# Condensed homotopy theory

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# Introduction

When we perform completion constructions, these involve operations – such as the formation of limits – that are not compatible with many colimits. As a result, these constructions produce derived functors.

**Example** Let $R$ be a commutative ring, and let $I \subset R$ be an ideal. Then the $I$-adic completion $C_I$ of modules has left derived functors $\boldsymbol{L}^n C_I$. These fit into short exact sequences

$$0 \to \lim_k{}^1 \operatorname{Tor}^R_{n+1}(R/I^k, M) \to \boldsymbol{L}^n C_I(M) \to \lim_k \operatorname{Tor}^R_n(R/I^k, M) \to 0\ .$$

# PART ONE

## PRELIMINARIES

# 1
# Foundations

## 1.1 Cardinals

Mathematicians' 'stock' set theory, ZFC (Zermelo–Fraenkel set theory ZF plus the Axiom of Choice AC) doesn't quite have the expressive power one needs for work with categories and higher categories. The issue ultimately comes down to Cantor's diagonal argument: there is no surjection of a set onto its powerset. This is ultimately why no one can contemplate a set of all sets, and it's also the key to Freyd's observation that if $C$ is a category and $\kappa$ is the cardinality of its set of arrows, then $C$ has all $\kappa$-indexed products only if $C$ is a poset. This, in turn, is what's behind the 'solution set condition' in representability theorems or the Adjoint Functor Theorem. Hence one really must distinguish between 'large' and 'small' objects.

One improves matters by passing to von Neumann–Bernays–Gödel set theory (NBG), which is a conservative extension of ZFC. In NBG, the formal language consists of the symbols $\in$ and $=$; a constant $V$; suitable variables; the usual connectives of first-order logic ($\neg, \wedge, \vee, \implies$, and $\iff$); and the quantifiers $\forall$ and $\exists$. The objects of the theory are called *classes*. A class $x$ is called a *set* if and only if $x \in V$; a *proper class* is a class $X$ such that $X \notin V$. We summarize the axioms of NBG in informal language:

**Extensionality** Classes $X$ and $Y$ are equal if and only if, for any $z$, one has $z \in X$ if and only if $z \in Y$.

**Regularity** For every class $X$, there exists an element $z \in X$ such that $z \cap X = \varnothing$.

**Infinity** There is an infinite set.

**Union** If $x$ is a set, then $\bigcup x = \bigcup_{z \in x} z$ is a set as well.

**Pairing** If $x$ and $y$ are sets, then $\{x, y\}$ is a set as well.

**Powerset** If $x$ is a set, then the powerset $P(x)$ is a set as well.

**Limitation of size** A class $X$ is a proper class if and only if there is a bijection between $X$ and $V$.

**Class comprehension** For every first-order formula $\phi(x)$ with a free variable $x$ in which the quantifiers are over sets, there exists a class $\{x \in V : \phi(x)\}$ whose elements are exactly those sets $x$ such that $\phi(x)$.

In matters of set theory, we will generally follow the notations and terminological conventions of the comprehensive monograph of Jech (2003). We will also refer to the texts of Drake (1974) and Kanamori (2009). In particular, **Ord** denotes the proper class of ordinal numbers. For any ordinal $\alpha$, the set $V_\alpha$ is defined recursively as follows:

1 If $\alpha = 0$, then $V_\alpha \coloneqq \varnothing$;
2 if $\alpha = \beta + 1$ for an ordinal number $\beta$, then $V_\alpha \coloneqq P(V_\beta)$;
3 if $\alpha$ is a limit ordinal, then $V_\alpha \coloneqq \bigcup_{\beta<\alpha} V_\beta$.

If $x$ is a set, then the *rank* of $x$ is the smallest ordinal number $\alpha$ such that $x \in V_\alpha$. The proper class $V$ is then the union $\bigcup_{\alpha \in \mathbf{Ord}} V_\alpha$.

In matters of (higher) category theory, we will generally follow the notations and terminological conventions of Lurie (2009). However, we will simplify some pieces of language and notation, and our set-theoretic conventions are slightly different:

**Definition 1.1.0.1** A *large-category*[1] $C$ consists of a sequence $(C_n)_{n\in \mathbf{N}_0}$ of classes, along with a family of class maps $\phi^* : C_m \to C_n$ for each map $\phi \colon n \to m$, subject to all the formulas that express the statement that $C_n$ is a simplicial class that satisfies the inner Kan condition.

If the large-category $C$ contains a full subcategory $C' \subseteq C$ such that each $C'_n$ is a set and such that every object of $C$ is equivalent to an object of $C'$, then we will call $C$ a *category* or, for emphasis, a *small category*.[2]

A large-category $C$ is said to be *locally small* if and only if, for every subset $C'_0 \subseteq C_0$, the full subcategory $C' \subseteq C$ spanned by the elements of $C'_0$ is small.

Limits and colimits are only considered for functors $J \to C$ in which the $J$ is a small category. Hence when we refer to *all limits* or *all colimits*, we mean all limits or colimits of diagrams indexed on small categories.

If $C$ is a category and $D$ is a large-category, then the large-category $\mathrm{Fun}(C, D)$ is defined in the usual way, so that $\mathrm{Fun}(C, D)_n$ is the class of simplicial maps $C \times \Delta^n \to D$.

[1] We use the term 'category' for what other authors might call '$\infty$-category', '$(\infty, 1)$-category', or 'quasicategory'. We will use the term '1-category' when the specification is needed.
[2] These are sometimes called *essentially small*.

**Notation 1.1.0.2** We shall write $\mathbf{Set}^V$ for the large-category of all sets. The objects of $\mathbf{Set}^V$ are thus precisely the elements of $V$.

We shall write $\mathbf{An}^V$ for the large-category of all animae.[3] The objects of $\mathbf{An}^V$ are thus precisely the Kan complexes $\Delta^{op} \to \mathbf{Set}^V$.

If $C$ is a category, then we shall write $\boldsymbol{P}^V(C) \coloneqq \mathrm{Fun}(C^{op}, \mathbf{An}^V)$ and $\tau_0\boldsymbol{P}^V(C) \coloneqq \mathrm{Fun}(C^{op}, \mathbf{Set}^V)$. We will write

$$よ : C \hookrightarrow \boldsymbol{P}^V(C)$$

for the Yoneda embedding $X \mapsto よ_X$. Thus if $X$ is an object of $C$, then $よ_X : C^{op} \to \mathbf{An}^V$ is the functor represented by $X$, so that $よ_X(U) = \mathrm{Map}(U, X)$.

We shall write $\mathbf{Cat}^V$ for the large-category of all categories. The objects of $\mathbf{Cat}^V$ are thus precisely the weak Kan complexes $\Delta^{op} \to \mathbf{Set}^V$.

The Class Comprehension Axiom Schema implies the *Axiom of Global Choice*, which ensures the existence of a choice function $\tau : V \to V$ such that $\tau(x) \in x$. One needs this to make sense of a construction like 'the' functor $- \times u : C \to C$ for a large-category $C$ with all finite products and a fixed object $u \in C$.

Here, we will work with NBG as our base theory, so that we may speak of proper classes and large-categories whenever the occasion arises.

However, the whole project of higher category theory turns on the principle that we want to be able to deal with the collections of all objects of a given kind as a mathematical object in its own right, and that the passage up and down these category levels is a fruitful way to understand even completely 'decategorified' objects.[4] If we have a large-category $C$, NBG provides us with no mechanism to view $C$ itself as an object of a still larger category of all categories. This limits the kinds of operations we are permitted to do to $C$.

To give ourselves the room to pass up and down category levels, we need to have a hierarchy of 'scales' at which we can work. These scales will be identified by inaccessible cardinals or, equivalently, Grothendieck universes. The existence of these cardinals is independent of NBG.

But now a different sort of concern arises. We will also want to have assurance that the results we obtain at one scale remain valid at other scales. We might prefer to prove sentences about *all* sets, animae, groups, etc. – not just those within a universe. This sort of 'scale-invariance of truth' is expressed by a *reflection principle* (1.2). The reflection principle we will use here states roughly that there is an inaccessible cardinal $\kappa$ such that statements of set theory hold in the universe $V_\kappa$ just in case they hold in $V$ itself. This is the *Lévy scheme* LÉVY,

[3] We follow Clausen and Scholze, and we use the term 'anima' for what other authors might call 'space', '$\infty$-groupoid', or '$(\infty, 0)$-category'.

[4] The *microcosm principle* of Baez–Dolan is a precise illustration of this principle.

which we will first formulate as a large cardinal axiom. In effect, this permits us to focus our attention on *categories* (as opposed to large-categories). This scheme has a strictly higher conistency strength than the existence of a proper class of inaccessible cardinals; however, its consistency strength is strictly lower than that of the existence of a single Mahlo cardinal.

Following beautiful work of Hamkins (2003), we will later prove that the Lévy scheme is equivalent to (not only equiconsistent with) a *maximality principle*, which states roughly that every sentence of set theory that is true for a sieve of forcing extensions is true.

### 1.1.1 Regularity and smallness

**Definition 1.1.1.1** If $\kappa$ is a cardinal, then a set $S$ is *$\kappa$-small* if and only if $|S| < \kappa$. We shall write $\mathbf{Set}^{\kappa} \subset \mathbf{Set}^{V}$ for the category of $\kappa$-small sets, Thus $\mathbf{Set}^{V}$ is the filtered union of the categories $\mathbf{Set}^{\kappa}$ over the proper class of regular cardinals.

A cardinal $\kappa$ is *regular* if and only if, for every map $f : S \to T$ in which $T$ and every fiber $f^{-1}\{t\}$ are all $\kappa$-small, the set $S$ is $\kappa$-small as well. Equivalently, $\kappa$ is regular if and only if $\mathbf{Set}^{\kappa} \subset \mathbf{Set}^{V}$ is stable under colimits indexed by $\kappa$-small posets.

**Example 1.1.1.2** Under this definition, 0 is a regular cardinal.[5] There are no 0-small sets.

**Example 1.1.1.3** The $\aleph$ family of cardinals is defined by a function from the class of ordinal numbers to the class of cardinal numbers, by transfinite induction:

1 The cardinal $\aleph_0$ is the ordinal number $\omega$ consisting of all finite ordinals.
2 For any ordinal $\alpha$, one defines $\aleph_{\alpha+1}$ to be the smallest cardinal number strictly greater than $\aleph_\alpha$.
3 For any limit ordinal $\alpha$, one defines $\aleph_\alpha \coloneqq \sup \left\{\aleph_\beta : \beta < \alpha\right\}$.

The countable cardinal $\aleph_0$ is regular. Every infinite successor cardinal is regular; consequently, $\aleph_n$ for $n \in \boldsymbol{N}$ is regular as well. The cardinal $\aleph_\omega$ is the smallest infinite cardinal that is not regular.

**Example 1.1.1.4** A set is $\aleph_0$-small if and only if it is finite.

**1.1.1.5** If $\kappa$ is a regular cardinal, then $\mathbf{Set}^{\kappa} \subset \mathbf{Set}^{V}$ is the full subcategory generated by the singleton $\{0\}$ under colimits over $\kappa$-small posets.

[5] Many texts require that a regular cardinal be infinite.

**Definition 1.1.1.6** If $\kappa$ is a regular cardinal, then we shall write $\mathbf{An}^{\kappa} \subset \mathbf{An}^{V}$ for the full subcategory generated by $\{0\}$ under colimits over $\kappa$-small posets. The objects of $\mathbf{An}^{\kappa}$ will be called *$\kappa$-small animae*.

Similarly, we shall write $\mathbf{Cat}^{\kappa} \subset \mathbf{Cat}^{V}$ for the full subcategory generated by $\{0\}$ and $\{0 < 1\}$ under colimits over $\kappa$-small posets. The objects of $\mathbf{Cat}^{\kappa}$ will be called *$\kappa$-small categories*.

Finally, a large-category $C$ is said to be *locally $\lambda$-small* if and only if, for every $\lambda$-small subset $C_0' \subseteq C_0$ of objects of $C$, the full subcategory $C' \subseteq C$ that it spans is $\lambda$-small.

**Example 1.1.1.7** This turn of phrase above is slightly ambiguous when $\kappa = 0$. In that case, we take the phrase 'subcategory generated by … under colimits over the empty collection of posets' to mean the empty category. With this convention, there are no 0-small animae or categories:

$$\mathbf{Set}^0 = \mathbf{An}^0 = \mathbf{Cat}^0 = \varnothing \, .$$

**Example 1.1.1.8** An anima is $\aleph_0$-small if and only if it is weak homotopy equivalent to a simplicial set with only finitely many nondegenerate simplices.

A category $C$ is $\aleph_0$-small if and only if it is Joyal equivalent to a simplicial set with only finitely many nondegenerate simplices.

**Example 1.1.1.9** Why does regularity arise so often in category theory? What role does this hypothesis play? Here is the sort of scenario that is often lurking in the background when we appeal to the regularity of a cardinal.

Let $C$ be a large-category. Suppose that we have a *diagram of diagrams* in $C$, in the following sense. We have a category $A$; a functor $B\colon A \to \mathbf{Cat}^{V}$; and for each $\alpha \in A$, a functor $X_\alpha\colon B_\alpha \to C$. Furthermore, the colimits of each of these functors organize themselves into a functor $A \to C$:

$$\alpha \mapsto \operatorname*{colim}_{\beta \in B_\alpha} X_\alpha(\beta) \, .$$

We will often be in situations in which we need to analyze the *colimit of colimits*:

$$\operatorname*{colim}_{\alpha \in A} \operatorname*{colim}_{\beta \in B_\alpha} X_\alpha(\beta) \, .$$

In this case, we may reorganize these data. We first construct the cocartesian fibration corresponding to the functor $B$, which we will abusively write $B \to A$, since the fibers are the categories $B_\alpha$. We now have a single functor $X\colon B \to C$ whose restriction to any fiber $B_\alpha$ is the functor $X_\alpha$. Now the colimit of colimits above is a single colimit:

$$\operatorname*{colim}_{\alpha \in A} \operatorname*{colim}_{\beta \in B_\alpha} X_\alpha(\beta) \simeq \operatorname*{colim}_{\gamma \in B} X(\gamma) \, .$$

Now let $\kappa$ be a cardinal. If $A$ is $\kappa$-small, and if each category $B_\alpha$ is $\kappa$-small, then what can we conclude about $B$? In general, nothing. However, if $\kappa$ is a regular cardinal, then $B$ is also $\kappa$-small.

The motto here, then, is that *if $\kappa$ is regular, then $\kappa$-small colimits of $\kappa$-small colimits are $\kappa$-small colimits.*

### 1.1.2 Accessibility and presentablity

**Definition 1.1.2.1** Let $\kappa$ be a regular cardinal. A category $\Lambda$ is *$\kappa$-filtered* if and only if it satisfies the following equivalent conditions:

1. For every $\kappa$-small category $J$, every functor $f\colon J \to \Lambda$ can be extended to a functor $F\colon J^{\triangleright} \to \Lambda$.
2. For every $\kappa$-small category $J$ and every functor $H\colon \Lambda \times J \to \mathbf{An}^V$, the natural morphism
$$\operatorname*{colim}_{\lambda\in\Lambda} \lim_{j\in J} H(\lambda, j) \to \lim_{j\in J} \operatorname*{colim}_{\lambda\in\Lambda} H(\lambda, j)$$
is an equivalence.
3. For every $\kappa$-small category $J$, the diagonal functor $\Lambda \to \mathrm{Fun}(J, \Lambda)$ is cofinal.

**Example 1.1.2.2** For any regular cardinal $\kappa$, the ordinal $\kappa$, regarded as a category, is $\kappa$-filtered.

More generally, a poset is $\kappa$-filtered if and only if every $\kappa$-small subset thereof is dominated by some element.

**Example 1.1.2.3** A $\kappa$-small category is $\kappa$-filtered if and only if it contains a terminal object.

**Example 1.1.2.4** Since no category is 0-small, every category is 0-filtered.

**Definition 1.1.2.5** Let $\kappa$ be a regular cardinal. A functor $f\colon C \to D$ between large-categories will be said to be *$\kappa$-continuous* if and only if it preserves $\kappa$-filtered colimits.

An object $X$ of a locally small large-category $C$ is said to be *$\kappa$-compact* if and only if the functor $\text{よ}^X\colon C \to \mathbf{An}^V$ corepresented by $X$ (i.e., the functor $Y \mapsto \mathrm{Map}_C(X, Y)$) is $\kappa$-continuous. We write $C^{(\kappa)} \subseteq C$ for the full subcategory of $\kappa$-compact objects.

A large-category $C$ is *$\kappa$-accessible* if and only if it satisfies the following conditions:

1. The category $C$ is locally small.
2. The category $C$ has all $\kappa$-filtered colimits.
3. The subcategory $C^{(\kappa)} \subseteq C$ is small.

4 The subcategory $C^{(\kappa)} \subseteq C$ generates $C$ under $\kappa$-filtered colimits.

A $\kappa$-accessible large-category $C$ is *$\kappa$-presentable*[6] if and only if $C^{(\kappa)}$ has all $\kappa$-small colimits.

**Example 1.1.2.6** A 0-continuous functor is one that preserves all colimits. Hence a 0-compact object $X$ is one in which the natural map

$$\mathrm{Map}(X, \operatorname*{colim}_{\alpha \in A} Y_\alpha) \simeq \operatorname*{colim}_{\alpha \in A} \mathrm{Map}(X, Y_\alpha)$$

is an equivalence, irrespective of the category $A$ or the diagram $Y\colon A \to C$.

The following are equivalent for a large-category $C$.

1 There exists a small full subcategory $D \subseteq C$ whose inclusion extends along the Yoneda embedding to an equivalence of categories (1.1.0.2)

$$\boldsymbol{P}^V(D) \simeq C\,.$$

2 The full subcategory $C^{(0)} \subseteq C$ of 0-compact objects is small, and its inclusion extends along the Yoneda embedding to an equivalence of categories

$$\boldsymbol{P}^V(C^{(0)}) \simeq C\,.$$

3 The large-category $C$ is 0-accessible.
4 The large-category $C$ is 0-presentable.

**Example 1.1.2.7** Let $\kappa$ be an uncountable regular cardinal. Then the following are equivalent for a category $C$.

1 The category $C$ is $\kappa$-small.
2 The set of equivalence classes of objects of $C$ is $\kappa$-small, and for every morphism $f\colon X \to Y$ of $C$ and every $n \in \boldsymbol{N}_0$, the set $\pi_n(\mathrm{Map}_C(X,Y), f)$ is $\kappa$-small.
3 The category $C$ is $\kappa$-compact as an object of $\mathbf{Cat}^V$; that is, $\mathbf{Cat}^\kappa = \mathbf{Cat}^{V,(\kappa)}$.

In particular, an anima $X$ is $\kappa$-small if and only if all its homotopy sets are $\kappa$-small, if and only if it is $\kappa$-compact as an object of $\mathbf{An}^V$.

**Example 1.1.2.8** The equivalence above is doubly false if $\kappa = \aleph_0$.

First, we certainly have a containment

$$\mathbf{Cat}^{\aleph_0} \subset \mathbf{Cat}^{V,(\aleph_0)}\,,$$

but this containment is proper. An $\aleph_0$-compact anima is a *retract* of an $\aleph_0$-small anima, but it may not be $\aleph_0$-small itself. If $X$ is $\aleph_0$-compact and *simply connected*, then $X$ is $\aleph_0$-small, but for non-simply-connected animae, we have

[6] Some authors use the phrase *$\kappa$-compactly generated* instead.

the *de Lyra–Wall finiteness obstruction*, which lies in the reduced $K_0$ of the group ring $\boldsymbol{Z}[\pi_1(X)]$.

Second, the homotopy sets of an $\aleph_0$-small $X$ anima are not generally finite. By a theorem of Serre, if each connected component $Y \subseteq X$ has finite fundamental group, then its homotopy groups are finitely generated. But if $\pi_1(X)$ isn't finite, this too fails; for example, $\pi_3(S^1 \vee S^2)$ is not finitely generated.

It is still true that the category $\mathbf{Cat}^V$ is $\aleph_0$-presentable.

**1.1.2.9** Let $\kappa \leq \lambda$ be regular cardinals. A $\kappa$-small category is $\lambda$-small. A $\lambda$-filtered category is $\kappa$-filtered. A $\kappa$-continuous functor is $\lambda$-continuous. In general, however, there are $\kappa$-accessible categories that are not $\lambda$-accessible.

**Definition 1.1.2.10** Let $\kappa$ and $\lambda$ be regular cardinals. We write $\kappa \ll \lambda$ if and only if, for every pair of cardinals $\kappa_0 < \kappa$ and $\lambda_0 < \lambda$, one has $\lambda_0^{\kappa_0} < \lambda$. Equivalently, $\kappa \ll \lambda$ if and only if, for every $\kappa$-small set $A$ and every $\lambda$-small set $B$, the set $\mathrm{Map}(A, B)$ is $\lambda$-small.

**Example 1.1.2.11** For every regular cardinal $\kappa$, one has $0 \ll \kappa$.

**Example 1.1.2.12** For every infinite regular cardinal $\kappa$, one has $\aleph_0 \ll \kappa$.

**1.1.2.13** Let $\kappa$ and $\lambda$ be regular cardinals. How is the condition $\kappa \ll \lambda$ used in practice? The answer comes down to the following pair of manoeuvres, which we can do whenever $\kappa \ll \lambda$.

If $J$ is a $\lambda$-small poset, then we can write

$$J = \bigcup_{\ell \in \Lambda} J_\ell\,,$$

where $\Lambda$ is a $\lambda$-small and $\kappa$-filtered poset, and each $J_\ell \subseteq J$ is a $\kappa$-small poset. In this way, we may express any $\lambda$-small colimit as a $\lambda$-small and $\kappa$-filtered colimit of $\kappa$-small colimits:

$$\operatorname*{colim}_{j \in J} X(j) \simeq \operatorname*{colim}_{\ell \in \Lambda} \operatorname*{colim}_{j \in J_\ell} X(j)$$

(Lurie, 2009, Corollary 4.2.3.11).

On the other hand, if $M$ is a $\kappa$-filtered poset, then we can write

$$M = \bigcup_{k \in K} M_k\,,$$

where $K$ is a $\lambda$-filtered poset, and each $M_k \subseteq M$ is $\lambda$-small and $\kappa$-filtered. In this way, we may express any $\kappa$-filtered colimit as a $\lambda$-filtered colimit of $\lambda$-small and $\kappa$-filtered colimits:

$$\operatorname*{colim}_{m \in M} Y(m) \simeq \operatorname*{colim}_{k \in K} \operatorname*{colim}_{m \in M_k} Y(m)$$

(Lurie, 2009, Lemma 5.4.2.10).

**Proposition 1.1.2.14** (Lurie, 2009, Proposition 5.4.2.11) *If $\kappa \ll \lambda$ are regular cardinals, then every $\kappa$-accessible category is $\lambda$-accessible. Similarly, every $\kappa$-presentable category is $\lambda$-presentable.*

**Definition 1.1.2.15** A large-category $C$ is *accessible* if and only if there exists a regular cardinal $\kappa$ such that $C$ is $\kappa$-accessible.

We shall say that $C$ is *presentable* if and only if there exists a regular cardinal $\kappa$ such that $C$ is $\kappa$-presentable.

**Example 1.1.2.16** A small category is accessible if and only if it is idempotent-complete (Lurie, 2009, Corollary 5.4.3.6).

**1.1.2.17** A large-category is presentable if and only if it is accessible and has all colimits. A presentable large-category automatically has all limits as well.

**Definition 1.1.2.18** A large-category $C$ is *locally presentable*[7] if and only if every object $X \in C$ is contained in a presentable full large-subcategory $C' \subseteq C$ such that the inclusion $C' \hookrightarrow C$ preserves colimits.

**1.1.2.19** In other words, a large-category $C$ is locally presentable just in case it can be expressed as a class-indexed union of presentable large-categories, each of which is embedded in $C$ via a colimit-preserving, fully faithful functor.

### 1.1.3 Presheaf categories

Let $C$ be a large-category. What happens if we seek to make sense in NBG of the category $\tau_0 \boldsymbol{P}^V(C)$ of presheaves of sets $C^{op} \to \mathbf{Set}^V$?

Right away we encounter a problem: if the objects of $C$ form a proper class $C_0$, then there is no class of class maps $\mathrm{Map}(C_0, V)$. Indeed, on one hand, in NBG, every element of a class is itself a set, and on the other hand, a class map $f : C_0 \to V$ cannot be a set.[8]

**1.1.3.1** If $C$ is a small category, then the large-category $\tau_0 \boldsymbol{P}^V(C)$ is locally small, and it enjoys many of the same good properties enjoyed by $\mathbf{Set}^V$ itself. For every regular cardinal $\kappa$, it is $\kappa$-presentable, and it is *cartesian closed*: for every pair of presheaves $X, Y : C^{op} \to \mathbf{Set}^V$, the morphisms $X \to Y$ form a presheaf $\mathrm{Mor}(X, Y) : D^{op} \to \mathbf{Set}^V$. The category $\tau_0 \boldsymbol{P}^V(C)$ is a *1-topos*.

[7] In the 1-category literature, the phrase *locally presentable category* is used for what we call *presentable category*.

[8] Worse still, the 'very large' category of classes is not cartesian closed, so there's no hope of defining $\mathrm{Map}(C_0, V)$ by means of some other artifice.

Similarly, the category $\boldsymbol{P}^V(C)$ of presheaves $C^{op} \to \mathbf{An}^V$ is a $\kappa$-presentable topos for every regular cardinal $\kappa$.

**Example 1.1.3.2** Let $C$ be a locally small category. If $Y \in C$ is an object, then $よ_Y : C^{op} \to \mathbf{Set}^V$ is the presheaf $X \mapsto \mathrm{Map}_C(X, Y)$ represented by $Y$.

Dually, if $X \in C$ is an objects, then $よ^X : C \to \mathbf{Set}^V$ is the functor $Y \mapsto \mathrm{Map}_C(X, Y)$ corepresented by $X$.

**Definition 1.1.3.3** Let $C$ be a locally small large-category. A *small presheaf* of sets on $C$ is a functor $C^{op} \to \mathbf{Set}^V$ that is left Kan extended from its restriction to some small full subcategory $D \subseteq C$. We write $\tau_0\boldsymbol{P}^V(C)$ for the locally small large-category of small presheaves of sets.

Similarly, a *small presheaf* (of animae) is a functor $C^{op} \to \mathbf{An}^V$ that is left Kan extended from its restriction to some small full subcategory $D \subseteq C$. We write $\boldsymbol{P}^V(C)$ for the locally small large-category of small presheaves.

**Example 1.1.3.4** Of course if $C$ is a small category, then every presheaf on $C$ is small. Thus the notation above does not conflict with the one established in Notation 1.1.0.2.

**1.1.3.5** Let $C$ be a locally small large-category. For any small full subcategory $D \subseteq C$, we may contemplate the large-category $\boldsymbol{P}^V(D)$ of presheaves $D^{op} \to \mathbf{An}^V$. If we have an inclusion of full subcategories $D' \subseteq D \subset C$, then left Kan extension identifies $\boldsymbol{P}^V(D')$ with a full subcategory of $\boldsymbol{P}^V(D)$.

The (class-indexed) filtered union $\bigcup_D \boldsymbol{P}^V(D)$ over the class of small full subcategories of $C$ is precisely the large-category $\boldsymbol{P}^V(C)$.

The categories $\tau_0\boldsymbol{P}^V(C)$ and $\boldsymbol{P}^V(C)$ are thus locally presentable large-categories.

**Example 1.1.3.6** Let $C$ be a locally small large-category. For any object $Y \in C$, the representable presheaf $よ_Y$ is left Kan extended from any full subcategory that contains $Y$. In particular, $よ_Y$ is small.

Thus the assignment $Y \mapsto よ_Y$ is the fully faithful *Yoneda embedding*

$$よ : C \hookrightarrow \boldsymbol{P}^V(C) .$$

**Example 1.1.3.7** Let $C$ be a locally small large-category. If $C^{op}$ is accessible, then $\tau_0\boldsymbol{P}^V(C)$ and $\boldsymbol{P}^V(C)$ are the categories of accessible functors $C^{op} \to \mathbf{Set}^V$ and $C^{op} \to \mathbf{An}^V$, respectively.

**1.1.3.8** Let $C$ be a locally small large-category. The categories $\tau_0\boldsymbol{P}^V(C)$ and $\boldsymbol{P}^V(C)$ may not enjoy all the same good features that $\mathbf{Set}^V$ and $\mathbf{An}^V$ have. The categories $\tau_0\boldsymbol{P}^V(C)$ and $\boldsymbol{P}^V(C)$ possess all colimits, but they do not generally have all limits. For example, if $C$ has no nonidentity arrows, then there is no terminal object in $\tau_0\boldsymbol{P}^V(C)$.

If $C^{op}$ is accessible or small, then $\tau_0 \boldsymbol{P}^V(C)$ and $\boldsymbol{P}^V(C)$ do have all limits.

**Definition 1.1.3.9** Let $A$ be a class of categories. Let $C$ be a locally small large-category, and let $C' \subseteq C$ be a full subcategory. Then we say that $C'$ *generates* $C$ *freely under* $A$*-shaped colimits* if and only if, for every large-category $D$ that has all $A$-shaped colimits, the following assertions obtain.

1 Every functor $C' \to D$ extends to a functor $C \to D$ that preserves $A$ -shaped colimits.
2 For every pair of functors $F, G \colon C \to D$ that preserve $A$-shaped colimits, the map $\mathrm{Map}(F, G) \to \mathrm{Map}(F|C', G|C')$ is an equivalence.

If $f : C'' \hookrightarrow C$ is a fully faithful functor, then we will say that $f$ *generates* $C$ *freely under* $A$*-shaped colimits* if and only if its image $f(C'') \subseteq C$ does so.

**Remark 1.1.3.10** If $C$ is not small, then in NBG we can make sense neither of $\mathrm{Fun}(C, D)$, nor of the full subcategory $\mathrm{Fun}^A(C, D) \subseteq \mathrm{Fun}(C, D)$ consisting of those functors that preserve $A$-shaped colimits. If however we are in a situation in which these objects *can* be made sensible, then $C'$ generates $C$ freely under $A$-shaped colimits if and only if the restriction induces an equivalence

$$\mathrm{Fun}^A(C, D) \xrightarrow{\sim} \mathrm{Fun}(C', D) .$$

**Proposition 1.1.3.11** *Let $C$ be a locally small large-category. Then the Yoneda embedding $よ \colon C \hookrightarrow \boldsymbol{P}^V(C)$ generates $\boldsymbol{P}^V(C)$ freely under all colimits.*

The theory of small presheaves can be relativized to a regular cardinal $\kappa$:

**Definition 1.1.3.12** Let $\kappa$ be a regular cardinal. Let $C$ be a locally $\kappa$-small large-category. A *$\kappa$-small presheaf* of sets on $C$ is a functor $C^{op} \to \mathbf{Set}^\kappa$ that is left Kan extended from its restriction to some $\kappa$-small full subcategory $D \subseteq C$. The large-category of $\kappa$-small presheaves of sets will be denoted $\tau_0 \boldsymbol{P}^\kappa(C)$.

Similarly, a *$\kappa$-small presheaf* (of animae) is a functor $C^{op} \to \mathbf{An}^\kappa$ that is left Kan extended from its restriction to some $\kappa$-small full subcategory $D \subseteq C$. The large-category of $\kappa$-small presheaves will be denoted $\boldsymbol{P}^\kappa(C)$.

**1.1.3.13** If $C$ is small, then so is $\boldsymbol{P}^\kappa(C)$.

**1.1.3.14** Since we have assumed that $C$ is locally $\kappa$-small, it follows that the Yoneda embedding lands in $\boldsymbol{P}^\kappa(C)$. We can therefore characterize $\boldsymbol{P}^\kappa(C)$ as the smallest full large-subcategory of $\boldsymbol{P}^V(C)$ containing $よ(C)$ and closed under $\kappa$-small colimits.

**Proposition 1.1.3.15** *Let $\kappa$ be a regular cardinal. Let $C$ be a locally $\kappa$-small large-category. Then the Yoneda embedding $よ \colon C \hookrightarrow \boldsymbol{P}^\kappa(C)$ generates $\boldsymbol{P}^\kappa(C)$ freely under $\kappa$-small colimits.*

**1.1.3.16** The category $\boldsymbol{P}^{\kappa}(C)$ has all $\kappa$-small colimits, but in general, it does not have $\kappa$-small limits, and it is not cartesian closed. To ensure these properties as well, we must turn to a discussion of inaccessible cardinals.

### 1.1.4 Strong limit and inaccessible cardinals

**Definition 1.1.4.1** One says that $\kappa$ is a *weak limit cardinal* if and only if, for every cardinal $\xi$, if $\xi < \kappa$, then $\xi^+ < \kappa$.

A cardinal $\kappa$ is said to be a *strong limit cardinal* if and only if, for every cardinal $\xi$, if $\xi < \kappa$, then $2^\xi < \kappa$ as well. Equivalently, $\kappa$ is a strong limit cardinal if and only if, for every pair of $\kappa$-small sets $X$ and $Y$, the set $\mathrm{Map}(X, Y)$ of maps $X \to Y$ is $\kappa$-small as well.

One says that $\kappa$ is *weakly inaccessible* if and only if it is a regular, uncountable, weak limit cardinal.

One says that $\kappa$ is *inaccessible*[9] if and only if it is a regular, uncountable,[10] strong limit cardinal. Equivalently, an uncountable cardinal $\kappa$ is inaccessible if and only if $\mathbf{Set}^{\kappa}$ has all $\kappa$-small colimits and is cartesian closed. Equivalently again, an uncountable cardinal $\kappa$ is inaccessible if and only if $\mathbf{Set}^{\kappa}$ has all $\kappa$-small colimits and all $\kappa$-small colimits.

The Generalized Continuum Hypothesis (GCH) is equivalent to the statement that the classes of strong and weak limit cardinals coincide, and similarly the classes of inaccessible and weakly inaccessible cardinal coincide.

**Example 1.1.4.2** A cardinal $\kappa$ is a weak limit cardinal if and only if, for some limit ordinal $\alpha$, one has $\kappa = \aleph_\alpha$.

**Example 1.1.4.3** The $\beth$ family of cardinals is defined by a function from the class of ordinal numbers to the class of cardinal numbers. It's defined by transfinite induction:

1 By definition, $\beth_0 = \aleph_0$.
2 For any ordinal $\alpha$, one defines $\beth_{\alpha+1} \coloneqq 2^{\beth_\alpha}$.
3 For any limit ordinal $\alpha$, one defines $\beth_\alpha \coloneqq \sup\left\{\beth_\beta : \beta < \alpha\right\}$.

The cardinal $\beth_\alpha$ is the cardinality of $V_{\omega+\alpha}$.

The Generalized Continuum Hypothesis (GCH) is equivalent to the statement that $\aleph_\alpha = \beth_\alpha$ for each ordinal $\alpha$,

A cardinal $\kappa$ is a strong limit cardinal if and only if, for some limit ordinal $\alpha$, one has $\kappa = \beth_\alpha$.

[9] Some authors say *strongly inaccessible* instead of *inaccessible*.
[10] We include the condition of uncountability only for convenience. It is not unreasonable to regard 0 and $\aleph_0$ as inaccessible as well.

The cardinal $\beth_\omega$ is the smallest uncountable strong limit cardinal. It is not inaccessible, however, because it is not regular.

An inaccessible cardinal $\kappa$ is a $\beth$-*fixed point*: that is, $\beth_\kappa = \kappa$.

**1.1.4.4** A regular uncountable cardinal $\kappa$ is inaccessible if and only if one has $\kappa \ll \kappa$.

**Definition 1.1.4.5** (SGA 4 I, Exposé I, §0 and Appendix) An uncountable set $U$ is a *Grothendieck universe* if it satisfies the following conditions.

1 The set $U$ is *transitive*: if $X \in Y \in U$, then $X \in U$ as well.
2 If $X, Y \in U$, then $\{X, Y\} \in U$ as well.
3 If $X \in U$, then the powerset $P(X) \in U$ as well.
4 If $A \in U$ and $X\colon A \to U$ is a map, then

$$\bigcup_{\alpha \in A} X(\alpha) \in U$$

as well.

Grothendieck universes are essentially the same thing as inaccessible cardinals. This was effectively proved by Tarski (1938). See also Bourbaki, SGA 4 I, Exposé I, Appendix.

**Proposition 1.1.4.6** *If $\kappa$ is an inaccessible cardinal, then the set $V_\kappa$ of all sets of rank less than $\kappa$ is a Grothendieck universe of rank and cardinality $\kappa$.*

*If $U$ is a Grothendieck universe, then there exists an inaccessible cardinal $\kappa$ such that $U = V_\kappa$.*

**Theorem 1.1.4.7** *If $\kappa$ is an inaccessible cardinal, then $V_\kappa \models$ ZFC, and $V_{\kappa+1} \models$ NBG.*

*Assuming that* ZFC *(respectively,* NBG*) is consistent, then the existence of inaccessible cardinals is not provable by methods formalizable in* ZFC *(resp.,* NBG*).*

**Axiom 1.1.4.8** The *Axiom of Universes* (AU) is the assertion that every cardinal is dominated by an inaccessible cardinal, or, equivalently, every set is an element of some Grothendieck universe. *Tarski–Grothendieck set theory* is the schema TG = NBG + AU.

Under AU, the proper class of inaccessible cardinals can be well ordered. It will be helpful for us to have a notation for this.

**Definition 1.1.4.9** Assume AU. Let us define the $\daleth$ family of cardinals as a function from the class of ordinal numbers to the class of cardinal numbers:

1 By definition, $\daleth_0 = \aleph_0$.

2 For any ordinal $\alpha$, one defines $\daleth_{\alpha+1}$ as the smallest inaccessible number greater than $\daleth_\alpha$.
3 For any limit ordinal $\alpha$, one defines $\daleth_\alpha \coloneqq \sup\left\{\daleth_\beta : \beta < \alpha\right\}$.

Thus $\daleth_0 = \aleph_0$, and for $\alpha \geq 1$ an ordinal, $\daleth_\alpha$ is the '$\alpha$-th inaccessible cardinal'.

### 1.1.5 Echelons of accessibility

The notions of smallness, accessibility, and presentability of categories can all be relativized to a Grothendieck universe.

**Definition 1.1.5.1** The *echelon* of a category $C$ is the smallest ordinal $\alpha$ such that $C$ is both locally $\daleth_\alpha$-small and $\daleth_{\alpha+1}$-small.

**Example 1.1.5.2** The category of finite sets is of echelon 0. More generally, for any ordinal number $\alpha$, the category of $\daleth_\alpha$-small sets is of echelon $\alpha$.

**Notation 1.1.5.3** Let $\alpha$ be an ordinal number. We will denote by $\mathbf{Cat}_\alpha$ the category of categories that are $\daleth_\alpha$-small.

Accordingly, we will denote by $\mathbf{Set}_\alpha$ and $\mathbf{An}_\alpha$ the categories of $\daleth_\alpha$-small sets and animae, respectively.

The categories $\mathbf{Cat}_\alpha$, $\mathbf{Set}_\alpha$, and $\mathbf{An}_\alpha$ are all of echelon $\alpha$.

**Definition 1.1.5.4** Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \daleth_\alpha$ be a regular cardinal. Let $C$ and $D$ be categories of echelon $\leq \alpha$.

A functor $f : C \to D$ is *$\kappa$-continuous of echelon* $\leq \alpha$ if and only if it preserves all $\daleth_\alpha$-small, $\kappa$-filtered colimits.

An object $X$ of $C$ is said to be *$\kappa$-compact of echelon* $\leq \alpha$ if and only if the functor $よ^X : C \to \mathbf{An}_\alpha$ corepresented by $X$ is $\kappa$-continuous of echelon $\leq \alpha$. We write $C_\alpha^{(\kappa)} \subseteq C$ for the full subcategory of $\kappa$-compact objects of echelon $\leq \alpha$.

A category $C$ is *$\kappa$-accessible of echelon* $\leq \alpha$ if and only if it satisfies the following quartet of conditions:

1 The category $C$ is of echelon $\leq \alpha$.
2 The category $C$ has all $\daleth_\alpha$-small, $\kappa$-filtered colimits.
3 The subcategory $C_\alpha^{(\kappa)} \subseteq C$ is $\daleth_\alpha$-small.
4 The subcategory $C_\alpha^{(\kappa)}$ generates $C$ under $\daleth_\alpha$-small and $\kappa$-filtered colimits.

A category $C$ is *$\kappa$-presentable of echelon* $\leq \alpha$ if and only if it is $\kappa$ accessible of echelon $\leq \alpha$, and $C_\alpha^{(\kappa)}$ has all $\kappa$-small colimits.

A category $C$ is *accessible of echelon* $\leq \alpha$ if and only if there exists a regular cardinal $\kappa < \daleth_\alpha$ such that $C$ is $\kappa$-accessible of echelon $\leq \alpha$. It is *presentable of echelon* $\leq \alpha$ if and only if there exists a regular cardinal $\kappa < \daleth_\alpha$ such that $C$ is $\kappa$-presentable of echelon $\leq \alpha$.

**Example 1.1.5.5** Let $\alpha \geq 1$ be an ordinal number. Let $C$ be a $\beth_\alpha$-small category. The category $\boldsymbol{P}^{\beth_\alpha}(C)$ is then 0-presentable of echelon $\leq \alpha$.

Conversely, if $D$ is a 0-presentable category of echelon $\leq \alpha$, then there exists a $\beth_\alpha$-small category $C$ and an equivalence $D \simeq \boldsymbol{P}^{\beth_\alpha}(C)$.

**Notation 1.1.5.6** Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \beth_\alpha$ be a regular cardinal. We shall write $\mathbf{Acc}^\alpha_\kappa \subset \mathbf{Cat}_{\alpha+1}$ for the following subcategory.

1 The objects of $\mathbf{Acc}^\alpha_\kappa$ are the $\kappa$-accessible categories of echelon $\leq \alpha$.
2 The morphisms $f : C \to D$ of $\mathbf{Acc}^\alpha_\kappa$ are the $\kappa$-continuous functors of echelon $\leq \alpha$ such that $f(C^{(\kappa)}_\alpha) \subseteq D^{(\kappa)}_\alpha$.

Similarly, we shall write $\mathbf{Pr}^{\alpha,L}_\kappa \subset \mathbf{Acc}^\alpha_\kappa$ for the following subcategory.

1 The objects of $\mathbf{Pr}^{\alpha,L}_\kappa$ are the $\kappa$-presentable categories of echelon $\leq \alpha$.
2 The morphisms of $\mathbf{Pr}^{\alpha,L}_\kappa$ are those functors in $\mathbf{Acc}^\alpha_\kappa$ that preserve all $\beth_\alpha$-small colimits.

Now we may write

$$\mathbf{Acc}^\alpha = \bigcup_{\kappa<\beth_\alpha} \mathbf{Acc}^\alpha_\kappa \qquad \text{and} \qquad \mathbf{Pr}^{\alpha,L} = \bigcup_{\kappa<\beth_\alpha} \mathbf{Pr}^{\alpha,L}_\kappa$$

for the category of accessible categories of echelon $\alpha$ and the category of presentable categories of echelon $\alpha$, respectively.

We specify some corresponding subcategories of $\mathbf{Cat}_\alpha$. Let $\mathbf{Cat}^{idem}_\alpha \subset \mathbf{Cat}_\alpha$ denote the full subcategory consisting of the idempotent-complete $\beth_\alpha$-small categories. Let $\mathbf{Cat}^\kappa_\alpha \subseteq \mathbf{Cat}_\alpha$ denote the subcategory whose objects are $\beth_\alpha$-small categories that possess all $\kappa$-small colimits and whose morphisms are functors that preserve $\kappa$-small colimits. Finally, let

$$\mathbf{Cat}^{\kappa,idem}_\alpha \coloneqq \mathbf{Cat}^\kappa_\alpha \cap \mathbf{Cat}_\alpha .$$

Please observe that if $\kappa$ in uncountable, then in fact $\mathbf{Cat}^{\kappa,idem}_\alpha = \mathbf{Cat}^\kappa_\alpha$, because one can split an idempotent using a colimit that is $\aleph_1$-small and $\aleph_0$-filtered (Lurie, 2009, Corollary 4.4.5.15 & Example 5.3.1.9)

**1.1.5.7** Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \beth_\alpha$ be a regular cardinal. The assignment $C \mapsto C^{(\kappa)}_\alpha$ defines a functor

$$\mathbf{Acc}^\alpha_\kappa \to \mathbf{Cat}^{idem}_\alpha .$$

This functor is an equivalence; furthermore, it restricts to an equivalence

$$\mathbf{Pr}^{\alpha,L}_\kappa \xrightarrow{\sim} \mathbf{Cat}^{\kappa,idem}_\alpha .$$

We will describe the inverses of these equivalences in the next section.

### 1.1.6 Indization

**Definition 1.1.6.1** Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \beth_\alpha$ be a regular cardinal. Let $C$ be a category of echelon $\leq \alpha$.

Then $\mathrm{Ind}_\kappa^\alpha(C)$ is the smallest full subcategory $D \subseteq \boldsymbol{P}^{\beth_\alpha}(C)$ such that $D$ contains the image of the Yoneda embedding $よ\colon C \hookrightarrow \boldsymbol{P}^{\beth_\alpha}(C)$, and $D$ is stable under $\beth_\alpha$-small, $\kappa$-filtered colimits.

Accordingly, if $C$ is a locally small large-category, then $\mathrm{Ind}_\kappa^V(C)$ is the smallest full subcategory $D \subseteq \boldsymbol{P}^V(C)$ such that $D$ contains the image of the Yoneda embedding $よ\colon C \hookrightarrow \boldsymbol{P}^V(C)$, and $D$ is stable under $\kappa$-filtered colimits.

**Example 1.1.6.2** The category $\mathrm{Ind}_0^\alpha(C)$ is equivalent to the presheaf category $\boldsymbol{P}^{\beth_\alpha}(C)$.

**Proposition 1.1.6.3** *Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \beth_\alpha$ be a regular cardinal. Let $C$ be a category of echelon $\leq \alpha$. Then the Yoneda embedding $よ\colon C \hookrightarrow \mathrm{Ind}_\kappa^\alpha(C)$, generates $\mathrm{Ind}_\kappa^\alpha(C)$ freely under $\beth_\alpha$-small, $\kappa$-filtered colimits.*

*Similarly, if $C$ is a locally small large-category, then the Yoneda embedding $よ\colon C \hookrightarrow \mathrm{Ind}_\kappa^V(C)$, generates $\mathrm{Ind}_\kappa^V(C)$ freely under $\kappa$-filtered colimits.*

**1.1.6.4** Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \beth_\alpha$ be a regular cardinal. Let $C$ and $D$ be categories that contain all $\beth_\alpha$-small, $\kappa$-filtered colimits. Denote by $\mathrm{Fun}_\kappa^\alpha(C, D)$ the full subcategory of $\mathrm{Fun}(C, D)$ consisting of the functors $C \to D$ that preserve all $\beth_\alpha$-small, $\kappa$-filtered colimits.

If $C'$ is a category of echelon $\leq \alpha$, then restriction along the Yoneda embedding induces an equivalence of categories

$$\mathrm{Fun}_\kappa^\alpha(\mathrm{Ind}_\kappa^\alpha(C'), D) \simeq \mathrm{Fun}(C', D)\,.$$

**Example 1.1.6.5** Let $\alpha \geq 1$ be an ordinal number, and let $\kappa < \beth_\alpha$ be a regular cardinal. Let $C$ be a $\beth_\alpha$-small category. Then the category $\mathrm{Ind}_\kappa^\alpha(C)$ is $\kappa$-accessible of echelon $\leq \alpha$.

Hence we obtain a functor

$$\mathrm{Ind}_\kappa^\alpha\colon \mathbf{Cat}_\alpha \to \mathbf{Acc}_\kappa^\alpha\,.$$

This functor exhibits $\mathbf{Acc}_\kappa^\alpha$ as a localization of $\mathbf{Cat}_\alpha$, and it restricts to the inverse

$$\mathbf{Cat}_\alpha^{idem} \simeq \mathbf{Acc}_\kappa^\alpha$$

of the equivalence $C \mapsto C_\alpha^{(\kappa)}$ constructed in (1.1.5.7). It also restricts further to the equivalence

$$\mathbf{Cat}_\alpha^{\kappa,idem} \simeq \mathbf{Pr}_\kappa^{\alpha,L}$$

inverse to the restriction of $C \mapsto C_\alpha^{(\kappa)}$.

**Construction 1.1.6.6** Let $\beta > \alpha \geq 1$ be two ordinal numbers. Then we can use indization to define a change-of-universe functor

$$\boldsymbol{I}_{\alpha}^{\beta} \coloneqq \mathrm{Ind}_{\beth_\alpha}^{\beta} \ .$$

If $\kappa < \beth_\alpha$ is a regular cardinal, then this is a fully faithful functor

$$\boldsymbol{I}_{\alpha}^{\beta} \colon \mathbf{Acc}_{\kappa}^{\alpha} \hookrightarrow \mathbf{Acc}_{\kappa}^{\beta} \ ,$$

which is equivalent to the inclusion $\mathbf{Cat}_{\alpha}^{idem} \hookrightarrow \mathbf{Cat}_{\beta}^{idem}$. The functor $\boldsymbol{I}_{\alpha}^{\beta}$ restricts to a fully faithful functor

$$\boldsymbol{I}_{\alpha}^{\beta} \colon \mathbf{Pr}_{\kappa}^{\alpha,L} \hookrightarrow \mathbf{Pr}_{\kappa}^{\beta,L} \ ,$$

which is equivalent to the inclusion $\mathbf{Cat}_{\alpha}^{\kappa,idem} \hookrightarrow \mathbf{Cat}_{\beta}^{\kappa,idem}$.

**Example 1.1.6.7** For any category $C$ of echelon $\alpha$, one has

$$\boldsymbol{I}_{\alpha}^{\beta}(\boldsymbol{P}^{\beth_\alpha}(C)) \simeq \boldsymbol{P}^{\beth_\beta}(C) \ .$$

**Notation 1.1.6.8** Let $\alpha \geq 1$ be an ordinal number. Then we will abberviate

$$\mathrm{Ind}^{\alpha} \coloneqq \mathrm{Ind}_{\aleph_0}^{\alpha} \ .$$

### 1.1.7 Higher inaccessibility

1.1.7.1 We shall endow an ordinal with its order topology. This may be described recursively as follows:

1. The ordinal 0 is the empty topological space.
2. For any ordinal $\alpha$ with its order topology, the order topology on the ordinal $\alpha + 1$ is the one-point compactification of $\alpha$.
3. For any limit ordinal $\alpha$, the order topology is the colimit topology $\mathrm{colim}_{\beta<\alpha} \beta$.

We will use terminology that treats **Ord** itself as a topological space, even though it is not small.

**Definition 1.1.7.2** If $W \subseteq$ **Ord** is a subclass, then a *limit point* of $A$ is an ordinal $\alpha$ such that $\alpha = \sup(W \cap \alpha)$. The class $W$ will be said to be *closed* if and only if it contains all its limit points.

An *ordinal function* is a class map $f \colon \mathbf{Ord} \to \mathbf{Ord}$. We say that $f$ is *continuous* if and only if its restriction to any subset is continuous. Equivalently, $f$ is continuous if and only if, for every subclass $W \subseteq$ **Ord** and every limit point $\alpha$ of $W$, the ordinal $f(\alpha)$ is a limit point of $f(W)$.

We say that $f$ is *normal* if and only if it is continuous and strictly increasing.

**1.1.7.3** If $f$ is a normal ordinal function, then its image is a closed and unbounded class[11] of ordinals. Conversely, if $W \subseteq \mathbf{Ord}$ is a closed and unbounded class, then we can define a normal ordinal function $f$ by

$$f(\alpha) = \min \{\gamma \in W : (\forall \beta < \alpha)(f(\beta) < \gamma)\} .$$

**Definition 1.1.7.4** Let $f$ be an ordinal function. A regular cardinal $\kappa$ is said to be $f$*-inaccessible* if and only if, for every ordinal $\alpha$, if $\alpha < \kappa$, then $f(\alpha) < \kappa$ as well.

**Example 1.1.7.5** If $f$ is the ordinal function that carries an ordinal $\alpha$ to the cardinal $2^{|\alpha|}$, then an $f$-inaccessible cardinal is precisely an inaccessible cardinal.

**Construction 1.1.7.6** Let $f$ be an increasing ordinal function such that for every ordinal $\beta$, one has $\beta < f(\beta)$. For every ordinal $\xi$, the normal ordinal function $\alpha \mapsto f^{\alpha}(\xi)$ in uniquely specified by the requirements that $f^0(\xi) = \xi$ and $f^{\alpha+1}(\xi) = f(f^{\alpha}(\xi))$.

Jorgensen (1970) proves that an $f$-inaccessible cardinal greater than an ordinal $\xi$ is precisely a regular cardinal that is a *fixed point* for the ordinal function $\alpha \mapsto f^{\alpha}(\xi)$.

**Example 1.1.7.7** If $f$ is the ordinal function $\beta \mapsto 2^{|\beta|}$, then $f^{\alpha}(\omega) = \beth_{\alpha}$. An inaccessible cardinal is thus precisely a regular $\beth$-fixed point.

If $f$ is the ordinal function $\beta \mapsto |\beta|^{+}$, then $f^{\alpha}(\omega) = \aleph_{\alpha}$. A weakly inaccessible cardinal is precisely a regular $\aleph$-fixed point.

**Example 1.1.7.8** Assume AU. Consider the ordinal function $f$ that carries an ordinal $\beta$ to the smallest inaccessible cardinal greater than $\beta$. For any ordinal $\alpha$, we have $\daleth_{\alpha} = f^{\alpha}(\omega)$.

An $f$-inaccessible cardinal is precisely a $\daleth$-fixed point. These are called *1-inaccessible cardinals*. If $\kappa$ is 1-inaccessible, then $V_{\kappa} \models$ (ZFC + AU). If ZFC + AU is consistent, then the existence of 1-inaccessible cardinals is not provable by methods formalizable in ZFC + AU.

Iterating this strategy, one can now proceed to define $\alpha$-inaccessibility for every ordinal $\alpha$. Iterating the iteration, one can define notions of hyperinaccessibility, hyper$^{\alpha}$inaccessibility, *etc.* We cut to the chase:

**Axiom 1.1.7.9** The *Lévy scheme* (LÉVY) is the assertion that for every ordinal function $f$ and every ordinal $\xi$, there exists an $f$-inaccessible cardinal $\kappa$ such that $\xi < \kappa$.

[11] This is often abbreviated *club class* in set theory literature.

**Theorem 1.1.7.10** (Lévy (1960); Montague (1962); Jorgensen (1970)) *The following are equivalent.*

1. *The Lévy scheme.*
2. *Every normal ordinal function has a regular cardinal in its image.*
3. *Every closed unbounded subclass $W \subseteq$ **Ord** contains a regular cardinal.*
4. *Every normal ordinal function has an inaccessible cardinal in its image.*
5. *Every closed unbounded subclass $W \subseteq$ **Ord** contains an inaccessible cardinal.*

**1.1.7.11** The Lévy scheme implies the Axiom of Universes, and the consistency strength of NBG + LÉVY is strictly greater than that of NBG + AU.

The consistency strength of the Lévy scheme is also strictly greater than the existence of $\alpha$-inaccessible, hyperinaccessible, hyper$^{\alpha}$inaccessible, *etc.*, cardinals.

**Definition 1.1.7.12** Let $\kappa$ be a regular cardinal. One says that $\kappa$ is *Mahlo* if and only if every closed unbounded subset $W \subseteq \kappa$ contains a regular cardinal.

**1.1.7.13** Assume that $\kappa$ is a Mahlo cardinal. Then $\kappa$ is $f$-inaccessible for every ordinal function $f$. Accordingly, $\kappa$ is a fixed point of every normal ordinal function.

Additionally, if $\kappa$ is a Mahlo cardinal, then $V_\kappa \models$ (ZFC + LÉVY), and similarly $V_{\kappa+1} \models$ (NBG + LÉVY). The consistency strength of the axiom 'a Mahlo cardinal exists' is strictly greater than the Lévy scheme.

**1.1.7.14** The Lévy scheme and its equivalents and slight variants have appeared under various names: 'Mahlo's principle' (Gloede, 1973), 'Axiom F' (Drake, 1974), '**Ord** is Mahlo' (Hamkins, 2003).

For our purposes, one of the main appeals of the Lévy scheme is the following.

**Theorem 1.1.7.15** *Assume* LÉVY.

### 1.1.8 Universe polymorphism

## 1.2 Reflection principles

### 1.2.1 Lévy hierarchy

### 1.2.2 Elementary embeddings

### 1.2.3 Reflection principles

### 1.2.4 Indescribable cardinals

## 1.3 Bicategories

## 1.4 Monads

### 1.4.1 Monoidal categories

**Definition 1.4.1.1** An *operator category* is a 1-category $\Phi$ of echelon 0 that satisfies the following conditions.

1 There exists a terminal object $* \in \Phi$. A map $i\colon * \to I$ will be called a *point* of $I$, and the set of points will be denoted $|I| \coloneqq \mathrm{Mor}_{\Phi}(*, I)$. Given a point $i \in |I|$, we will write $\{i\}$ for the corresponding object of $\Phi_{/I}$.
2 For every morphism $f\colon J \to I$ and for every point $i \in |I|$, there exists a fiber $J_i \coloneqq J \times_I \{i\}$.

A functor $f\colon \Psi \to \Phi$ is *admissible* if and only if it preserves the terminal object and the formation of fibers. We call $f$ an *operator morphism* if and only if, in addition, for every object $I \in \Phi$, the map $|I| \to |f(I)|$ is a bijection.

**Example 1.4.1.2** The one-point category $*$ is an operator category. In fact, it is a zero object in the category of operator categories and admissible functors, and it is an initial object in the category of operator categories and operator morphisms.

**Example 1.4.1.3** The category $F$ of finite sets is an operator category. It is terminal in the category of operator categories and operator morphisms. Every object of $F$ is isomorphic to a set of the form

$$\langle n \rangle \coloneqq \{1, \ldots, n\},$$

where $n \in \boldsymbol{N}_0$.

**Example 1.4.1.4** The category $E$ of totally ordered finite sets is an operator category. Every object of $E$ is isomorphic to a set of the form[12]

$$[n] := \{1, \dots, n\},$$

where $n \in \boldsymbol{N}_0$.

**Example 1.4.1.5** Let $n \in \boldsymbol{N}$. If $\Phi$ is an operator category, then so is the full subcategory $\Phi_{\leq n} \subseteq \Phi$ consisting of those objects $I \in \Phi$ such that the cardinality of $|I|$ is no greater than $n$.

**Construction 1.4.1.6** If $\Phi$ and $\Psi$ are operator categories, then we may define an operator category $\Phi \wr \Psi$ as follows. The objects are pairs $(I, M_I)$ consisting of an object $I \in \Phi$ and an indexed collection $M_I = \{M_i\}_{i \in |I|}$ of objects $M_i \in \Psi$. A morphism $(\phi, \psi_J)\colon (J, N_J) \to (I, M_J)$ consists of a morphism $\phi\colon J \to I$ of $\Phi$ and, for every $j \in |J|$, a morphism $\psi_j\colon N_j \to M_{\phi(j)}$ of $\Psi$.

This is the *wreath product* of operator categories. One can show that this is a monoidal structure on the category of operator categories and operator morphisms with unit $*$.

The functor $\Phi \wr \Psi \to \Phi$ given by the assignment $(I, M_I) \mapsto I$ is admissible. The functors $\Phi = \Phi \wr * \to \Phi \wr \Psi$ and $\Psi = * \wr \Psi \to \Phi \wr \Psi$ are both operator morphisms.

**Example 1.4.1.7** Perhaps the most interesting examples of operator categories are the iterated wreath products of $E$: we let $E^0 := 0$, and for every $n \in \boldsymbol{N}_0$, we let $E^{n+1} := E \wr E^n$.

Thus the objects of $E^2$ are tuples of the form $([i]; [m_1], \dots, [m_i])$. It may be tempting to think of such an object as a suitable partition of the totally ordered set $[m_1 + \cdots + m_i]$, but this overall ordering need not be respected by a morphism

$$(\phi; \psi_1, \dots, \psi_i)\colon ([i]; [m_1], \dots, [m_i]) \to ([j]; [n_1], \dots, [n_j])$$

of $E^2$, which consists of a morphism $\phi\colon [i] \to [j]$ and morphisms

$$\psi_1\colon [m_1] \to [n_{\phi(1)}],\ \dots,\ \psi_i\colon [m_i] \to [n_{\phi(i)}].$$

**Definition 1.4.1.8** An operator category $\Phi$ is said to be *perfect* if and only if it satisfies the following hypotheses.

1 There exist an object $T \in \Phi$ and a point $t \in |T|$ such that, for every object $I \in \Phi$ and every point $i \in |I|$, there exists a unique morphism $\chi_i\colon I \to T$ such that $I_t = \{i\}$. We call the pair $(T, t)$ a *point classifier*.

[12] This notation may engender some bemusement. Regarded as an object of the simplicial category $\Delta$, the object $[n]$ normally refers to a totally ordered set with $n + 1$ elements. However, there is method in our madness: we will construct a category $\Theta(E)$, which has the same objects as $E$, and which is equivalent to the category $\Delta^{op}$; under this equivalence, our $[n]$ corresponds to the usual $[n]$.

2 The functor $\Phi_{/T} \to \Phi$ given by the assignment $I \mapsto I_t$ admits a right adjoint $T\colon \Phi \to \Phi_{/T}$.

**1.4.1.9** We have abused notation by using $T$ for both the point classifier and the right adjoint of the functor $I \mapsto I_t$. This abuse is partially justified by the observation that $T = T(*)$.

Let $I \in \Phi$, and let $J \in \Phi_{/T}$. There is a natural unit morphism $I \to TI$, and its universal property states that every morphism $J_t \to I$ extends in a unique fashion to a morphism $J \to TI$ over $T$.

**Example 1.4.1.10** The operator category $F$ is perfect: the point classifier is the set $T = \{t, 0\}$, and the functor $T$ carries a finite set $I$ to the finite set

$$TI = I \sqcup \{0\}\,,$$

viewed as an object over $T$ via the map $\chi_I\colon TI \to T$ with $\chi_I^{-1}\{t\} = I$.

Thus $T \cong \langle 2\rangle$, and $T\langle n\rangle \cong \langle n+1\rangle$.

**Example 1.4.1.11** The operator category $E$ is also perfect: the point classifier is the totally ordered finite set $T = \{-\infty < t < +\infty\}$, and the functor $T$ carries a totally ordered finite set $J$ to the finite set

$$TJ \coloneqq \{-\infty\} \sqcup J \sqcup \{+\infty\}\,,$$

where, for every $j \in J$, both $-\infty < j$ and $j < +\infty$, viewed as an object over $T$ via the map $\chi_J\colon TJ \to T$ with $\chi_J^{-1}\{t\} = J$.

Thus $T \cong [3]$, and $T[n] \cong [n+2]$.

**Example 1.4.1.12** The operator categories $F_{\leq n}$ and $E_{\leq n}$ are not perfect unless $n = 1$.

**Example 1.4.1.13** If $\Phi$ and $\Psi$ are two perfect operator categories, then the wreath product $\Phi \wr \Psi$ is perfect as well. The point classifier is the pair $(I, M_I)$, where $I = T_\Phi$ is the point classifier for $\Phi$, and $M_I = \{M_i\}_{i \in |T_\Phi|}$ is the indexed collection in which

$$M_i = \begin{cases} * & \text{if } i \neq t\,; \\ T_\Psi & \text{if } i = t\,. \end{cases}$$

**Example 1.4.1.14** In particular, $E^2$ is a perfect operator category. Abstractly, its point classifier is $([3]; [1], [3], [1])$, and

$$T([i]; [m_1], \dots, [m_i]) \cong ([i+2]; [1], [m_1+2], \dots, [m_i+2], [1])\,.$$

**Definition 1.4.1.15** Let $\Phi$ be a perfect operator category, and let $I, J \in \Phi$. Then a morphism $TJ \to TI$ of $\Phi$ is *algebraic* if and only if the induced morphism $J \times_{TI} I \hookrightarrow TJ \times_{TI} I$ is an isomorphism.

**1.4.1.16** Let $TJ \to TI$ be an algebraic morphism of $\Phi$. Consider $TI$ as an object over $T$ via the structure map, and consider $TJ$ as an object over $T$ via the composite $TJ \to TI \to T$. The universal property of $TJ$ then states that $TJ \to TI$ is uniquely determined by its restriction $TJ \times_{TI} I \to I$. Since the morphism $J \times_{TI} I \hookrightarrow TJ \times_{TI} I$ is an isomorphism, it follows that an algebraic morphism $TJ \to TI$ is uniquely determined by its restriction $J \times_{TI} I \to I$.

**Example 1.4.1.17** If $\psi\colon J \to I$ is a morphism of $\Phi$, then the induced morphism $T(\psi)\colon TJ \to TI$ is algebraic.

**Example 1.4.1.18** If $I \in \Phi$ is an object and $i \in |I|$, then there is a classifying morphism $I \to T$ of $\Phi$ whose fiber over $t$ is $i$. The classifying morphism in turn extends to an algebraic morphism $\chi_i\colon TI \to T$.

**Construction 1.4.1.19** Let $\Phi$ be a perfect operator category.

If $I, J, K \in \Phi$, then the composition of an algebraic morphism $TK \to TJ$ along with another algebraic morphism $TJ \to TI$ is again algebraic.

We are therefore entitled to construct the following category $\Theta(\Phi)$. The objects of $\Theta(\Phi)$ are the objects of $\Phi$. A morphism $I \to J$ in $\Theta(\Phi)$ is an algebraic morphism $TJ \to TI$. Please note the reversal of direction!

**Example 1.4.1.20** The category $\Theta(F)$ is Segal's category $\boldsymbol{\Gamma}$; that is, $\Theta(F)$ is opposite to the category of pointed finite sets.

**Example 1.4.1.21** The category $\Theta(E)$ is the simplicial category $\boldsymbol{\Delta}$. André Joyal constructed the equivalence $\Theta(E) \xrightarrow{\sim} \boldsymbol{\Delta}$ as the assignment $I \mapsto \mathrm{Mor}_{\Theta(E)}(\varnothing, I)$, where the set of algebraic morphisms $TI \to T(\varnothing)$ is given its natural ordering. This equivalence carries $[n] = \{1 < \cdots < n\} \in \Theta(E)$ to the object

$$\{0 < \cdots < n\} \in \boldsymbol{\Delta},$$

which justifies our surprising notation for objects of $E$.

**Example 1.4.1.22** For every $n \in \boldsymbol{N}_0$, the category $\Theta(E^n)$ is Joyal's category $\Theta_n$. This follows from a theorem of Clemens Berger, which identifies $\Theta_n$ as an iterated wreath product of $\Theta_1$.

**1.4.1.23** On a perfect operator $\Phi$, the assignment $I \mapsto TI$ defines an endofunctor $T\colon \Phi \to \Phi$. In fact, this endofunctor always admits the structure of a monad.

We have a natural transformation $\epsilon\colon \mathrm{id} \to T$ between endofunctors on $\Phi$, which for any $I \in \Phi$ gives the inclusion $I \hookrightarrow TI$ that is pulled back from the inclusion $\{t\} \to T$.

Let us describe another key natural transformation $\mu\colon T^2 \to T$. For every $I \in$

$\Phi$, we may compose the map $T^2I \to TT$ with the classifying map $\chi_t : TT \to T$ of $t \in |T| \subset |TT|$. The result is a map $T^2I \to T$ whose fiber over $t$ is $I$. Now the universal property of the inclusion $I \hookrightarrow TI$ ensures that the identity $\mathrm{id}: I \to I$ extends uniquely to a map $\mu_I : T^2I \to TI$.

The triple $(T, \epsilon, \mu)$ is a monad on $\Phi$. The category $\Theta(\Phi)$ is opposite to the Kleisli category of this monad. In [CITE], we wrote $\Lambda(\Phi) = \Theta(\Phi)^{op}$, and we called this the *Leinster category* of the operator category $\Phi$.

**Definition 1.4.1.24** Let $\Phi$ be a perfect operator category. Let $\phi: I \to J$ be a morphism of $\Theta(\Phi)$, that is, an algebraic morphism $TJ \to TI$ of $\Phi$.

We shall say that $\phi$ is *inert* if and only if the projection $J \times_{TI} I \to I$ is an isomorphism.

Dually, we shall say that $\phi$ is *active* if and only if the projection $J \times_{TI} I \to J$ is an isomorphism.

**Example 1.4.1.25** If $I \in \Phi$ is an object and $i \in |I|$, then the algebraic morphism $\chi_i : TI \to T$, is inert as a morphism $* \to I$ of $\Theta(\Phi)$.

**Example 1.4.1.26** Let $I$ be a finite set, and let $K \subseteq I$ be a subset. Then we may define an inert morphism $\chi: K \to I$ of $\Theta(F)$ as the algebraic morphism $\phi: TI \to TK$ given by

$$\phi(x) = \begin{cases} x & \text{if } x \in K \\ * & \text{if } x \notin K\,. \end{cases}$$

**Lemma 1.4.1.27** *Let $\Phi$ be a perfect operator category. The following are equivalent for a morphism $\phi: I \to J$ of $\Theta(\Phi)$.*

1. *The morphism $\phi$ corresponds to an algebraic morphism $TJ \to TI$ of the form $T(\psi)$ for some morphism $\psi: J \to I$ of $\Phi$.*
2. *The morphism $\phi$ is active.*

*Proof* On one hand, $T(\psi): TJ \to TI$ restricts to the morphism $\psi: J \to I$. On the other hand, the algebraic morphism $TJ \to TI$ is uniquely determined by its restriction to the morphism $J \times_{TI} I \to I$. □

**Notation 1.4.1.28** If $p: X \to \Theta(\Phi)$ is a functor, then for every $I \in \Theta(\Phi)$, we shall write $X_I$ for the fiber $p^{-1}\{I\}$.

**Definition 1.4.1.29** Let $\Phi$ be a perfect operator category. A *$\Phi$-monoidal category* is a cocartesian fibration

$$C^{\otimes} \to \Theta(\Phi)^{op}$$

such that, for every $I \in \Theta(\Phi)$, the functor

$$P_I \coloneqq \prod_{i\in|I|} \chi_i^* : C_I^\otimes \to \prod_{i\in|I|} C_{\{i\}}^\otimes$$

is an equivalence.

In particular, an $E$-monoidal category is called a *monoidal category*. For any natural number $n \in \boldsymbol{N}_0$, an $E^n$-monoidal category is called an *n-monoidal category*. An $F$-monoidal category is called a *symmetric monoidal category*.

**Notation 1.4.1.30** Let $\Phi$ be a perfect operator category, and let $C^\otimes$ be a $\Phi$-monoidal category. We will denote by $C$ the fiber $C_*^\otimes$ over the terminal object $*$, regarded as an object of $\Theta(\Phi)^{op}$. For example, if $I \in \Phi$ is an object, then the map $P_I$ of Definition 1.4.1.29 is an equivalence $C_I^\otimes \xrightarrow{\sim} C^{|I|}$.

Any active morphism $\phi \colon I \to J$ of $\Theta(\Phi)$ induces a functor $\phi^* : C_J^\otimes \to C_I^\otimes$. We may use the equivalences $P_J$ and $P_I$ to identify this functor with a functor

$$\otimes_{J/I} : C^{|J|} \to C^{|I|} .$$

In particular, when $I = *$, we obtain a functor

$$\otimes_J : C^{|J|} \to C ,$$

which we may call the *tensor product*. The functor $\otimes_{J/I}$ can then be identified with the product over $i \in |I|$ of the tensor products

$$\otimes_{J_i} : C^{|J_i|} \to C .$$

**Example 1.4.1.31** If $C^\otimes$ is a monoidal category, then for every $n \in \boldsymbol{N}_0$, the functor $\otimes_{[n]} : C^n \to C$ can be written

$$(X_1, \dots, X_n) \mapsto X_1 \otimes \cdots \otimes X_n .$$

In particular, when $n = 0$, this functor picks out a *unit object* $1 \in C$.

**Example 1.4.1.32** Similarly, if $C^\otimes$ is a symmetric monoidal category, then for every $n \in \boldsymbol{N}_0$, the functor $\otimes_{\langle n\rangle} : C^n \to C$ can be written

$$(X_1, \dots, X_n) \mapsto X_1 \otimes \cdots \otimes X_n .$$

**Example 1.4.1.33** If $C^\otimes$ is a 2-monoidal category, then for every $i \in \boldsymbol{N}_0$ and every $m_1, \dots, m_i \in \boldsymbol{N}_0$, we have a functor

$$\otimes_{([i];[m_1],\dots,[m_i])} : C^{m_1+\cdots+m_i} \to C .$$

In particular, for every $n \in \boldsymbol{N}_0$, we obtain a *horizontal tensor product*

$$\otimes_{([n];[1],\dots,[1])} : C^n \to C ,$$

and well as a *vertical tensor product*

$$\otimes_{([1];[n])} : C^n \to C\,,$$

How are these tensor products related?

To fix ideas, let's consider the case $n = 2$, and let's write $(X, Y) \mapsto X \otimes^h Y$ and $(X, Y) \mapsto X \otimes^v Y$ for the horizontal and vertical tensor products. We have two maps of $E^2$

$$([2]; [1], [1]) \to ([1], [2])$$

that induce bijections $\langle 2\rangle \simeq \langle 2\rangle$ on the underlying finite sets. These functors induce natural equivalences

$$X \otimes^h Y \simeq X \otimes^v Y \qquad \text{and} \qquad X \otimes^h Y \simeq Y \otimes^v X\,.$$

If we think of the first equivalence as an identification of the horizontal and vertical tensor products, then the second equivalence provides a *braiding*. Indeed, one may prove that 2-monoidal 1-categories are precisely *braided monoidal 1-categories* in the classical sense.

**Definition 1.4.1.34** Let $\Phi$ be a perfect operator category. Let $p\colon C^{\otimes} \to \Theta(\Phi)^{op}$ be a $\Phi$-monoidal category. Then a *$\Phi$-monoid in $C^{\otimes}$* is a section $M\colon \Theta(\Phi)^{op} \to C^{\otimes}$ of $p$ that carries inert morphisms of $\Theta(\Phi)$ to cocartesian morphisms of $C^{\otimes}$.

The value $M(*) \in C$ is the *underlying object* of the $\Phi$-monoid $M$.

### 1.4.2 Monads and modules

### 1.4.3 Monadicity

### 1.4.4 Free modules

### 1.4.5 Codensity monads

The codensity monad of a functor $f\colon A \to B$ is the right Kan extension $\beta(f)$ of $f$ along itself, when it exists. For formal reasons, this is always a monad on $B$.

The full functoriality of the construction $f \mapsto \beta(f)$ is relevant to us. In effect, we regard functors as the objects of a category, and the morphisms are lax-commutative squares.

**Definition 1.4.5.1** Let $A$ and $B$ be categories. Then a bifibration (Lurie, 2009, §§2.4.7) $X \to B \times A$ is *representable* if and only if, for every object $a \in A$, the fiber $X_a$ has a terminal object.

**1.4.5.2** Let $A$ and $B$ be categories. A bifibration $X \to B \times A$ corresponds to a functor $B^{op} \times A \to \boldsymbol{S}_V$, or equivalently to a functor $\Xi\colon A \to \boldsymbol{P}(B)$. A representable bifibration is one in which each presheaf $\Xi(a)$ is representable. In this way, the category of representable bifibrations to $B \times A$ is equivalent to the category $\mathrm{Fun}(A, B)$.

One can be explicit about the correspondence: if $f\colon A \to B$ is a functor, then the corresponding representable bifibration is

$$\mathrm{Fun}(\Delta^1, B) \times_B A \to B \times A\,,$$

and every representable fibration is of this form.

**Construction 1.4.5.3** Let **LaxCat** be the full subcategory of $\mathrm{Fun}(\Lambda^1_0, \mathbf{Cat})$ spanned by those diagrams $A \leftarrow X \to B$ such that $X \to B \times A$ is a representable bifibration.

The objects can be identified with functors $f\colon A \to B$, but **LaxCat** is not equivalent to the category $\mathrm{Fun}(\Delta^1, \mathbf{Cat})$. If $f\colon A \to B$ and $g\colon C \to D$ are functors, then a morphism $\sigma\colon f \to g$ of **LaxCat** determines a functor

$$\mathrm{Fun}(\Delta^1, B) \times_B A \to \mathrm{Fun}(\Delta^1, D) \times_D C\,.$$

If $a \in A$ and $b \in B$ are objects, then $\sigma$ determines a map

$$\mathrm{Map}_B(b, f(a)) \to \mathrm{Map}_D(\psi(b), g(\phi(a)))\,.$$

When $b = f(a)$, the image of the identity under this map is thus a morphism $\sigma_a\colon \psi(f(a)) \to g(\phi(a))$. Thus the morphism $\sigma$ amounts to a lax-commutative square:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\swArrow\sigma"] & B \arrow[d, "\psi"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \tag{1.4.5.1}$$

We have two functors $s, t\colon \mathbf{LaxCat} \to \mathbf{Cat}$ which carry a diagram $[A \leftarrow X \to B]$ to $A$ and $B$, respectively. We have an equivalence

$$\{A\} \times_{\mathbf{Cat}} \mathbf{LaxCat} \times_{\mathbf{Cat}} \{B\} \simeq \mathrm{Fun}(A, B)\,.$$

In fact, the functor $H_B\colon \mathbf{Cat}^{op} \to \mathbf{Cat}$ represented by $B$ and the functor $H^A\colon \mathbf{Cat} \to \mathbf{Cat}$ corepresented by $A$ correspond under straightening/unstraightening to the cartesian fibration

$$\mathbf{LaxCat} \times_{\mathbf{Cat}} \{B\} \to \mathbf{Cat}$$

and the cocartesian fibration

$$\{A\} \times_{\mathbf{Cat}} \mathbf{LaxCat} \to \mathbf{Cat}\,,$$

respectively.

**Definition 1.4.5.4** We call **LaxCat** the *lax arrow category of categories*. If $C$ is a fixed category, then we call

**Construction 1.4.5.5** Let $C$ be a category. We write $\mathbf{End}(C)$ for the monoidal category of endofunctors of $C$, with the monoidal structure given by composition.

acts on the left on the category $\mathrm{Fun}(D, C)$. Both the monoidal structure and the left module structure are given by composition.

We consider the category $\boldsymbol{LMod}_{\mathbf{End}(C)}(\mathrm{Fun}(D, C))$ whose objects can be regarded as pairs $(T, f)$ consisting of an algebra $T \in \mathbf{Alg}(\mathbf{End}(C))$ and a $T$-module $f$ in $\mathrm{Fun}(D, C)$. Thus

For any functor $f : D \to C$, we consider the monoidal category $\mathbf{End}(C)[f]$ constructed in (Lurie, 2017, Definition 4.7.1.1). The objects of the category $\mathbf{End}(C)[f]$ are pairs $(T, \eta)$ consisting of an object $T \in \mathbf{End}(C)$ and a natural transformation $T \circ f \to f$. The assignment $(T, \eta) \mapsto T$ defines a monoidal forgetful functor $\mathbf{End}(C)[f] \to \mathbf{End}(C)$.

The terminal object (if it exists) of $\mathbf{End}(C)[f]$ is automatically an algebra object $B(f) = (\beta(f), \epsilon)$. The image of $B(f)$ under the forgetful functor $\mathbf{End}(C)[f] \to \mathbf{End}(C)$ is the algebra object

$$\beta(f) \in \mathbf{Alg}(\mathbf{End}(C))\,;$$

in other words, $\beta(f)$ is a monad on $C$.

If $\mathbf{End}(C)[f]$ has a terminal object $B(f) = (\beta(f), \epsilon)$, then the monad $\beta(f)$ will be called the *codensity monad* attached to $f : D \to C$.

If the category $\mathbf{End}(C)[f]$ has a terminal object, then $\epsilon$ exhibits $\beta(f)$ as the right Kan extension of $f$ along itself. Conversely, if the right Kan extension of $f$ along itself exists, then that Kan extension defines a terminal object of the category $\mathbf{End}(C)[f]$.

(Lurie, 2017, §§4.7.1) identifies three categories:

$$\boldsymbol{LMod}(\mathrm{Fun}(D, C)) \times_{\mathrm{Fun}(D,C)} \{f\} \simeq \mathbf{Alg}(\mathbf{End}(C)[f]) \simeq \mathbf{Alg}(\mathbf{End}(C))_{/\beta(f)}\,.$$

More informally, we may say that a morphism of monads $T \to \beta(f)$ is the same thing as a $T$-module structure on $f$.

# 2
# Topology

## 2.1 Ultrafilters

### 2.1.1 Ultrafilters on sets

Ultrafilters on sets are masters of disguise. They manage to appear exceptionally natural in several different outfits:

1 Most simply, an ultrafilter on a set $S$ provides an integral that assigns, to a map $f$ from $S$ to a finite set $I$, an element $\int f \, d\mu \in I$.

2 In functional analysis, one is used to the idea that the data of an integral operator for functions on a suitable space is equivalent to the data of a measure of that space. Accordingly, an ultrafilter on $S$ can be regarded as a probability measure $\mu$ on $S$, for which every subset is either measure 0 ('thin') or measure 1 ('thick').

3

**2.1.1.1** Recall that $\mathbf{Set}_0$ denotes the category of finite sets.

**Definition 2.1.1.2** Let $S$ be a set. We shall write $S_+ \coloneqq S \sqcup \{\infty\}$. An *ultrafilter* is a section of the forgetful functor

$$S_+/\mathbf{Set}_0 \to S/\mathbf{Set}_0 \,.$$

**2.1.1.3** Let $i \colon \mathbf{Set}_0 \hookrightarrow \mathbf{Set}^V$ denote the inclusion. The forgetful functor $S/\mathbf{Set}_0 \to \mathbf{Set}_0$ is a cocartesian fibration classified by the restriction $よ^S \circ i$ of the functor corepresented by $S$. Consequently, an ultrafilter on $S$ is tantamount to a natural transformation

$$よ^S \circ i \to i \,.$$

**Notation 2.1.1.4** Let $S$ be a set, and let $\mu$ be an ultrafilter on $S$. We find it expressive to write

$$\int_S (\cdot)\, d\mu$$

for the functor $S/\mathbf{Set}_0 \to S_+/\mathbf{Set}_0$. So if $I$ is a finite set, and if $f\colon S \to I$ is a map, then this natural transformation specifies an element of $I$ that we shall write as

$$\int_S f\, d\mu \quad \text{or} \quad \int_{s\in S} f(s)\, d\mu\,.$$

The naturality condition ensures that for a map $\phi\colon I \to J$ of finite sets, we have

$$\phi\left(\int_S f\, d\mu\right) = \int_S \phi \circ f\, d\mu\,.$$

**Example 2.1.1.5** Let $S$ be a set, and let $s \in S$ be an element. The *principal ultrafilter* $\delta_s$ is then defined so that

$$\int_S f\, d\delta_s = f(s)\,.$$

By Yoneda, every ultrafilter on a finite set is principal, but as we shall see, infinite sets have ultrafilters that are not principal.

**Notation 2.1.1.6** If $S$ is a set, then we write $\beta(S)$ for the set of ultrafilters on $S$:

$$\beta(S) \coloneqq \mathrm{Map}(よ^S \circ i, i)\,.$$

If $\alpha$ is an ordinal number such that $S \in \mathbf{Set}_\alpha$, then $よ^S$ takes values in $\mathbf{Set}_\alpha$, so $\beta(S) \in \mathbf{Set}_\alpha$ as well. Thus $\beta$ defines a functor $\mathbf{Set}_\alpha \to \mathbf{Set}_\alpha$.

The formation of the principal ultrafilter defines a natural transformation $\delta\colon \mathrm{id} \to \beta$.

**2.1.1.7** Let $\phi\colon S \to T$ be a map, and let $\mu$ be an ultrafilter on $S$. The induced ultrafilter $\phi_*\mu$ on $T$ is then given by the change-of-variables formula:

$$\int_T f\, d(\phi_*\mu) = \int_S (f \circ \phi)\, d\mu\,.$$

**2.1.1.8** Let $S$ be a set. We can express $\beta(S)$ as an end:

$$\beta(S) = \oint_{I\in\mathbf{Set}_0} I^{I^S}\,.$$

(Here we are writing $Y^X$ for the set $\mathrm{Map}(X, Y)$ of maps $X \to Y$.) Hence for an ordinal number $\alpha$, the functor $\beta\colon \mathbf{Set}_\alpha \to \mathbf{Set}_\alpha$ is the right Kan extension of $i\colon \mathbf{Set}_0 \hookrightarrow \mathbf{Set}_\alpha$ along itself. Equivalently, we can express $\beta(S)$ as the limit over all finite sets to which $S$ maps:

$$\beta(S) = \lim_{I\in(\mathbf{Set}_0)_{S/}} I\,.$$

Thus far, there is nothing to guarantee that we aren't simply speaking about the identity functor; we have not seen an example of an ultrafilter that is not principal. It turns out that an infinite set $S$ has plenty of non-principal ultrafilters. To prove the existence of these, let us look at a more conventional way of describing ultrafilters.

The connection between the definition of ultrafilters in 2.1.1.2 and their more conventional definition is a primitive variant of the Riesz–Markov–Kakutani Representation Theorem. If an ultrafilter on a set $S$ is a functional $f \mapsto \int_S f\, d\mu$, then we can look for the *measure* $\mu$ on $S$ relative to which this functional is the integral. As in the functional analysis setting, one connects these two perspectives through characteristic functions.

**Definition 2.1.1.9** Let $S$ be a set, and let $T \subseteq S$ be a subset. The *characteristic map* $\chi_T : S \to \{0, 1\}$ is defined by the formula

$$\chi_T(s) = \begin{cases} 1 & \text{if } s \in T\,; \\ 0 & \text{if } s \notin T\,. \end{cases}$$

Now let $\mu$ be an ultrafilter on $S$. Let us write

$$\mu(T) \coloneqq \int_S \chi_T\, d\mu \in \{0, 1\}\,.$$

We shall say that $T$ is *$\mu$-thick* if and only if $\mu(T) = 1$. Accordingly, we say that $T$ is *$\mu$-thin* if and only if $\mu(T) = 0$. We let $\mathscr{F}_\mu \subseteq P(S)$ denote the subset consisting of all $\mu$-thick subsets of $S$.

**Example 2.1.1.10** For an element $s \in S$, the principal ultrafilter $\delta_s$ is the unique ultrafilter relative to which the singleton $\{s\}$ is thick. That is, $\{s\} \in \mathscr{F}_\mu$ if and only if $\mu = \delta_s$.

**Example 2.1.1.11** Let $\phi \colon S \to T$ be a map, let $\mu \in \beta(S)$ be an ultrafilter, and let $U \subseteq T$ be a subset. One has

$$(\phi_* \mu)(U) = \mu(\phi^{-1}(U))\,.$$

Hence $U$ is $\phi_*\mu$-thick if and only if $\phi^{-1}U$ is $\mu$-thick. In other words,

$$\mathscr{F}_{\phi_*\mu} = (\phi^{-1})^{-1}\mathscr{F}_\mu\,.$$

The proof of the following is routine.

**Lemma 2.1.1.12** *Let $S$ be a set, and let $\mu \in \beta(S)$ be an ultrafilter.*

1 *The set $S$ is $\mu$-thick.*
2 *Supersets of $\mu$-thick sets are $\mu$-thick.*

3 *The intersection of two $\mu$-thick sets is $\mu$-thick.*
4 *The complement of a $\mu$-thick set is $\mu$-thin.*

**Example 2.1.1.13** Let $\mu$ be an ultrafilter on $S$. A subset $T \subseteq S$ is $\mu$-thick if and only if it intersects any $\mu$-thick subset of $S$. One direction follows from the fact that $\mu$-thick subsets are closed under finite intersection. In the other direction, if $T$ intersects every $\mu$-thick subset, then $S \setminus T$ cannot be $\mu$-thick, so $T$ must be.

**Definition 2.1.1.14** Let $S$ be a set. A *finitely additive measure* on $S$ is a collection $\mathscr{F} \subseteq P(S)$ of subsets of $S$ (the subsets of positive measure) such that for every partition

$$S = S_1 \sqcup \cdots \sqcup S_n ,$$

there is a unique $i$ such that $S_i \in \mathscr{F}$.

Equivalently, $\mathscr{F} \subseteq P(S)$ is a finitely additive measure if and only the following conditions obtain.

1 $S \in \mathscr{F}$.
2 If $U \subseteq T \subseteq S$ and if $U \in \mathscr{F}$, then $T \in \mathscr{F}$.
3 If $T, U \in \mathscr{F}$, then $T \cap U \in \mathscr{F}$.
4 If $T \in \mathscr{F}$, then $S \setminus T \in \mathscr{F}$.

Equivalently, $\mathscr{F} \subseteq P(S)$ is a finitely additive measure if and only if $\mathscr{F} = \mu^{-1}\{1\}$ for some map $\mu\colon P(S) \to \{0,1\}$ satisfying the following.

1 $\mu(S) = 1$.
2 For every family $T_1, \ldots, T_n$ of pairwise disjoint subsets of $S$, one has

$$\mu(T_1 \sqcup \cdots \sqcup T_n) = \mu(T_1) + \cdots + \mu(T_n) .$$

**Construction 2.1.1.15** Let $S$ be a set. Let $m(S)$ be the set of finitely additive measures on $S$.

Attached to an ultrafilter $\mu$ on $S$ is the collection $\mathscr{F}_\mu$ of $\mu$-thick subsets. In the other direction, attached to a finitely additive measure $\mathscr{F}$ is the ultrafilter $\mu_{\mathscr{F}}$ that carries a finite set $I$ and a map $f : S \to I$ to the unique element $i = \int_S f \, d\mu \in I$ such that $S_i \in \mathscr{F}$. The assignments $\mu \mapsto \mathscr{F}_\mu$ and $\mathscr{F} \mapsto \mu_{\mathscr{F}}$ together define a bijection $\beta(S) \cong m(S)$.

**Example 2.1.1.16** Let $T \subseteq S$ be a subset. Then the inclusion induces an inclusion $\beta(T) \hookrightarrow \beta(S)$ that identifies ultrafilters on $T$ with ultrafilters on $S$ relative to which $T$ is thick.

**Definition 2.1.1.17** Let $S$ be a set, and let $\mathscr{G} \subseteq P(S)$. We say that $\mathscr{G}$ has the *finite intersection property* if and only if no finite intersection of elements of $\mathscr{G}$ is empty.

An ultrafilter $\mu$ on $S$ is said to be *supported on* $\mathscr{G}$ if and only if every element of $\mathscr{G}$ is $\mu$-thick, that is, $\mathscr{G} \subseteq \mathscr{F}_\mu$.

**Example 2.1.1.18** Let $\mu$ be an ultrafilter on $S$. Since $\mu$-thick subsets are closed under finite intersections, if $\mathscr{G} \subseteq P(S)$ is a finite collection, then $\mu$ is supported on $\mathscr{G}$ if and only if it is supported on the intersection of the elements of $\mathscr{G}$. In other words, the set $\beta(T_1 \cap \cdots \cap T_n)$ is naturally identified with the set of ultrafilters on $S$ that are supported on $\{T_1, \ldots, T_n\}$.

Thus the notion of an ultrafilter supported on a family $\mathscr{G} \subseteq P(S)$ is really only interesting if $\mathscr{G}$ is infinite.

If there exists an ultrafilter supported on $\mathscr{G}$, then certainly $\mathscr{G}$ has the finite intersection property. We now prove that the converse is true, and this will provide us with a good supply of ultrafilters on an infinite set.

**Lemma 2.1.1.19** *Let $S$ be a set, and let $\mathscr{G} \subseteq P(S)$ be a family that has the finite intersection property. Then there exists an ultrafilter on $S$ supported on $\mathscr{G}$.*

*Proof* Consider the families $\mathscr{F} \subseteq P(S)$ that contain $\mathscr{G}$ and have the finite intersection property. By Zorn's lemma there is a maximal such family, $\mathscr{F}$. We claim that $\mathscr{F}$ is a finitely additive measure.

Let $S = S_1 \sqcup \cdots \sqcup S_n$ be a finite partition of $S$. The finite intersection property ensures that at most one of the summands $S_i$ lies in $\mathscr{F}$.

Now suppose that none of the summands $S_i$ lies in $\mathscr{F}$. Consider, for each $i$, the family $\mathscr{F} \cup \{S_i\} \subseteq P(S)$; the maximality of $\mathscr{F}$ implies that this family fails to have the finite intersection property. Thus for each $i$, there is a finite intersection $T_i = T_{i1} \cap \cdots \cap T_{im_i}$ of elements of $\mathscr{F}$ such that

$$S_i \cap T_i = \varnothing\,.$$

But this implies that

$$T_1 \cap \cdots \cap T_n = T_{11} \cap \cdots \cap T_{1m_1} \cap \cdots T_{n1} \cap \cdots \cap T_{nm_n} = \varnothing\,,$$

which contradicts the finite intersection property for $\mathscr{F}$ itself. Hence one of the $S_i$ lies in $\mathscr{F}$.

Thus $\mathscr{F}$ is a finitely additive measure. □

**Example 2.1.1.20** Let $S$ be an infinite set. Let $\mathscr{G}$ be the collection of cofinite subsets of $S$ – those subsets $T \subseteq S$ such that $S \setminus T$ is finite. Since $\mathscr{G}$ has the finite intersection property, it follows that there is an ultrafilter on $S$ relative to which every cofinite subset is thick. Such an ultrafilter is necessarily nonprincipal.

**2.1.1.21** It is not quite accurate to say that the Axiom of Choice is *necessary* to produce nonprincipal ultrafilters, but it is true that their existence is independent of Zermelo–Fraenkel set theory.

**2.1.1.22** What colimits does $\beta$ preserve? We can easily rule out the possibility that $\beta$ preserves arbitrary coproducts: if $S$ is an infinite set, then we have seen that $\beta(S) \neq S$, but on the other hand

$$\coprod_{s \in S} \beta(\{s\}) = \coprod_{s \in S} \{s\} = S\,.$$

On the other hand, $\beta$ does preserve finite coproducts: if $\mu$ is an ultrafilter on $S = S_1 \sqcup \cdots \sqcup S_n$, then $\mu$ is supported on exactly one of the summands $S_i$. Thus $\mu$ is induced by a unique ultrafilter on this summand $S_i$. Consequently, the natural map

$$\beta(S_1) \sqcup \cdots \sqcup \beta(S_n) \to \beta(S)$$

is a bijection.

Since all coequalizers in **Set** are split (AC), it follows that any functor out of **Set** preserves them. We thus conclude:

**Proposition 2.1.1.23** *The functor* $\beta\colon \mathbf{Set}^V \to \mathbf{Set}^V$ *preserves finite colimits.*

**Example 2.1.1.24** Let $S$ be a set, and let $R \rightrightarrows S$ be an equivalence relation. Applying $\beta$, we obtain a relation $\beta(R) \rightrightarrows \beta(S)$, which induces the natural bijection $\beta(S)/\beta(R) \xrightarrow{\sim} \beta(S/R)$. Let us try to understand the relation $\beta(R)$ in explicit terms.

Let $\mu$ and $\nu$ be ultrafilters on $S$. Let us write $\mu R \nu$ if the pair $(\mu, \nu) \in \beta(S) \times \beta(S)$ lies in the image of $\beta(R)$. Thus $\mu R \nu$ if and only if there exists an ultrafilter $\tau$ on $R$ such that:

- for any $\mu$-thick subset $U \subseteq S$, the set $(U \times S) \cap R$ is $\tau$-thick, and
- for any $\nu$-thick subset $V \subseteq S$, the set $(S \times V) \cap R$ is $\tau$-thick.

Combining these conditions, we see that $\mu R \nu$ if and only if there exists an ultrafilter $\tau$ on $R$ supported on the family

$$\mathscr{G} := \{(U \times V) \cap R : (U \in \mathscr{F}_\mu) \wedge (V \in \mathscr{F}_\nu)\}\,.$$

By Lemma 2.1.1.19, $\mu R \nu$ if and only if $\mathscr{G}$ enjoys the finite intersection property. Since $\mathscr{F}_\mu$ and $\mathscr{F}_\nu$ are closed under finite intersection, so is $\mathscr{G}$. So $\mu R \nu$ if and only if $\mathscr{G}$ does not contain the empty set, which in turn holds if and only if, for every $\mu$-thick $U \subseteq S$ and every $\nu$-thick $V \subseteq S$, we have $(U \times V) \cap R \neq \varnothing$.

For any subset $W \subseteq S$, let us write $\overline{W} \subseteq S$ for the saturation of $W$ under the equivalence relation $R$. We have shown that $\mu R \nu$ if and only if for every $\mu$-thick subset $U \subseteq S$ and every $\nu$-thick subset $V \subseteq S$, the set $\overline{U} \cap V \neq \varnothing$, or equivalently, $U \cap \overline{V} \neq \varnothing$.

Using the observation of Example 2.1.1.13, we conclude that the following are equivalent:

1 $\mu R \nu$;
2 the saturation of every $\mu$-thick subset is $\nu$-thick;
3 the saturation of every $\nu$-thick subset is $\mu$-thick;
4 a saturated subset is $\mu$-thick if and only if it is $\nu$-thick;
5 $\mu$ and $\nu$ induce the same ultrafilter on $S/R$.

**2.1.1.25** What *limits* does $\beta$ preserve? As it happens, not very many.

Clearly $\beta$ preserves the terminal object. If $S$ and $T$ are sets, then we can contemplate the natural map $(\mathrm{pr}_{1,*}, \mathrm{pr}_{2,*}) \colon \beta(S \times T) \to \beta(S) \times \beta(T)$. If $\mu$ is an ultrafilter on $S$, and $\nu$ is an ultrafilter on $T$, then the fiber of this map over $(\mu, \nu)$ is the set of those ultrafilters $\tau$ on $S \times T$ such that

$$\mathrm{pr}_{1,*}\, \tau = \mu \qquad \text{and} \qquad \mathrm{pr}_{2,*}\, \tau = \nu\,.$$

These fibers are exactly the ultrafilters supported on

$$\mathscr{G} \coloneqq \{U \times V : (U \in \mathscr{F}_\mu) \wedge (V \in \mathscr{F}_\nu)\}\,.$$

Since $\mathscr{G}$ enjoys the finite intersection property, this fiber is certainly nonempty.

Let us construct two ultrafilters supported on $\mathscr{G}$.

1 We have an ultrafilter $\mu \ltimes \nu$, which is defined by the formula

$$\int_{X \times Y} f\, d(\mu \ltimes \nu) = \int_{x \in X} \int_{y \in Y} f(x, y)\, d\nu\, d\mu\,.$$

A subset $W \subseteq X \times Y$ is $\mu \ltimes \nu$-thick if and only if

$$\{x \in X : \{y \in Y : (x, y) \in W\} \in \mathscr{F}_\nu\} \in \mathscr{F}_\mu\,.$$

2 We can also exchange the order of integration in this formula. This results in an ultrafilter $\mu \rtimes \nu$ given by the formula

$$\int_{X \times Y} f\, d(\mu \rtimes \nu) = \int_{y \in Y} \int_{x \in X} f(x, y)\, d\mu\, d\nu\,.$$

A subset $W \subseteq X \times Y$ is $\mu \rtimes \nu$-thick if and only if

$$\left\{y \in Y : \{x \in X : (x, y) \in W\} \in \mathscr{F}_\mu\right\} \in \mathscr{F}_\nu\,.$$

If one of $\mu$ or $\nu$ is a principal ultrafilter, then $\mu \ltimes \nu = \mu \rtimes \nu$, but in general these two are distinct. Indeed, let $\mu, \nu \in \beta(\boldsymbol{N})$ be two non-principal ultrafilters. Then the set $\{(x, y) \in \boldsymbol{N} \times \boldsymbol{N} : x < y\}$ is $(\mu \ltimes \nu)$-thick but $(\mu \rtimes \nu)$-thin, whereas the set $\{(x, y) \in \boldsymbol{N} \times \boldsymbol{N} : x > y\}$ is $(\mu \ltimes \nu)$-thin but $(\mu \rtimes \nu)$-thick.

From this, we deduce that $\beta$ does not preserve products or pullbacks in general.

**Construction 2.1.1.26** Let $\alpha$ be an ordinal number. For entirely formal reasons, the functor $\beta\colon \mathbf{Set}_\alpha \to \mathbf{Set}_\alpha$ is a monad with unit $\delta\colon \mathrm{id} \to \beta$.

To explain this, note that $\delta$ restricts to a natural isomorphism $i^*\delta\colon i \xrightarrow{\sim} \beta \circ i$. Furthermore, $\beta$ enjoys the following universal property: for every functor $\phi\colon \mathbf{Set}_\alpha \to \mathbf{Set}_\alpha$, every natural transformation $\eta\colon \phi \circ i \to i$, extends to a unique natural transformation $\overline{\eta}\colon \phi \to \beta$, by which we mean that $i^*\overline{\eta} = i^*\delta \circ \eta$. In other words, restriction induces a natural bijection

$$\mathrm{Map}(\phi, \beta) = \mathrm{Map}(\phi \circ i, i) .$$

The identification $i^*\delta$ gives rise to an identification $\beta^2 \circ i \xrightarrow{\sim} i$, which in turn extends uniquely to a natural transformation $\lambda\colon \beta^2 \to \beta$.

Let's unpack this natural transformation. For the sake of brevity, if $X$ and $Y$ are sets, let us write $Y^X$ for $\mathrm{Map}(X, Y)$. Let $X$ be a set, and let $I$ be a finite set. The key operation is the evaluation map

$$\epsilon_X\colon X \to I^{I^X} .$$

Of course this is the composite

$$X \xrightarrow{\delta_X} \beta(X) = \oint_{I \in F} I^{I^X} \xrightarrow{\mathrm{pr}_I} I^{I^X}$$

The map $\epsilon_X$ induces a map

$$\epsilon_X^*\colon I^{I^{I^X}} \to I^X ,$$

and we apply it to the set $X = I^S$ for a set $S$. This produces a natural map

$$\epsilon_{I^S}^*\colon I^{I^{I^{I^S}}} \to I^{I^S} .$$

Explicitly, if $f\colon I^{I^{I^S}} \to I$ is a map, then its image under $\epsilon_{I^S}^*$ is the map $I^S \to I$ that carries $\phi$ to $f(\epsilon_S(\phi))$. Now the map $\lambda_S\colon \beta^2(S) \to \beta(S)$ is the composite

$$\beta(\beta(S)) = \oint_{J \in F} J^{J^{\oint_{I \in F} I^{I^S}}} \xrightarrow{c} \oint_{J \in F} \oint_{I \in F} J^{J^{I^{I^S}}} \xrightarrow{p} \oint_{I \in F} I^{I^{I^{I^S}}} \xrightarrow{\oint_I \epsilon_{I^S}^*} \oint_{I \in F} I^{I^S} = \beta(S) ,$$

where $c$ and $p$ are the canonical maps.

Now let $\tau$ be an ultrafilter on $\beta(S)$. The induced ultrafilter $\lambda_S(\tau)$ on $S$ is defined so that

$$\int_S f \, d\lambda_S(\tau) = \int_{\mu\in\beta(S)} \left( \int_S f \, d\mu \right) d\tau \, .$$

Said differently, if $T \subseteq S$ is a subset, then the induced map $\beta(T) \to \beta(S)$ identifies $\beta(T)$ with the set of ultrafilters on $S$ supported on $\{T\}$, and the ultrafilter $\lambda(\tau)$ on $S$ is defined so that

$$\lambda(\tau)(T) = \tau(\beta(T)) \, .$$

### 2.1.2 Completeness of ultrafilters

### 2.1.3 Ultrafilters on posets

### 2.1.4 Ultraproducts

## 2.2 Topoi

### 2.2.1 Topoi

### 2.2.2 Sheaves and hypersheaves

### 2.2.3 Postnikov completeness

### 2.2.4 Coherence

### 2.2.5 Stone duality

### 2.2.6 Spectral duality

### 2.2.7 Classifying topoi

## 2.3 Compacta

### 2.3.1 Compacta and $\beta$-algebras

**Construction 2.3.1.1** Let **Top** denote the category of tiny topological spaces. If $S$ is a set, we can introduce a topology on $\beta(S)$ simply by forming the inverse limit $\lim_{I\in\mathbf{Fin}_{S/}} I$ in **Top**. That is, we endow $\beta(S)$ with the coarsest topology such that all the projections $\beta(S) \to I$ are continuous. We call this the *Stone topology* on $\beta(S)$. By Tychonoff, this limit is a compact Hausdorff topological space. This lifts $\beta$ to a functor **Set** $\to$ **Top**.

**2.3.1.2** Let's be more explicit about the topology on $\beta(S)$. The topology on $\beta(S)$ is generated by the sets $T^\dagger$ (for $T \subseteq S$). In fact, since the sets $T^\dagger$ are stable under finite intersections, they form a base for the Stone topology on $\beta(S)$. Additionally, since the sets $T^\dagger$ are stable under the formation of complements, they even form a base of clopens of $\beta(S)$.

**Definition 2.3.1.3** A *compactum* is an algebra for the monad $\beta$. Hence a compactum consists of a set $K$ and a map $\lambda_K : \beta(K) \to K$, which is required to satisfy the usual identities:

$$\lambda_K(\lambda_{K,*}\tau) = \lambda_K(\mu_\tau) \qquad \text{and} \qquad \lambda_K(\delta_s) = s\,,$$

for any ultrafilter $\tau$ on $\beta(S)$ and any point $s \in S$. The image $\lambda_K(\mu)$ will be called the *limit* of the ultrafilter $\mu$. We write **Comp** for the category of compacta, and write **Free** $\subset$ **Comp** for the full subcategory spanned by the *free compacta* – *i.e.*, free algebras for $\beta$.

**Construction 2.3.1.4** If $K$ is a compactum, then we use the limit map $\lambda_K : \beta(K) \to K$ to topologise $K$ as follows. For any subset $T \subseteq K$, we define the closure of $T$ as the image $\lambda_K(T^\dagger)$.

A subset $Z \subseteq K$ is thus closed if and only if the limit of any ultrafilter relative to which $Z$ is thick lies in $Z$. Dually, a subset $U \subseteq K$ is open if and only if it is thick with respect to any ultrafilter whose limit lies in $U$.

We denote the resulting topological space $K^{top}$. The assignment $K \mapsto K^{top}$ defines a lift **Alg**$(\beta) \to$ **Top** of the forgetful functor **Alg**$(\beta) \to$ **Set**.

**Proposition 2.3.1.5** *The functor $K \mapsto K^{top}$ identifies the category of compacta with the category of compact Hausdorff topological spaces.*

We will spend the remainder of this section proving this claim. Please observe first that $K \mapsto K^{top}$ is faithful. What we will do now is prove:

1. that for any compactum $K$, the topological space $K^{top}$ is compact Hausdorff;
2. that for any compact Hausdorff topological space $X$, there is a $\beta$-algebra structure $K$ on the underlying set of $X$ such that $X \cong K^{top}$; and
3. that for any compacta $K$ and $L$, any continuous map $K^{top} \to L^{top}$ lifts to a $\beta$-algebra homomorphism $K \to L$.

To do this, it is convenient to describe a related idea: that of *convergence* of ultrafilters on topological spaces.

**Definition 2.3.1.6** Let $X$ be a topological space, and let $x \in X$. We say that $x$ is a *limit point* of an ultrafilter $\mu$ on (the underlying set of) $X$ if and only if every open neighbourhood of $x$ is $\mu$-thick. In other words, $x$ is a limit point of $\mu$ if and only if, for every open neighbourhood $U$ of $x$, one has $\mu \in U^\dagger$.

**Lemma 2.3.1.7** *Let $X$ be a topological space, and let $U \subseteq X$ be a subset. Then $U$ is open if and only if it is thick with respect to any ultrafilter with limit point in $U$.*

*Proof* If $U$ is open, then $U$ is by definition thick with respect to any ultrafilter with limit point in $U$.

Conversely, assume that $U$ is thick with respect to any ultrafilter with limit point in $U$. Let $u \in U$. Consider the set $G \coloneqq N(u) \cup \{X \smallsetminus U\}$, where $N(u)$ is the collection of open neighbourhoods of $u$. If $U$ does not contain any open neighbourhood of $u$, then no finite intersection of elements of $G$ is empty. By **??** there is an ultrafilter $\mu$ supported on the $N(u) \cup \{X \smallsetminus U\}$, whence $u$ is a limit point of $\mu$, but $U$ is not $\mu$-thick. This contradicts our assumption, and so we deduce that $U$ contains an open neighbourhood of $u$. □

**Lemma 2.3.1.8** *Let $X$ and $Y$ be topological spaces, and let $\phi \colon X \to Y$ be a map. Then $\phi$ is continuous if and only if, for any ultrafilter $\mu$ on $X$ with limit point $x \in X$, the point $\phi(x)$ is a limit point of $\phi_* \mu$.*

*Proof* Assume that $\phi$ is continuous, and let $\mu$ be an ultrafilter on $X$, and assume that $x \in X$ is a limit point of $\mu$. Now assume that $V$ is an open neighbourhood of $\phi(x)$. Since $\phi^{-1}V$ is an open neighbourhood of $x$, so it is $\mu$-thick, whence $V$ is $\phi_* \mu$-thick. Thus $\phi(x)$ is a limit point of $\phi_* \mu$.

Assume now that if $x \in X$ is a limit point of an ultrafilter $\mu$, then $\phi(x)$ is a limit point of $\phi_* \mu$. Let $V \subseteq Y$ be an open set. Let $x \in \phi^{-1}(V)$, and let $\mu$ be an ultrafilter on $X$ with limit point $x$. Then $\phi(x)$ is a limit point of $\phi_* \mu$, so $V$ is $\phi_* \mu$-thick, whence $\phi^{-1}(V)$ is $\mu$-thick. It follows from 2.3.1.7 that $\phi^{-1}(V)$ is open. □

**Lemma 2.3.1.9** *Let $X$ be a topological space. Then $X$ is quasicompact if and only if every ultrafilter on $X$ has at least one limit point.*

*Proof* Assume first that $X$ is quasicompact. Let $\mu$ be an ultrafilter on $X$, and assume that $\mu$ has no limit point. Select, for every point $x \in X$, an open neighbourhood $U_x$ thereof that is not $\mu$-thick. Quasicompactness implies that there is a finite collection $x_1, \dots, x_n \in X$ such that $\left\{U_{x_1}, \dots, U_{x_n}\right\}$ covers $X$. But at least one of $U_{x_1}, \dots, U_{x_n}$ must be $\mu$-thick. This is a contradiction.

Now assume that $X$ is not quasicompact. Then there exists a collection $G \subseteq \boldsymbol{P}(X)$ of closed subsets of $X$ such that the intersection all the elements of $G$ is empty, but no finite intersection of elements of $G$ is empty. In light of **??**, there is an ultrafilter $\mu$ with the property that every element of $G$ is thick. For any $x \in X$, there is an element $Z \in G$ such that $x \in X \smallsetminus Z$. Since $Z$ is $\mu$-thick, $X \smallsetminus Z$ is not. Thus $\mu$ has no limit points. □

**Lemma 2.3.1.10** *Let $X$ be a topological space. Then $X$ is Hausdorff if and only if every ultrafilter on $X$ has at most one limit point.*

*Proof* Assume that $\mu$ is an ultrafilter with two distinct limit points $x_1$ and $x_2$. Choose open neighbourhoods $U_1$ of $x_1$ and $U_2$ of $x_2$. Since they are both $\mu$-thick, they cannot be disjoint; hence $X$ is not Hausdorff.

Conversely, assume that $X$ is not Hausdorff. Select two points $x_1$ and $x_2$ such that every open neighbourhoods $U_1$ of $x_1$ and $U_2$ of $x_2$ intersect. Now the set $G$ consisting of open neighbourhoods of either $x_1$ *or* $x_2$ has the property that no finite intersection of elements of $G$ is empty. In light of **??**, there is an ultrafilter $\mu$ with the property that every element of $G$ is thick. Thus $x_1$ and $x_2$ are limit points of $\mu$. □

Let us now return to our functor $K \mapsto K^{top}$.

**Lemma 2.3.1.11** *Let $K$ be a compactum, and let $\mu$ be an ultrafilter on $K$. Then a point of $K^{top}$ is a limit point of $\mu$ in the sense of 2.3.1.6 if and only if it is the limit of $\mu$ in the sense of 2.3.1.3.*

*Proof* Let $x \coloneqq \lambda_K(\mu)$. The open neighbourhoods $U$ of $x$ are by definition thick (relative to $\mu$), so certainly $x$ is a limit point of $\mu$.

Now assume that $y \in K^{top}$ is a limit point of $\mu$. To prove that the limit of $\mu$ is $y$, we shall build an ultrafilter $\tau$ on $\beta(K)$ with the following properties:

1 under the multiplication $\beta^2 \to \beta$, the ultrafilter $\tau$ is sent to $\mu$; and
2 under the map $\lambda_* : \beta^2 \to \beta$, the ultrafilter $\tau$ is sent to $\delta_y$.

Once we have succeeded, it will follow that

$$\lambda_K(\mu) = \lambda_K(\mu_\tau) = \lambda_K(\lambda_{K,*}\tau) = \lambda_K(\delta_y) = y\ ,$$

and the proof will be complete.

Consider the family $G'$ of subsets of $\beta(K)$ of the form $T^\dagger$ for a $\mu$-thick subset $T \subseteq S$; since these are all nonempty and they are stable under finite intersections, it follows that no finite intersection of elements of $G'$ is empty.

Now consider the set $G \coloneqq G' \cup \{\lambda_K^{-1}\{y\}\}$. If $T$ is $\mu$-thick, then we claim that there is an ultrafilter $\nu \in \lambda_K^{-1}\{y\} \cap T^\dagger$. Indeed, consider the set $N(y) \cup \{T\}$, where $N(y)$ is the collection of open neighbourhoods of $y$. Since every open neighbourhood of $y$ is $\mu$-thick, no intersection of an open neighbourhood of $y$ with $T$ is empty. By **??** there is an ultrafilter supported on $N(y) \cup \{T\}$, which implies that no finite intersection of elements of $G$ is empty.

Applying **??** again, we see that $G$ supports an ultrafilter $\tau$ on $\beta(K)$. For any $T \subseteq K$,

$$\mu_\tau(T) = \tau(T^\dagger)\ ,$$

so since $\tau$ is supported on $G'$, it follows that $\mu_\tau = \mu$. At the same time, since $\tau$ is supported on $\{\lambda_K^{-1}\{y\}\}$, it follows that $\{y\}$ is thick relative to $\lambda_{K,*}\tau$, whence $\lambda_{K,*}\tau = \delta_y$. □

*Proof of 2.3.1.5* Let $K$ be a compactum. Combine **??** to conclude that $K^{top}$ is a compact Hausdorff topological space.

Let $X$ be a compact Hausdorff topological space with underlying set $K$. Define a map $\lambda_K : \beta(K) \to K$ by carrying an ultrafilter $\mu$ to its unique limit point in $X$. This is a $\beta$-algebra structure on $X$, and it follows from 2.3.1.11 and the definition of the topology together imply that $X \cong K^{top}$.

Finally, let $K$ and $L$ be compacta, and let $\phi\colon K^{top} \to L^{top}$ be a continuous map. To prove that $\phi$ is a $\beta$-algebra homomorphism, it suffices to confirm that if $\mu$ is an ultrafilter on $K$, then

$$\lambda_L(\phi_*\mu) = \phi(\lambda_K(\mu)) ,$$

but this follows exactly from 2.3.1.8. □

**2.3.1.12** We opted in 2.3.1.4 to define the topology on a compactum $K$ in very explicit terms, but note that the map $\lambda_K : \beta(K) \to K^{top}$ is a continuous surjection between compact Hausdorff topological spaces. Thus $K^{top}$ is endowed with the quotient topology relative to $\lambda_K$.

### 2.3.2 Boolean algebras

### 2.3.3 Stone topological spaces

### 2.3.4 Projective compacta

# PART TWO

# CONDENSED SETS

# PART THREE

## CONDENSED GROUPS

# PART FOUR

# CONDENSED ABELIAN GROUPS

# PART FIVE

## CONDENSED SPACES

# PART SIX

## CONDENSED SPECTRA

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