Regular Expressions

A notation to describe "finite-automaton" patterns

E.g. Binary strings that "begin with a string of 0's followed by a string of 1's".

Binary strings that "start and end with the same symbol".

Regular expressions are just such a compact notation to describe these patterns, which are described respectively as $0^* \cdot 1^*$ and
$0 + 1 + (0(0+1)^*0) + (1(0+1)^*1)$.

Regular expressions have many important applications in CS:

- Lexical analysis in compiler construction.
- Search facilities provided by text editors and databases; utilities such as `awk` and `grep` in Unix.
- Programming languages such as Perl and XML.

Regular expressions and their denotations

Fix a Σ. We define simultaneously *regular expression* E and the *language denoted* by E, written $L(E)$, by induction over the following rules:

- The constants ϵ and $\emptyset$ are regular expressions; $L(\epsilon) \stackrel{\text{def}}{=} \{\, \epsilon \,\}$ and $L(\emptyset) \stackrel{\text{def}}{=} \emptyset$.
- For $a \in \Sigma$, a is a regular expression; $L(a) \stackrel{\text{def}}{=} \{\, a \,\}$.
- If E and F are regular expressions, then so are $(E + F)$, $(E \cdot F)$ and (E^*); we have

union	$L((E+F))$	$\stackrel{\text{def}}{=}$	$L(E) \cup L(F)$
concatenation	$L((E \cdot F))$	$\stackrel{\text{def}}{=}$	$L(E) \cdot L(F)$
star	$L((E^*))$	$\stackrel{\text{def}}{=}$	$(L(E))^*$

Notations

$+$ is sometimes written $\cup$ or $|$, and $(E \cdot F)$ is sometimes simply written (EF).

Parentheses may sometimes be omitted. We assume:

(i) The regular expression operators have the following *order of precedence* (in decreasing order): star, concatenation, union.

E.g. 01^* means $0(1^*)$, not $(01)^*$; $0 + 10$ means $0 + (10)$, not $(0 + 1)0$.

(ii) Union and concatenation associate to the left i.e. $E \cdot F \cdot G$ means $(E \cdot F) \cdot G$. (Since union and concatenation are associative, the choice of left or right association does not really matter.)

Examples

(i) $01^* + 1$ is formally $((0 \cdot (1^*)) + 1)$.

(ii) $(0 + 1)01^*0$ is formally $((((0 + 1) \cdot 0) \cdot (1^*)) \cdot 0)$.

Examples: Languages over $\{0, 1\}$ denoted by regular expressions

1. 0^*10^* denotes words that have exactly one 1.
2. $(0+1)^*1(0+1)^*$ denote words that have at least one 1.
3. $0(0+1)^*0 + 1(0+1)^*1 + 0 + 1$ denotes words that start and end with the same symbol.

We shall say that a word w *matches* E just in case $w \in L(E)$.

Equivalence of regular expressions

We say that E and F are *equivalent*, written $E \equiv F$, just in case $L(E) = L(F)$.

Note that $\equiv$ is an equivalence relation.

Some identities

1. Associativity: $(E + F) + G \equiv E + (F + G)$ and $(EF)G \equiv E(FG)$.
2. Commutativity: $E + F \equiv F + E$
3. $E\,\emptyset \equiv \emptyset$.
4. $\emptyset^* \equiv \{\,\epsilon\,\}$.
5. $E + \emptyset \equiv E \equiv \emptyset + E$ and $E \cdot \epsilon \equiv E \equiv \epsilon \cdot E$.
6. But *in general* $E + \epsilon \not\equiv E$, and $E \cdot \emptyset \not\equiv E$.

   For which E do the equivalences hold?

Example: Verify $(a+b)^* \equiv a^*\,(b\,a^*)^*$

Proof. Observe that $L((a+b)^*)$ is the set of all strings over $\{\,a,b\,\}$, thus $L(a^*\,(b\,a^*)^*) \subseteq L((a+b)^*)$.

Note that any $s \in L((a+b)^*)$ can be written *uniquely* as

$$a^{n_0}\,b\,a^{n_1}\,b\,\cdots\,a^{n_{r-1}}\,b\,a^{n_r} \tag{1}$$

where a^n means $\underbrace{a\cdots a}_{n}$, each $n_i \geq 0$ and $r \geq 0$. (r is just the number of occurrences of b in s. E.g. in case s is b, $n_0 = n_1 = 0$; in case s is ϵ, $n_0 = 0$.)

Any string in $L((ba^*)^*)$ has the shape $\underbrace{(b\,a^{n_1})\;\cdots\;(b\,a^{n_r})}_{r}$, where each $n_i \geq 0$ and $r \geq 0$. It follows that any string in $L(a^*(ba^*)^*)$ has the shape $a^{n_0}\,b\,a^{n_1}\;\cdots\;b\,a^{n_r}$ where each $n_i \geq 0$ and $r \geq 0$ i.e. of the shape (1). $\square$

Question: Is there a finite set of (equivalence) axioms and rules such that $(a+b)^* \equiv a^*\,(b\,a^*)^*$ (indeed any valid equivalence) is a theorem?

Kozen's Axioms for the Algebra of Regular Expressions

1. $E + (F + G) \equiv (E + F) + G$
2. $E + F \equiv F + E$
3. $E + \emptyset \equiv E$
4. $E + E \equiv E$
5. $(EF)G \equiv E(FG)$
6. $\epsilon E \equiv E\epsilon \equiv E$
7. $E(F + G) \equiv EF + EG$
8. $(E + F)G \equiv EG + FG$
9. $\emptyset E \equiv E\emptyset \equiv \emptyset$
10. $\epsilon + EE^* \equiv E^*$
11. $\epsilon + E^*E \equiv E^*$

and two rules:

12. $F + EG \leq G \Rightarrow E^*F \leq G$
13. $F + GE \leq G \Rightarrow FE^* \leq G$

Note: $E \leq F$ means $L(E) \subseteq L(F)$.

(Sound and) Complete Axiomatisation of Equivalence

Soundness: Each axiom is a valid equivalence between regular expressions, and each rule is sound (i.e. if the premise is a valid equivalence, so is the conclusion).

E.g. to say that rule (13) is sound is to say that for any E, F and G, if $L(F + GE) \subseteq L(G)$, then $L(FE^*) \subseteq L(G)$.

Completeness: Further the axiomatisation is *complete* i.e.

Kozen's Theorem. All valid equivalences between regular expressions can be derived from Kozen's axioms and rules, using the laws of (in)equational logic i.e.

if $E \leq F$ then $E \oplus G \leq F \oplus G, G \oplus E \leq G \oplus F$ and $E^* \leq F^*$ where $\oplus = +$ and $\cdot$.

On proving Kozen's Theorem

Soundness proof: For an illustration, we prove (13).

Suppose $L(F + GE) \subseteq L(G)$. Any $w \in L(FE^*)$ has the form $fe_1 \cdots e_n$ where $n \geq 0$, $f \in L(F)$ and $e_i \in L(E)$. We prove that $w \in L(G)$ by induction on n.

The base case of $n = 0$ follows from $L(F) \subseteq L(G)$.

For the inductive case, we need to show that $fe_1 \cdots e_{n+1} \in L(G)$. Now $fe_1 \cdots e_n \in L(FE^*)$; by the IH we have $fe_1 \cdots e_n \in L(G)$; and so $fe_1 \cdots e_n e_{n+1} \in L(GE)$ which is contained in $L(G)$ by supposition.

Completeness proof: beyond the scope of this course. $\square$

Example: $(a+b)^* \equiv a^*\,(b\,a^*)^*$ revisited

We prove the harder direction "$\leq$" using Kozen's system.

Note that $\epsilon, a, b \leq a^*(ba^*)^*$. [Ex. Prove it using $\epsilon \leq E^*$!]

We have $a\,(a^*(ba^*)^*) \equiv (a\,a^*)(ba^*)^* \leq a^*(ba^*)^*$ by (5) and (10). Similarly

$$b\,(a^*(ba^*)^*) \equiv (ba^*)(ba^*)^* \leq (ba^*)^* \equiv \epsilon(ba^*)^* \leq a^*(ba^*)^*.$$

The last two steps follow from (6) and (10). Because of the preceding, by (7), we have $(a+b) + (a+b)(a^*(ba^*)^*) \leq a^*(ba^*)^*$, and so, by rule (12) – taking E and F to be $(a+b)$ – we have

$$(a+b)^*(a+b) \leq a^*(ba^*)^*.$$

Since $\epsilon \leq a^*(ba^*)^*$, by rule (10) we have $(a+b)^* \leq a^*\,(b\,a^*)^*$ as desired. $\square$

Equivalence of regular expressions and finite automata

The equivalence of regular expressions and finite automata is a fundamental result in Automata Theory.

Kleene's Theorem. Let $L \subseteq \Sigma^*$. The following are equivalent:

(i) L is regular i.e. for some finite automaton M, $L = L(M)$.

(ii) L is denoted by some regular expression E i.e. $L = L(E)$.

Proof of Kleene's Theorem: "(i) $\Rightarrow$ (ii)"

We show that there is a systematic way to transform a regular expression E to an equivalent NFA N_E – so that $L(E) = L(N_E)$ – by recursion on the structure of E.

Base cases: For each of the three cases, namely $E = \epsilon, \emptyset$ and a where $a \in \Sigma$, there is an NFA N_E that accepts $L(E)$.

Inductive cases: Take regular expressions E and F. Suppose N_E and N_F are NFAs that accept $L(E)$ and $L(F)$ respectively. We have proved that regular languages are closed under union, concatenation and star by constructing NFAs that accept $L(N_E) \cup L(N_F), L(N_E) \cdot L(N_F)$ and $(L(N_E))^*$ respectively. By definition, these NFAs are equivalent to $E + F$, $E \cdot F$ and E^* respectively.

$\square$

Example: Transforming regular expressions to NFAs

We construct the NFA that accepts $(0 + 01^*0)^*0$.

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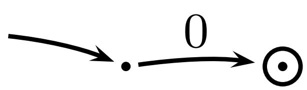

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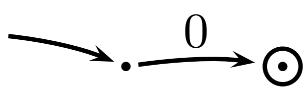

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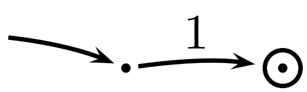

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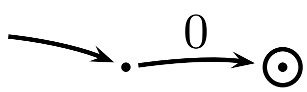

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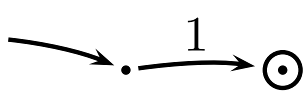

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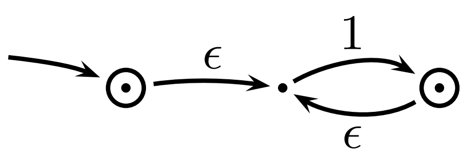

Example: Transforming regular expressions to NFAs

We construct the NFA that accepts $(0 + 01^*0)^*0$.

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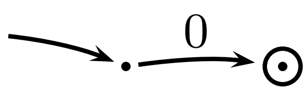

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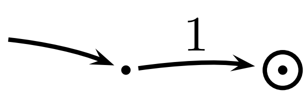

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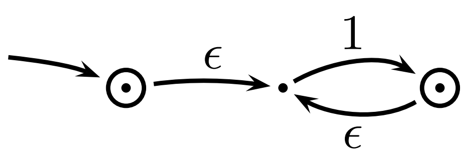
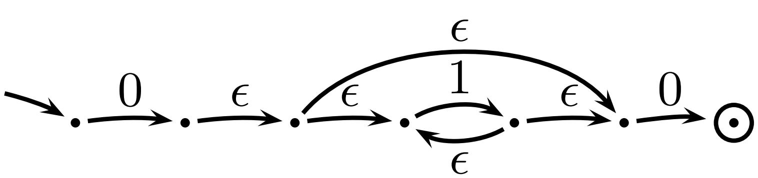


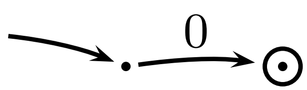
$0\,1^*\,0$



Example: Transforming regular expressions to NFAs

We construct the NFA that accepts $(0 + 01^*0)^*0$.

0



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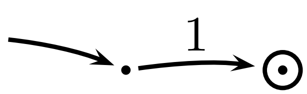

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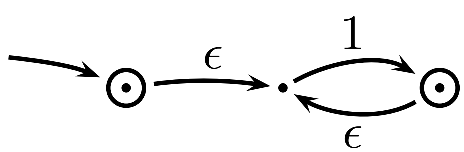

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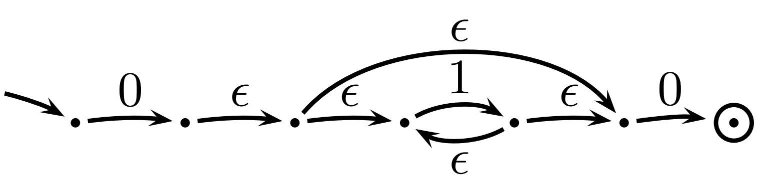
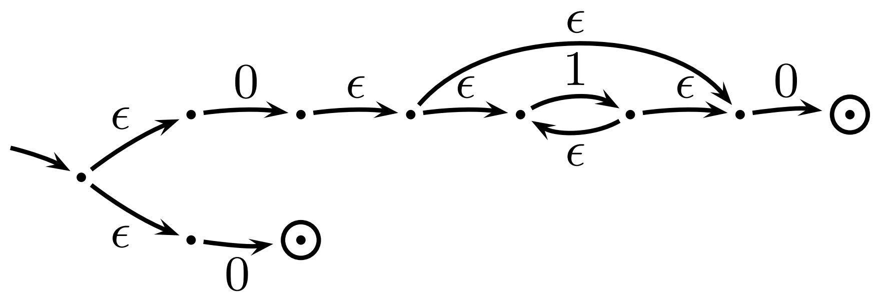


$(0 + 0\,1^*\,0)$



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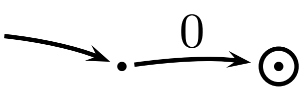
0



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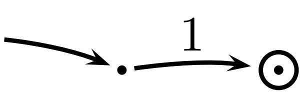

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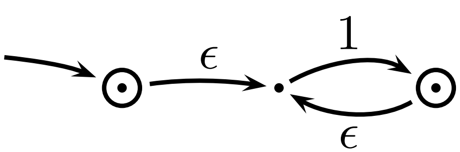
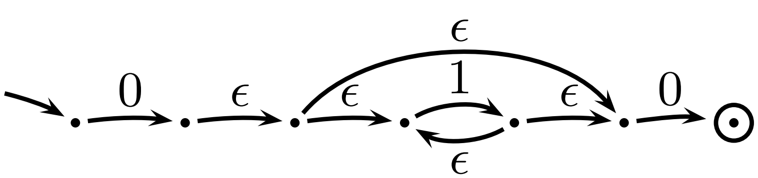
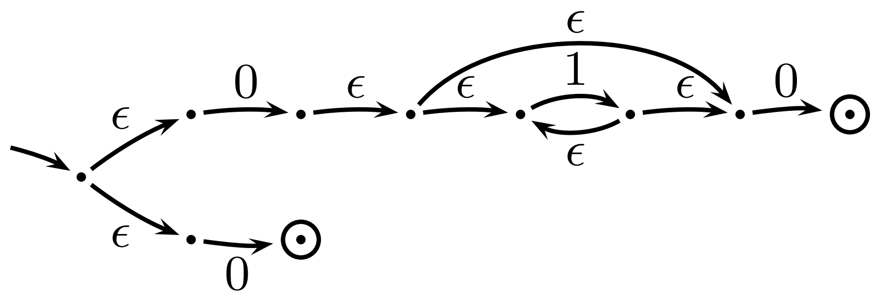


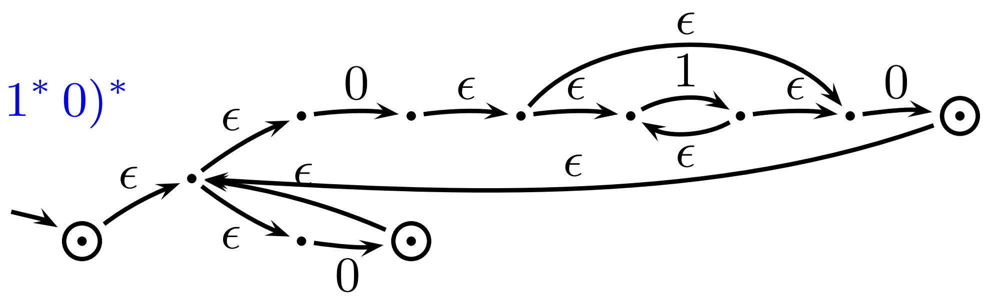
$0\,1^*\,0$



$(0 + 0\,1^*\,0)$



$(0 + 0\,1^*\,0)^*$



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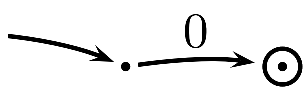

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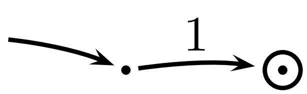

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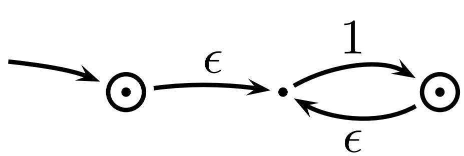
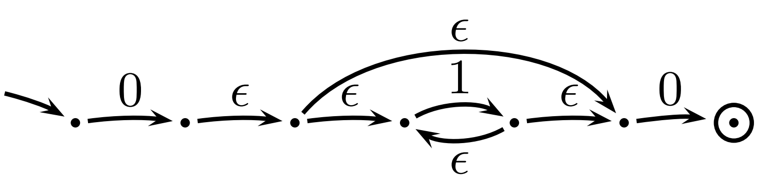
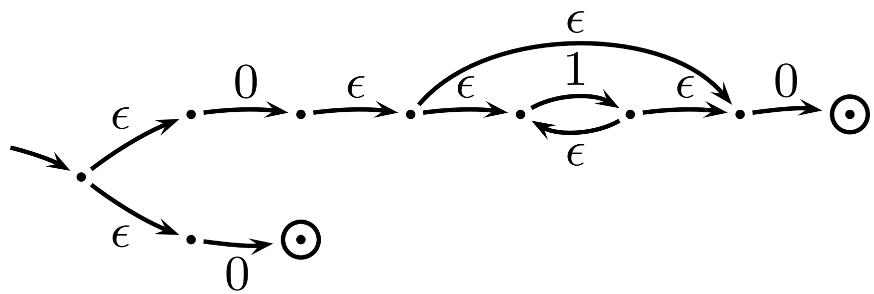


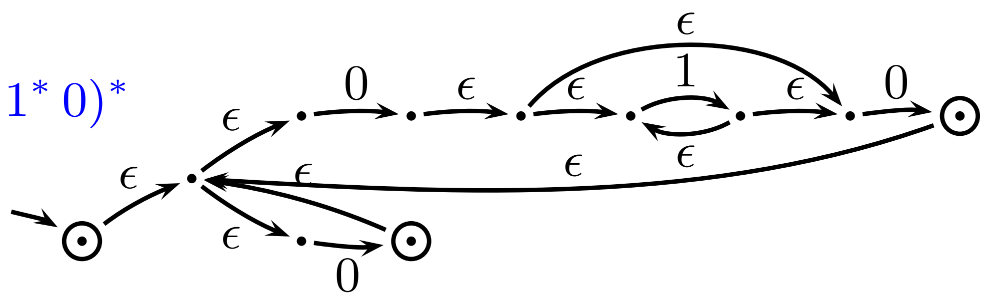
$0\,1^*\,0$



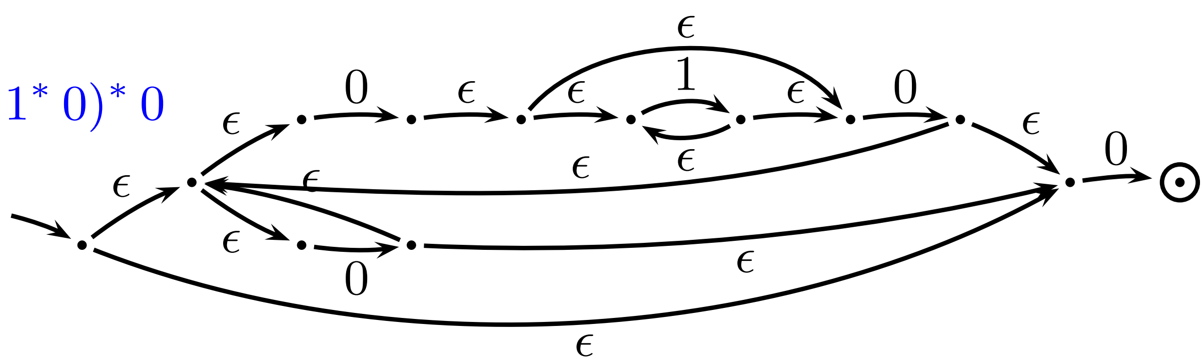
$(0 + 0\,1^*\,0)$



$(0 + 0\,1^*\,0)^*$



$(0 + 0\,1^*\,0)^*\,0$



Proof of Kleene's Theorem: "(ii) $\Rightarrow$ (i)"

Given an NFA $M = (Q, \Sigma, \delta, q_0, F)$, for $X \subseteq Q$ and $q, q' \in Q$, we construct, by induction on the size of X, a regular expression

$$E^X_{q,q'}$$

whose denotation is the set of all strings w such that

> there is a path from q to q' in M labelled by w (i.e. $q \stackrel{w}{\Longrightarrow} q'$) such that all intermediate states along that path lie in X.

It suffices to prove:

Lemma. For any $X \subseteq Q$, for any $q, q' \in Q$, there is a regular expression $E^X_{q,q'}$ satisfying $L(E^X_{q,q'}) =$

$$\{\, w \in \Sigma^* : q \stackrel{w}{\Longrightarrow} q' \text{ in } M \text{ with all intermediate states of seq. in } X \,\}$$

We prove the Lemma by induction on the size of X.

Basis: $X = \emptyset$. Let $a_1, \cdots, a_k$ be all the symbols in $\Sigma \cup \{\, \epsilon \,\}$ such that $q' \in \delta(q, a_i)$. For $q \neq q'$, take

$$E^{\emptyset}_{q,q'} \stackrel{\mathrm{def}}{=} \begin{cases} a_1 + \cdots + a_k & \text{if } k \geq 1 \\ \emptyset & \text{if } k = 0 \end{cases}$$

and for $q = q'$, take

$$E^{\emptyset}_{q,q'} \stackrel{\mathrm{def}}{=} \begin{cases} a_1 + \cdots + a_k + \epsilon & \text{if } k \geq 1 \\ \epsilon & \text{if } k = 0 \end{cases}$$

Inductive step: For a nonempty X, choose an element $r \in X$ - call it the *separating state*. Now any path from q to q' with all intermediate states in X, *either*

(1) never visits r, or

(2) visits r for the first time, followed by a finite number of loops from r back to itself without visiting r in between but staying in X, and finally followed by a path from r to q'.

Thus we take

$$E^{X}_{q,q'} \stackrel{\text{def}}{=} \underbrace{E^{X-\{r\}}_{q,q'}}_{(1)} + \underbrace{E^{X-\{r\}}_{q,r} \cdot (E^{X-\{r\}}_{r,r})^{*} \cdot E^{X-\{r\}}_{r,q'}}_{(2)}$$

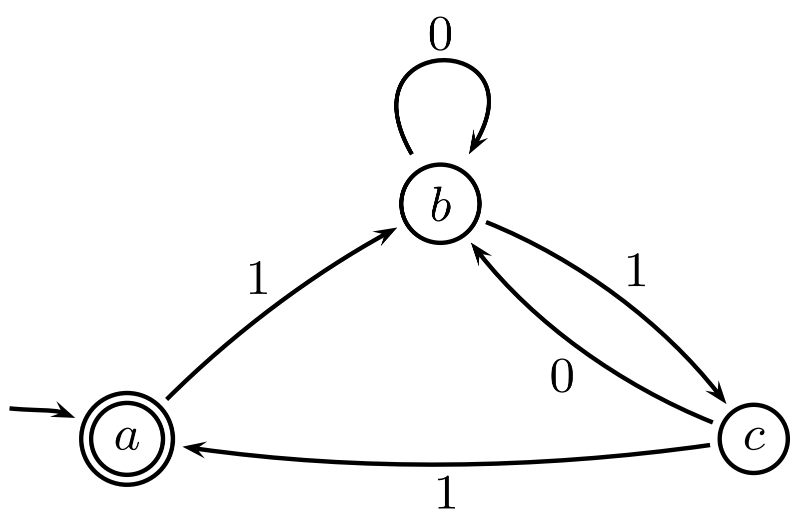
Finally the expression $\sum_{f \in F} E^{Q}_{q_0,f}$ has denotation $L(M)$. $\square$

Heuristic: it is best to choose a separating state r that disconnects the automaton as much as possible.

Example: Transforming NFAs to regular expressions

Consider the NFA $M = (\{ a, b, c \}, \{ 0, 1 \}, \delta, a, \{ a \})$ where δ is given by

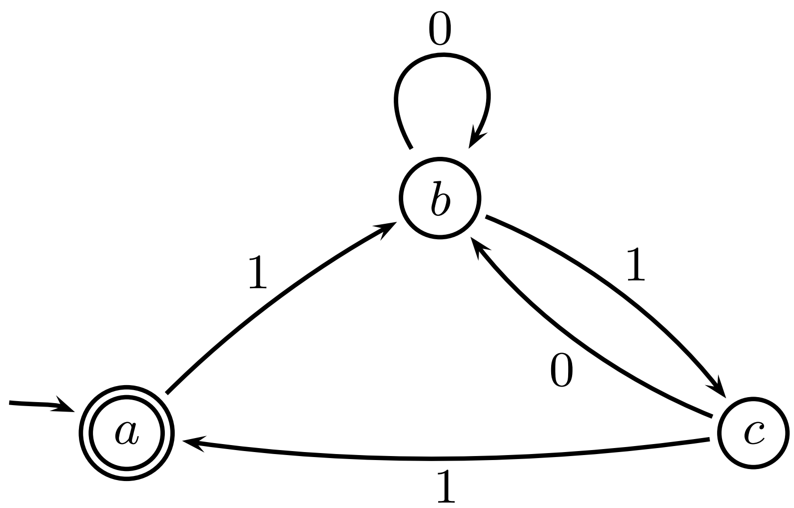
	0	1	ϵ
a	$\emptyset$	$\{ b \}$	$\emptyset$
b	$\{ b \}$	$\{ c \}$	$\emptyset$
c	$\{ b \}$	$\{ a \}$	$\emptyset$



We pick b as the separating state: the resulting regular expression is

$$E_{a,a}^{\{ a,b,c \}} = E_{a,a}^{\{ a,c \}} + E_{a,b}^{\{ a,c \}} \cdot (E_{b,b}^{\{ a,c \}})^* \cdot E_{b,a}^{\{ a,c \}}.$$

By inspection $E_{a,a}^{\{ a,c \}} = \epsilon$, $\quad E_{a,b}^{\{ a,c \}} = 1$ and $E_{b,a}^{\{ a,c \}} = 11$.



Picking c as the separating state, we have

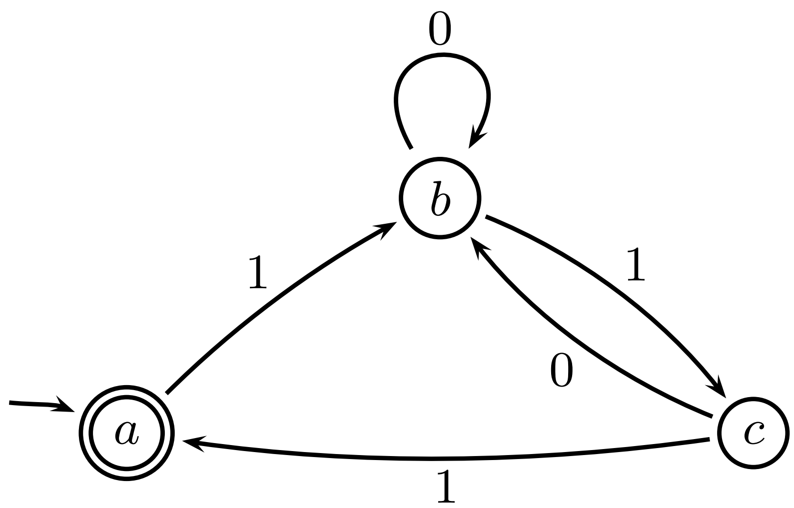
$$E_{b,b}^{\{a,c\}} = E_{b,b}^{\{a\}} + E_{b,c}^{\{a\}} \cdot (E_{c,c}^{\{a\}})^* \cdot E_{c,b}^{\{a\}}$$

where $E_{b,b}^{\{a\}} = 0 + \epsilon$, $E_{b,c}^{\{a\}} = 1$, $E_{c,c}^{\{a\}} = \epsilon$ and $E_{c,b}^{\{a\}} = 0 + 11$.

Hence putting it all together we have

$$E_{a,a}^{\{a,b,c\}} = \epsilon + 1(0 + \epsilon + 1\epsilon^*(0 + 11))^*11 \equiv \epsilon + 1\,(0 + 10 + 111)^*\,11$$

I.e. $L(M) = L(\epsilon + 1\,(0 + 10 + 111)^*\,11)$.

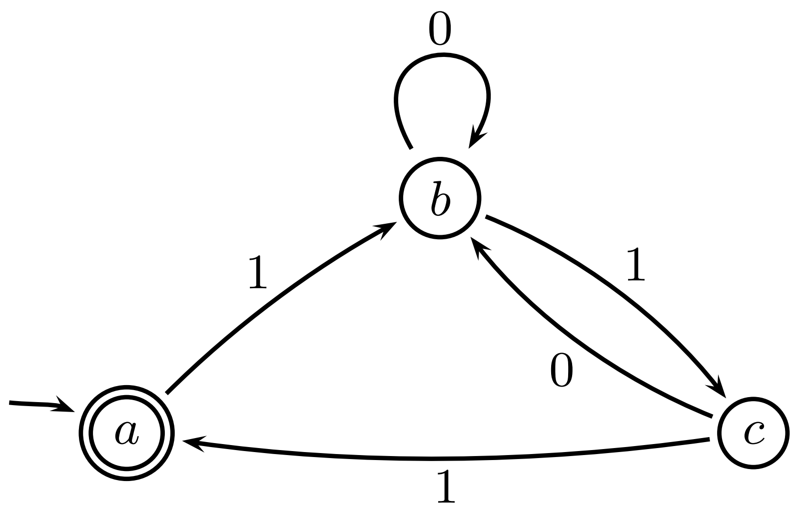


But regular expressions are not uniquely determined by this process. It depends on the choices of seperating states.

If we start with c as the separating state: the resulting regular expression is

$$E_{a,a}^{\{a,b,c\}} = E_{a,a}^{\{a,b\}} + E_{a,c}^{\{a,b\}} \cdot (E_{c,c}^{\{a,b\}})^* \cdot E_{c,a}^{\{a,b\}}.$$

By inspection $E_{a,a}^{\{a,b\}} = \epsilon$, $E_{a,c}^{\{a,b\}} = 10^*1$ and $E_{c,a}^{\{a,b\}} = 1$.



Picking b as the next separating state, we have

$$E^{\{\,a,b\,\}}_{c,c} = E^{\{\,a\,\}}_{c,c} + E^{\{\,a\,\}}_{c,b} \cdot (E^{\{\,a\,\}}_{b,b})^* \cdot E^{\{\,a\,\}}_{b,c}$$

where $E^{\{\,a\,\}}_{c,c} = \epsilon$, $E^{\{\,a\,\}}_{c,b} = 0 + 11$, $E^{\{\,a\,\}}_{b,b} = \epsilon + 0$ and $E^{\{\,a\,\}}_{b,c} = 1$.

Hence putting it all together we have

$$E^{\{\,a,b,c\,\}}_{a,a} = \epsilon{+}10^*1(\epsilon{+}(0{+}11)(\epsilon{+}0)^*1)^*1 \equiv \epsilon{+}10^*1\,((0{+}11)\,0^*\,1)^*1$$

I.e. $L(M) = L(\epsilon + 10^*1\,((0 + 11)\,0^*\,1)^*1)$.

Therefore the two resulting regular expressions have to be equivalent, which can be shown equationally as follows:

$$
\begin{array}{rlr}
 & \epsilon + 1\,0^*1\,((0+11)\,0^*\,1)^*1 & \\
\equiv & \epsilon + 1\,0^*(1\,(0+11)\,0^*)^*\,11 & A(BA)^* \equiv (AB)^*A \\
\equiv & \epsilon + 1\,0^*((10+111)\,0^*)^*\,11 & A(B+C) \equiv AB+AC \\
\equiv & \epsilon + 1\,(0+10+111)^*\,11 & A^*(B\,A^*)^* \equiv (A+B)^*
\end{array}
$$

Question

Is there a procedure (algorithm) that, given a string s and a regular expression E, will decide whether or not s matches E?