COMPLEXITY, an important concept in everyday discourse, also used in various technical senses in biology, economics, and other sciences, especially computer science. The basic meaning of complex is *not simple:* roughly speaking, a thing is complex if it resists simple explanations, if its structure or behaviour cannot be adequately explained in a few words. In discussing the complexity of concrete objects (as opposed to mathematical entities like numbers or bit sequences) it is of course necessary to specify which aspects of the object one is trying to describe, as opposed to other aspects considered irrelevant. For example a block of wood may appear macroscopically simple, but its microscopic details reveal a lot about the history of a particular tree, and indeed of life on earth. If the block were scanned and digitized on a coarse mesh, the resulting data would be simple, but if it were scanned on a sufficiently fine mesh, the resulting data would be complex.

What constitutes an adequate explanation, and how does one assess the simplicity of an explanation? In the philosophy of science, the principle of *Ockham's Razor* rates alternative explanations by the economy of their assumptions, preferring a theory with fewer, simpler assumptions even if the deductive path connecting them to the phenomena they explain is long and complicated. In chemistry, for example, atomic weights are no longer regarded as arbitrary, independent constants of nature as they were in the nineteenth century, but as phenomena to be explained, at the cost of considerable numerical computation, from the assumed properties of a small number of elementary particles. As with many other sciences, the explanations offered by chemistry have become more plausible and successful, while at the same time becoming mathematically more complicated.

The theory of computation has a special role to play in formalizing notions of complexity because of the existence of *universal computers*, such as the universal *Turing machine* which can be programmed to perform any computation or follow any chain of deductive logic. To put it more grandly, the input:output relation of a universal computer is a microcosm of all truths that can be demonstrated by numerical computation or deductive logic.

Computation and Complexity

The word "complexity" is used in two different technical senses in computer science, (both discussed further under *Turing machine*):

- The *program size complexity* (also called *Kolmogorov complexity* or *algorithmic information* content) of an object is the size in bits of its *minimal program*, the smallest input to a standard universal Turing machine that causes it to generate precisely that object as output. It represents the minimal information required to specify the object uniquely, without regard to the difficulty of the generation process. This kind of complexity is greatest for patternless objects, such as a typical bit string 01110111111001010010…. generated by coin tossing. Such objects are termed *algorithmically random* or *incompressible*, because they have no description more concise than the object itself.

- *Computational complexity* (also called dynamic or space/time complexity) is the time (number of steps) or space (number of bits of memory) required to perform a computation that has already been specified. It is usually expressed in relation to the size of the input to the computation. For example the complexity class **P** consists of all problems solvable using time bounded by some polynomial of the size of the input. Such problems are generally regarded as tractable, because the difficulty of solving them does not increase too rapidly with input size. Other complexity classes such as **NP** (problems for which solutions can be tested in polynomial time, even though they may be hard to find) or **PSPACE** (like **P**, but with the polynomial bound on space rather than time) include many famous problems widely thought to lie outside **P**, but, frustratingly, this has not been proved. The possibility remains, though most computer scientists doubt it, that **P=NP,** or even **P=NP=PSPACE,** which would mean that most problems we think of as hard have easy algorithms we haven't discovered yet. Other important complexity classes include **BPP,** the class of problems efficiently solvable in polynomial time with the help of coin tossing, and **BQP,** problems efficiently solvable in polynomial time on a quantum computer. **BPP** is widely believed to coincide with **P**, whereas **BQP** is thought to be properly larger, including some problems such as integer factoring for which no efficient classical algorithm is known. But since **BQP** is known to lie within **PSPACE,** the classical/quantum distinction would also vanish if **P=PSPACE**.

While these two notions of complexity have proved quite useful technically within computer science and information theory, neither alone corresponds to the intuitive distinction between a complex and a simple object. That is better captured by *logical depth,* defined as the time required for a universal Turing machine to generate an object from its minimal program (or more generally from a program of near minimal size). In terms of Ockham's Razor, the minimal program represents the object's most probable causal origin, and the process of generating the object from this input is the object's most plausible causal history. Logically deep objects, then, are those containing internal evidence of having had a long causal history.

Manifestations of Complexity in other Fields

Outside computer science, the terms "complexity" and "complex system" are used to describe a variety of phenomena, especially those occurring in systems with many similar, interacting parts. Any given complex system may exhibit several of these properties, or others not listed here:

- *emergent behaviour*:
- *nonlinearity*
- *chaotic dynamics*
- *self-similar or fractal structure*

- *criticality and self-organized criticality*
- *multistability and frozen accidents*
- *self-organization and evolution*

Emergent behaviour occurs when a system of many parts exhibits collective properties qualitatively different, and not easily predictable, from the properties of the parts. Old examples of emergent behaviour include phase transitions such melting and order-disorder transitions in equilibrium statistical mechanics. By now the theory of equilibrium phase transitions is well enough understood that the term emergent is usually used for less well understood cases, typically involving interacting parts that are more complicated than atoms and/or far from equilibrium, for example neurons, aggregating animals or competing investors. Fruitful comparisons can be made nonetheless.

Non-linearity. Even in systems with a few degrees of freedom, non-linearity makes equations of motion hard to solve, and one cannot easily combine solutions to get other solutions as one can with linear systems, or systems linearized by considering small deviations from equilibrium. A nonlinearly evolving system is typically full of surprises, and one must explore it thoroughly to find all the stable points, limit cycles, and other attractors—sets of points toward which trajectories converge. Nonlinear equations in hydrodynamics, including the famous Navier-Stokes equation describing the behaviour of viscous fluids, have kept generations of applied mathematicians busy, with no end in sight.

Chaos (cf chaos theory) A dynamical system is said to be chaotic if it exponentially amplifies small differences in its conditions, so that its macroscopic behavior appears random, even though the underlying evolution is entirely deterministic. Chaos is what makes long-range weather forecasting infeasible: atmospheric dynamics is sufficiently chaotic that a difference too small to measure in today's atmosphere can grow into completely different weather after a week or so, or as one of the founders of chaos theory put it, the flap of a butterfly's wings in Brazil can eventually set off a tornado in Texas. Even though chaotic dynamics would seem to be one of the most discouraging kinds of complexity for scientists wishing to make accurate predictions, chaotic systems have been found to exhibit common features associated with the fractal structure of their attractors. Population biologists and ecologists once sought to find evidence of deterministic chaos in the irregularly fluctuating populations of interacting species, but lately hybrid explanations have been more successful, involving both nonlinear effects within the ecosystem and random influences (eg weather) from outside.

A *fractal* is a geometric shape of anomalous dimension, for example a line so wiggly that it has infinite length, or a surface so rough that it has infinite area. Despite their complexity, fractals exhibit a kind of regularity called self-similarity, each small piece, when magnified, resembles the whole. As their discoverer Benoit Mandelbrot likes to point out, approximately fractal shapes are the rule rather than the exception in nature, and understanding them greatly extends the range of natural phenomena that can be treated in a quantitative manner.

Criticality refers to situations where a system becomes exquisitely sensitive to small changes in its environmental parameters. The classic examples occur in equilibrium thermodynamic systems, for example a pure liquid in equilibrium with its own vapour. If a sealed tube about one third full of water is heated, the liquid phase becomes less dense (by thermal expansion) while the vapour phase becomes more dense (by evaporation of liquid into it). Eventually, at about 374C, the distinction between the two phases disappears. At this so-called critical point the fluid develops long-range density fluctuations with a power law distribution of sizes, giving it a milky appearance, and its compressibility becomes infinite, making the system exquisitely sensitive to weak external forces such as gravity. Equilibrium criticality occurs only at exceptional points in the phase diagram, requiring that both temperature and pressure be tuned to exact values. At typical points in the diagram (ie if the temperature and pressure are chosen arbitrarily), the system's structure and behavior are much more prosaic: there will be only one stable phase (liquid, vapor, or one of the several forms of ice), properties such as compressibility will be finite, and fluctuations will be short-ranged, dying off exponentially rather than by a power law. The exceptionality of critical points and the presence of only one stable phase under typical values of external parameters such as temperature and pressure is a manifestation of the *Gibbs phase rule* obeyed by systems at thermal equilibrium.

Self-organized criticality is a phenomenon analogous to criticality that occurs in nonequilibrium driven systems, but requires no tuning of external parameters: the external driving force causes the system to tune itself into a condition of exquisite sensitivity. The classic example is an idealized sand pile. Sand grains are added to the middle of a table one grain at a time, forming a conical pile. Eventually the pile reaches the edge of the table and builds up to a critical slope, where a single additional grain can trigger an avalanche. The avalanches come in a broad power-law distribution of sizes from tiny one-grain avalanches up to full scale avalanches reaching all the way to the edge and dumping sand off onto the floor. This must be so because, if there were only small avalanches the sand would accumulate and become too steep, and if there were only large avalanches too much sand would be removed and the slope would become too flat. Self-organized criticality has been suggested to explain a wide range of natural phenomena, from earthquakes to traffic jams to stock market crashes, where spontaneous exquisite sensitivity and power-law distributions have been observed.

Multistability and frozen accidents: As noted two paragraphs ago, under typical external conditions (eg. temperature and pressure) a system at thermal equilibrium will have only one thermodynamically stable phase, whose structure is spatially uniform, uniquely determined by the external conditions, and rather insensitive to small variations in them. This boring behaviour contrasts greatly with the complexity and sensitivity displayed by nonequilibrium driven systems, such as the earth, whose atmosphere, oceans, and biosphere are driven by the sun, and whose core and mantle are driven by the heat of radioactive decay. Energy from both sources eventually leaves the earth in the form of infrared radiation into space, but in doing so it removes *entropy* , keeping the earth's surface and interior away from thermal equilibrium. The most important consequence of all this external driving is the earth's capacity for self-organization, including the origin of life. A less dramatic consequence is the tendency of the earth, and other driven systems, to develop *frozen accidents*: random information that apparently serves no useful purpose but is nonetheless persistent over time and widely replicated over space. Non-coding "junk" DNA is a prime example, but frozen accidents also occur in the inanimate world, for example the seafloor's fossil record of geomagnetic field reversals. Frozen accidents are probably not essential to the function of complex systems, but rather are a byproduct of conditions under which complexity typically arises, namely the simultaneous presence of random events (e.g. mutations or field reversals) and a dynamical process (reproduction or seafloor spreading) able to preserve them.

Self-organization and Life: The most conspicuous examples of complexity, of course, are the origin and *evolution* of life, and several billion years later, of human culture. These supremely complex phenomena will probably indefinitely resist the kind of comprehensive understanding we have of statistical mechanical systems at thermal equilibrium, or dynamical processes governed by linear differential equations. It would be foolhardy to hope for a comprehensive "theory of complex systems". Nonetheless an important lesson from the scientific study of complex systems is that some aspects of their behaviour do obey simple laws. The most successful of these is *natural selection*, which explains and predicts a great many phenomena in biology, while leaving others underdetermined and still able to surprise us.

Complexity in the Universe

Displacement from thermal equilibrium has long been recognized as a necessary condition for the generation and maintenance of complexity, but it is probably not sufficient. The surface of the sun is a very nonequilibrium environment, but it is probably not a hospitable place for life to evolve. Besides displacement from nonequilibrium, another probable prerequisite for self-organization, which is probably lacking on the sun, is the existence of structures stable enough to store information at least for a while, until it can be amplified and reinforced by other nonequilibrium processes. It is this storage ability that has as its side effect the production of frozen accidents alluded to above. In the 19th and early 20th centuries, after the discovery of thermodynamic laws but before the discovery of the expansion of the universe, it was generally thought that life and complexity would not go on forever, but would be extinguished by the eventual approach to thermal equilibrium, an event colorfully referred to as "the heat death of the universe". Nowadays it appears *(cf cosmology, entropy)* that the universe will go on expanding forever, and never reach thermal equilibrium. But the hopes for perpetual complexity are hardly better, since it appears that in the far future all matter will be reprocessed by black holes into a dilute expanding sea of radiation, offering few opportunities for stable storage of frozen accidents or other information.

Suggestions for further reading:

Science magazine, issue on Complex Systems, Vol 284 No 5411 (1999)
http://www.sciencemag.org/content/vol284/issue5411/index.shtml

Wikipedia articles on Complexity
http://en.wikipedia.org/wiki/Complexity
http://en.wikipedia.org/wiki/Algorithmic_information_theory