UNIVERSITETET I OSLO
Det matematisk-naturvitenskapelige fakultet

Take home exam in: FYS3110 - Quantum mechanics.

Due: October 12. 2018 at 14:30 on the Inspera system

Pages: 4

Some subproblems have more than one question, be sure to answer them all.

Max score: 50 points

Problem 1

The operators $\hat{A}$ and $\hat{B}$ are defined in terms of two harmonic oscillator ladder operators $\hat{a}_x$ and $\hat{a}_y$ as follows

$$\hat{A} = \frac{1}{\sqrt{2}}\left(\hat{a}_x + i\hat{a}_y\right), \qquad \hat{B} = \frac{1}{\sqrt{2}}\left(\hat{a}_x - i\hat{a}_y\right)$$

The operators $\hat{a}_x$ and $\hat{a}_y$ satisfiy $[\hat{a}_x, \hat{a}_x^\dagger] = [\hat{a}_y, \hat{a}_y^\dagger] = 1$ and $[\hat{a}_x, \hat{a}_y] = [\hat{a}_x, \hat{a}_y^\dagger] = 0$.

1.1(4 points) Calculate the commutators $[\hat{A}, \hat{A}^\dagger]$, $[\hat{B}, \hat{B}^\dagger]$, $[\hat{A}, \hat{B}]$ and $[\hat{A}, \hat{B}^\dagger]$.

Two other operators $\hat{C}$ and $\hat{D}$ are defined as

$$\hat{C} = \hat{A}^\dagger\hat{A} + \hat{B}^\dagger\hat{B}, \qquad \hat{D} = \hat{A}^\dagger\hat{A} - \hat{B}^\dagger\hat{B}$$

1.2(4 points) What are the possible eigenvalues of $\hat{C}$? What are the possible eigenvalues of $\hat{D}$ for eigenstates which also are eigenstates of $\hat{C}$ with a particular eigenvalue c? Explain your answers.

The harmonic oscillator ladder operators can be expressed as (dimensionless) linear differential operators as

$$\hat{a}_x = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right), \qquad \hat{a}_y = \frac{1}{\sqrt{2}}\left(y + \frac{\partial}{\partial y}\right)$$

where we for simplicity use reduced units where we have set $\hbar = m = \omega = 1$ and x, y are dimensionless coordinates.

1.3(6 points) Express the operators $\hat{C}$ and $\hat{D}$ as (dimensionless) linear differential operators in cartesian coordinates. Then transfer these expressions to two dimensional polar coordinates r, φ where $x = r\cos\varphi$ and $y = r\sin\varphi$. (there is no need for further justification of this transformation to polar coordinates, just use formulae from Rottmann or elsewhere.)

The dimensionless stationary Schrödinger equation for a particular one-dimensional potential takes the form

$$\left[-\frac{\partial^2}{\partial x^2} + \left(\frac{s+1}{2}\right)^2 \left(e^{-2x} - 2e^{-x}\right)\right]\psi(x) = E\psi(x) \tag{1}$$

where we use dimensionless coordinates for simplicity. The constant s is restricted to be a *non-negative integer*. E is the (dimensionless) energy eigenvalue.

1.4(6 points) Find all the (negative) energies E of the Schrödinger equation above for a given integer value of s. How many different (non-positive) energy levels are there? (The negative energies are characterized by having wave functions that goes to zero at infinity $\psi(x \pm \infty) = 0$.) Hint: Transform to a new variable : $r = \sqrt{s+1}e^{-x/2}$.

Problem 2

Two spin-1/2s are far away from each other. One is at a location labelled 2 and the other at another location 3 far away from 2 (we will reserve location 1 and 4 for later use). The two spin-1/2s at location 2 and 3 are in a state

$$\frac{1}{\sqrt{2}}\Big(|\uparrow_2\downarrow_3\rangle - |\downarrow_2\uparrow_3\rangle\Big)$$

where the subscript indicates the location of the spin. $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of the z-component of the spin-1/2 operator $\hat{S}_z$ with eigenvalues $+\hbar/2$, and $-\hbar/2$ respectively.

A third spin in the state $\alpha|\downarrow_1\rangle + \beta|\uparrow_1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$ is present at location 1, which lies close to location 2 so that it is possible to make simultaneous measurements on the spins at 1 and 2. The total state of the three spins can then be written

$$|\psi\rangle = \frac{\alpha}{\sqrt{2}}\Big(|\downarrow_1\uparrow_2\downarrow_3\rangle - |\downarrow_1\downarrow_2\uparrow_3\rangle\Big) + \frac{\beta}{\sqrt{2}}\Big(|\uparrow_1\uparrow_2\downarrow_3\rangle - |\uparrow_1\downarrow_2\uparrow_3\rangle\Big)$$

Assume that an (ideal) measurement is made on the two spins at location 1 and 2. The observable $\hat{O}$ being measured on those two spins gives a measurement value v_1 which correponds to the eigenstate

$$|\chi_1\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle\Big)$$

of $\hat{O}$. The spectrum of $\hat{O}$ is not degenerate.

2.1(6 points) What is the probability of getting this measurement value v_1?

2.2(6 points) What is the (normalized) state of the spin at location 3 just after the measurement value v_1 was obtained?

Now a fourth spin-1/2 is added at location 4 which is close to location 3. The four spins are in a new quantum state $|\chi\rangle$ which is a product of two spin-0 states, one involving the spins at 2 and 3 and the other one involving spins at 1 and 4.

2.3(6 points) Show that the state $|\chi\rangle$ can be written

$$|\chi\rangle = \frac{1}{2}\Big(|\uparrow\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle \Big)$$

where $|\uparrow\downarrow\downarrow\uparrow\rangle \equiv |\uparrow_1\downarrow_2\downarrow_3\uparrow_4\rangle$, i.e. the state of the spin at location i is denoted by the i'th symbol from the left.

The total spin operator is $\vec{S}_{tot} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 + \vec{S}_4$, where $\vec{S}_i$ is the spin-1/2 operator acting on the spin-1/2 at location i.

2.4(4 points) Find the energy eigenvalues and their degeneracies for the following Hamiltonian involving the four spin-1/2s:

$$\hat{H} = \frac{J}{24\hbar^4}\left(S_{tot}^2 - 2\hbar^2\right) S_{tot}^2$$

where J is a positive constant with dimensions of energy.

Two spins-1/2's can combine to give a total spin-1 state or a total spin-0 state. Writing the three spin-1 states as $|+\rangle = |\uparrow\uparrow\rangle$, $|0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ and $|-\rangle = |\downarrow\downarrow\rangle$, the projection operator that acts on two spin-1/2s on sites i and j and project onto the spin-1 subspace can be written as

$$\hat{P}_{ij} = |+\rangle\langle\uparrow_i\uparrow_j| + |0\rangle\frac{1}{\sqrt{2}}\Big(\langle\uparrow_i\downarrow_j| + \langle\downarrow_i\uparrow_j|\Big) + |-\rangle\langle\downarrow_i\downarrow_j|$$

where we have made the mapping from two spin-1/2s to a spin-1 clearer by writing all the kets in the spin-1 notation and the bras in the spin-1/2 notation.

2.5(6 points) Write the state $c\hat{P}_{12}\hat{P}_{34}|\chi\rangle$ in terms of two spin-1 states, one composed of the spin-1/2s at location 1 and 2, and the other one composed of the spin-1/2s at location 3 and 4. Determine the constant c so you get a normalized state. Also determine the total spin of the state $c\hat{P}_{12}\hat{P}_{34}|\chi\rangle$. Hint: You may like to consult the Clebsch-Gordan tables.

2.6(2 points) Find the energy eigenvalues and their degeneracies for the following Hamiltonian

$$\hat{H} = \frac{J}{6\hbar^4}\left[\left(\vec{S}_1 \cdot \vec{S}_2\right)^2 + 3\hbar^2 \vec{S}_1 \cdot \vec{S}_2 + 2\hbar^4\right]$$

where J is a positive constant with dimensions of energy and $\vec{S}_i$ is a spin-1 operator acting on the spin-1 on site i.