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A RESEARCH DIARY

CONTENTS

WHY ARE YOU READING THIS?

If you aren't me, and you're reading this, you should be aware that pretty much everything here is speculative. I'll try on a lot of ideas, and I probably won't do a good job of saying which ones turned out to be bullshit. You can rest assured that that will include most of them.

Hell, I probably won't even manage to really say what I mean. If any of this is legible, it's not because I was trying to make it so.

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Absolute schemes in characteristic p

—11 October 2021—

Mike Hill suggested that I try to keep a regular research diary. It seemed like a good idea.

Here's a basic construction. If **Perf** denotes the category of perfect schemes of characteristic p, then there is an action of the circle $\boldsymbol{T}$ on this category, which on objects is given by the action of $\boldsymbol{Z}$ by the Frobenius F. I want to think about the orbits for this action $\mathbf{Perf}/\boldsymbol{T}$. That's the 2-category whose objects are perfect schemes in characteristic p, whose 1-morphisms are morphisms of such, and whose 2-morphisms $f \to g$ are integers m such that $g = fF^m$.

The reason I like this is that the product $X \times Y$ in sheaves[1] on $\mathbf{Perf}/\boldsymbol{T}$ is well-behaved with respect to étale homotopy. For instance, $\mathrm{Gal}(X \times Y) = \mathrm{Gal}(X) \times \mathrm{Gal}(Y)$, which doesn't usually happen in characteristic p.

[1] for the v topology, I guess

This is all very well, but how should one make sense of the geometry of the objects of $\mathrm{Sh}(\mathbf{Perf}/\boldsymbol{T}) = \mathrm{Sh}(\mathbf{Perf})/\boldsymbol{T}$?[2] One possibility is to think of it as introducing a homotopy between the identity and the Frobenius, which is trivial on the stratified homotopy type. So should one think of the underlying space of X in this category as $X \times S^1$?

[2] In categories, that's the fixed points, but in topoi, it's the orbits.

On the other hand, one of the things I like about the category $\mathrm{Sh}(\mathbf{Perf}/\boldsymbol{T})$ is that the (stratified) étale homotopy type factors through it.

Absolute schemes in characteristic p

—12 October 2021—

I guess the question is which approach to Drinfel'd's lemma do I want to take seriously?

Sometimes people want to regard it as a statement about the product $(X/F^{\mathbf{Z}}) \times (Y/F^{\mathbf{Z}})$; others want to see it as a statement about the product $X \times_{\boldsymbol{F}_1} Y$, where that's now my notation for the product in sheaves on $\mathbf{Perf}/\boldsymbol{T}$.

Maybe what I want to say is that the étale picture of these absolute schemes is not different from the étale picture of them as ordinary schemes, but there might be some finer topology that "sees" the extra loop coming from the Frobenius.

This seems connected to the idea that Ben Zvi suggested: he pointed out that we should probably expect some dimensional reduction from 4 to 3 when considering field theories on $\mathrm{Spec}\, O_K$. As BZ suggested, the geometrization of global Langlands is probably an equivalence of two 4-dimensional field theories, and the way that this makes sense on our 3-dimensional number rings is a dimensional reduction. That dimensional reduction, in effect, is probably a product with a circle.

This is just a naïve dimensional analysis, but it might be made sensible. For instance, we could start by understanding something very elementary – like Spec E for E an algebraic closure of a finite field.

It seems as though I'm asserting that there's a more refined topology on Spec E whose homotopy type is a circle. What then are the points of this new topology? Some additional structures that see that F is nontrivial on E, despite the fact that it's trivial on the Zariski or étale topoi.

Such a structure must, in some sense, measure the size of a field.

Anabelian stratified spaces

—14 October 2021—

Let G be a pyknotic group, and let $X \to BG$ be a pyknotic stratified space over BG. In other words, the fiber $\overline{X}$ over "the" basepoint of BG is a pyknotic stratified space with a homotopy action of G.

From this data, we can construct a genuine G-space. For the orbit category O_G of G, I guess I'll take[3] the category of connected pyknotic spaces over BG whose fibers are finite and discrete. So given my X, the G-space I want to consider carries $T \to BG$ to the space $N\,\mathrm{Fun}_{BG}(T, X)$. If we think of $T = BH$, we're saying the H-fixed points are $N(\overline{X}^{hH})$.

Since the nerve[4] doesn't preserve homotopy fixed points, there is no guarantee that this G-space is cofree – i.e., that the H-fixed points are homotopy fixed points.[5]

So here's the definition. I'll call X ***anabelian*** if and only if this G-space is cofree.

Grothendieck's Section Conjecture says that if $X = \mathrm{Gal}(C)$ for a high-genus complete curve over a number field K, then $X \to BG_K$ is anabelian.

This rhymes with the Sullivan conjecture. I don't know how to make that sentiment precise. But if I understood the proof of the Sullivan conjectures, could I in fact understand this?

[3] One question is whether this is really the correct orbit category. I think it depends on the kinds of examples one wants. In my case, I think this will do.

[4] AKA the "invert-everything" functor

[5] What G-spaces can one get by this sort of move? My temptation is to say that you can model *any* G-space in this way. But I don't really have any idea. It feels kind of Thomason-ish.