Primitive and Infinite Integration

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Primitive

Definition

A function F is called a **primitive** or **anti-derivative** of f if

$$F'(x) = f(x)$$

since, the differential coefficient of a constant is zero. we denote and define the integration of f by

$$\int f(x)\,dx = F(x) + c, \qquad c \in \mathbb{R}$$

f is called the integrand, dx indicates that x is a variable of the integration.

Integration is the inverse process of differentiation. This helps to find the function whose differential coefficient is known

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Basic Formulae

Theorem

Suppose, c is the constant $\in \mathbb{R}$, *then*

1. $\displaystyle\int 0\,dx = c$
2. $\displaystyle\int k\,dx = kx + c, k \neq 0$
3. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
4. $\displaystyle\int (ax+b)^n\,dx = \frac{1}{a}.\frac{x^{n+1}}{n+1}; a, b \in \mathbb{R}$
5. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + c$
6. $\displaystyle\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|x| + c$
7. $\displaystyle\int e^x\,dx = e^x + c$
8. $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$

Basic Formulae

Theorem (Infinite Integration for Trigonometric Function)

Suppose $c \in \mathbb{R}$, *then*

1. $\displaystyle\int \sin(x)\,dx = -\cos(x) + c$
2. $\displaystyle\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$
3. $\displaystyle\int \cos(x)\,dx = \sin(x) + c$
4. $\displaystyle\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$
5. $\displaystyle\int \frac{1}{\cos^2 x}\,dx = \tan(x) + c$
6. $\displaystyle\int \frac{1}{\sin^2(x)}\,dx = -\cot(x) + c$

Integration Properties

Theorem

1. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
2. $\int kf(x)\,dx = k\int f(x)\,dx, k \in \mathbb{R} - \{0\}$

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Integration by Substitution

Theorem

$$\int f[g(x)]g'(x)\,dx = \int f[g(x)]\,d(g(x)) = \int f(t)\,dt, \qquad \textit{where } \; t = g(x)$$

Theorem

$$\int f(ax+b)\,dx$$

then letting new variable by substituting $u = ax + b$

Example: Compute the following integral

1. $\displaystyle\int e^{3x+1}\,dx$
2. $\displaystyle\int (2x+1)(x^2+x+1)^3\,dx$
3. $\displaystyle\int 2x\sin(x^2+3)\,dx$
4. $\displaystyle\int \frac{x}{1+x^2}\,dx$

Integration by Substitution

Theorem

Integral of the form

$$\int \sin^m x \cos^n x \, dx$$

1. *if m is odd, put $t = \cos x$*
2. *If n is odd, put $t = \sin(x)$*
3. *If m and n both are even, use the formula*

$$1 + \cos^2 x = 2\cos^2 x, \qquad 1 - \cos(2x) = 2\sin^2 x$$

4. *If $m + n$ is negative even integer, put $t = \tan(x)$ or $t = \cot(x)$*

Integration by Substitution

Example: compute the following integral

1. $\int \cos^2 x \, dx$
2. $\int \sin^2 x + 2x \, dx$
3. $\int \tan(x) \, dx$
4. $\int \cos^3 x \, dx$
5. $\int \sin^5 x \, dx$
6. $\int \sin^4 x cos^3 x \, dx$

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Integration by Part

Theorem (Integration by Part)

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx \text{ or } \int u\,dv = uv - \int v\,du$$

Proof: We have.

$$\begin{aligned}
(f(x)g(x))' &= f'(x)g(x) + g'(x)f(x) \\
\Longrightarrow f(x)g(x) &= \int (f'(x)g(x) + g'(x)f(x))dx \\
f(x)g(x) &= \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx \\
\text{or } \int f(x)g'(x)\,dx &= f(x)g(x) - \int g(x)f'(x)\,dx
\end{aligned}$$

$\square$

Integration by Part

Integral by part is the integral of combination two function that's integrand contains inverse trigonometric function, logarithm function, algebraic function, trigonometric function and exponential function **(ILATE)**. Then we set $u(x)$ by following this order and the rest is $v'(x)$

Exanple: Compute the following integral

1. $\displaystyle\int (x+1)\sin(x)\,dx$
2. $\displaystyle\int (2x+1)\cos(2x)\,dx$
3. $\displaystyle\int e^x \sin(x)\,dx$
4. $\displaystyle\int \ln(x)\,dx$
5. $\displaystyle\int x\ln(x)\,dx$
6. $\displaystyle\int \cos(2x)\ln(x)\,dx$

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Integration by Partial Fractions

Suppose that the integrand $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are two polynomial of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
$$Q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$$

- If $n \geq m$ then write $f(x) = \dfrac{P(x)}{Q(x)} = R(x) + \dfrac{P_1(x)}{Q(x)}$ where $\deg(P_1(x)) < \deg(Q(x))$
- If $n < m$ and when the $Q(x)$ is expressible as the product of non-repeating linear factors. That is

  $$Q(x) = (x - a_1)(x - a_2)\cdots(x - a_n)$$

  Write

  $$f(x) = \frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_n}{x - a_n}$$

  where $A_i; i = 1, 2, \cdots, n$ are the constant that's required to find

Integral by Partial Fraction

- If $n < m$ and when $Q(x)$ is expressible as the product of linear factors such that some of them are repeating.That is

$$Q(x) = (x - a_1)^k (x - a_2) \cdots (x - a_n)$$

write

$$f(x) = \frac{P(x)}{Q(x)} = \frac{A_1}{(x - a_1)} + \frac{A_2}{(x - a_1)^2} + \cdots + \frac{A_k}{(x - a_1)^k} + \frac{B_2}{x - a_2} + \cdots + \frac{B_n}{x - a_n}$$

- If $n < m$ and when $Q(x)$ is expressible as

$$Q(x) = (x - a_1)^k (x^2 + bx + c)^l$$

write

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x - a_1)} + \frac{A_2}{(x - a_1)^2} + \cdots + \frac{A_k}{(x - a_1)^k} + \frac{B_1 x + C_1}{x^2 + bx + c}$$

The constant can bee determined by equating the numerator on Right hand side (RHS) to the numerator on Left Hand Side (LHS) and then substituting some value of x

Integration by partial Fractions

Example: Compute the following integral

1. $$\int \frac{x^3 + x^2 + 4}{(x-1)(x+3)}\, dx$$
2. $$\int \frac{2x + 1}{x^2 - 7x + 12}\, dx$$
3. $$\int \frac{1}{(x-1)^2(x^2 + 2x + 4)}\, dx$$