Problem Set 8: Visualizing Data (Quantile-Quantile Plots)

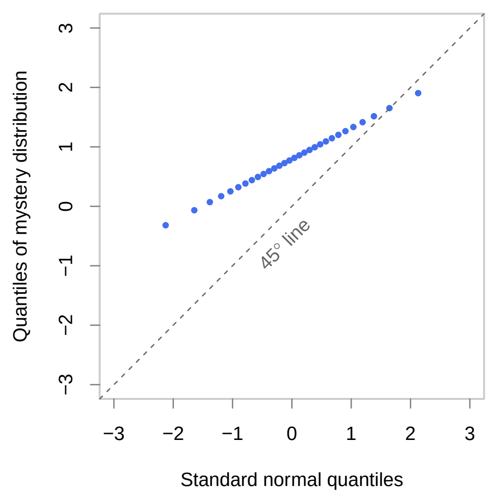
Solutions that require a discussion or an explanation should be type-written in a 12-point font and submitted in class. If you would rather write than type equations, please attach handwritten equations as an appendix.

1. a. [5 pts] Let Z be a continuous random variable and $Y = aZ + b$, where a and b are constants with $a \neq 0$. Let y_p and z_p denote the p-quantile of the distribution of Y and Z, respectively, where $0 < p < 1$. Show that

$$y_p = az_p + b.$$

b. [5 pts] Suppose Z has a standard normal distribution. Let $Y = aZ + b$ where $a > 0$, $a \neq 1$, and $b \neq 0$. Let $p_1 < \cdots < p_m$ with $0 < p_i < 1$. Let y_{p_i} and z_{p_i} denote the p_i-quantile of the distribution of Y and Z, respectively. Would the points (y_{p_i}, z_{p_i}), $i = 1, \ldots, m$, fall on a $45°$ line? Justify your answer (mathematically, rather than intuitively).

c. [2 pts] The graph below shows a plot of the quantiles of a mystery distribution against the corresponding quantiles of a standard normal distribution. Does the departure of the points from the $45°$ line indicate that the mystery distribution is not a normal distribution? Explain.



2. Let Y be a random variable with cumulative distribution function

$$F(a;\theta) = \begin{cases} 1 - \mathrm{e}^{-\theta a} & \text{for } a > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is a parameter. Let Z be a random variable with cumulative distribution $F(a; 1)$.

a. [5 pts] Find the quantile functions of Y and Z, denoted Q_Y and Q_Z, respectively.

b. [5 pts] Let $p_1 < \cdots < p_m$ with $0 < p_i < 1$. Let y_{p_i} and z_{p_i} denote the p_i-quantile of the distribution of Y and Z, respectively. Would the points (y_{p_i}, z_{p_i}), $i = 1, \ldots, m$, fall on a straight line? If so, give the intercept and slope of the line. If not, explain why.

3. Let Y be a random variable with cumulative distribution function

$$F(a; \lambda, \alpha) = \begin{cases} 1 - \mathrm{e}^{-\lambda a^{\gamma}} & a > 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $\lambda > 0$ and $\gamma > 0$ are parameters. Let Z be a random variable with cumulative distribution function $F(a; 1, 1)$.

a. [5 pts] Find the quantile functions of Y and Z, denoted Q_Y and Q_Z, respectively.

b. [5 pts] Let $p_1 < \cdots < p_m$ with $0 < p_i < 1$. Let y_{p_i} and z_{p_i} denote the p_i-quantile of the distribution of Y and Z, respectively. Would the points (y_{p_i}, z_{p_i}), $i = 1, \ldots, m$, fall on a straight line? If so, give the intercept and slope of the line. If not, explain why.

4. [Extra credit, 5 pts] Discuss the implications of the results in problems 1–3 with regard to the use of quantile-quantile plots for comparing two distributions.