CLARK BARWICK

LA LONGUE MARCHE

A RESEARCH DIARY

CONTENTS

2021

October

Absolute schemes in characteristic p —11 October 2021—

Mike Hill suggested that I try to keep a regular research diary. It seemed like a good idea.

Here's a basic construction. If **Perf** denotes the category of perfect schemes of characteristic p, then there is an action of the circle $\boldsymbol{T}$ on this category, which on objects is given by the action of $\boldsymbol{Z}$ by the Frobenius F. I want to think about the orbits for this action $\mathbf{Perf}/\boldsymbol{T}$. That's the 2-category whose objects are perfect schemes in characteristic p, whose 1-morphisms are morphisms of such, and whose 2-morphisms $f \to g$ are integers m such that $g = fF^m$.

The reason I like this is that the product $X \times Y$ in sheaves[1] on $\mathbf{Perf}/\boldsymbol{T}$ is well-behaved with respect to étale homotopy. For instance, $\mathrm{Gal}(X \times Y) = \mathrm{Gal}(X) \times \mathrm{Gal}(Y)$, which doesn't usually happen in characteristic p.

[1] for the v topology, I guess

This is all very well, but how should one make sense of the geometry of the objects of $\mathrm{Sh}(\mathbf{Perf}/\boldsymbol{T}) = \mathrm{Sh}(\mathbf{Perf})/\boldsymbol{T}$?[2] One possibility is to think of it as introducing a homotopy between the identity and the Frobenius, which is trivial on the stratified homotopy type. So should one think of the underlying space of X in this category as $X \times S^1$?

[2] In categories, that's the fixed points, but in topoi, it's the orbits.

On the other hand, one of the things I like about the category $\mathrm{Sh}(\mathbf{Perf}/\boldsymbol{T})$ is that the (stratified) étale homotopy type factors through it.

Absolute schemes in characteristic p —12 October 2021—

I guess the question is which approach to Drinfel'd's lemma do I want to take seriously?

Sometimes people want to regard it as a statement about the product $(X/F^{\mathbf{Z}}) \times (Y/F^{\mathbf{Z}})$; others want to see it as a statement about the product $X \times_{\boldsymbol{F}_1} Y$, where that's now my notation for the product in sheaves on $\mathbf{Perf}/\boldsymbol{T}$.

Maybe what I want to say is that the étale picture of these absolute schemes is not different from the étale picture of them as ordinary schemes, but there might be some finer topology that "sees" the extra loop coming from the Frobenius.

This seems connected to the idea that Ben Zvi suggested: he pointed out that we should probably expect some dimensional reduction from 4 to 3 when considering field theories on $\mathrm{Spec}\, O_K$. As BZ suggested, the geometrization of global Langlands is probably an equivalence of two 4-dimensional field theories, and the way that this makes sense on our 3-dimensional number rings is a dimensional reduction. That dimensional reduction, in effect, is probably a product with a circle.

This is just a naïve dimensional analysis, but it might be made sensible. For instance, we could start by understanding something very elementary – like Spec E for E an algebraic closure of a finite field.

It seems as though I'm asserting that there's a more refined topology on Spec E whose homotopy type is a circle. What then are the points of this new topology? Some additional structures that see that F is nontrivial on E, despite the fact that it's trivial on the Zariski or étale topoi.

Such a structure must, in some sense, measure the size of a field.