Condensed homotopy theory

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PART ONE

PRELIMINARIES

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Foundations

1.1 Cardinals

Mathematicians' 'stock' set theory, ZFC (Zermelo–Fraenkel set theory ZF plus the Axiom of Choice AC) doesn't quite have the expressive power one needs for work with categories and higher categories. The issue ultimately comes down to Cantor's diagonal argument: there is no surjection of a set onto its powerset. This is ultimately why no one can contemplate a set of all sets, and it's also the key to Freyd's observation that if C is a category and κ is the cardinality of its set of arrows, then C has all κ-indexed products only if C is a poset. This, in turn, is what's behind the 'solution set condition' in representability theorems or the Adjoint Functor Theorem. Hence one really must distinguish between 'large' and 'small' objects.

One improves matters by passing to von Neumann–Bernays–Gödel set theory (NBG), which is a conservative extension of ZFC. In NBG, the formal language consists of the symbols $\in$ and $=$; a constant V; suitable variables; the usual connectives of first-order logic ($\neg, \wedge, \vee, \implies$, and $\iff$); and the quantifiers $\forall$ and $\exists$. The objects of the theory are called *classes*. A class x is called a *set* if and only if $x \in V$; a *proper class* is a class X such that $X \notin V$. We summarize the axioms of NBG in informal language:

Extensionality Classes X and Y are equal if and only if, for any z, one has $z \in X$ if and only if $z \in Y$.

Regularity For every class X, there exists an element $z \in X$ such that $z \cap X = \varnothing$.

Infinity There is an infinite set.

Union If x is a set, then $\bigcup x = \bigcup_{z \in x} z$ is a set as well.

Pairing If x and y are sets, then $\{x, y\}$ is a set as well.

Powerset If x is a set, then the powerset $P(x)$ is a set as well.

Limitation of size A class X is a proper class if and only if there is a bijection between X and V.

Class comprehension For every first-order formula $\phi(x)$ with a free variable x in which the quantifiers are over sets, there exists a class $\{x \in V : \phi(x)\}$ whose elements are exactly those sets x such that $\phi(x)$.

In matters of set theory, we will generally follow the notations and terminological conventions of the comprehensive monograph of Jech (2003). We will also refer to the texts of Drake (1974) and Kanamori (2009) In particular, **Ord** denotes the proper class of ordinal numbers. For any ordinal α, the set V_α is defined recursively as follows:

1 If $\alpha = 0$, then $V_\alpha \coloneqq \varnothing$;
2 if $\alpha = \beta + 1$ for an ordinal number β, then $V_\alpha \coloneqq P(V_\beta)$;
3 if α is a limit ordinal, then $V_\alpha \coloneqq \bigcup_{\beta<\alpha} V_\beta$.

If x is a set, then the *rank* of x is the smallest ordinal number α such that $x \in V_\alpha$. The proper class V is then the union $\bigcup_{\alpha\in\mathbf{Ord}} V_\alpha$.

Definition 1.1.0.1 A *large-category*[1] C consists of a sequence $(C_n)_{n\in\mathbf{N}_0}$ of classes, along with a family of class maps $\phi^* : C_m \to C_n$ for each map $\phi\colon n \to m$, subject to all the formulas that express the statement that C_n is a simplicial class that satisfies the inner Kan condition.

If the large-category C contains a full subcategory $C' \subseteq C$ such that each C'_n is a set and such that every object of C is equivalent to an object of C', then we will call C a *category* or, for emphasis, a *small category*.[2]

A large-category C is said to be *locally small* if and only if, for every subset $C'_0 \subseteq C_0$, the full subcategory $C' \subseteq C$ spanned by the elements of C'_0 is small.

Limits and colimits are only considered for functors $J \to C$ in which the J is a small category. Hence when we refer to *all limits* or *all colimits*, we mean all limits or colimits of diagrams indexed on small categories.

Example 1.1.0.2 We shall write $\mathbf{Set}^V$ for the large-category of all sets. The objects of $\mathbf{Set}^V$ are thus precisely the elements of V.

We shall write $\mathbf{An}^V$ for the large-category of all animae.[3] The objects of $\mathbf{An}^V$ are thus precisely the Kan complexes $\Delta^{op} \to \mathbf{Set}^V$.

We shall write $\mathbf{Cat}^V$ for the large-category of all categories. The objects of $\mathbf{Cat}^V$ are thus precisely the weak Kan complexes $\Delta^{op} \to \mathbf{Set}^V$.

[1] We use the term 'category' for what other authors might call '∞-category', '$(\infty, 1)$-category', or 'quasicategory'. We will use the term '1-category' when the specification is needed.
[2] These are sometimes called *essentially small.*
[3] We follow Clausen and Scholze, and we use the term 'anima' for what other authors might call 'space', '∞-groupoid', or '$(\infty, 0)$-category'.

The Class Comprehension Axiom Schema implies the *Axiom of Global Choice*, which ensures the existence of a choice function $\tau\colon V \to V$ such that $\tau(x) \in x$. One needs this to make sense of a construction like 'the' functor $- \times u\colon C \to C$ for a large-category C with all finite products and a fixed object $u \in C$.

Here, we will work with NBG as our base theory, so that we may speak of proper classes and large-categories whenever the occasion arises.

However, the whole project of higher category theory turns on the principle that we want to be able to deal with the collections of all objects of a given kind as a mathematical object in its own right, and that the passage up and down these category levels is a fruitful way to understand even completely 'decategorified' objects.[4] If we have a large-category C, NBG provides us with no mechanism to view C itself as an object of a still larger category of all categories. This limits the kinds of operations we are permitted to do to C.

To give ourselves the room to pass up and down category levels, we need to have a hierarchy of 'scales' at which we can work. These scales will be identified by inaccessible cardinals or, equivalently, Grothendieck universes. The existence of these cardinals is independent of NBG.

But now a rather different concern arises. We will also want to have reasonable assurance that the results we obtain at one scale remain valid at other scales. We might prefer to prove sentences about *all* sets, animae, groups, etc. – not just those within a universe. This sort of 'scale-invariance of truth' is expressed by a *reflection principle* (1.2). The reflection principle we will use here states roughly that there is an inaccessible cardinal κ such that statements of set theory hold in the universe V_κ just in case they hold in V itself. This is the *Lévy scheme* LÉVY, which we will first formulate as a large cardinal axiom. In effect, this permits us to focus our attention on *categories* (as opposed to large-categories). This scheme has a strictly higher conistency strength than the existence of a proper class of inaccessible cardinals; however, its consistency strength is strictly lower than that of the existence of a single Mahlo cardinal.

Following beautiful work of Hamkins (2003), we will later prove that the Lévy scheme is equivalent to (not only equiconsistent with) a *maximality principle*, which states roughly that every sentence of set theory that is true for a sieve of forcing extensions is true.

[4] The *microcosm principle* of Baez–Dolan is a precise illustration of this principle.

1.1.1 Regularity and smallness

Definition 1.1.1.1 If κ is a cardinal, then a set S is *κ-small* if and only if $|S| < \kappa$. We shall write $\mathbf{Set}^{\kappa} \subset \mathbf{Set}^{V}$ for the category of κ-small sets, Thus $\mathbf{Set}^{V}$ is the filtered union of the categories $\mathbf{Set}^{\kappa}$ over the proper class of regular cardinals.

A cardinal κ is *regular* if and only if, for every map $f : S \to T$ in which T and every fiber $f^{-1}\{t\}$ are all κ-small, the set S is κ-small as well. Equivalently, κ is regular if and only if $\mathbf{Set}^{\kappa} \subset \mathbf{Set}^{V}$ is stable under colimits indexed by κ-small posets.

Example 1.1.1.2 Under this definition, 0 is a regular cardinal. Many texts require that a regular cardinal be infinite.

Example 1.1.1.3 The $\aleph$ family of cardinals is defined by a function from the class of ordinal numbers to the class of cardinal numbers, by transfinite induction:

1. The cardinal $\aleph_0$ is the ordinal number ω consisting of all finite ordinals.
2. For any ordinal α, one defines $\aleph_{\alpha+1}$ to be the smallest cardinal number strictly greater than $\aleph_\alpha$.
3. For any limit ordinal α, one defines $\aleph_\alpha \coloneqq \sup\left\{\aleph_\beta : \beta < \alpha\right\}$.

The countable cardinal $\aleph_0$ is regular. Every successor cardinal is regular; consequently, $\aleph_n$ for $n \in \boldsymbol{N}$ is regular as well. The cardinal $\aleph_\omega$ is the smallest infinite cardinal that is not regular.

1.1.1.4 If κ is a regular cardinal, then $\mathbf{Set}^{\kappa} \subset \mathbf{Set}^{V}$ is the full subcategory generated by the singleton $\{0\}$ under colimits over κ-small posets.

Definition 1.1.1.5 If κ is a regular cardinal, then we shall write $\mathbf{An}^{\kappa} \subset \mathbf{An}^{V}$ for the full subcategory generated by $\{0\}$ under colimits over κ-small posets. The objects of $\mathbf{An}^{\kappa}$ will be called *κ-small animae*.

Similarly, we shall write $\mathbf{Cat}^{\kappa} \subset \mathbf{Cat}^{V}$ for the full subcategory generated by $\{0\}$ and $\{0 < 1\}$ under colimits over κ-small posets. The objects of $\mathbf{Cat}^{\kappa}$ will be called *κ-small categories*.

Finally, a large-category C is said to be *locally λ-small* if and only if, for every λ-small subset $C_0' \subseteq C_0$ of objects of C, the full subcategory $C' \subseteq C$ that it spans is λ-small.

Example 1.1.1.6 A set is $\aleph_0$-small if and only if it is finite.

An anima is $\aleph_0$-small if and only if it is weak homotopy equivalent to a simplicial set with only finitely many nondegenerate simplices.

A category C is $\aleph_0$-small if and only if it is Joyal equivalent to a simplicial set with only finitely many nondegenerate simplices.

Example 1.1.1.7 Why does regularity arise so often in category theory? What role does this hypothesis play? Here is the sort of scenario that is often lurking in the background when we appeal to the regularity of a cardinal.

Let C be a large-category. Suppose that we have a *diagram of diagrams* in C, in the following sense. We have a category A; a functor $B\colon A \to \mathbf{Cat}^V$; and for each $\alpha \in A$, a functor $X_\alpha\colon B_\alpha \to C$. Furthermore, the colimits of each of these functors organize themselves into a functor $A \to C$:

$$\alpha \mapsto \operatorname*{colim}_{\beta\in B_\alpha} X_\alpha(\beta)\ .$$

We will often be in situations in which we need to analyze the *colimit of colimits*:

$$\operatorname*{colim}_{\alpha\in A}\operatorname*{colim}_{\beta\in B_\alpha} X_\alpha(\beta)\ .$$

In this case, we may reorganize these data. We first construct the cocartesian fibration corresponding to the functor B, which we will abusively write $B \to A$, since the fibers are the categories B_α. We now have a single functor $X\colon B \to C$ whose restriction to any fiber B_α is the functor X_α. Now the colimit of colimits above is a single colimit:

$$\operatorname*{colim}_{\alpha\in A}\operatorname*{colim}_{\beta\in B_\alpha} X_\alpha(\beta) \simeq \operatorname*{colim}_{\gamma\in B} X(\gamma)\ .$$

Now let κ be a cardinal. If A is κ-small, and if each category B_α is κ-small, then what can we conclude about B? In general, nothing. However, if κ is a regular cardinal, then B is also κ-small.

The motto here, then, is that *if κ is regular, then κ-small colimits of κ-small colimits are κ-small colimits.*

1.1.2 Accessibility and presentablity

Definition 1.1.2.1 Let κ be a regular cardinal. A category Λ is *κ-filtered* if and only if it satisfies the following equivalent conditions:

1. For every κ-small category J, every functor $f\colon J \to \Lambda$ can be extended to a functor $F\colon J^{\triangleright} \to \Lambda$.
2. For every κ-small category J and every functer $H\colon \Lambda\times J \to \mathbf{An}^V$, the natural morphism
$$\operatorname*{colim}_{\lambda\in\Lambda}\lim_{j\in J} H(\lambda, j) \to \lim_{j\in J}\operatorname*{colim}_{\lambda\in\Lambda} H(\lambda, j)$$
is an equivalence.
3. For every κ-small category J, the diagonal functor $\Lambda \to \operatorname{Fun}(J, \Lambda)$ is cofinal.

Example 1.1.2.2 For any regular cardinal κ, the ordinal κ, regarded as a category, is κ-filtered.

More generally, a poset is κ-filtered if and only if every κ-small subset thereof is dominated by some element.

Example 1.1.2.3 A κ-small category is κ-filtered if and only if it contains a terminal object.

Definition 1.1.2.4 Let κ be a regular cardinal. A functor $f : C \to D$ between large-categories will be said to be *κ-continuous* if and only if it preserves κ-filtered colimits.

An object X of a locally small large-category C is said to be *κ-compact* if and only if the functor $よ^X : C \to \mathbf{An}^V$ corepresented by X (i.e., the functor $Y \mapsto \mathrm{Map}_C(X, Y)$) is κ-continuous. We write $C^{(\kappa)} \subseteq C$ for the full subcategory of κ-compact objects.

A large-category C is *κ-accessible* if and only if it satisfies the following conditions:

1 The category C is locally small.
2 The category C has all κ-filtered colimits.
3 The subcategory $C^{(\kappa)}$ generates C under κ-filtered colimits.

A κ-accessible large-category C is *κ-presentable*[5] if and only if $C^{(\kappa)}$ has all κ-small colimits.

Example 1.1.2.5 Let κ be an uncountable regular cardinal. Then the following are equivalent for a category C.

1 The category C is κ-small.
2 The set of equivalence classes of objects of C is κ-small, and for every morphism $f : X \to Y$ of C and every $n \in \mathbf{N}_0$, the set $\pi_n(\mathrm{Map}_C(X, Y), f)$ is κ-small.
3 The category C is κ-compact as an object of $\mathbf{Cat}^V$; that is, $\mathbf{Cat}^\kappa = \mathbf{Cat}^{V,(\kappa)}$.

In particular, an anima X is κ-small if and only if all its homotopy sets are κ-small, if and only if it is κ-compact as an object of $\mathbf{An}^V$.

Example 1.1.2.6 The equivalence above is doubly false if $\kappa = \aleph_0$.

First, an $\aleph_0$-compact anima is a *retract* of an $\aleph_0$-small anima, but it may not be $\aleph_0$-small itself. If X is $\aleph_0$-compact and *simply connected*, then X is $\aleph_0$-small. The obstruction is the *de Lyra–Wall finiteness obstruction*, which lies in the reduced K_0 of the group ring $\mathbf{Z}[\pi_1(X)]$.

[5] Some authors use the phrase *κ-compactly generated* instead.

Second, the homotopy sets of an $\aleph_0$-small X anima are not generally finite. By a theorem of Serre, if each connected component $Y \subseteq X$ has finite fundamental group, then its homotopy groups are finitely generated. But if $\pi_1(X)$ isn't finite, this too fails; for example, $\pi_3(S^1 \vee S^2)$ is not finitely generated.

1.1.2.7 Let $\kappa \leq \lambda$ be regular cardinals. A κ-small category is λ-small. A λ-filtered category is κ-filtered. A κ-continuous functor is λ-continuous. In general, however, there are κ-accessible categories that are not λ-accessible.

Definition 1.1.2.8 Let κ and λ be regular cardinals. We write $\kappa \ll \lambda$ if and only if, for every pair of cardinals $\kappa_0 < \kappa$ and $\lambda_0 < \lambda$, one has $\lambda_0^{\kappa_0} < \lambda$. Equivalently, $\kappa \ll \lambda$ if and only if, for every κ-small set A and every λ-small set B, the set $\mathrm{Map}(A, B)$ is λ-small.

Example 1.1.2.9 For every infinite regular cardinal κ, one has $\aleph_0 \ll \kappa$.

1.1.2.10 If $\kappa \ll \lambda$ are regular cardinals, then every κ-accessible category is λ-accessible. Similarly, every κ-presentable category is λ-presentable.

Definition 1.1.2.11 A large-category C is *accessible* if and only if there exists a regular cardinal κ such that C is κ-accessible.

We shall say that C is *presentable* if and only if there exists a regular cardinal κ such that C is κ-presentable.

1.1.2.12 A large-category is presentable if and only if it is accessible and has all colimits. A presentable category automatically admits all limits:

Definition 1.1.2.13 A large-category C is *locally presentable*[6] if and only if, every object $X \in C$ is contained in a presentable full subcategory $C' \subseteq C$ such that the inclusion $C' \hookrightarrow C$ preserves colimits.

1.1.3 Presheaf categories

Let C be a large-category. What happens if we seek to make sense in NBG of the category $\tau_0 \boldsymbol{P}(C)$ of presheaves of sets $C^{op} \to \mathbf{Set}^V$?

Right away we encounter a problem: if the objects of C form a proper class C_0, then there is no class of class maps $\mathrm{Map}(C_0, V)$. Indeed, on one hand, in NBG, every element of a class is itself a set, and on the other hand, a class map $f : C_0 \to V$ cannot be a set.[7]

[6] In the 1-category literature, the phrase *locally presentable category* is used for what we call *presentable category*.

[7] Worse still, the 'very large' category of classes is not cartesian closed, so there's no hope of defining $\mathrm{Map}(C_0, V)$ by means of some other artifice.

1.1.3.1 If C is a small category, then the large-category $\boldsymbol{P}_0(C)$ is locally small, and it enjoys many of the same good properties enjoyed by $\mathbf{Set}^V$ itself. For every regular cardinal κ, it is κ-presentable, and it is *cartesian closed*: for every pair of presheaves $X, Y : C^{op} \to \mathbf{Set}^V$, the morphisms $X \to Y$ form a presheaf $\mathrm{Mor}(X, Y) : D^{op} \to \mathbf{Set}^V$. The category $\boldsymbol{P}_0(C)$ is a *1-topos*.

Similarly, the category $\boldsymbol{P}(C)$ of presheaves $C^{op} \to \mathbf{An}^V$ is a κ-presentable topos for every regular cardinal κ.

Example 1.1.3.2 Let C be a locally small category. If $Y \in C$ is an object, then $よ_Y : C^{op} \to \mathbf{Set}^V$ is the presheaf $X \mapsto \mathrm{Map}_C(X, Y)$ represented by Y.

Dually, if $X \in C$ is an objects, then $よ^X : C \to \mathbf{Set}^V$ is the functor $Y \mapsto \mathrm{Map}_C(X, Y)$ corepresented by X.

Definition 1.1.3.3 Let C be a locally small large-category. A *small presheaf* of sets on C is a functor $C^{op} \to \mathbf{Set}^V$ that is left Kan extended from its restriction to some small full subcategory $D \subseteq C$. We write $\boldsymbol{P}_0^{sm}(C)$ for the locally small large-category of small presheaves of sets.

Similarly, a *small presheaf* (of animae) is a functor $C^{op} \to \mathbf{An}^V$ that is left Kan extended from its restriction to some small full subcategory $D \subseteq C$. We write $\boldsymbol{P}^{sm}(C)$ for the locally small large-category of small presheaves.

1.1.3.4 For any small full subcategory $D \subseteq C$, we may contemplate the large-category $\boldsymbol{P}(D)$ of presheaves $D^{op} \to \mathbf{An}^V$. If we have an inclusion of full subcategories $D' \subseteq D \subset C$, then left Kan extension identifies $\boldsymbol{P}(D')$ with a full subcategory of $\boldsymbol{P}(D)$.

The (class-indexed) filtered union $\bigcup_D \boldsymbol{P}_0(D)$ over the class of small full subcategories of C is precisely the large-category $\boldsymbol{P}^{sm}(C)$.

Example 1.1.3.5 Assume that C is locally small. For any object $Y \in C$, the representable presheaf $よ_Y$ is left Kan extended from any full subcategory that contains Y. In particular, $よ_Y$ is small.

Thus the assignment $Y \mapsto よ_Y$ is the fully faithful *Yoneda embedding*

$$よ : C \hookrightarrow \boldsymbol{P}^{sm}(C) .$$

Example 1.1.3.6 If C^{op} is accessible, then $\boldsymbol{P}_0^{sm}(C)$ and $\boldsymbol{P}^{sm}(C)$ are the categories of accessible functors $C^{op} \to \mathbf{Set}^V$ and $C^{op} \to \mathbf{An}^V$, respectively.

1.1.3.7 The categories $\boldsymbol{P}_0^{sm}(C)$ and $\boldsymbol{P}^{sm}(C)$ may not enjoy all the same good features that $\mathbf{Set}^V$ and $\mathbf{An}^V$ have. The categories $\boldsymbol{P}_0^{sm}(C)$ and $\boldsymbol{P}^{sm}(C)$ possess all colimits, but they do not generally have all limits. For example, if C has no nonidentity arrows, then there is no terminal object in $\boldsymbol{P}_0^{sm}(C)$.

If C^{op} is accessible or small, then $\boldsymbol{P}_0^{sm}(C)$ and $\boldsymbol{P}^{sm}(C)$ do have all limits.

Definition 1.1.3.8 Let A be a class of categories. Let C be a locally small large-category, and let $C^0 \subseteq C$ be a full subcategory. Then we say that C^0 *generates* C *freely under* A*-shaped colimits* if and only if, for every large-category D that has all A-shaped colimits, the following assertions obtain.

1. Every functor $C^0 \to D$ extends to a functor $C \to D$ that preserves A -shaped colimits.
2. For every pair of functors $F, G\colon C \to D$ that preserve A-shaped colimits, the map $\mathrm{Map}(F, G) \to \mathrm{Map}(F|C^0, G|C^0)$ is an equivalence.

If $f : C' \hookrightarrow C$ is a fully faithful functor, then we will say that f *generates* C *freely under* A*-shaped colimits* if and only if its image $f(C') \subseteq C$ does so.

Remark 1.1.3.9 If C is not small, then in NBG we can make sense neither of $\mathrm{Fun}(C, D)$, nor of the full subcategory $\mathrm{Fun}^A(C, D) \subseteq \mathrm{Fun}(C, D)$ consisting of those functors that preserve A-shaped colimits. If however we are in a situation in which these objects *can* be made sensible, then C^0 generates C freely under A-shaped colimits if and only if the restriction induces an equivalence

$$\mathrm{Fun}^A(C, D) \xrightarrow{\sim} \mathrm{Fun}(C^0, D) .$$

Proposition 1.1.3.10 *Let C be a locally small large-category. Then the Yoneda embedding* $よ : C \hookrightarrow \boldsymbol{P}^{sm}(C)$ *generates* $\boldsymbol{P}^{sm}(C)$ *freely under all colimits.*

The theory of small presheaves can be relativized to a regular cardinal κ:

Definition 1.1.3.11 Let κ be a regular cardinal. Let C be a locally κ-small large-category. A κ*-small presheaf* of sets on C is a functor $C^{op} \to \mathbf{Set}^{\kappa}$ that is left Kan extended from its restriction to some κ-small full subcategory $D \subseteq C$. The large-category of κ-small presheaves of sets will be denoted $\tau_0 \boldsymbol{P}^{\kappa}(C)$.

Similarly, a κ*-small presheaf* (of animae) is a functor $C^{op} \to \mathbf{An}^{\kappa}$ that is left Kan extended from its restriction to some κ-small full subcategory $D \subseteq C$. The large-category of κ-small presheaves will be denoted $\boldsymbol{P}^{\kappa}(C)$.

1.1.3.12 If C is small, then so is $\boldsymbol{P}^{\kappa}(C)$.

Since we have assumed that C is locally κ-small, it follows that the Yoneda embedding lands in $\boldsymbol{P}^{\kappa}(C)$.

Proposition 1.1.3.13 *Let κ be a regular cardinal. Let C be a locally κ-small large-category. Then the Yoneda embedding* $よ : C \hookrightarrow \boldsymbol{P}^{\kappa}(C)$ *generates* $\boldsymbol{P}^{\kappa}(C)$ *freely under κ-small colimits.*

1.1.3.14 The category $\boldsymbol{P}^{\kappa}_0(C)$ has all κ-small colimits, but in general, it does not have κ-small limits, and it is not cartesian closed. To ensure these properties as well, we must turn to a discussion of inaccessible cardinals.

1.1.4 Strong limit and inaccessible cardinals

Definition 1.1.4.1 One says that κ is a *weak limit cardinal* if and only if, for every cardinal ξ, if $\xi < \kappa$, then $\xi^+ < \kappa$.

A cardinal κ is said to be a *strong limit cardinal* if and only if, for every cardinal ξ, if $\xi < \kappa$, then $2^\xi < \kappa$ as well. Equivalently, κ is a strong limit cardinal if and only if, for every pair of κ-small sets X and Y, the set $\mathrm{Map}(X, Y)$ of maps $X \to Y$ is κ-small as well.

One says that κ is *weakly inaccessible* if and only if it is a regular, uncountable, weak limit cardinal.

One says that κ is *inaccessible*[8] if and only if it is a regular, uncountable, strong limit cardinal. Equivalently, an uncountable cardinal κ is inaccessible if and only if $\mathbf{Set}^\kappa$ has all κ-small colimits and is cartesian closed. Equivalently again, an uncountable cardinal κ is inaccessible if and only if $\mathbf{Set}^\kappa$ has all κ-small colimits and all κ-small colimits.

The Generalized Continuum Hypothesis (GCH) is equivalent to the statement that the classes of strong and weak limit cardinals coincide, and similarly the classes of inaccessible and weakly inaccessible cardinal coincide.

Example 1.1.4.2 A cardinal κ is a weak limit cardinal if and only if, for some limit ordinal α, one has $\kappa = \aleph_\alpha$.

Example 1.1.4.3 The $\beth$ family of cardinals is defined by a function from the class of ordinal numbers to the class of cardinal numbers. It's defined by transfinite induction:

1 By definition, $\beth_0 = \aleph_0$.
2 For any ordinal α, one defines $\beth_{\alpha+1} \coloneqq 2^{\beth_\alpha}$.
3 For any limit ordinal α, one defines $\beth_\alpha \coloneqq \sup\left\{\beth_\beta : \beta < \alpha\right\}$.

The cardinal $\beth_\alpha$ is the cardinality of $V_{\omega+\alpha}$.

The Generalized Continuum Hypothesis (GCH) is equivalent to the statement that $\aleph_\alpha = \beth_\alpha$ for each ordinal α,

A cardinal κ is a strong limit cardinal if and only if, for some limit ordinal α, one has $\kappa = \beth_\alpha$.

The cardinal $\beth_\omega$ is the smallest uncountable strong limit cardinal. It is not inaccessible, however, because it is not regular.

An inaccessible cardinal κ is a $\beth$-*fixed point*: that is, $\beth_\kappa = \kappa$.

1.1.4.4 A regular uncountable cardinal κ is inaccessible if and only if one has $\kappa \ll \kappa$.

[8] Some authors say *strongly inaccessible* instead of *inaccessible*.

Definition 1.1.4.5 (SGA 4 I, Exposé I, §0 and Appendix) An uncountable set U is a *Grothendieck universe* if it satisfies the following conditions.

1. The set U is *transitive*: if $X \in Y \in U$, then $X \in U$ as well.
2. If $X, Y \in U$, then $\{X, Y\} \in U$ as well.
3. If $X \in U$, then the powerset $P(X) \in U$ as well.
4. If $A \in U$ and $X : A \to U$ is a map, then

$$\bigcup_{\alpha \in A} X(\alpha) \in U$$

as well.

Grothendieck universes are essentially the same thing as inaccessible cardinals. This was effectively proved by Tarski (1938). See also Bourbaki, SGA 4 I, Exposé I, Appendix.

Proposition 1.1.4.6 *If κ is an inaccessible cardinal, then the set V_κ of all sets of rank less than κ is a Grothendieck universe of rank and cardinality κ.*

If U is a Grothendieck universe, then there exists an inaccessible cardinal κ such that $U = V_\kappa$.

Theorem 1.1.4.7 *If κ is an inaccessible cardinal, then $V_\kappa \models$ ZFC, and $V_{\kappa+1} \models$ NBG.*

Assuming that ZFC (respectively, NBG) is consistent, then the existence of inaccessible cardinals is not provable by methods formalizable in ZFC (resp., NBG).

Axiom 1.1.4.8 The *Axiom of Universes* (AU) is the assertion that every cardinal is dominated by an inaccessible cardinal, or, equivalently, every set is an element of some Grothendieck universe. *Tarski–Grothendieck set theory* is the schema TG = NBG + AU.

Under AU, the proper class of inaccessible cardinals can be well ordered. It will be helpful for us to have a notation for this.

Definition 1.1.4.9 Assume AU. Let us define the $\daleth$ family of cardinals as a function from the class of ordinal numbers to the class of cardinal numbers:

1. By definition, $\daleth_0 = \aleph_0$.
2. For any ordinal α, one defines $\daleth_{\alpha+1}$ as the smallest inaccessible number greater than $\daleth_\alpha$.
3. For any limit ordinal α, one defines $\daleth_\alpha \coloneqq \sup\left\{\daleth_\beta : \beta < \alpha\right\}$.

Thus $\daleth_0 = \aleph_0$, and $\daleth_\alpha$ is the 'α-th inaccessible cardinal'.

The notions of smallness, accessibility, and presentability of categories can all be relativized to a Grothendieck universe.

Definition 1.1.4.10 Let α be an ordinal number. A category C is *of rank* α if and only if it is locally $\beth_\alpha$-small and $\beth_{\alpha+1}$-small. We will say that C is *small of rank* α if and only if it is of rank α and $\beth_\alpha$-small.

Example 1.1.4.11 Let α be an ordinal number, and let $\kappa \coloneqq \beth_\alpha$. The categories $\mathbf{Set}^\kappa$ and $\mathbf{An}^\kappa$ are of rank α.

More generally, if C is a κ-small category, then $\boldsymbol{P}^\kappa_0(C)$ and $\boldsymbol{P}^\kappa(C)$ are each locally λ-small.

Definition 1.1.4.12 Let λ be an inaccessible cardinal, and let κ be a regular cardinal less than λ. A functor between categories is *κ-continuous below λ* if and only if it preserves all λ-small, κ-filtered colimits.

Let C be a locally λ-small category. An object X of C is said to be *κ-compact below λ* if and only if the functor $\text{よ}^X : C \to \mathbf{An}^\lambda$ corepresented by X is κ-continuous below λ. We write $C^{\kappa<\lambda} \subseteq C$ for the full subcategory of κ-compact objects below λ.

A category C is *κ-accessible below λ* if and only if it satisfies the following trio of conditions:

1. The category C is locally small below λ.
2. The category C has all λ-small, κ-filtered colimits.
3. The subcategory $C^{\kappa<\lambda}$ generates C under λ-small and κ-filtered colimits.

A category C is *κ-presentable below λ* if and only if it is κ accessible below λ, and $C^{\kappa<\lambda}$ has all κ-small colimits.

1.1.5 Ind and Pro

Definition 1.1.5.1 Let $\kappa \leq \lambda$ be regular cardinals. Let C be a locally λ-small category. Then $\mathrm{Ind}^\lambda_\kappa(C)$ is the smallest full subcategory $D \subseteq \boldsymbol{P}^\lambda(C)$ such that D contains the image of the Yoneda embedding $\text{よ} : C \hookrightarrow \boldsymbol{P}^\lambda(C)$, and D is stable under λ-small, κ-filtered colimits.

Accordingly, if C is a locally small large-category, then $\mathrm{Ind}^{\boldsymbol{V}}_\kappa$ is the smallest full subcategory $D \subseteq \boldsymbol{P}^{sm}(C)$ such that D contains the image of the Yoneda embedding $\text{よ} : C \hookrightarrow \boldsymbol{P}^{sm}(C)$, and D is stable under κ-filtered colimits.

Example 1.1.5.2 The category $\mathrm{Ind}^\kappa_\kappa(C)$ is equivalent to C itself.

Proposition 1.1.5.3 *Let $\kappa \leq \lambda$ be regular cardinals. Let C be a locally λ-small category. Then the Yoneda embedding $\text{よ} : C \hookrightarrow \mathrm{Ind}^\lambda_\kappa(C)$, generates $\mathrm{Ind}^\lambda_\kappa(C)$ freely under λ-small, κ-filtered colimits.*

Similarly, if C is a locally small large-category, then the Yoneda embedding $\text{よ} : C \hookrightarrow \mathrm{Ind}^V_\kappa(C)$, generates $\mathrm{Ind}^V_\kappa(C)$ freely under κ-filtered colimits.

Notation 1.1.5.4 Let $\kappa \leq \lambda$ be regular cardinals. Let C and D be locally λ-small categories that contain all λ-small, κ-filtered colimits. Denote by $\mathrm{Fun}^{\kappa\leq\lambda}(C, D)$ the full subcategory of $\mathrm{Fun}(C, D)$ consisting of the functors $C \to D$ that preserve all λ-small, κ-filtered colimits.

If C^0 is a locally λ-small category, then restriction along the Yoneda embedding induces and equivalence of categories

$$\mathrm{Fun}^{\kappa\leq\lambda}(\mathrm{Ind}^{\lambda}_{\kappa}(C^0), D) \xrightarrow{\sim} \mathrm{Fun}(C^0, D) .$$

Example 1.1.5.5 Let $\kappa \leq \lambda \leq \mu$ be regular cardinals. Then for any

1.1.6 Higher inaccessibility

1.1.6.1 We shall endow an ordinal with its order topology. This may be described recursively as follows:

1 The ordinal 0 is the empty topological space.
2 For any ordinal α with its order topology, the order topology on the ordinal $\alpha + 1$ is the one-point compactification of α.
3 For any limit ordinal α, the order topology is the colimit topology $\mathrm{colim}_{\beta<\alpha} \beta$.

We will use terminology that treats **Ord** itself as a topological space, even though it is not small.

Definition 1.1.6.2 If $W \subseteq \mathbf{Ord}$ is a subclass, then a *limit point* of A is an ordinal α such that $\alpha = \sup(W \cap \alpha)$. The class W will be said to be *closed* if and only if it contains all its limit points.

An *ordinal function* is a class map $f : \mathbf{Ord} \to \mathbf{Ord}$. We say that f is *continuous* if and only if its restriction to any subset is continuous. Equivalently, f is continuous if and only if, for every subclass $W \subseteq \mathbf{Ord}$ and every limit point α of W, the ordinal $f(\alpha)$ is a limit point of $f(W)$.

We say that f is *normal* if and only if it is continuous and strictly increasing.

1.1.6.3 If f is a normal ordinal function, then its image is a closed and unbounded class[9] of ordinals. Conversely, if $W \subseteq \mathbf{Ord}$ is a closed and unbounded class, then we can define a normal ordinal function f by

$$f(\alpha) = \min\left\{\gamma \in W : (\forall\beta < \alpha)(f(\beta) < \gamma)\right\} .$$

Definition 1.1.6.4 Let f be an ordinal function. A regular cardinal κ is said to be *f-inaccessible* if and only if, for every ordinal α, if $\alpha < \kappa$, then $f(\alpha) < \kappa$ as well.

[9] This is often abbreviated *club class* in set theory literature.

Example 1.1.6.5 If f is the ordinal function that carries an ordinal α to the cardinal $2^{|\alpha|}$, then an f-inaccessible cardinal is precisely an inaccessible cardinal.

Construction 1.1.6.6 Let f be an increasing ordinal function such that for every ordinal β, one has $\beta < f(\beta)$. For every ordinal ξ, the normal ordinal function $\alpha \mapsto f^{\alpha}(\xi)$ in uniquely specified by the requirements that $f^{0}(\xi) = \xi$ and $f^{\alpha+1}(\xi) = f(f^{\alpha}(\xi))$.

Jorgensen (1970) proves that an f-inaccessible cardinal greater than an ordinal ξ is precisely a regular cardinal that is a *fixed point* for the ordinal function $\alpha \mapsto f^{\alpha}(\xi)$.

Example 1.1.6.7 If f is the ordinal function $\beta \mapsto 2^{|\beta|}$, then $f^{\alpha}(\omega) = \beth_{\alpha}$. An inaccessible cardinal is thus precisely a regular $\beth$-fixed point.

If f is the ordinal function $\beta \mapsto |\beta|^{+}$, then $f^{\alpha}(\omega) = \aleph_{\alpha}$. A weakly inaccessible cardinal is precisely a regular $\aleph$-fixed point.

Example 1.1.6.8 Assume AU. Consider the ordinal function f that carries an ordinal β to the smallest inaccessible cardinal greater than β. For any ordinal α, we have $\daleth_{\alpha} = f^{\alpha}(\omega)$.

An f-inaccessible cardinal is precisely a $\daleth$-fixed point. These are called 1-*inaccessible cardinals*. If κ is 1-inaccessible, then $V_{\kappa} \vDash (\text{ZFC} + \text{AU})$. If ZFC + AU is consistent, then the existence of 1-inaccessible cardinals is not provable by methods formalizable in ZFC + AU.

Iterating this strategy, one can now proceed to define α-inaccessibility for every ordinal α. Iterating the iteration, one can define notions of hyperinaccessibility, hyper$^{\alpha}$inaccessibility, *etc*. We cut to the chase:

Axiom 1.1.6.9 The *Lévy scheme* (LÉVY) is the assertion that for every ordinal function f and every ordinal ξ, there exists an f-inaccessible cardinal κ such that $\xi < \kappa$.

Theorem 1.1.6.10 (Lévy (1960); Montague (1962); Jorgensen (1970)) *The following are equivalent.*

1 *The Lévy scheme.*
2 *Every normal ordinal function has a regular cardinal in its image.*
3 *Every closed unbounded subclass $W \subseteq$ **Ord** contains a regular cardinal.*
4 *Every normal ordinal function has an inaccessible cardinal in its image.*
5 *Every closed unbounded subclass $W \subseteq$ **Ord** contains an inaccessible cardinal.*

1.1.6.11 The Lévy scheme implies the Axiom of Universes, and the consistency strength of NBG + LÉVY is strictly greater than that of NBG + AU.

The consistency strength of the Lévy scheme is also strictly greater than the existence of α-inaccessible, hyperinaccessible, hyper$^{\alpha}$inaccessible, *etc.*, cardinals.

Definition 1.1.6.12 Let κ be a regular cardinal. One says that κ is *Mahlo* if and only if every closed unbounded subset $W \subseteq \kappa$ contains a regular cardinal.

1.1.6.13 Assume that κ is a Mahlo cardinal. Then κ is f-inaccessible for every ordinal function f. Accordingly, κ is a fixed point of every normal ordinal function.

Additionally, if κ is a Mahlo cardinal, then $V_\kappa \models (\text{ZFC} + \text{LÉVY})$, and similarly $V_{\kappa+1} \models (\text{NBG} + \text{LÉVY})$. The consistency strength of the axiom 'a Mahlo cardinal exists' is strictly greater than the Lévy scheme.

1.1.6.14 The Lévy scheme and its equivalents and slight variants have appeared under various names: 'Mahlo's principle' (Gloede, 1973), 'Axiom F' (Drake, 1974), '**Ord** is Mahlo' (Hamkins, 2003).

For our purposes, one of the main appeals of the Lévy scheme is the following.

Theorem 1.1.6.15 *Assume* LÉVY.

1.1.7 Universe polymorphism

1.2 Reflection principles

1.2.1 Lévy hierarchy

1.2.2 Elementary embeddings

1.2.3 Reflection principles

1.2.4 Indescribable cardinals

1.3 Bicategories

2
Topology

2.1 Ultrafilters

2.1.1 Codensity monads

The codensity monad of a functor $f : A \to B$ is the right Kan extension $\beta(f)$ of f along itself, when it exists. For formal reasons, this is always a monad on B.

The full functoriality of the construction $f \mapsto \beta(f)$ is relevant to us. In effect, we regard functors as the objects of a category, and the morphisms are lax-commutative squares.

Definition 2.1.1.1 Let A and B be categories. Then a bifibration (Lurie, 2009, §§2.4.7) $X \to B \times A$ is *representable* if and only if, for every object $a \in A$, the fiber X_a has a terminal object.

2.1.1.2 Let A and B be categories. A bifibration $X \to B \times A$ corresponds to a functor $B^{op} \times A \to \boldsymbol{S}_V$, or equivalently to a functor $\Xi : A \to \boldsymbol{P}(B)$. A representable bifibration is one in which each presheaf $\Xi(a)$ is representable. In this way, the category of representable bifibrations to $B \times A$ is equivalent to the category $\mathrm{Fun}(A, B)$.

One can be explicit about the correspondence: if $f : A \to B$ is a functor, then the corresponding representable bifibration is

$$\mathrm{Fun}(\Delta^1, B) \times_B A \to B \times A\,,$$

and every representable fibration is of this form.

Construction 2.1.1.3 Let **LaxCat** be the full subcategory of $\mathrm{Fun}(\Lambda^1_0, \mathbf{Cat})$ spanned by those diagrams $A \leftarrow X \to B$ such that $X \to B \times A$ is a representable bifibration.

The objects can be identified with functors $f : A \to B$, but **LaxCat** is not equivalent to the category $\mathrm{Fun}(\Delta^1, \mathbf{Cat})$. If $f\colon A \to B$ and $g\colon C \to D$ are functors, then a morphism $\sigma\colon f \to g$ of **LaxCat** determines a functor

$$\mathrm{Fun}(\Delta^1, B) \times_B A \to \mathrm{Fun}(\Delta^1, D) \times_D C\ .$$

If $a \in A$ and $b \in B$ are objects, then σ determines a map

$$\mathrm{Map}_B(b, f(a)) \to \mathrm{Map}_D(\psi(b), g(\phi(a)))\ .$$

When $b = f(a)$, the image of the identity under this map is thus a morphism $\sigma_a\colon \psi(f(a)) \to g(\phi(a))$. Thus the morphism σ amounts to a lax-commutative square:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\swarrow\sigma"] & B \arrow[d, "\psi"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \tag{2.1.1.1}$$

We have two functors $s, t\colon \mathbf{LaxCat} \to \mathbf{Cat}$ which carry a diagram $[A \leftarrow X \to B]$ to A and B, respectively. We have an equivalence

$$\{A\} \times_{\mathbf{Cat}} \mathbf{LaxCat} \times_{\mathbf{Cat}} \{B\} \simeq \mathrm{Fun}(A, B)\ .$$

In fact, the functor $H_B\colon \mathbf{Cat}^{op} \to \mathbf{Cat}$ represented by B and the functor $H^A\colon \mathbf{Cat} \to \mathbf{Cat}$ corepresented by A correspond under straightening/unstraightening to the cartesian fibration

$$\mathbf{LaxCat} \times_{\mathbf{Cat}} \{B\} \to \mathbf{Cat}$$

and the cocartesian fibration

$$\{A\} \times_{\mathbf{Cat}} \mathbf{LaxCat} \to \mathbf{Cat}\ ,$$

respectively.

Definition 2.1.1.4 We call **LaxCat** the *lax arrow category of categories*. If C is a fixed category, then we call

Construction 2.1.1.5 Let C be a category. We write $\mathbf{End}(C)$ for the monoidal category of endofunctors of C, with the monoidal structure given by composition.

acts on the left on the category $\mathrm{Fun}(D, C)$. Both the monoidal structure and the left module structure are given by composition.

We consider the category $\boldsymbol{LMod}_{\mathbf{End}(C)}(\mathrm{Fun}(D, C))$ whose objects can be regarded as pairs (T, f) consisting of an algebra $T \in \mathbf{Alg}(\mathbf{End}(C))$ and a T-module f in $\mathrm{Fun}(D, C)$. Thus

For any functor $f\colon D \to C$, we consider the monoidal category $\mathbf{End}(C)[f]$

constructed in (Lurie, 2017, Definition 4.7.1.1). The objects of the category $\mathbf{End}(C)[f]$ are pairs (T, η) consisting of an object $T \in \mathbf{End}(C)$ and a natural transformation $T \circ f \to f$. The assignment $(T, \eta) \mapsto T$ defines a monoidal forgetful functor $\mathbf{End}(C)[f] \to \mathbf{End}(C)$.

The terminal object (if it exists) of $\mathbf{End}(C)[f]$ is automatically an algebra object $B(f) = (\beta(f), \epsilon)$. The image of $B(f)$ under the forgetful functor $\mathbf{End}(C)[f] \to \mathbf{End}(C)$ is the algebra object

$$\beta(f) \in \mathbf{Alg}(\mathbf{End}(C))\,;$$

in other words, $\beta(f)$ is a monad on C.

If $\mathbf{End}(C)[f]$ has a terminal object $B(f) = (\beta(f), \epsilon)$, then the monad $\beta(f)$ will be called the *codensity monad* attached to $f\colon D \to C$.

If the category $\mathbf{End}(C)[f]$ has a terminal object, then ϵ exhibits $\beta(f)$ as the right Kan extension of f along itself. Conversely, if the right Kan extension of f along itself exists, then that Kan extension defines a terminal object of the category $\mathbf{End}(C)[f]$.

(Lurie, 2017, §§4.7.1) identifies three categories:

$$\boldsymbol{L}\mathbf{Mod}(\mathrm{Fun}(D, C)) \times_{\mathrm{Fun}(D,C)} \{f\} \simeq \mathbf{Alg}(\mathbf{End}(C)[f]) \simeq \mathbf{Alg}(\mathbf{End}(C))_{/\beta(f)}\,.$$

More informally, we may say that a morphism of monads $T \to \beta(f)$ is the same thing as a T-module structure on f.

2.1.2 Ultrafilters on sets

Notation 2.1.2.1 Let us write $\mathbf{Fin} \subset \mathbf{Set}$ for the full subcategory of finite sets, and let us write i for the inclusion $\mathbf{Fin} \hookrightarrow \mathbf{Set}$.

Definition 2.1.2.2 Let S be a set. An *ultrafilter* on S is a natural transformation

$$よ^S \circ i \to i\,.$$

Notation 2.1.2.3 Let S be a set, and let μ be an ultrafilter on S. We find it expressive to write

$$\int_S (\cdot)\, d\mu\colon よ^S \circ i \to i$$

for the natural transformation. If I is a finite set, and if $f\colon S \to I$ is a map, then this natural transformation permits us to specify an element

$$\int_S f\, d\mu \in I\,.$$

The naturality is the condition that if $\phi\colon I \to J$ is a map of finite sets, then

$$\phi\left(\int_S f\,d\mu\right) = \int_S \phi \circ f\,d\mu\,.$$

Example 2.1.2.4 Let S be a set, and let $s \in S$ be an element. The *principal ultrafilter* δ_s is then defined so that

$$\int_S f\,d\delta_s = f(s)\,.$$

By the Yoneda lemma, every ultrafilter on a finite set is principal, but as we shall see, infinite sets have ultrafilters that are not principal.

Notation 2.1.2.5 If S is a set, then let us write $\beta(S)$ for the set of ultrafilters on S:

$$\beta(S) \coloneqq \mathrm{Map}(よ^S \circ i, i)\,.$$

Write $\beta(S)$ for the set of ultrafilters on S. For any set S, the set $\beta(S)$ is the set

$$\beta(S) = \lim_{I \in \mathbf{Fin}_{S/}} I\,.$$

The functor

$$\beta\colon \mathbf{Set} \to \mathbf{Set}$$

is thus the right Kan extension of the inclusion $\mathbf{Fin} \hookrightarrow \mathbf{Set}$ along itself.

To prove the existence of these, let us look at a more traditional way of defining an ultrafilter on a set.

Definition 2.1.2.6 Let S be a set, $T \subseteq S$, and μ an ultrafilter on S. There is a unique *characteristic map* $\chi_T\colon S \to \{0,1\}$ such that $\chi_T(s) = 1$ if and only if $s \in T$. Let us write

$$\mu(T) \coloneqq \int_S \chi_T\,d\mu\,.$$

We say that T *is μ-thick* if and only if $\mu(T) = 1$. Otherwise (that is, if $\mu(T) = 0$), then we say that T is *μ-thin*.

For any $s \in S$, the principal ultrafilter δ_s is the unique ultrafilter relative to which $\{s\}$ is thick.

Scholium 2.1.2.7 *If S is a set and μ is an ultrafilter on S, then we can observe the following facts about the collection of thick and thin subsets (relative to μ):*

1. *The empty set is thin.*
2. *Complements of thick sets are thin.*
3. *Every subset is either thick or thin.*
4. *Subsets of thin sets are thin.*

5 *The intersection of two thick sets is thick.*

In other words, if S is a set, then an ultrafilter on S is tantamount to a Boolean algebra homomorphism $\boldsymbol{P}(S) \to \{0, 1\}$.

It is possible to define ultrafilters on more general posets, and if P is a Boolean algebra, then an ultrafilter is precisely a Boolean algebra homomorphism $P \to \{0, 1\}$.

Scholium 2.1.2.8 *Ultrafilters are functorial in maps of sets. Let $\phi\colon S \to T$ be a map, and let μ be an ultrafilter on S. The ultrafilter $\phi_*\mu$ on T given by*

$$\int_T f\, d(\phi_*\mu) = \int_S (f \circ \phi)\, d\mu\,.$$

For any $U \subseteq T$, one has in particular

$$(\phi_*\mu)(U) = \mu(\phi^{-1}(U))\,.$$

Thus U is $\phi_*\mu$-thick if and only if $\phi^{-1}U$ is μ-thick.

Definition 2.1.2.9 A *system of thick subsets* of S is a collection $F \subseteq \boldsymbol{P}(S)$ such that for any finite set I and any partition

$$S = \coprod_{i \in I} S_i\,,$$

there is a unique $i \in I$ such that $S_i \in F$.

Construction 2.1.2.10 We have seen that an ultrafilter μ specifies the system F_μ of μ-thick subsets. In the other direction, attached to any system F of thick subsets is an ultrafilter μ_F: for any finite set I and any map $f\colon S \to I$, the element $i = \int_S f\, d\mu \in I$ is the unique one such that $S_i \in F$.

The assignments $\mu \mapsto F_\mu$ and $F \mapsto \mu_F$ together define a bijection between ultrafilters on S and systems of thick subsets.

Definition 2.1.2.11 If S is a set, and if $G \subseteq \boldsymbol{P}(S)$, then an ultrafilter μ is said to be *supported on* G if and only if every element of G is μ-thick, that is, $G \subseteq F_\mu$.

Lemma 2.1.2.12 *Let S be a set, and let $G \subseteq \boldsymbol{P}(S)$. Assume that no finite intersection of elements of G is empty. Then there exists an ultrafilter μ on S supported on G.*

Proof Consider all the families $A \subseteq \boldsymbol{P}(S)$ with the following properties:

1 A contains G;

2 no finite intersection of elements of A is empty.

By Zorn's lemma there is a maximal such family, F.

We claim that F is a system of thick subsets. For this, let $S = \coprod_{i \in I} S_i$ be a finite partition of S. Condition 2 ensures that at most one of the summands S_i can lie in F. Now suppose that none of the summands S_i lies in F. Consider, for each $i \in I$, the family $F \cup \{S_i\} \subseteq \boldsymbol{P}(S)$; the maximality of F implies that none of these families can satisfy Condition 2. Thus for each $i \in I$, there is an empty finite intersection $S_i \cap \bigcap_{j=1}^{n_i} T_{ij} = \varnothing$. But this implies that the intersection $\bigcap_{i \in I} \bigcap_{j=1}^{n_i} T_{ij}$ is empty, contradicting Condition 2 for F itself. Hence at least one – and thus exactly one – of the summands S_i lies in F. Thus F is a system of thick subsets of S. □

2.1.2.13 It is not quite accurate to say that the Axiom of Choice is *necessary* to produce nonprincipal ultrafilters, but it is true that their existence is independent of Zermelo–Fraenkel set theory.

2.1.2.14 If ϕ is a functor **Set** $\to$ **Set**, then a natural transformation $\phi \to \beta$ is the same thing as a natural transformation $\phi \circ i \to i$. Please observe that we have a canonical identification $\beta \circ i = i$.

It follows readily that the functor β is a monad: the unit $\delta\colon \mathrm{id} \to \beta$ corresponds to the identification $\mathrm{id} \circ i = i$, and the multiplication $\mu\colon \beta^2 \to \beta$ corresponds to the identification $\beta^2 \circ i = i$.

The unit for the monad β structure is the assignment $s \mapsto \delta_s$ that picks out the principal ultrafilter at a point.

To describe the multiplication $\tau \mapsto \mu_\tau$, let us write $T^\dagger$ for the set of ultrafilters supported on $\{T\}$. Now if τ is an ultrafilter on $\beta(S)$, then μ_τ is the ultrafilter on S such that

$$\mu_\tau(T) = \tau(T^\dagger)\,.$$

2.1.3 Completeness of ultrafilters

2.1.4 Ultrafilters on posets

2.1.5 Ultraproducts

2.2 Topoi

2.2.1 Topoi

2.2.2 Sheaves and hypersheaves

2.2.3 Postnikov completeness

2.2.4 Coherence

2.2.5 Stone duality

2.2.6 Spectral duality

2.2.7 Classifying topoi

2.3 Compacta

2.3.1 Compacta and β-algebras

Construction 2.3.1.1 Let **Top** denote the category of tiny topological spaces. If S is a set, we can introduce a topology on $\beta(S)$ simply by forming the inverse limit $\lim_{I \in \mathbf{Fin}_{S/}} I$ in **Top**. That is, we endow $\beta(S)$ with the coarsest topology such that all the projections $\beta(S) \to I$ are continuous. We call this the *Stone topology* on $\beta(S)$. By Tychonoff, this limit is a compact Hausdorff topological space. This lifts β to a functor **Set** $\to$ **Top**.

2.3.1.2 Let's be more explicit about the topology on $\beta(S)$. The topology on $\beta(S)$ is generated by the sets $T^\dagger$ (for $T \subseteq S$). In fact, since the sets $T^\dagger$ are stable under finite intersections, they form a base for the Stone topology on $\beta(S)$. Additionally, since the sets $T^\dagger$ are stable under the formation of complements, they even form a base of clopens of $\beta(S)$.

Definition 2.3.1.3 A *compactum* is an algebra for the monad β. Hence a compactum consists of a set K and a map $\lambda_K \colon \beta(K) \to K$, which is required to satisfy the usual identities:

$$\lambda_K(\lambda_{K,*}\tau) = \lambda_K(\mu_\tau) \qquad \text{and} \qquad \lambda_K(\delta_s) = s\,,$$

for any ultrafilter τ on $\beta(S)$ and any point $s \in S$. The image $\lambda_K(\mu)$ will be called the *limit* of the ultrafilter μ. We write **Comp** for the category of compacta, and write **Free** $\subset$ **Comp** for the full subcategory spanned by the *free compacta – i.e.*, free algebras for β.

Construction 2.3.1.4 If K is a compactum, then we use the limit map $\lambda_K : \beta(K) \to K$ to topologise K as follows. For any subset $T \subseteq K$, we define the closure of T as the image $\lambda_K(T^\dagger)$.

A subset $Z \subseteq K$ is thus closed if and only if the limit of any ultrafilter relative to which Z is thick lies in Z. Dually, a subset $U \subseteq K$ is open if and only if it is thick with respect to any ultrafilter whose limit lies in U.

We denote the resulting topological space K^{top}. The assignment $K \mapsto K^{top}$ defines a lift $\mathbf{Alg}(\beta) \to \mathbf{Top}$ of the forgetful functor $\mathbf{Alg}(\beta) \to \mathbf{Set}$.

Proposition 2.3.1.5 *The functor $K \mapsto K^{top}$ identifies the category of compacta with the category of compact Hausdorff topological spaces.*

We will spend the remainder of this section proving this claim. Please observe first that $K \mapsto K^{top}$ is faithful. What we will do now is prove:

1 that for any compactum K, the topological space K^{top} is compact Hausdorff;
2 that for any compact Hausdorff topological space X, there is a β-algebra structure K on the underlying set of X such that $X \cong K^{top}$; and
3 that for any compacta K and L, any continuous map $K^{top} \to L^{top}$ lifts to a β-algebra homomorphism $K \to L$.

To do this, it is convenient to describe a related idea: that of *convergence* of ultrafilters on topological spaces.

Definition 2.3.1.6 Let X be a topological space, and let $x \in X$. We say that x is a *limit point* of an ultrafilter μ on (the underlying set of) X if and only if every open neighbourhood of x is μ-thick. In other words, x is a limit point of μ if and only if, for every open neighbourhood U of x, one has $\mu \in U^\dagger$.

Lemma 2.3.1.7 *Let X be a topological space, and let $U \subseteq X$ be a subset. Then U is open if and only if it is thick with respect to any ultrafilter with limit point in U.*

Proof If U is open, then U is by definition thick with respect to any ultrafilter with limit point in U.

Conversely, assume that U is thick with respect to any ultrafilter with limit point in U. Let $u \in U$. Consider the set $G \coloneqq N(u) \cup \{X \smallsetminus U\}$, where $N(u)$ is the collection of open neighbourhoods of u. If U does not contain any open neighbourhood of u, then no finite intersection of elements of G is empty. By 2.1.2.12 there is an ultrafilter μ supported on the $N(u) \cup \{X \smallsetminus U\}$, whence u is a limit point of μ, but U is not μ-thick. This contradicts our assumption, and so we deduce that U contains an open neighbourhood of u. □

Lemma 2.3.1.8 *Let X and Y be topological spaces, and let $\phi\colon X \to Y$ be a map. Then ϕ is continuous if and only if, for any ultrafilter μ on X with limit point $x \in X$, the point $\phi(x)$ is a limit point of $\phi_*\mu$.*

Proof Assume that ϕ is continuous, and let μ be an ultrafilter on X, and assume that $x \in X$ is a limit point of μ. Now assume that V is an open neighbourhood of $\phi(x)$. Since $\phi^{-1}V$ is an open neighbourhood of x, so it is μ-thick, whence V is $\phi_*\mu$-thick. Thus $\phi(x)$ is a limit point of $\phi_*\mu$.

Assume now that if $x \in X$ is a limit point of an ultrafilter μ, then $\phi(x)$ is a limit point of $\phi_*\mu$. Let $V \subseteq Y$ be an open set. Let $x \in \phi^{-1}(V)$, and let μ be an ultrafilter on X with limit point x. Then $\phi(x)$ is a limit point of $\phi_*\mu$, so V is $\phi_*\mu$-thick, whence $\phi^{-1}(V)$ is μ-thick. It follows from 2.3.1.7 that $\phi^{-1}(V)$ is open. □

Lemma 2.3.1.9 *Let X be a topological space. Then X is quasicompact if and only if every ultrafilter on X has at least one limit point.*

Proof Assume first that X is quasicompact. Let μ be an ultrafilter on X, and assume that μ has no limit point. Select, for every point $x \in X$, an open neighbourhood U_x thereof that is not μ-thick. Quasicompactness implies that there is a finite collection $x_1, \ldots, x_n \in X$ such that $\{U_{x_1}, \ldots, U_{x_n}\}$ covers X. But at least one of $U_{x_1}, \ldots, U_{x_n}$ must be μ-thick. This is a contradiction.

Now assume that X is not quasicompact. Then there exists a collection $G \subseteq \boldsymbol{P}(X)$ of closed subsets of X such that the intersection all the elements of G is empty, but no finite intersection of elements of G is empty. In light of 2.1.2.12, there is an ultrafilter μ with the property that every element of G is thick. For any $x \in X$, there is an element $Z \in G$ such that $x \in X \smallsetminus Z$. Since Z is μ-thick, $X \smallsetminus Z$ is not. Thus μ has no limit points. □

Lemma 2.3.1.10 *Let X be a topological space. Then X is Hausdorff if and only if every ultrafilter on X has at most one limit point.*

Proof Assume that μ is an ultrafilter with two distinct limit points x_1 and x_2. Choose open neighbourhoods U_1 of x_1 and U_2 of x_2. Since they are both μ-thick, they cannot be disjoint; hence X is not Hausdorff.

Conversely, assume that X is not Hausdorff. Select two points x_1 and x_2 such that every open neighbourhoods U_1 of x_1 and U_2 of x_2 intersect. Now the set G consisting of open neighbourhoods of either x_1 *or* x_2 has the property that no finite intersection of elements of G is empty. In light of 2.1.2.12, there is an ultrafilter μ with the property that every element of G is thick. Thus x_1 and x_2 are limit points of μ. □

Let us now return to our functor $K \mapsto K^{top}$.

Lemma 2.3.1.11 *Let K be a compactum, and let μ be an ultrafilter on K. Then a point of K^{top} is a limit point of μ in the sense of 2.3.1.6 if and only if it is the limit of μ in the sense of 2.3.1.3.*

Proof Let $x \coloneqq \lambda_K(\mu)$. The open neighbourhoods U of x are by definition thick (relative to μ), so certainly x is a limit point of μ.

Now assume that $y \in K^{top}$ is a limit point of μ. To prove that the limit of μ is y, we shall build an ultrafilter τ on $\beta(K)$ with the following properties:

1 under the multiplication $\beta^2 \to \beta$, the ultrafilter τ is sent to μ; and
2 under the map $\lambda_* : \beta^2 \to \beta$, the ultrafilter τ is sent to δ_y.

Once we have succeeded, it will follow that

$$\lambda_K(\mu) = \lambda_K(\mu_\tau) = \lambda_K(\lambda_{K,*}\tau) = \lambda_K(\delta_y) = y\ ,$$

and the proof will be complete.

Consider the family G' of subsets of $\beta(K)$ of the form $T^\dagger$ for a μ-thick subset $T \subseteq S$; since these are all nonempty and they are stable under finite intersections, it follows that no finite intersection of elements of G' is empty.

Now consider the set $G \coloneqq G' \cup \{\lambda_K^{-1}\{y\}\}$. If T is μ-thick, then we claim that there is an ultrafilter $\nu \in \lambda_K^{-1}\{y\} \cap T^\dagger$. Indeed, consider the set $N(y) \cup \{T\}$, where $N(y)$ is the collection of open neighbourhoods of y. Since every open neighbourhood of y is μ-thick, no intersection of an open neighbourhood of y with T is empty. By 2.1.2.12 there is an ultrafilter supported on $N(y) \cup \{T\}$, which implies that no finite intersection of elements of G is empty.

Applying 2.1.2.12 again, we see that G supports an ultrafilter τ on $\beta(K)$. For any $T \subseteq K$,

$$\mu_\tau(T) = \tau(T^\dagger)\ ,$$

so since τ is supported on G', it follows that $\mu_\tau = \mu$. At the same time, since τ is supported on $\{\lambda_K^{-1}\{y\}\}$, it follows that $\{y\}$ is thick relative to $\lambda_{K,*}\tau$, whence $\lambda_{K,*}\tau = \delta_y$. □

Proof of 2.3.1.5 Let K be a compactum. Combine **??** to conclude that K^{top} is a compact Hausdorff topological space.

Let X be a compact Hausdorff topological space with underlying set K. Define a map $\lambda_K : \beta(K) \to K$ by carrying an ultrafilter μ to its unique limit point in X. This is a β-algebra structure on X, and it follows from 2.3.1.11 and the definition of the topology together imply that $X \cong K^{top}$.

Finally, let K and L be compacta, and let $\phi\colon K^{top} \to L^{top}$ be a continuous map. To prove that ϕ is a β-algebra homomorphism, it suffices to confirm that if μ is an ultrafilter on K, then

$$\lambda_L(\phi_*\mu) = \phi(\lambda_K(\mu)),$$

but this follows exactly from 2.3.1.8. □

2.3.1.12 We opted in 2.3.1.4 to define the topology on a compactum K in very explicit terms, but note that the map $\lambda_K\colon \beta(K) \to K^{top}$ is a continuous surjection between compact Hausdorff topological spaces. Thus K^{top} is endowed with the quotient topology relative to λ_K.

2.3.2 Boolean algebras

2.3.3 Stone topological spaces

2.3.4 Projective compacta

PART TWO

CONDENSED SETS

PART THREE

CONDENSED GROUPS

PART FOUR

CONDENSED ABELIAN GROUPS

PART FIVE

CONDENSED SPACES

PART SIX

CONDENSED SPECTRA

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