Differentiation

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Derivatives at a Point

Definition 1

Let $f : I \to \mathbb{R}$ and $x_0 \in I$. The **derivative** of a function f at x_0 is the value of the limit.

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{1}$$

or

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{2}$$

Theorem 1

- f is said to be **differentiable** at x_0 if that limit exists.
- f is called **differentiable** on the interval $[a, b] \in I$ if it is differentiable at every point $x \in [a, b]$

  We denote by y' or $f'(x)$ or $\dfrac{df}{dx}(x)$

Derivatives Defined

Other notation

from (2): we can find the derivative with x instead of x_0 and Δx instead of h, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{3}$$

If we write $y = f(x)$, the change of the function$f(x)$ for small increasing amount Δx, then $\Delta y = f(x + \Delta x) - f(x)$. So

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \tag{4}$$

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Differentiability

Definition 7

let $x_0 \in I$ and $f : I \to \mathbb{R}$. then

1. We denote f has right-hand derivative by $f'_r(x_0)$ at x_0 iff $\lim\limits_{h \to 0^+} \dfrac{f(x_0+h) - f(x_0)}{h}$
2. We denote f has left hand derivative by $f'_l(x_0)$ at x_0 iff $\lim\limits_{h \to 0^-} \dfrac{f(x_0+h) - f(x_0)}{h}$

Theorem 3

Let $x_0 \in I$ and $f : I \to \mathbb{R}$. Then, f is differentiable at x_0 iff $f'_l(x_0) = f'_r(x_0)$

Theorem 4

If f is differentiable at x_0, then f is continuous at x_0

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Direct computation of derivatives

1. The derivative of any constant function is zero. Let $f(x) = c$, where c is a constant in the set of real numbers. Then,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

   .

2. Consider the derivative of $f(x) = x$. Using definition of derivatives we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h} = 1$$

Direct computation of derivatives

3. Derivative of $f(x) = kx$ is $f'(x) = k$, where $k \in \mathbb{R}$. Consider

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k(x+h) - kx}{h} = k \lim_{h \to 0} \frac{h}{h} = k$$

4. Derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$. Consider

$$\begin{aligned} f'(x) = \frac{df(x)}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \to 0} \frac{(x+h-x)[(x+h)^{n-1} + \cdots + (x+h)x^{n-2} + x^{n-1}]}{h} = nx^{n-1} \end{aligned}$$

Proposition

If $y = u^n$, where u is the function of x then $y'(x) = \dfrac{dy}{dx} = \dfrac{du^n}{dx} = nu'u^{n-1}$

Derivative of Trigonometric Function

1. Derivative of $f(x) = \sin(x)$ is $f'(x) = \cos(x)$. Consider

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - sin(x)}{h} \\
&= \lim_{h\to 0} \frac{2\cos[(x+h+x)/2]\sin[((x+h)-x)/2]}{h} \\
&= \lim_{h\to 0} \frac{2\cos(x+h/2)\sin(h/2)}{h} = \lim_{h\to 0} 2\cos(x+h/2) \times \frac{\sin(h/2)}{2(h/2)} \\
&= \lim_{h\to 0} \cos(x+h/2) = \cos(x)
\end{aligned}
$$

2. Derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$
3. Derivative of $f(x) = \tan(x)$ is $f'(x) = 1 + \tan^2(x) = \dfrac{1}{\cos^2(x)}$
4. Derivatives of $f(x) = \cot(x)$ is $f'(x) = -\dfrac{1}{\sin^2(x)} = -(1 + \cot^2(x))$

Derivatives of Trigonometric Function

Proposition

If u is the function of x, then

- $y = \sin(u)$ then $y' = u' \cos(u)$
- $y = \cos(u)$ then $y' = -u' \sin(u)$
- $y = \tan(u)$ then $y' = \dfrac{u'}{\cos^2(x)} = u'(1 + \tan^2 u)$
- $y = \cot u$ then $y' = -\dfrac{u'}{\sin^2 u} = -u'(1 + \cot^2 u)$

Derivatives of Exponential and Logarithmic Function

1. Derivative $y = e^x$ is $y' = e^x$. Consider
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x$$
2. Derivatives of $y = \ln(x)$ is $y' = \dfrac{1}{x}$. Consider
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h\to 0} \frac{\ln((x+h)/x)}{h}$$
$$= \lim_{h\to 0} \frac{\ln(1 + h/x)}{x(h/x)} = \frac{1}{x}$$

Proposition

If u is the function of x. then

- $y = e^u$ then $y' = u'e^u$
- $y = \ln(u)$ then $y' = \dfrac{u'}{u}$

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Differentiation Rules

Let $\lambda \in \mathbb{R}$ and $u, v : I \to \mathbb{R}$.. Suppose u, v are differentiable. Then

- Constant Multiple Rule: $(\lambda u(x))' = \lambda u'(x)$ or
$$\frac{d(\lambda u(x))}{dx} = \lambda \frac{du(x)}{dx}$$
- Sum Rule: $(u \pm v)' = u' \pm v'$ or
$$\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$
- Product rule$(u.v)' = u'v + v'u$ or
$$\frac{d(uv)}{dx} = \frac{du}{dx}v + \frac{dv}{dx}u$$
- Quotient rule: $(\frac{u}{v})' = \frac{u'v - u.v'}{v^2}$ or
$$\frac{d}{dx}(\frac{u}{v}) = \frac{\frac{du}{dx}v - \frac{dv}{dx}u}{v^2}$$

Example

Find the derivative of the following function

1. $y = 2x + 3$
2. $y = (x^2 + x + 1)^8$
3. $y = 4e^{2x}$
4. $y = 2e^{x^3+1}$
5. $y = \ln(2x^2 + 1)$
6. $y = \cos(2x)$
7. $y = \sin(3x^3 + 2x)$
8. $y = \tan(e^{x+1})$
9. $y = (x^3 + 1)\cos(2x)$
10. $y = \dfrac{\sin(x)}{x^2 + 1}$

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Derivatives of Composition Function

Theorem (chain rule)

Let f and g be the functions, $f, g : I \to \mathbb{R}$. $\forall x \in I$ and g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function $h(x) = (f \circ g)(x) = f(g(x))$ is given by

$$h'(x) = f'(g(x)).g'(x)$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

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Higher Order Derivatives

Definition 8

Let $f : I \to R$ we defined $f^{(n)}(x_0)$ is the derivative of $f^{(n-1)}$ at x_0 if it exists, for $n = 1, 2, \cdots$

- $f^(n)$ is called the derivative of $f^{(n-1)}$
- $f^{(n)}$ is called the n-th derivative, or derivative of order n, of f.
- We say that f is n **times differentiable** on $\mathbb{I}$ iff $f^{(n)}$ id defined on I.
- We say that f is infinitely differentiable on I iff f is n times differentiable on I, $\forall n \in \mathbb{N}$

Notation

The nth derivative of f is denoted $f^{(n)}$. Thus

- Zero derivative of f is : $f^{(0)} = f$
- First derivative of f is : $f'(x)$
- Second derivative of f is: $f''(x)$
- Third derivative is $f^{(3)}(x)$
- Fourth derivative is $f^{(4)}$
- $\cdots$

Operation Notation

Operation notation. A common variation on Leibniz' notation for derivatives is called operator notation, as in

$$\frac{d(x^4 - 2x)}{dx} = \frac{d}{dx}(x^4 - 2x) = 4x^3 - 2$$

For higher derivatives one can write

$$\frac{d^n y}{dx^n} = f^{(n)}(x)$$

As Example

$$\frac{d^2 y}{dx^2} = \left(\frac{d}{dx}\right)^2 y$$

Note

$$\frac{d^2 y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$$

Higher Order Derivatives

Example: Find the nth derivative of $y = \sin(x)$

$$
\begin{aligned}
y' &= \cos(x) = \sin(\pi/2 + x) \\
y'' &= -\sin(x) = \sin(2\pi/2 + x) \\
y''' &= -\cos(x) = \sin(3\pi/2 + x) \\
y^{(4)} &= \sin(x) = \sin(4\pi/2 + x) \\
y^{(5)} &= \cos(x) = \sin(5\pi/2 + x) \\
y^{(n)} &= \sin(n\pi/2 + x)
\end{aligned}
$$

Higher Order derivatives

Leibniz's rule for higher derivatives

Let $\lambda \in \mathbb{R}, n \in \mathbb{N}, f : g : I \to \mathbb{R}$ are n times differentiable on I. Then

$$(fg)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n-k)}$$

Example: Find nth derivatives of the foll0wing function.

1. $(2x^3 + x + 1)\sin(x)$
2. $\dfrac{2x^2 + x + 1}{1 - x}$

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Extreme Value

Definition 9

A constant c is called a **critical point** of f if one of following satisfy:

1. $f'(c) = 0$
2. $f'(x)$ does not exists

The tangent of a curve

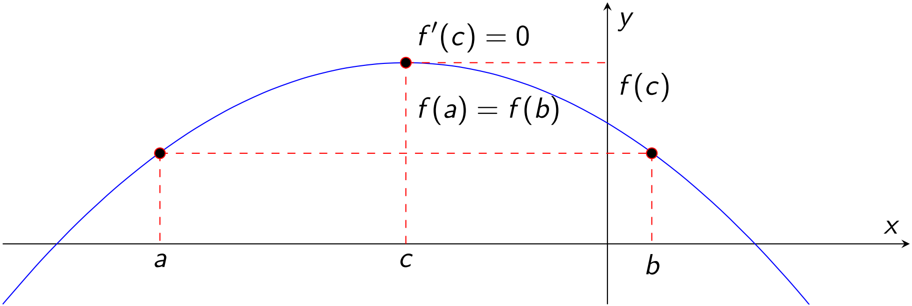
The line equation is defined by $y = mx + b$ or $(y - y_0 = f'(x_0)(x - x_0))$, where $f'(x_0)$ is the slope and y_0 is the intercept.

L'Hôpital Rule

Suppose $f, g : I \to \mathbb{R}$ and $a \in I$ then for the following indeterminate form $(\frac{0}{0})$ or $(\frac{\infty}{\infty})$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}, \text{ where } g'(x) \neq 0$$

Rolle's Theorem



Rolle' Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$ then there is ta least one number $c \in (a, b)$ such that $f'(c) = 0$

Monotone Functions

1. f is said to be **increasing** on I if:
$$\forall (x_1, x_2) \in I^2 : x_1 < x_2 \Longrightarrow f(x_1) \leq f(x_2)$$
.
2. f is said to be **strictly increasing** on I if :
$$\forall (x_1, x_2) \in I^2 : x_1 < x_2 \Longrightarrow f(x_1) < f(x_2).$$
3. f is said to be **decreasing** on I if:
$$\forall (x_1, x_2) \in I^2 : x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2)$$
4. f is said to be **strictly decreasing** on I if:
$$\forall (x_1, x_2) \in \mathbb{R}^2 : x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)$$
.
5. f is said to be **monotone** (strictly monotone) on I if f is either decreasing or increasing (stricty decreasing or strictly increasing) on I.

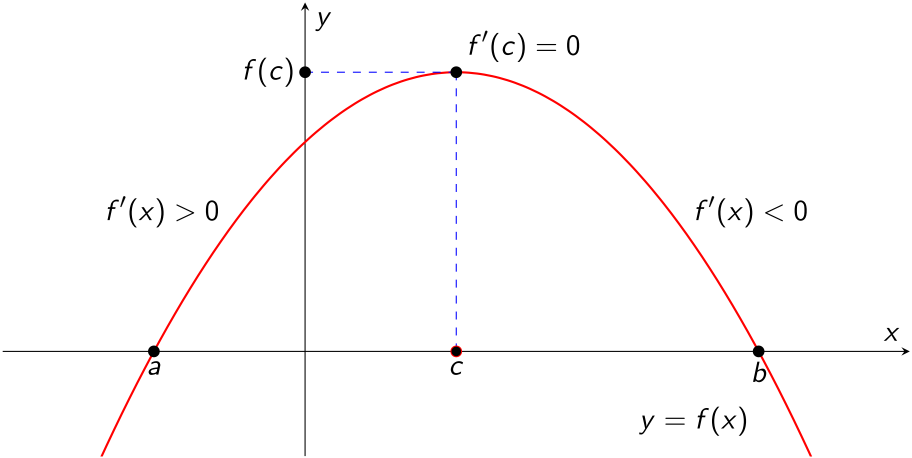
Monotone Functions

Theorem 5

Let f be a function that is continuous on$[a, b]$ and differentiable on (a, b).

1. If $f'(x) \geq 0, \forall x \in (a, b) \Longrightarrow f$ is increasing on $[a, b]$
2. If $f'(x) \leq 0, \forall x \in (a, b) \Longrightarrow f$ is decreasing on $[a, b]$
3. If $f'(x) = 0, \forall x \in (a, b) \Longrightarrow f$ is constant on $[a, b]$.

Graph

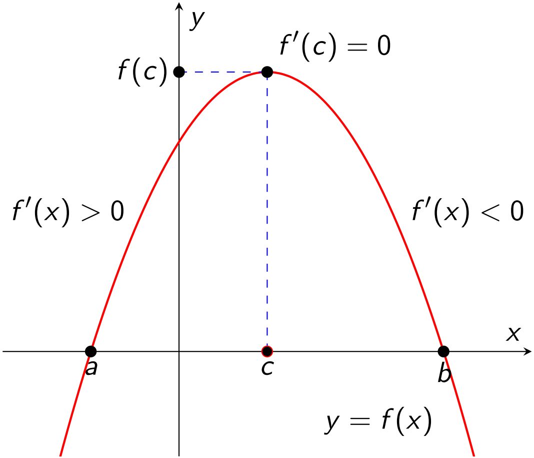


Local Maximum and Local Minimum of the Function

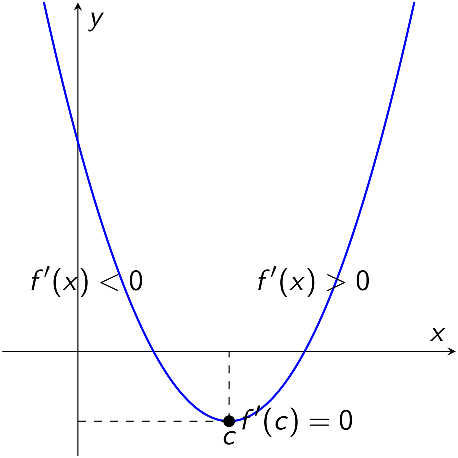
Definition (Local Maximum, Local Minimum and First Derivatives)

- we say f has the local maximum at x_0 if $\begin{cases} f'(x) > 0 & \text{if} \quad x < x_0 \\ f'(x) = 0 & \text{if} \quad x = x_0 \\ f'(x) < 0 & \text{if} \quad x > x_0 \end{cases}$
- we say f has the local minimum at x_0 if $\begin{cases} f'(x) < 0 & \text{if} \quad x < x_0 \\ f'(x) = 0 & \text{if} \quad x = x_0 \\ f'(x) > 0 & \text{if} \quad x > x_0 \end{cases}$

Local Maximum and Local Minimum



Local Maximum



Local Minimum

Local Maximum and Local Minimum

Technique to find the Relative Extreme

- After we do the $f'(x)$ then we set $f'(x) = 0$ after that we consider its sign
  1. If its sign change from $(+)$ to $(-)$, then f has the local maximum
  2. If its sign change from $(-)$ to $(+)$, then f has the local minimum

Example 1 Find the Relative Extreme value of $y = f(x) = -x^3 + 3x^2 + 1$
Example 2: What's the value m such that $y = x^3 + 3mx^2 - mx + 2$ has both local maximum and local minimum.

Local Maximum and Local Minimum and Second Derivative

Definition

Relative Extreme and Second Derivatives The function $y = f(x)$ has two times derivation at x_0

- We say f has the local maximum at x_0 if $\begin{cases} f'(x_0) = 0 \\ f''(x_0) < 0 \end{cases}$
- We say f has the local minimum at x_0 if $\begin{cases} f'(x_0) = 0 \\ f''(x_0) > 0 \end{cases}$