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Kandidatnr:

1

1.1

2 Second task

2.1

Probability given by

$$P = |\langle \chi_1 | \psi \rangle|^2 = \langle \psi | \chi_1 \rangle \langle \chi_1 | \psi \rangle.$$

Starting by calculating $\langle \chi_1 | \psi \rangle$, where we use tensor notation for the states $|\chi_1\rangle$ and $|\psi\rangle$. For $|\chi_1\rangle$ we get

$$|\chi_1\rangle = \frac{1}{\sqrt{2}} \left(| \uparrow_1 \downarrow_2 \rangle - | \downarrow_1 \uparrow_2 \rangle = | \uparrow_1 \rangle \otimes | \downarrow_2 \rangle \otimes I - | \downarrow_1 \rangle \otimes | \uparrow_2 \rangle \otimes I \right),$$

where I is the unity matrix. We then get

$$\begin{aligned}
\langle \chi_1 | \psi \rangle = & \frac{1}{\sqrt{2}} \left(\langle \uparrow_1 | \otimes \langle \downarrow_2 | \otimes I - \langle \downarrow_1 | \otimes \langle \uparrow_2 | \otimes I \right) \\
& \times \frac{\alpha}{\sqrt{2}} \left(| \downarrow_1 \rangle \otimes | \uparrow_2 \rangle \otimes | \downarrow_3 \rangle \otimes | \downarrow_3 \rangle - | \downarrow_1 \rangle \otimes | \downarrow_2 \rangle \right) \\
& + \frac{1}{\sqrt{2}} \left(\langle \uparrow_1 | \otimes \langle \downarrow_2 | \otimes I - \langle \downarrow_1 | \otimes \langle \uparrow_2 | \otimes I \right) \\
& \times \frac{\beta}{\sqrt{2}} \left(| \uparrow_1 \rangle \otimes | \uparrow_2 \rangle \otimes | \downarrow_3 \rangle - | \uparrow_1 \rangle \otimes | \downarrow_2 \rangle \otimes | \uparrow_3 \rangle \right).
\end{aligned}$$

When evaluating the terms above we get zero unless the spins-1/2's are the same for the bra and the ket (e.g $\langle \uparrow | \downarrow \rangle = 0$, for all combinations) while we get one when the spins-1/2's are the same (e.g. $\langle \uparrow | \uparrow \rangle = 1$, for all combinations). We then get

$$\begin{aligned}
\langle \chi_1 | \psi \rangle &= \frac{\alpha}{2} \left(- \langle \downarrow_1 | \downarrow_1 \rangle \otimes \langle \uparrow_2 | \uparrow_2 \rangle \otimes | \downarrow_3 \rangle \right) + \frac{\beta}{2} \left(- \uparrow_1 | \uparrow_1 \rangle \otimes \langle \downarrow_2 | \downarrow_2 \rangle \otimes | \uparrow_3 \rangle \right) \\
&= - \frac{\alpha}{2} | \downarrow_3 \rangle - \frac{\beta}{2} | \uparrow_3 \rangle.
\end{aligned}$$

We then conjugate the above and get

$$(\langle\chi_1|\psi\rangle)^* = \langle\psi|\chi_1\rangle = -\frac{\alpha^*}{2}|\uparrow_3\rangle - \frac{\beta^*}{2}|\downarrow_3\rangle,$$

which leads to the probability

$$\begin{aligned}
P = \langle\psi|\chi_1\rangle\langle\chi_1|\psi\rangle &= \left(-\frac{\alpha^*}{2}|\downarrow_3\rangle - \frac{\beta^*}{2}|\uparrow_3\rangle\right)\left(-\frac{\alpha}{2}|\uparrow_3\rangle - \frac{\beta}{2}|\downarrow_3\rangle\right) \\
&= \frac{\alpha^2}{4}\langle\downarrow_3|\downarrow_3\rangle + \frac{\alpha^*\beta}{4}\langle\downarrow_3|\uparrow_3\rangle + \frac{\alpha\beta^*}{4}\langle\downarrow_3|\uparrow_3\rangle + \frac{\beta^2}{4}\langle\uparrow_3|\uparrow_3\rangle \\
&= \frac{1}{4}\left(\alpha^2+\beta^2\right) \\
&= \frac{1}{4},
\end{aligned}$$

where we used the fact that $|\alpha|^2 + |\beta|^2 = 1$ and evaluated the brakets as stated above. The probability of getting the measurement v_1 is $1/4$.

2.2

The state of the spin at location three is given by the projection of the state $|\chi_1\rangle$ onto the state $|\psi\rangle$, which is given by

$$\begin{aligned}
|\chi_1\rangle\langle\chi_1|\psi\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle\right)\left(-\frac{\alpha}{2}|\downarrow_3\rangle - \frac{\beta}{2}|\uparrow_3\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(-\frac{\alpha}{2}|\uparrow_1\downarrow_2\downarrow_3\rangle + \frac{\beta}{2}|\downarrow_1\uparrow_2\uparrow_3\rangle - \frac{\beta}{2}|\uparrow_1\downarrow_2\uparrow_3\rangle + \frac{\alpha}{2}|\downarrow_1\uparrow_2\downarrow_3\right) \\
&= \frac{1}{2\sqrt{2}}\left[\left(\alpha(|\downarrow_1\uparrow_2\downarrow_3\rangle - |\uparrow_1\downarrow_2\downarrow_3\rangle\right) + \beta\left(|\downarrow_1\uparrow_2\uparrow_2\rangle - |\uparrow_1\downarrow_2\uparrow_3\rangle\right)\right] \\
&= |\psi'\rangle,
\end{aligned}$$

where we have named the new state ψ'. The indices of the spin-1/2's are now dropped, consider them to be in the order as mentioned in the problem set unless it is specified to be otherwise. We then normalize for a normalization constant A

$$\begin{aligned}
1 = |A|^2\langle\psi'|\psi'\rangle &= \frac{1}{16}\big(\alpha^*\left(\langle\downarrow\uparrow\downarrow| - \langle\uparrow\downarrow\downarrow|\right) + \beta^*\left(\langle\downarrow\uparrow\uparrow| - \langle\uparrow\downarrow\uparrow|\right)\big) \\
&\times\big(\alpha\left(|\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\rangle\right) + \beta\left(|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle\right)\big),
\end{aligned}$$

multiplying the terms, notice that the terms with $\alpha^*\beta$ and $\alpha\beta^*$ does not share a bra and ket with identical spin-1/2 state, and therefore the braket is zero. We then get

$$1 = \frac{1}{16}\left(|\alpha|^2\left(\langle\downarrow\uparrow\downarrow| - \langle\uparrow\downarrow\downarrow|\right)\left(|\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\rangle\right) + |\beta|^2\left(\langle\downarrow\uparrow\uparrow| - \langle\uparrow\downarrow\uparrow|\right)\left(|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle\right)\right),$$

which by previous arguments regarding the equality of the spin-1/2 states for the bra and the ket gives

$$\begin{aligned}
1 &= |A|^2 \frac{1}{16} |\alpha|^2 \left(\langle \downarrow\uparrow\downarrow | \downarrow\uparrow\downarrow \rangle + \langle \uparrow\downarrow\downarrow | \uparrow\downarrow\downarrow \rangle \right) + |\beta|^2 \left(\langle \downarrow\uparrow\uparrow | \downarrow\uparrow\uparrow \rangle + \langle \uparrow\downarrow\uparrow | \uparrow\downarrow\uparrow \rangle \right) \\
&= \frac{|A|^2}{16} \left(2|\alpha|^2 + 2|\beta|^2 \right) \\
&= \frac{|A|^2}{8}.
\end{aligned}$$

The normalization constant is $A = \sqrt{8} = 2\sqrt{2}$. So the normalized state is

$$|\psi''\rangle = \alpha \left(|\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\rangle \right) + \beta \left(|\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\uparrow\rangle \right).$$

2.3

We write the state $|\chi\rangle$ as a product of two spin-0 states

$$|\chi\rangle = |\chi_{14}\rangle \otimes |\chi_{23}\rangle\,,$$

where we know that the spin-0 state

$$|\chi_{23}\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow_2\downarrow_3\rangle - |\downarrow_2\downarrow_3\rangle \right).$$

We construct a spin-0 state

$$|\chi_{14}\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow_1\downarrow_4\rangle - |\downarrow_1\uparrow_4\rangle \right).$$

This yields

$$|\chi\rangle = \frac{1}{2} \left(|\uparrow_2\downarrow_3\rangle - |\downarrow_2\uparrow_3\rangle \right) \left(|\uparrow_1\downarrow_4\rangle - |\downarrow_1\uparrow_4\rangle \right),$$

which when multiplying out yields

$$\begin{aligned}
|\chi\rangle &= \frac{1}{2} \left(|\uparrow_2\downarrow_3\rangle \otimes |\uparrow_1\downarrow_4\rangle - |\uparrow_2\downarrow_3\rangle \otimes |\downarrow_1\uparrow_4\rangle - |\downarrow_2\uparrow_3\rangle \otimes |\uparrow_1\downarrow_4\rangle + |\downarrow_2\uparrow_3\rangle \otimes |\downarrow_1\uparrow_4\rangle \right) \\
&= \frac{1}{2} \left(|\uparrow\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle \right)
\end{aligned}$$

2.4

The spin states are written as $|sm_s\rangle$, where s is the quantum number for total spin and m_s is the quantum number for magnetic spin. For a state of four spin-1/2's we have $s = 2$, 1 or 0. The allowed values for m_s are $m = -s, -s+1, \ldots, 0, \ldots s-1, s$. The eigenvalues for the spin operator S_{tot} are $\hbar^2 s(s+1)$, which when applied on a spin state gives $S_{tot}^2 |sm_s\rangle = \hbar^2 s(s+1) |sm_s\rangle$. For $s = 2$ we get

$$\hat{H} |2m_s\rangle = \frac{J}{24\hbar^4} \left(S_{tot}^2 - 2\hbar^2 \right) S_{tot}^2 |2m_s\rangle$$

where we now get the eigenvalues when applying the spin operator S^2_{tot} on the state $|2m_s\rangle$, which yields

$$\begin{aligned}
\hat{H}\,|2m_s\rangle &= \frac{J}{24\hbar^4}\left(S^2_{tot} - 2\hbar^2\right)\hbar^2 2(2+1)\,|2m_s\rangle \\
&= \frac{J}{24\hbar^4}6\hbar^2\left(S^2_{tot}\,|2m_s\rangle - 2\hbar^2\,|2m_s\rangle\right) \\
&= \frac{J}{4\hbar^2}\left(\hbar^2 2\,(2+1) - 2\hbar^2\right)|2m_s\rangle \\
&= \frac{J}{4\hbar^2}\hbar^2\,(6-2)\,|2m_s\rangle \\
&= J\,|2m_s\rangle\,,
\end{aligned}$$

the energy For $s = 1$ we get

$$\begin{aligned}
\hat{H}\,|1m_s\rangle &= \frac{J}{24\hbar^4}\left(S^2_{tot} - 2\hbar^2\right)S^2_{tot}\,|1m_s\rangle \\
&= \frac{J}{24\hbar^4}\left(S^2_{tot} - 2\hbar^2\right)\hbar^2 1\,(1+1)\,|1m_s\rangle \\
&= \frac{J}{12\hbar^2}\left(S^2_{tot}\,|1m_s\rangle - 2\hbar^2\,|1m_s\rangle\right) \\
&= \frac{J}{12\hbar^2}2\hbar^2\,(1-1)\,|1m_s\rangle \\
&= 0\,|0m_s\rangle\,.
\end{aligned}$$

For $s = 0$ we see that the equation is multiplied by 0, which yields the energy eigenvalue 0. For the energy eigenvalue J we have 5 possible values for m_s with $s = 2$, the possible states are $|2-2\rangle$, $|2-1\rangle$, $|20\rangle$, $|21\rangle$ and $|22\rangle$, and the degeneracy is 5 fold. For the energy eigenvalue 0 we have 3 states for $s = 1$, which are $|1-1\rangle$, $|10\rangle$ and $|11\rangle$ and further there is one state for $s = 0$, that is $|00\rangle$, so the degeneracy for the energy eigenvalue 0 is 4 fold.

2.5

We have the projection operator for spin-1/2's at 1 and 2 as

$$P_{12} = |+\rangle_{12}\langle\uparrow_1\uparrow_2| + |0\rangle_{12}\frac{1}{\sqrt{2}}\left(\langle\uparrow_1\downarrow_2| + \langle\downarrow_1\uparrow_2|\right) + |-\rangle_{12}\langle\downarrow_1\uparrow_2|\,,$$

and for the spin-1/2's at location 3 and 4

$$P_{34} = |+\rangle_{34}\langle\uparrow_3\uparrow_4| + |0\rangle_{34}\frac{1}{\sqrt{2}}\left(\langle\uparrow_3\downarrow_4| + \langle\downarrow_3\uparrow_4|\right) + |-\rangle_{34}\langle\downarrow_3\uparrow_4|\,.$$

First calculating $P_{34}\,|\chi\rangle$:

$$\begin{aligned}
P_{34}\,|\chi\rangle = &\left(|+\rangle_{34}\langle\uparrow_3\uparrow_4| + |0\rangle_{34}\frac{1}{\sqrt{2}}\left(\langle\uparrow_3\downarrow_4| + \langle\downarrow_3\uparrow_4|\right) + |-\rangle_{34}\langle\downarrow_3\uparrow_4|\right) \\
&\times\left(\frac{1}{2}\left(|\uparrow\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle\right)\right),
\end{aligned}$$

again there are terms that are equal to zero and ones when setting specific bra's and ket's together. The specifics of the calculations are done as the following example

$$\begin{aligned}
\langle\uparrow_3\uparrow_4|\uparrow\uparrow\downarrow\downarrow\rangle &= \left(I \otimes I \otimes \langle\uparrow_3| \otimes \langle\uparrow_4|\right)\left(|\uparrow_1\rangle \otimes |\uparrow_2\rangle \otimes |\downarrow_3\rangle \otimes |\downarrow_4\rangle\right) \\
&= I\,|\uparrow_1\rangle \otimes I\,|\uparrow_2\rangle \otimes \langle\uparrow_3|\downarrow_3\rangle \otimes \langle\uparrow_4|\downarrow_4\rangle \\
&= |\uparrow_1\uparrow_2\rangle\,.
\end{aligned}$$

Going further with the calculation we get

$$P_{34}\,|\chi\rangle = \frac{1}{2}\left(|+\rangle\,\langle\uparrow_3\uparrow_4|\downarrow\downarrow\uparrow\uparrow\rangle - \frac{1}{\sqrt{2}}\left(\langle\uparrow_3\downarrow_4|\uparrow\downarrow\uparrow\downarrow\rangle + \langle\downarrow_3\uparrow_4|\downarrow\uparrow\downarrow\uparrow\rangle\right)\right)$$