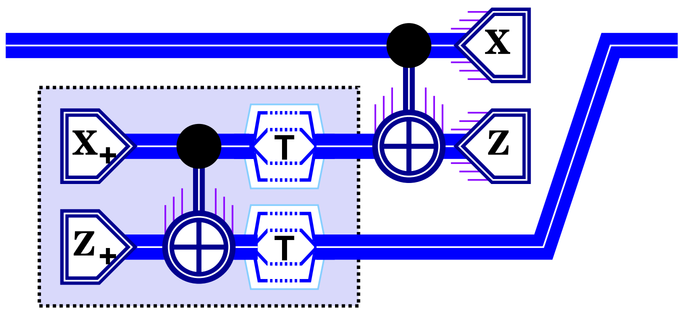

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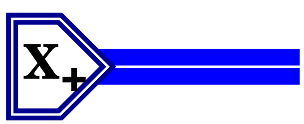

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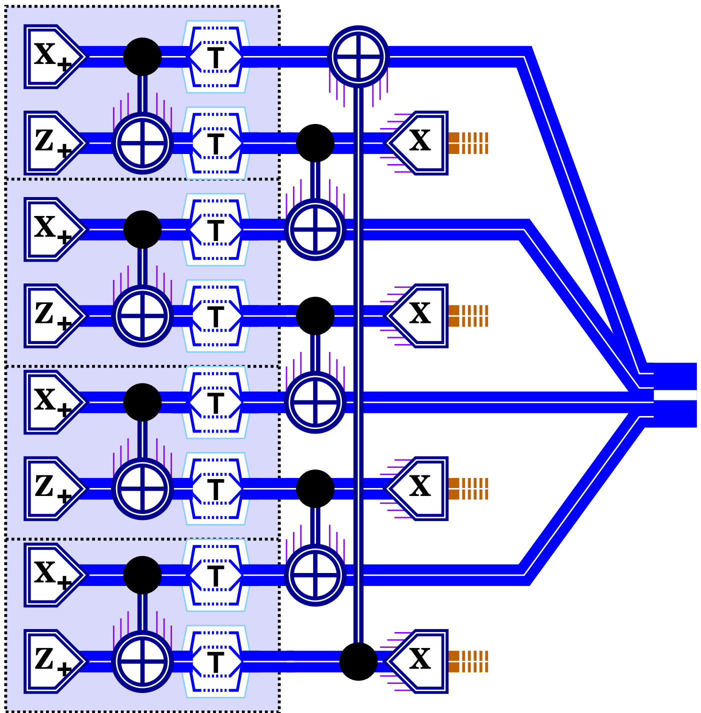

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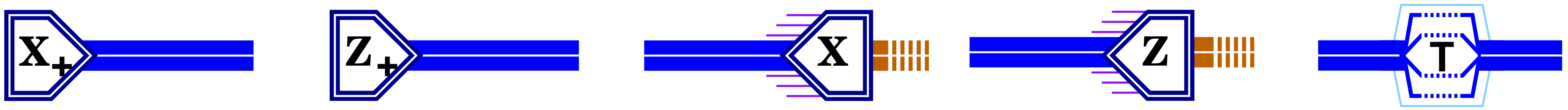

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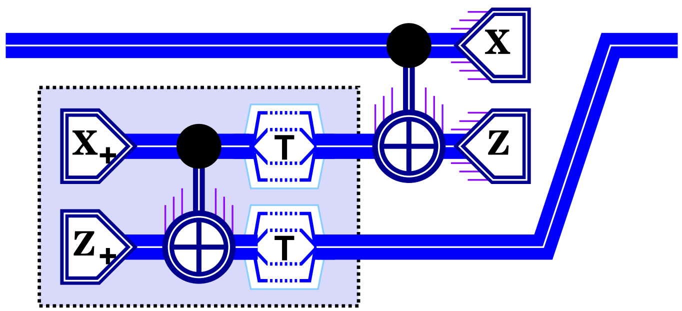

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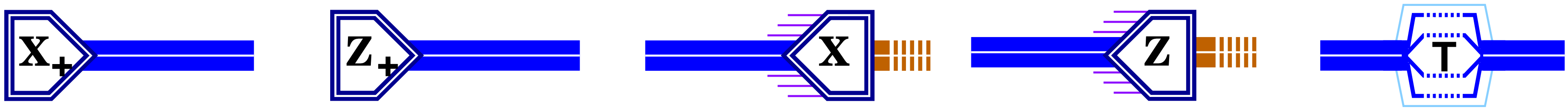

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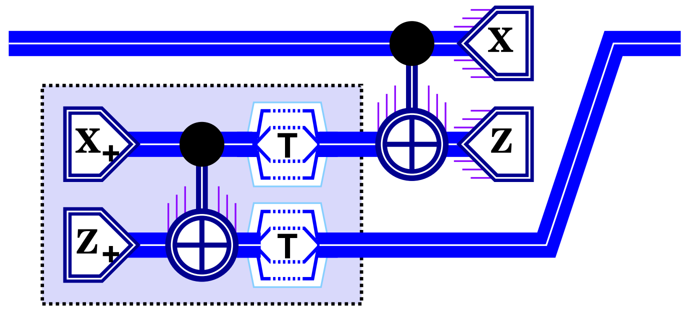

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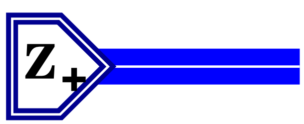

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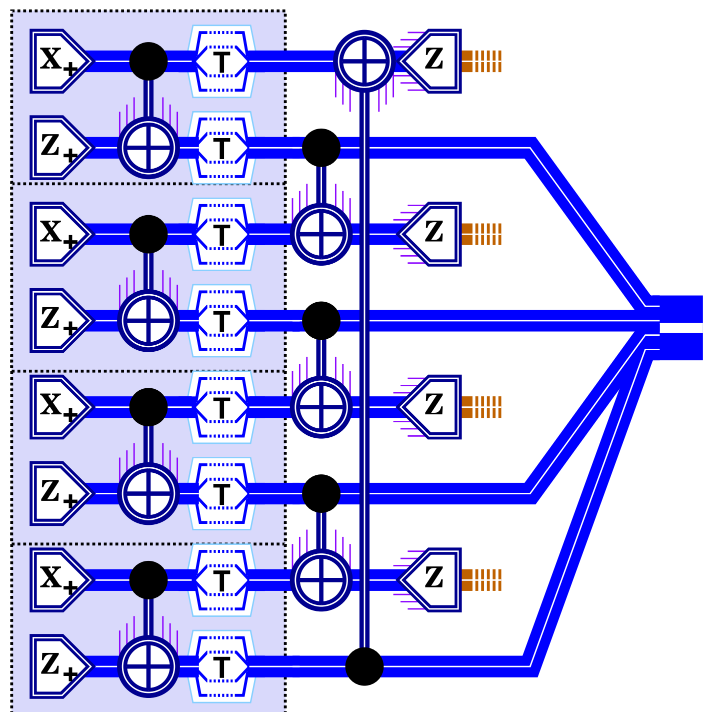

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  ...Nearly: Use postselected f.-t. to prepare key states.

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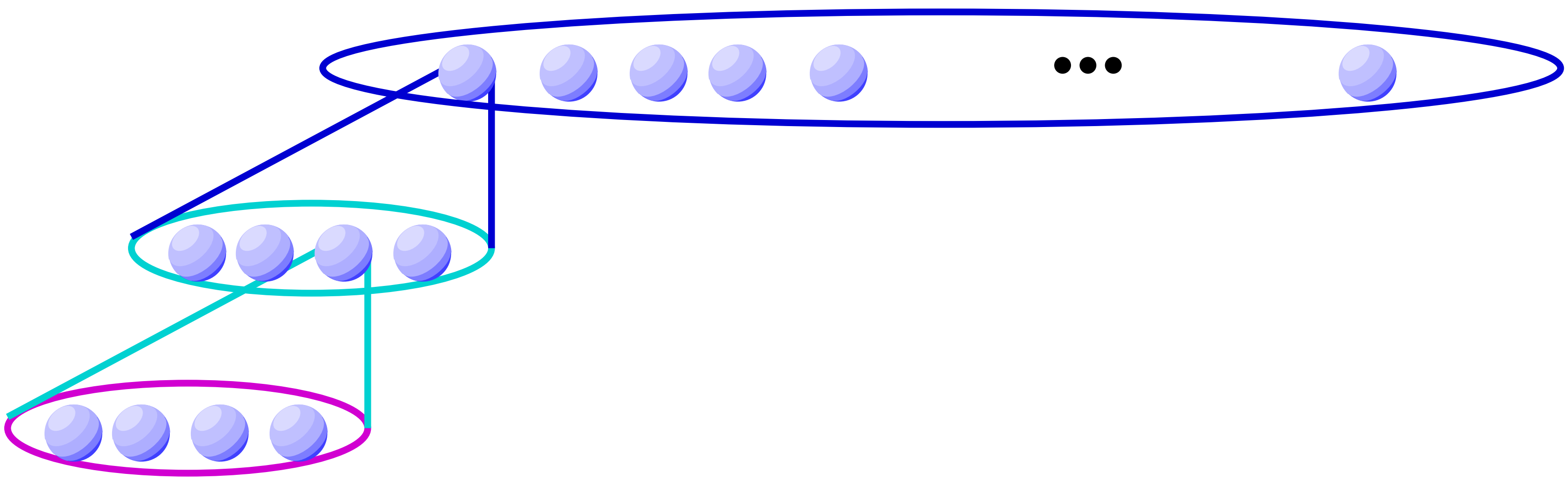

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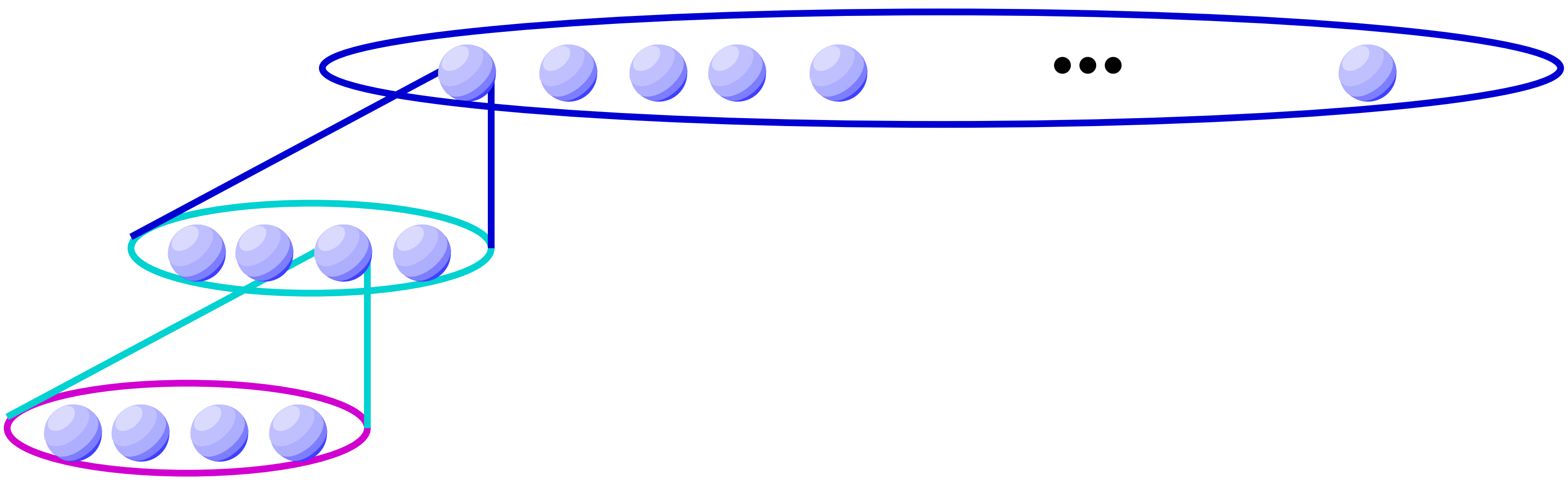

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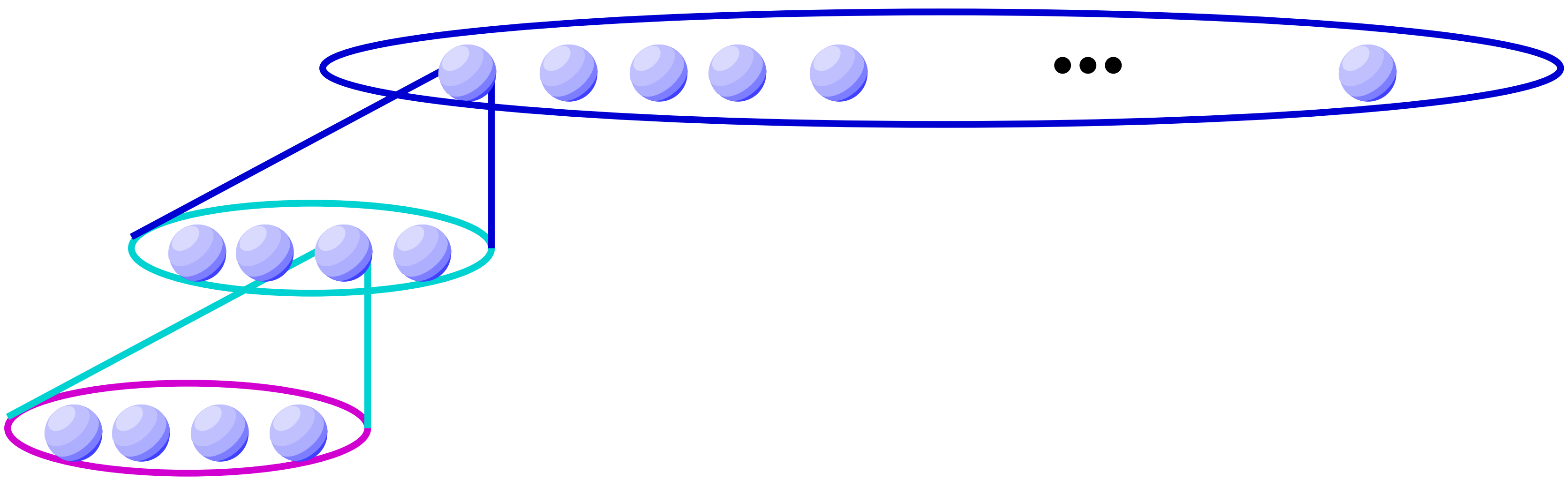

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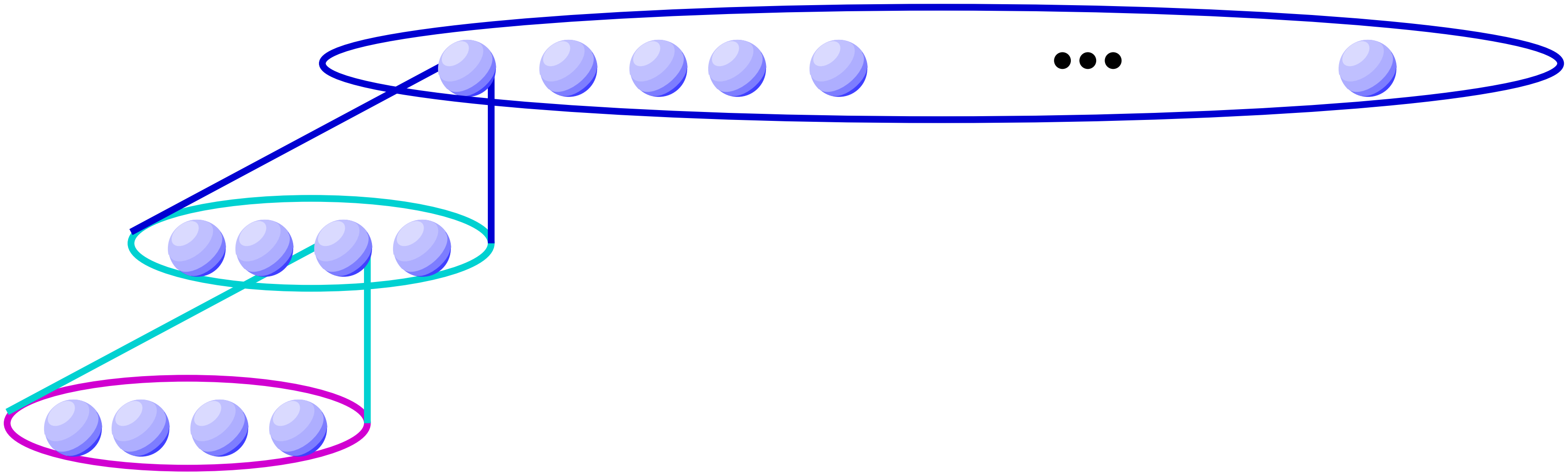

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$+\text{“}\delta\text{”}$: $|\pi/8\rangle$ purification using good $\mathcal{CSS} + \text{”}\epsilon\text{”}$
Bravyi&Kitaev (2004) [15], Knill (2004) [16]

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$$\text{QC} = \overbrace{\mathcal{CSS} + \underbrace{\text{``}\epsilon\text{''}}_{H}}^{\subseteq\mathcal{N}} + \underbrace{\text{``}\delta\text{''}}_{|\pi/8\rangle}.$$

- A fault-tolerant computation strategy:
  1. Implement a fault tolerant CSS computer, i.e. arbitrarily accurate logical $\mathcal{CSS}$ with feedforward.
  2. $+\text{``}\epsilon\text{''} + \text{``}\delta\text{''}\ldots$

  $+\text{``}\delta\text{''}$: $|\pi/8\rangle$ purification using good $\mathcal{CSS} + \text{''}\epsilon\text{''}$
  Bravyi&Kitaev (2004) [15], Knill (2004) [16]

- F.-t. $\mathcal{CSS}$ and ($|\pi/8\rangle$ error) $\leq$ ($|0\rangle$, $|+\rangle$ error) $\Rightarrow$ f.-t. QC?

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    Prepared state is $\{(1-e_p) : |\mathfrak{o}\rangle, e_p : \sigma_x|\mathfrak{o}\rangle\}$.
  - Measurement with noise:
    $|\psi\rangle \rightarrow \{(1-e_m) : \mathbb{1}, e_m : \sigma_x\}$ before σ_z measurement.
  - No operation (memory) with noise:
    $\{(1-e_n) : \mathbb{1}, e_n/3 : \sigma_x, e_n/3 : \sigma_y, e_n/3 : \sigma_z\}H$.
  - Hadamard with noise:
    $\{(1-e_h) : \mathbb{1}, e_h/3 : \sigma_x, e_h/3 : \sigma_y, e_h/3 : \sigma_z\}H$.
  - Cnot with noise:
    $\{(1-e_c) : \mathbb{1}, e_c/15 : {\sigma_x}^{(1)}, \ldots, e_c/15 : {\sigma_z}^{(1)}{\sigma_z}^{(2)}\}$**cnot**$^{(12)}$.

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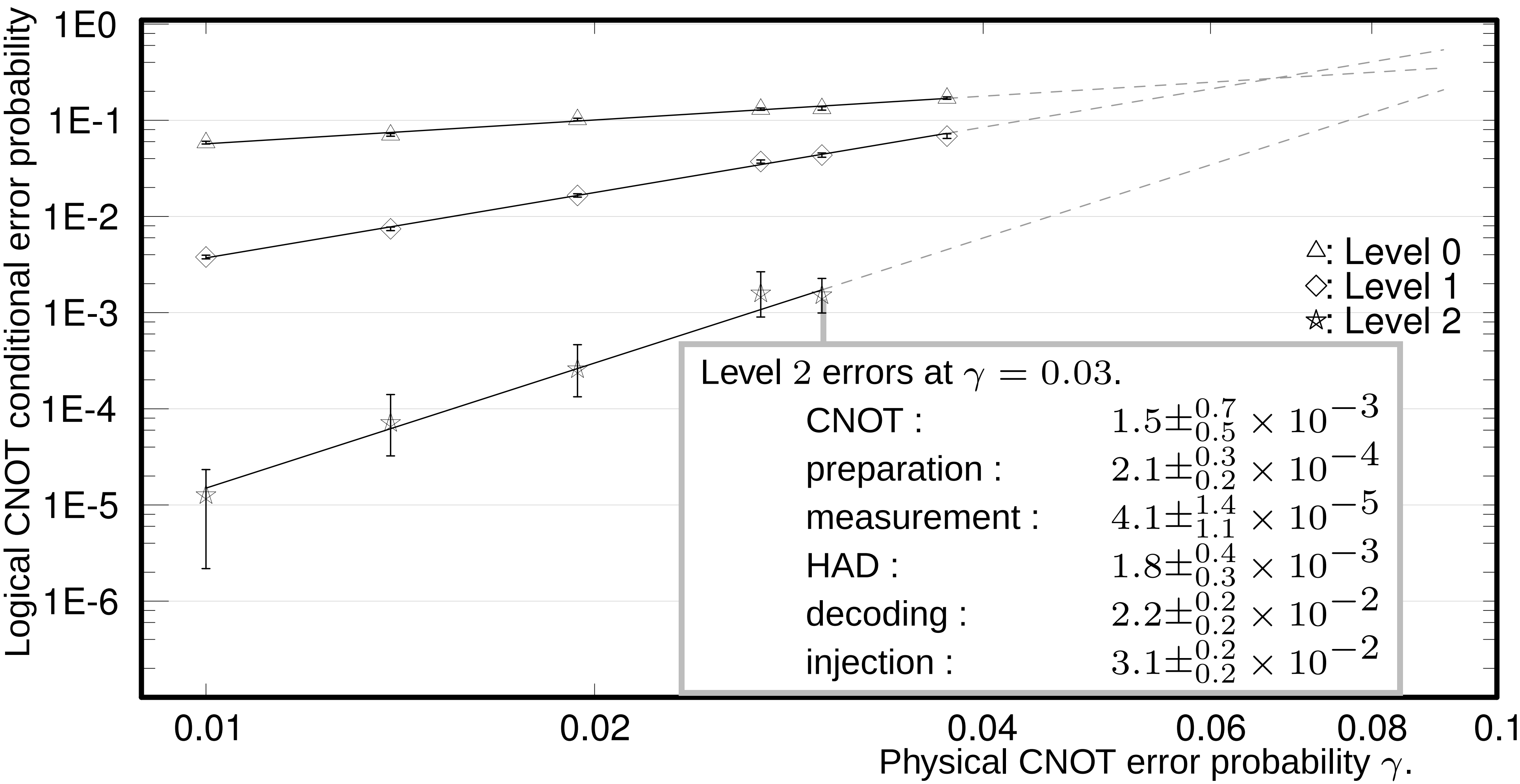
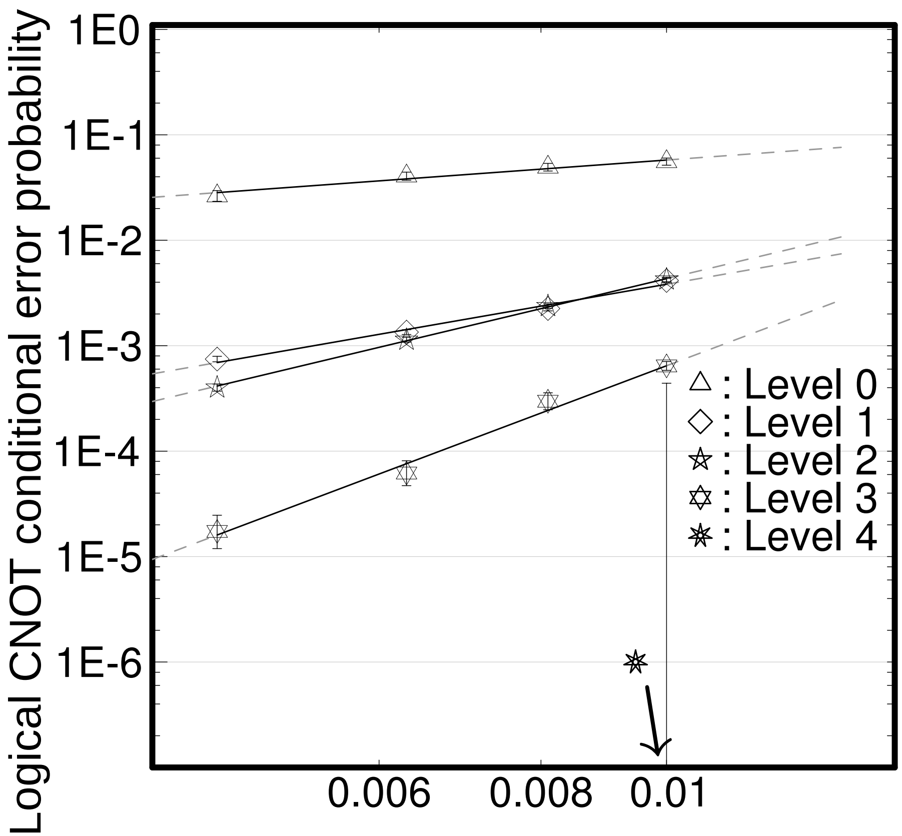
- Agnostic choice for $e_{p,m,h,c}$?

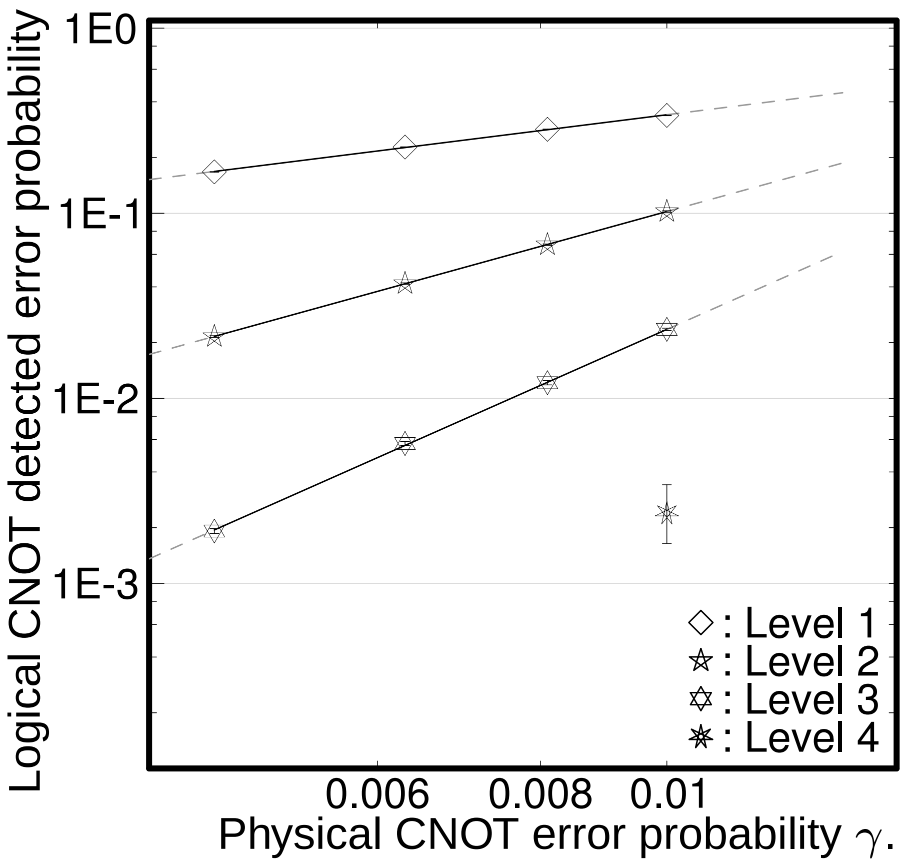
$$e_c = \epsilon, \quad e_h = \frac{4}{5}\epsilon, \quad e_p = \frac{4}{15}\epsilon, \quad e_m = \frac{4}{15}\epsilon, \quad e_n \leq e_h.$$

Conditional Logical Errors with Postselection

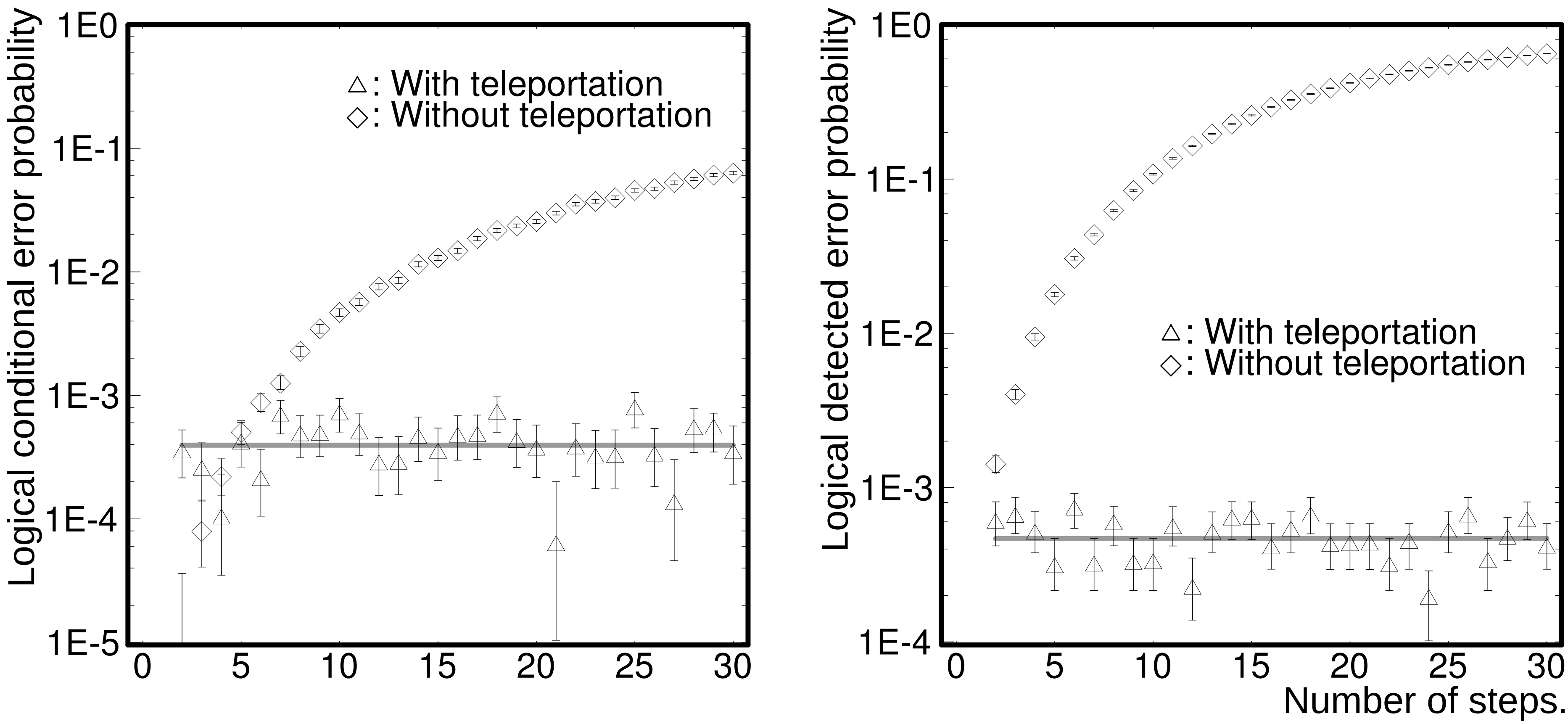


Error Probabilities for Scalable QC



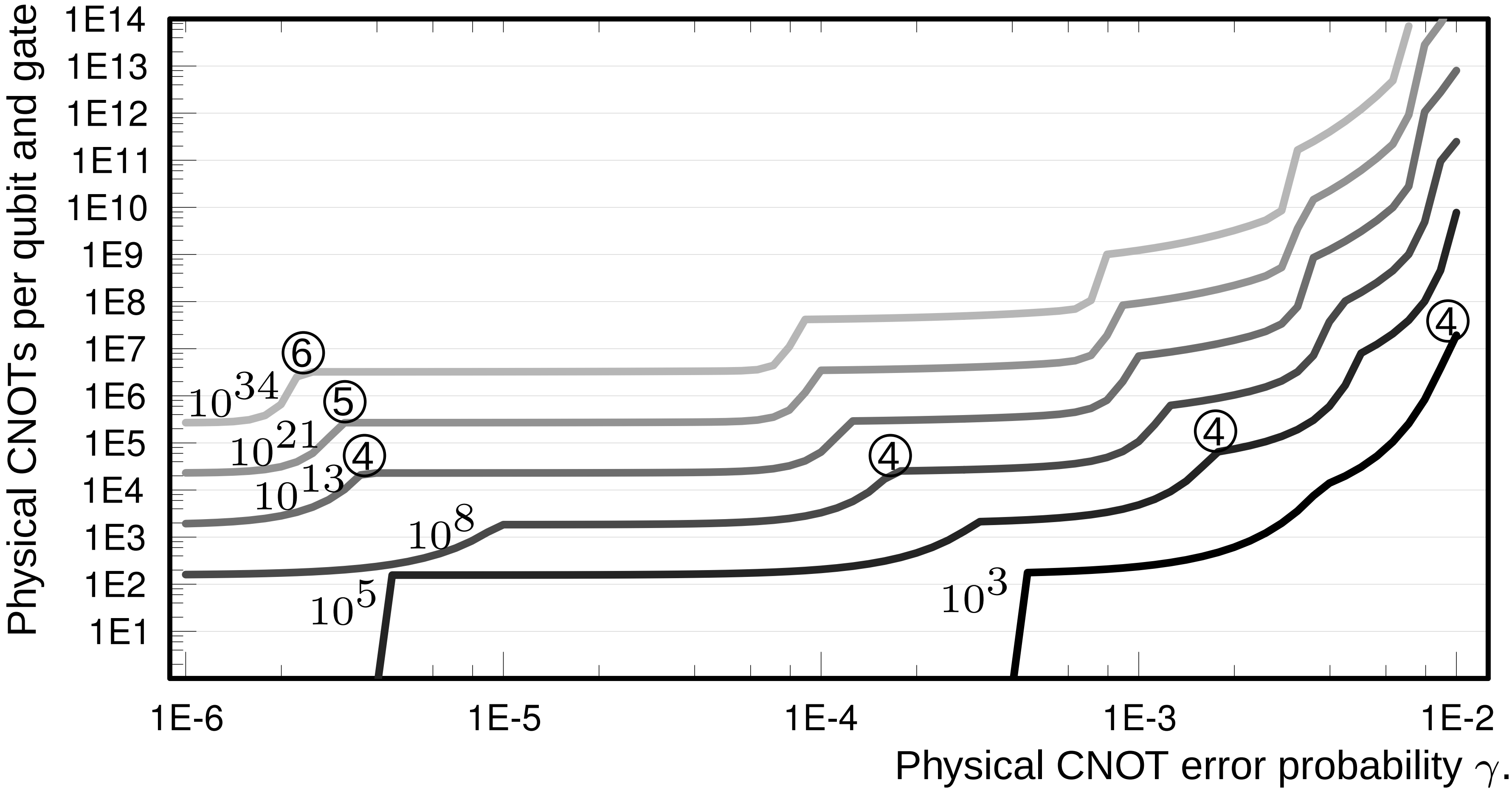


Error Compounding



- 30 applications of the HAD gate at level 3.

Conclusion



Physical CNOTs per qubit and gate
1E14
1E13
1E12
1E11
1E10
1E9
1E8
1E7
1E6
1E5
1E4
1E3
1E2
1E1
1E-6
1E-5
1E-4
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Physical CNOT error probability γ.
10^{34}
6
10^{21}
5
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4
10^{8}
10^{5}
4
10^{3}
4
4

The Clifford-Pauli Group

- Pauli matrix notation.

$$I = \mathbb{1}, X = \sigma_x, Y = \sigma_y, Z = \sigma_z$$
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$\mathcal{P}_n$ is the set of $\pm 1 *$Pauli products on n qubits.

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- **Theorem.** Any quantum computation using Z-eigenstate preparation, operators in $\mathcal{N}_n$, Z-measurements and feedforward can be efficiently classically simulated.

Gottesman (1997) [17]

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Fault-Tolerant Quantum Computing I

- **Requirement 3 for scalable QC[a] implementation:**
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Fault-Tolerance Threshold Theorem: *Given: Noisy qubits and gates. If the error rates are sufficiently low, then it is possible to efficiently process quantum information arbitrarily accurately.*

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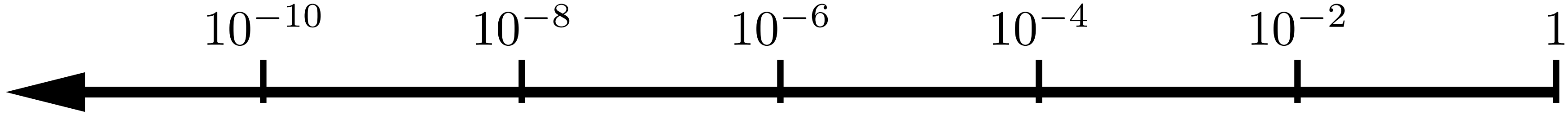
Error Thresholds: Proofs and Estimates

- Error thresholds in model-dependent “error probability”.



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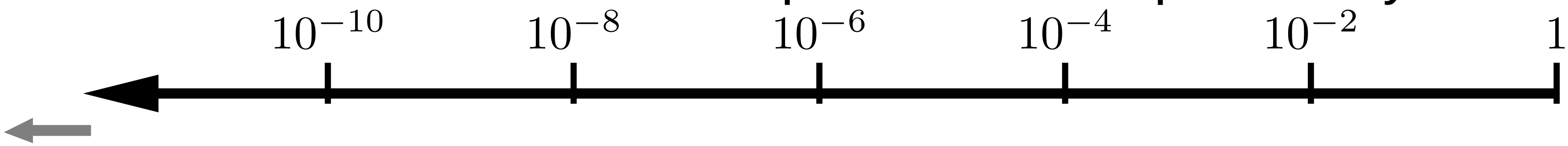

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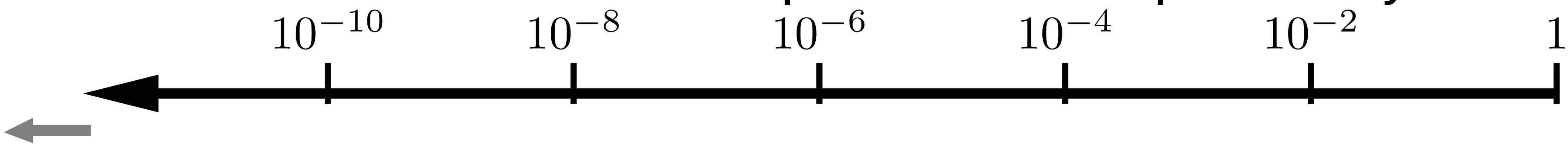

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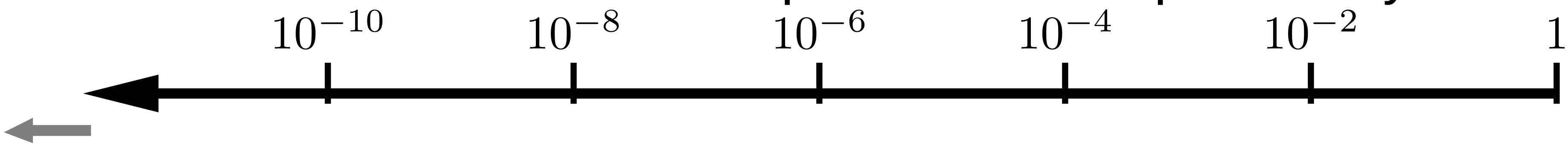

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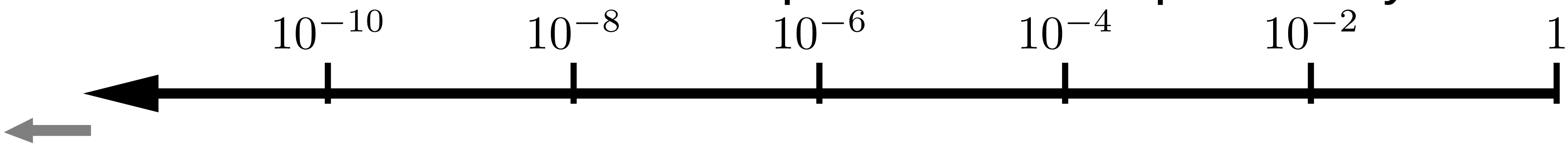

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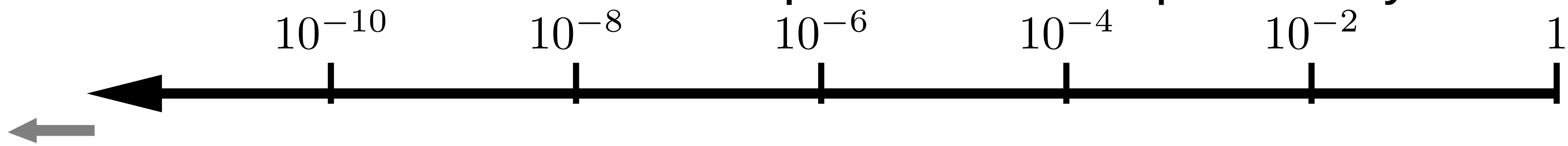

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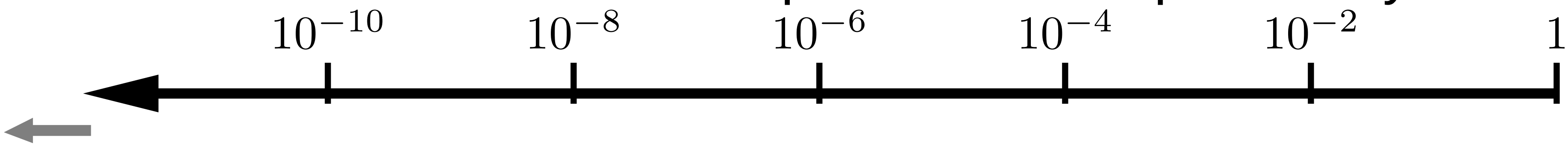

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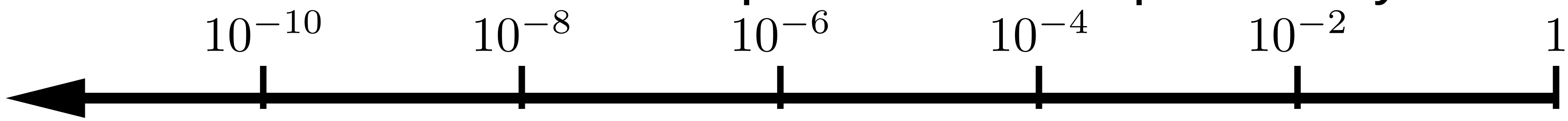

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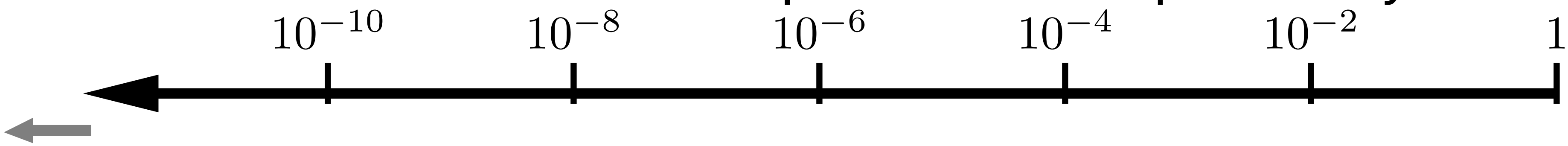

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  - Errors are generic, with no known exploitable biases.

Structural Assumptions

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- Arbitrarily many "physical" qubits can be called on.

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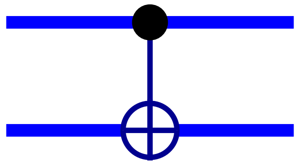

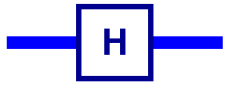

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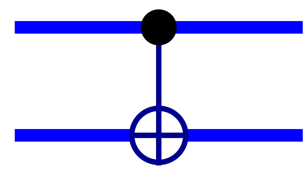

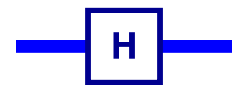

- At any time, can prepare qubits.
  - $|0\rangle$
  - $|+\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big)$.
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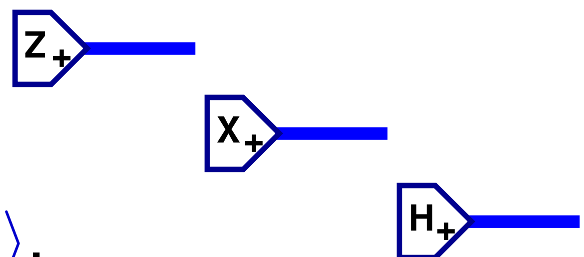

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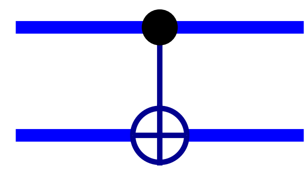

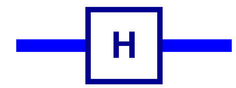

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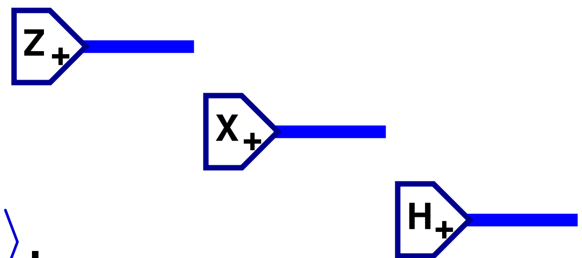
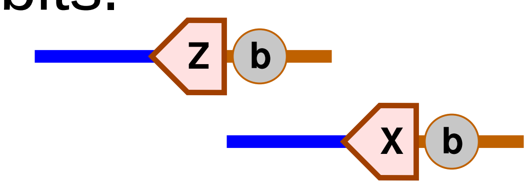


- At any time, can measure qubits.
  - $|0\rangle/|1\rangle$ measurement.
  - $|+\rangle/|-\rangle$ measurement.



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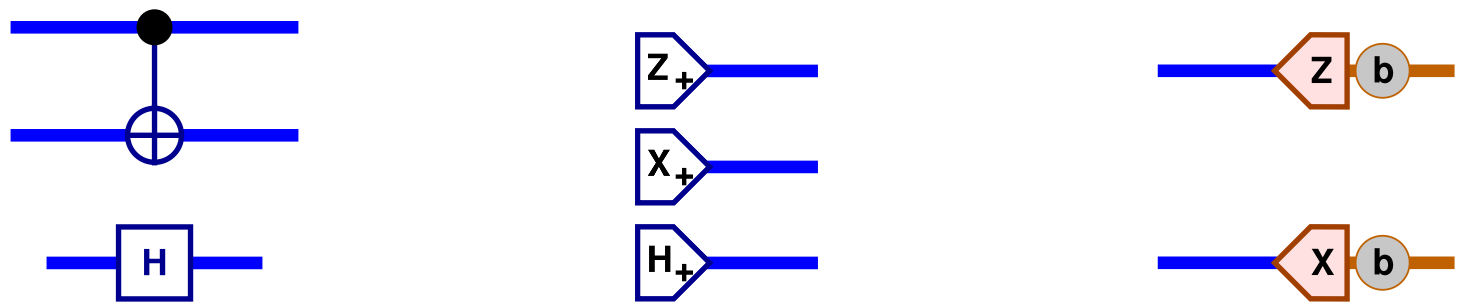

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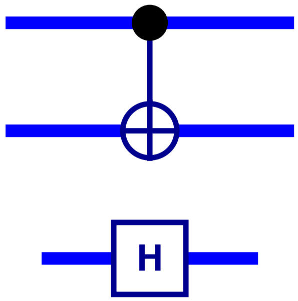

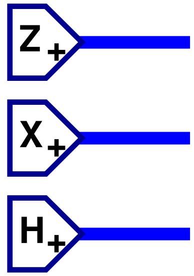

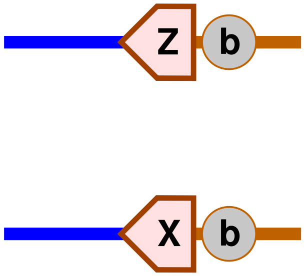

Global control capabilities:

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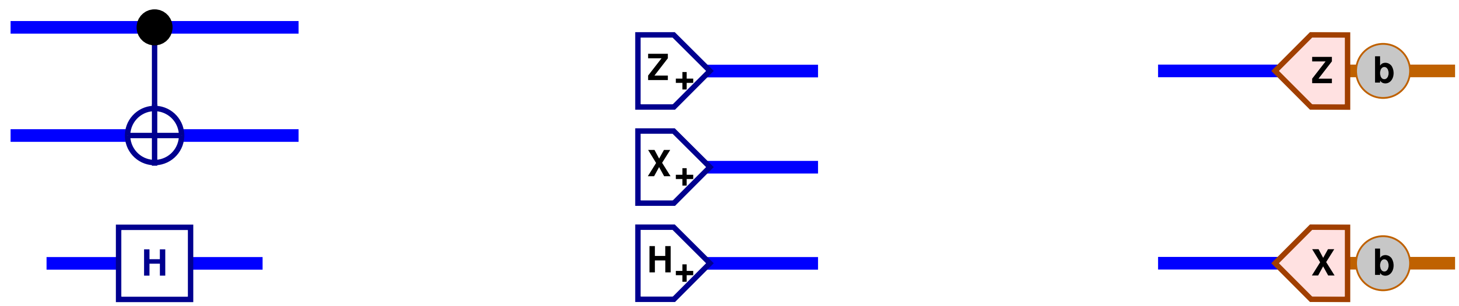

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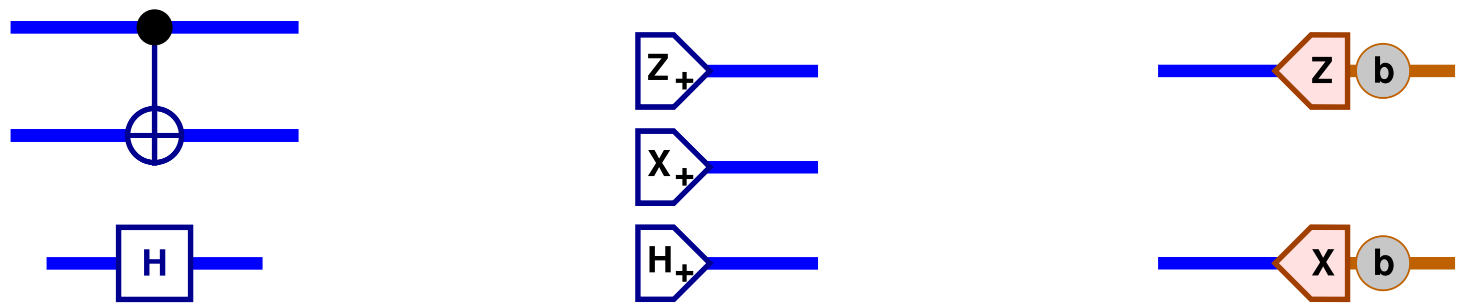

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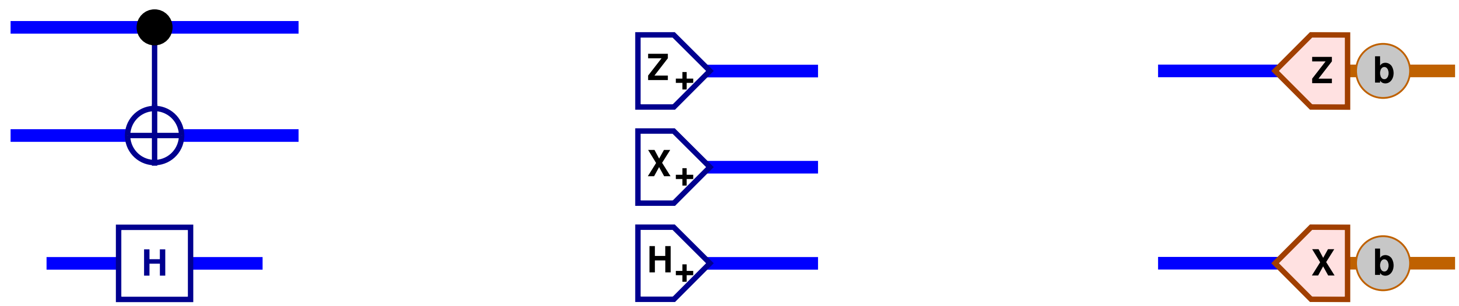

Global control capabilities:

- Massive parallelism or no memory error.
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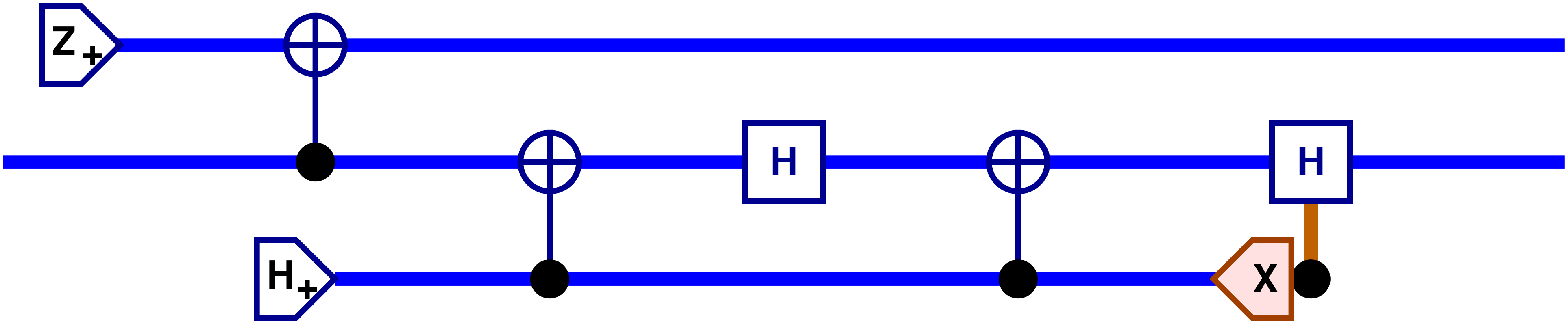
- Negligible quantum communication latency
  (for non-local two-qubit gates).

Error Models



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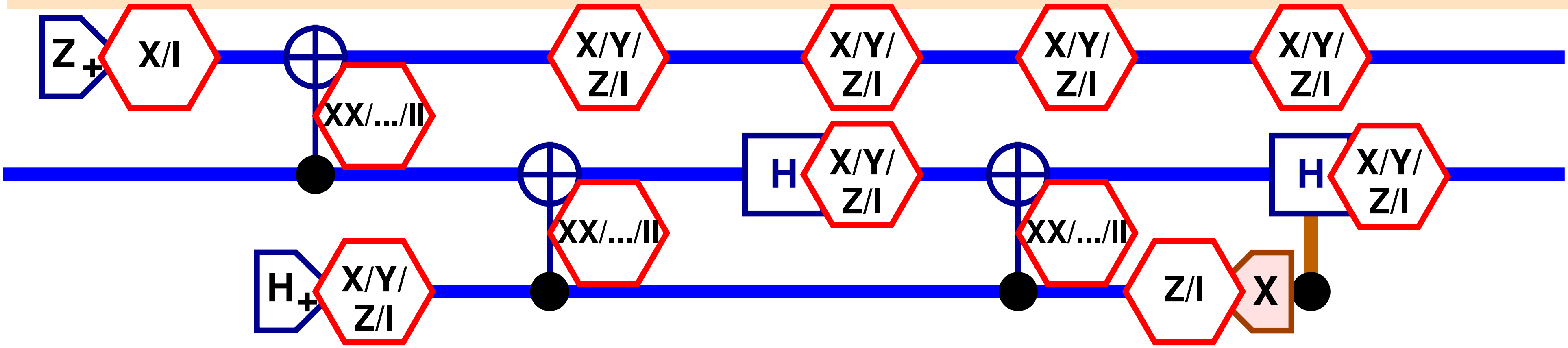

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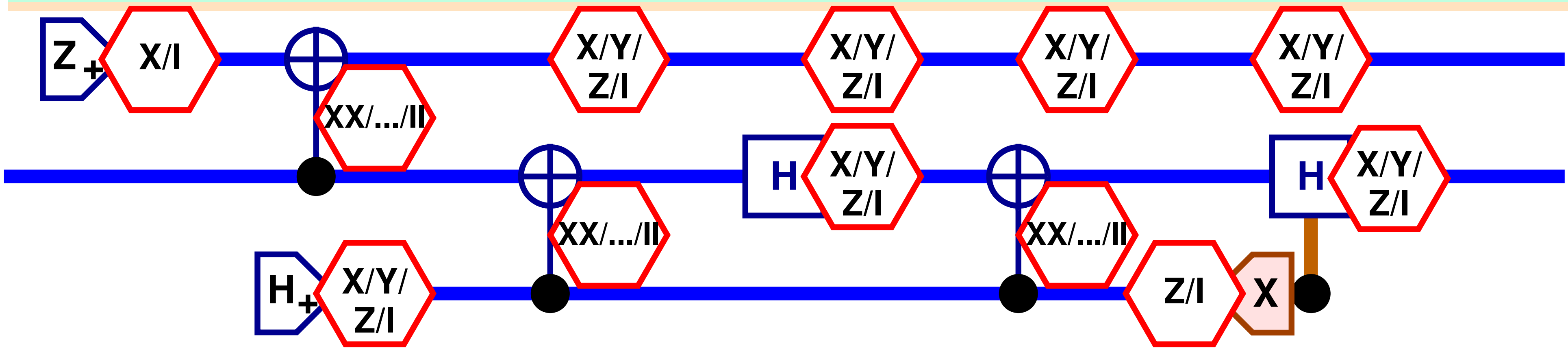

- General error expansion:

$$\sum_{e} \sum_{\mathbf{p}=(p_i)_i} \underbrace{|e_{\mathbf{p}}\rangle_{\text{Env}}}_{\text{Unnormalized “environment” state}} \prod_{i} \overbrace{G_i}^{\text{Gate } i} \underbrace{E_i(p_i)}_{\text{Pauli product at location } i}$$

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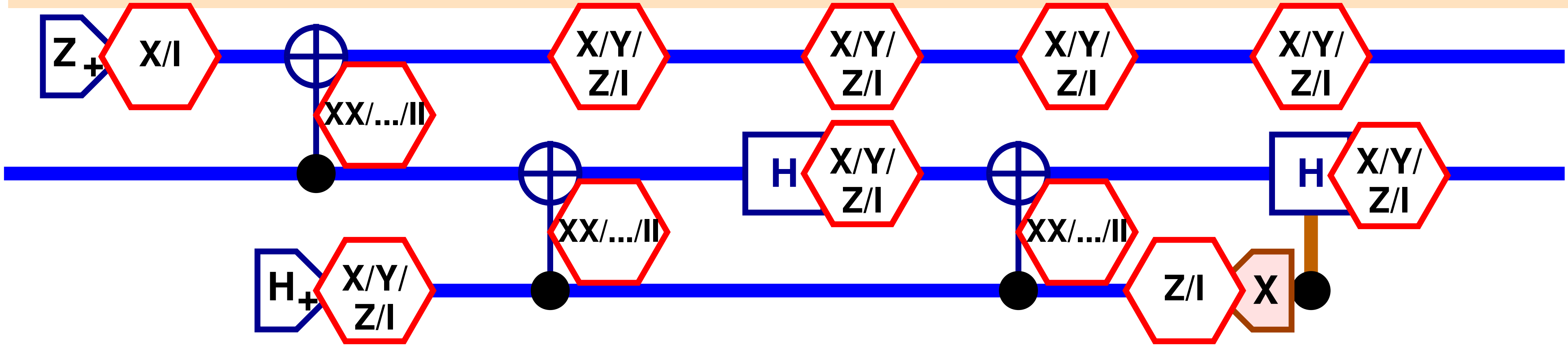

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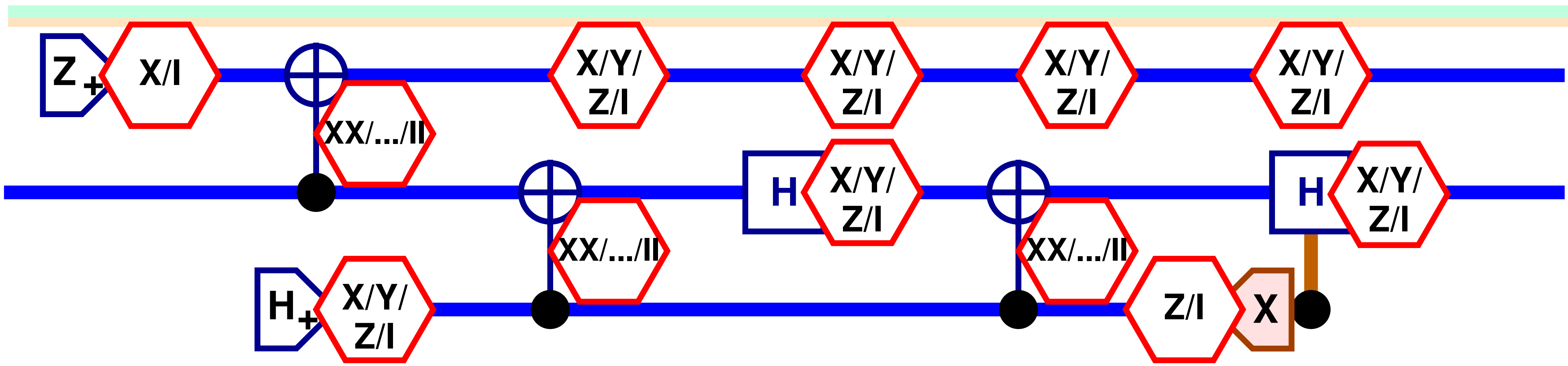

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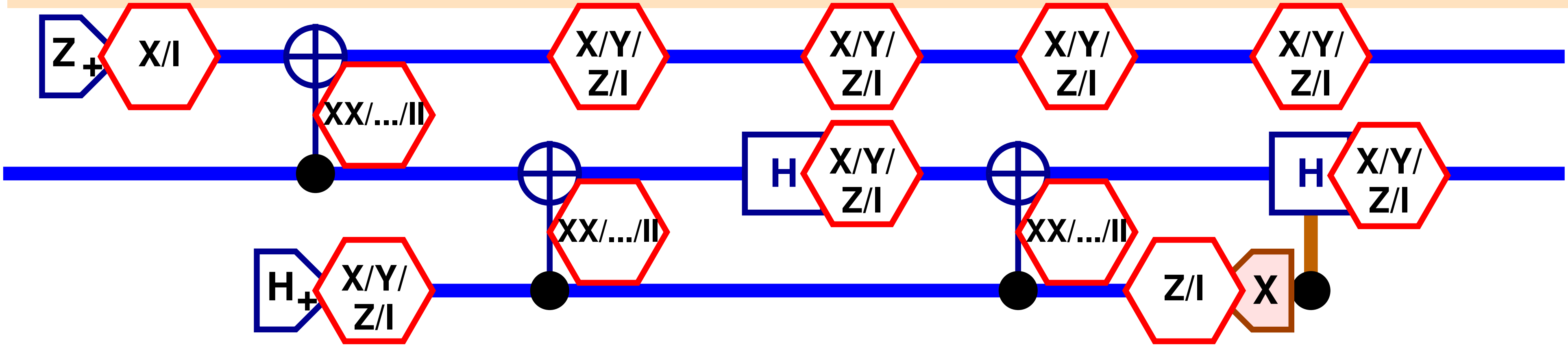

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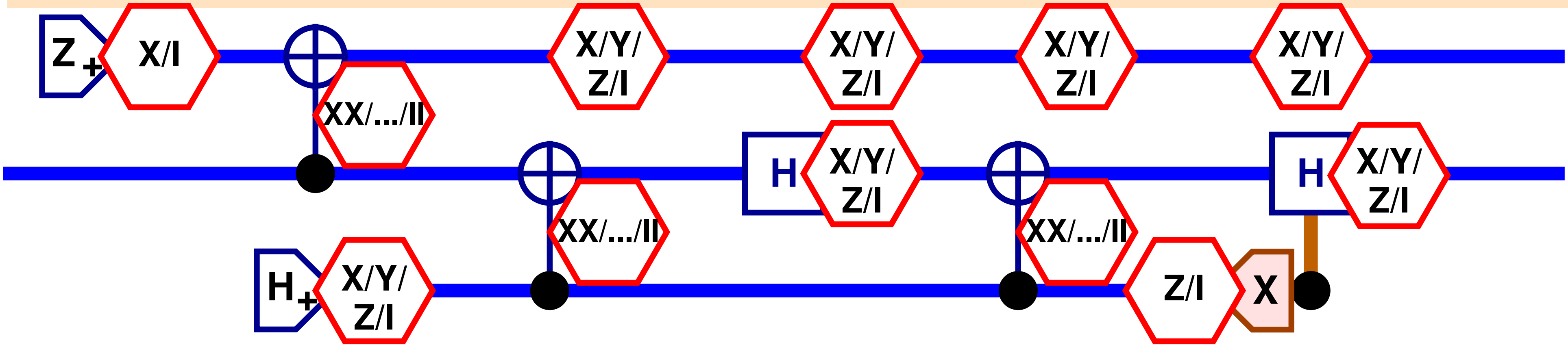

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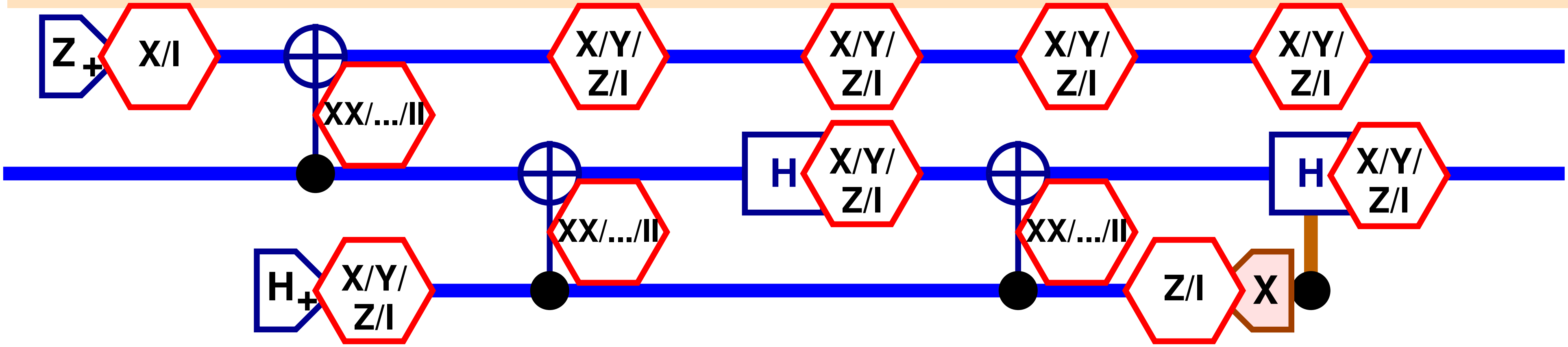

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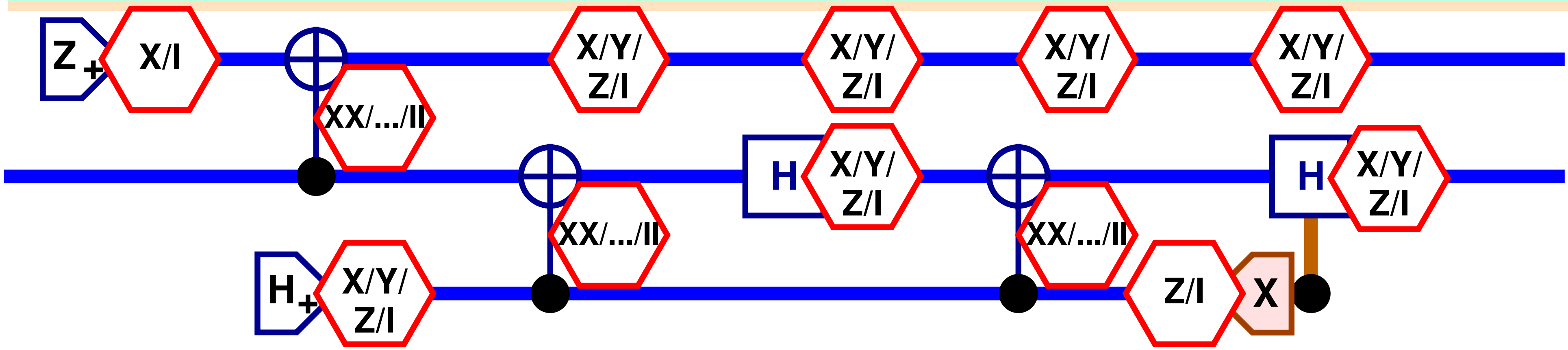

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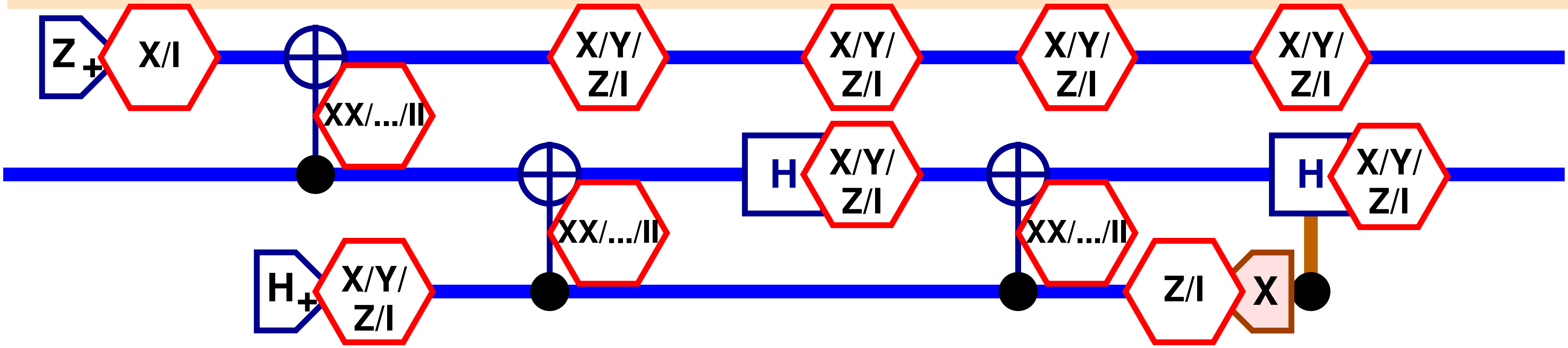

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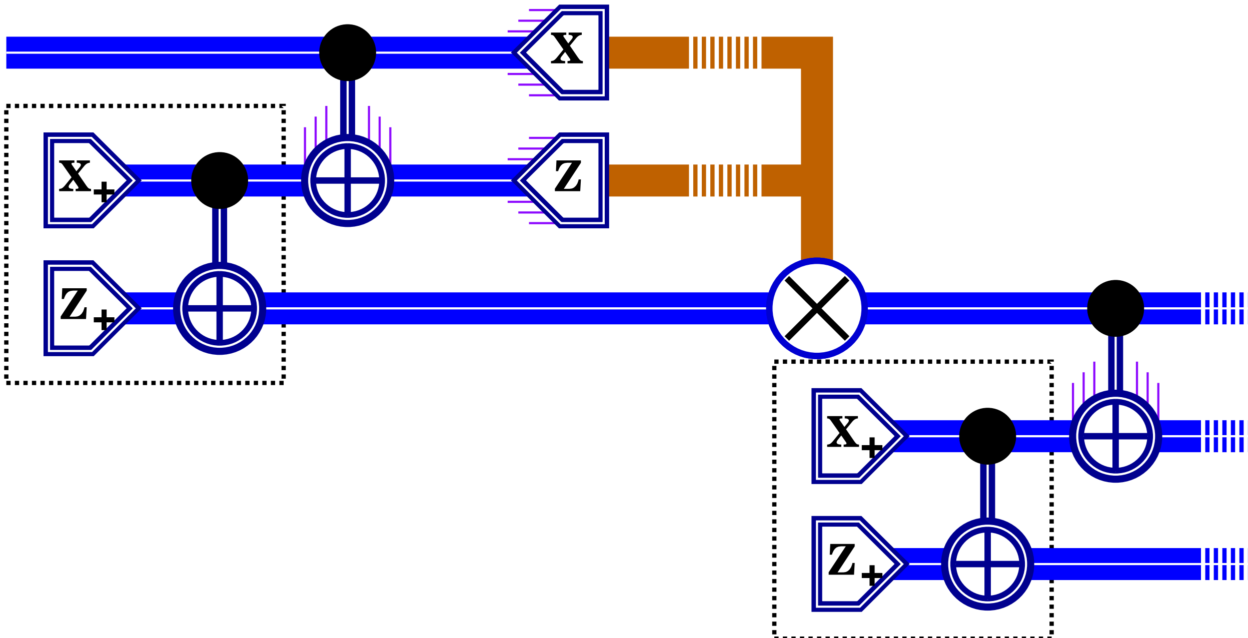
  - Stabilizer code implementations imply short lifetimes of unwanted coherences between Pauli errors.
  - Random Pauli pulses can further reduce these lifetimes.
  - Correlations are usually local.

Memory, Measurement, Parallelism Tradeoffs

- Problem: Long measurements req. good quantum memory.



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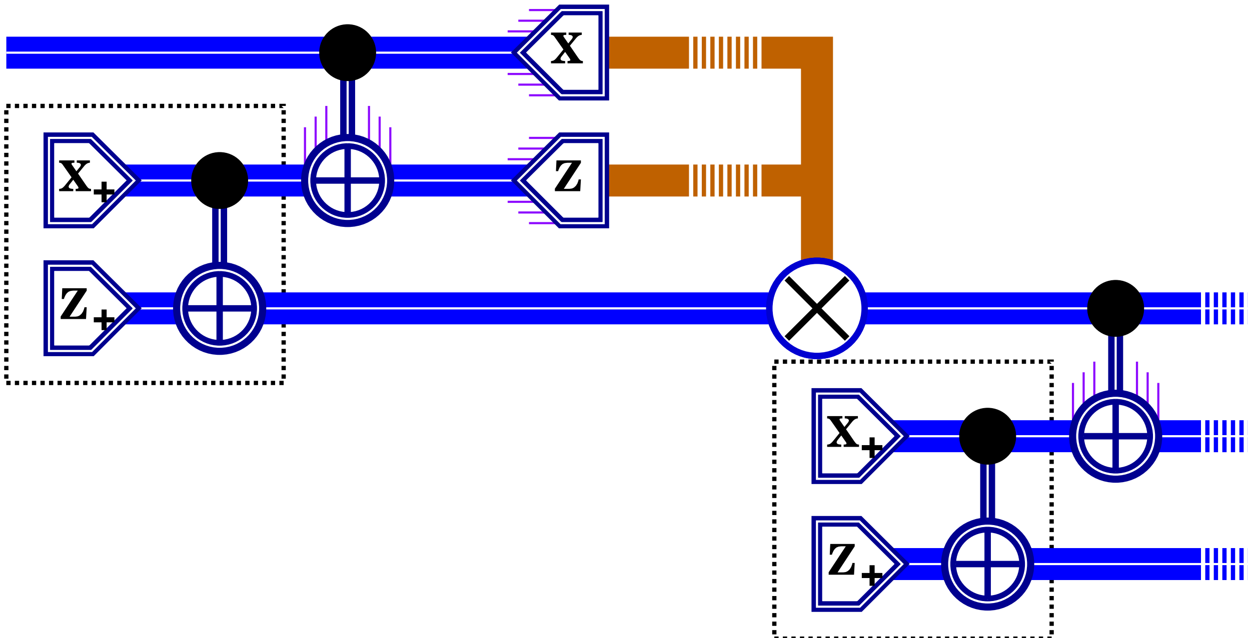

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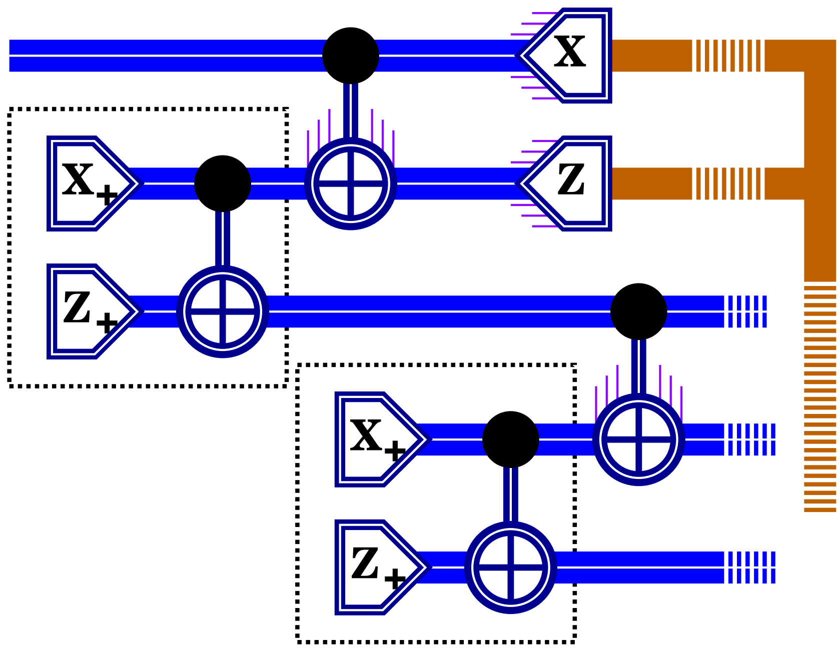

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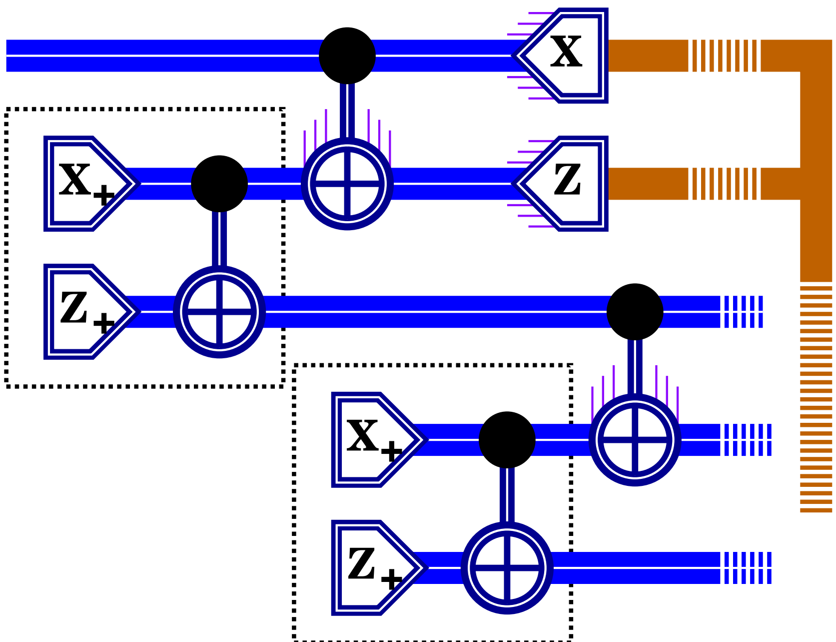

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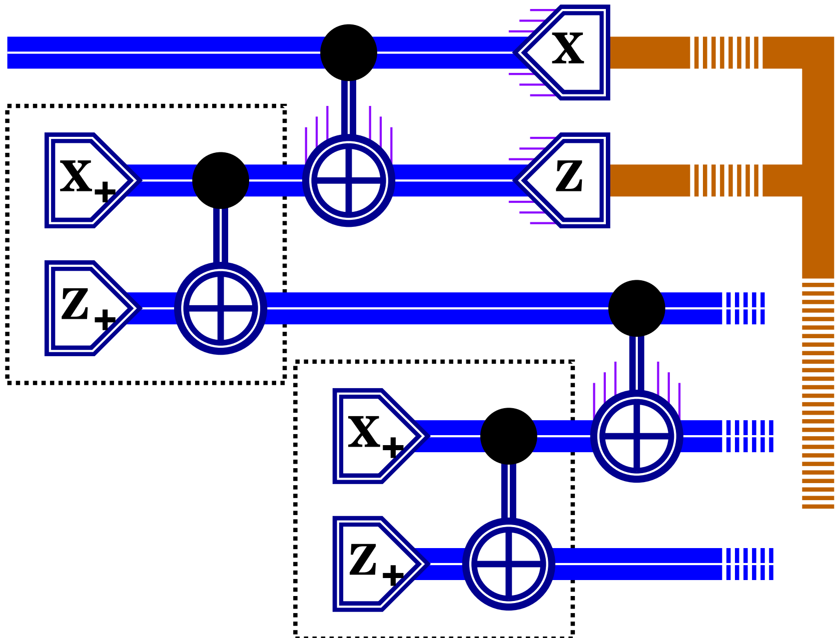

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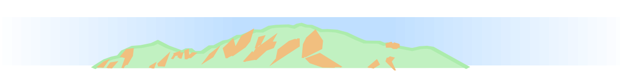
- Requirement:

  (memory error rate)$*$(measurement delay) $=$ small.



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