Function and its Graph

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Domain of Function

Domain of a function $f(x)$ refers to the set of all possible input values (independent variable) for which the function is defined. Denoted by D_f

Remark: Some of domain function you need to know

- For Polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, n \geq 0$ has the domain $D = \mathbb{R}$
- For the Rational function $f(x) = \dfrac{g(x)}{h(x)}$ has the domain of function is $h(x) \neq 0$
- For the irrational function $f(x) = \sqrt[n]{p(x)}$ has the domain of function:

$$\begin{cases} p(x) \geq 0 & n \text{ is even} \\ \text{significant depend on } p(x) & n \text{ is odd} \end{cases}$$

Domain of Function

- For the exponential function $f(x) = a^x$ is $x \in \mathbb{R}$ and $a > 0 \neq 1$
- For the exponential function $f(x) = e^x$ is $D = \mathbb{R}$
- For the exponential function $f(x) = e^{f(x)}$ has domain when $f(x)$ is significant
- For the logarithmic function $f(x) = \log_a x$ is $a > 0 \neq 1$ and $x > 0$
- For the logarithmic function $f(x) = \ln(f(x))$ has domain is $f(x) > 0$

Example Find the domain of function from following function

1. $f(x) = 2x^3 + x + 1$
2. $f(x) = \sqrt{\dfrac{2x+1}{x^2+2x-3}}$
3. $f(x) = \ln\left(\dfrac{x+1}{x^3-1}\right)$
4. $f(x) = \dfrac{e^x - 2}{e^x - 1}$
5. $f(x) = x + 1 + \dfrac{e^x}{e^x + 1}$
6. $f(x) = \dfrac{1}{x-1} + \ln\left(\dfrac{x-1}{x+1}\right)$

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Reviews

1. $\lim\limits_{x\to+\infty} e^x = +\infty$
2. $\lim\limits_{x\to-\infty} e^x = 0$
3. $\lim\limits_{x\to+\infty} \dfrac{e^x}{x^n} = +\infty, \forall n > 0$
4. $\lim\limits_{x\to+\infty} \dfrac{x^n}{e^x} = 0, \forall n > 0$
5. $\lim\limits_{x\to+\infty} \ln(x) = +\infty$
6. $\lim\limits_{x\to 0^+} \ln(x) = -\infty$
7. $\lim\limits_{x\to+\infty} \dfrac{\ln^\beta(x)}{x^\alpha} = 0, \forall(\alpha,\beta) \in \mathbb{R}_+^*$
8. $\lim\limits_{x\to 0^+} x^\alpha \ln^\beta(x) = 0, \forall(\alpha,\beta) \in \mathbb{R}_+{}^*$
9. $\lim\limits_{x\to 0^+} \dfrac{1}{x} = +\infty$
10. $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$

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Horizontal Asymptotes

Definition 1

The line $y = L, \in \mathbb{R}$ is called a horizontal asymptote of the curve $y = f(x)$ if

$$\lim_{x \to \pm\infty} f(x) = L$$

Example: Find the horizontal asymptote for the given function

1. $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$
2. $f(x) = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

Vertical Asymptote

Definition 2

The line $x = k, \in \mathbb{R}$ is called vertical asymptote of the curve $y = f(x)$ if

$$\lim_{x \to k} f(x) = \pm\infty$$

If $x = k$ is the vertical asymptote of $y = f(x)$ then k **is Not** present in the **domain of the function**. A function can have any number of vertical asymtote.

Example: Find the vertical asymptote in the following function.

1. $f(x) = \dfrac{x^2}{x+1}$
2. $f(x) = \dfrac{3x^2}{x^2 - 5x + 6}$

Slant Asysmtote (Oblique Asymptote)

Definition 3

The line $y = ax + b$ is the oblique asymptote of the curve $y = f(x)$ if

$$\lim_{x \to \pm\infty} [f(x) - (ax + b)] = \lim_{x \to \pm\infty} \epsilon(x) = 0$$

Proposition 1

The function $y = f(x) = ax + b + \epsilon(x)$ is the oblique asymptote of the curve if

$$\lim_{x \to \pm\infty} \epsilon(x) = 0$$

Oblique Asymptote

From definition (3), we have

$$f(x) - (ax + b) = \epsilon(x) \quad (\epsilon(x) \to 0; x \to \pm\infty)$$

$$\frac{f(x)}{x} - \frac{(ax + b)}{x} = \frac{\epsilon(x)}{x}$$

$$\iff \frac{f(x)}{x} = a + \frac{b}{x} + \frac{\epsilon(x)}{x} \to a \quad ; (x \to \infty)$$

Moreover, $\quad f(x) - ax = b + \epsilon(x) \to b; \quad x \to \infty$

Proposition 2

If $\displaystyle\lim_{x\to\pm\infty} \frac{f(x)}{x} = a$ and $\displaystyle\lim_{x\to\pm\infty} [f(x) - ax] = b$ then we say $y = ax + b$ is the oblique asymptote.

Example 1: Find the oblique asymptote of the following function

1. $y = \sqrt{4x^2 + x + 5}$
2. $y = \dfrac{x^2 + 1}{x}$
3. $g(x) = \dfrac{x^2 + 3x + 4}{x + 2}$

Example 2 Show that the equation $y = x - 3$ is the oblique asymptote of the curve $f(x) = \dfrac{x^2 - 5x + 7}{x - 2}$

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Proposition

- *$f(x)$ is called increasing function on interval I if* $f'(x) \geq 0, \forall x \in I$
- *$f(x)$ is called decreasing function on I if* $f'(x) \leq 0, \forall x \in I$

Example: Given $f(x) = \dfrac{x+1}{x-1}$

1. Find the domain of $f(x)$
2. Show that $f(x)$ is the strictly decreasing

Extrema Value and Relative Extrema

Definition

A constant c is called a **critical piont** of f if one of following satify:

- $f'(c) = 0$
- $f'(c)$ does not exist.

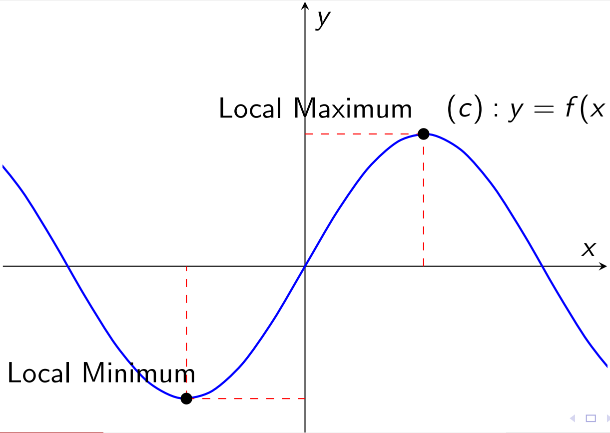
Definition

1. A function f is said to have **local maximum** or **relative maximum** at c if there is an open interval I such that $c \in I$ and
$$f(x) \leq f(c), \forall x \in I$$
2. A function f is said to have **local minimum** or **relative minimum** at c if there is open interval I such that $c \in I$ and
$$f(x) \geq f(c), \forall x \in I$$

Extrema Value and Relative Extrema

Definition

3. A function f is said to ahve **local extremum** or **relative extremum** at c if it has local maximum or local minimum at c.



Technique to find the Relative Extrema

Proposition

Suppose f be the function defined at x_0

- *We call f has local maximum at x_0 iff its graph is increasing until the extreme point x_0 then decreasing. Similarly, f has the local maximum at x_0 iff $f'(x)$ changes the sign from $(+)$ to $(-)$ and the local maximum value is $f(x_0)$.*
- *We call f has the local minimum at x_0 iff its graph is decreasing until the extreme value x_0 then increasing. Similarly, f has the local minimum at x_0 iff $f'(x)$ changes sign from $(-)$ to $(+)$ at x_0 and the local minimum value is $f(x_0)$*

Example Find the extreme value of f that is $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$

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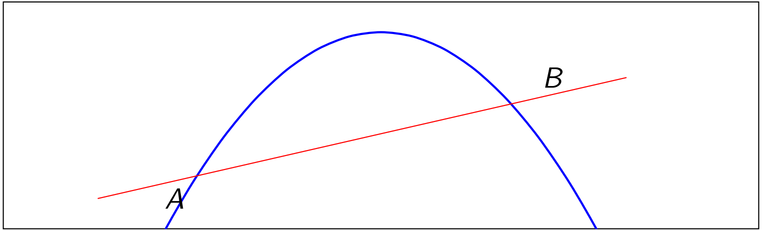
Concavity (Concave Up)

Definition

We call f is the concave (non convex) if the line segment between any two distinct points on the graph of the function lies above the graph between the two points.



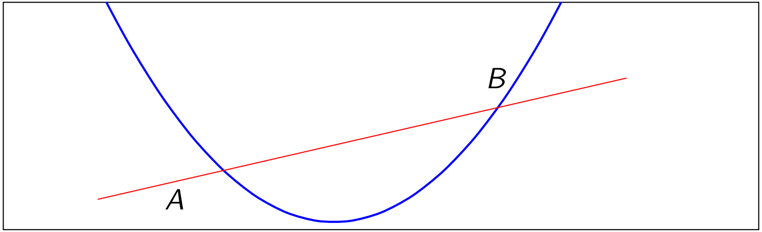
Theorem

We said f is the concave function on I iff $f''(x) < 0, \forall x \in I$.

Convexity (Concave Down)

Proposition

We call f is the convex if the line segment between any two distinct points on the graph of the function lies below the graph between the two points.



Theorem

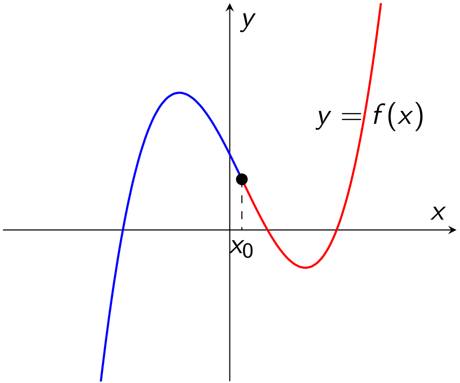
We said f is the convex function on I iff $f''(x) > 0, \forall x \in I$

Theorem

Reflection of a function f is the point where the graph is change from concave to convex or convex to concave. Similarly, it is equivalent that reflection is on its point at x-axis $f''(x)$ changes the sign from $(+)$ to $(-)$ **or** *$(-)$ to $(+)$.*

Method to find the reflection

1. find $f''(x)$
2. Solving equation $f''(x) = 0$
3. Study the sign of $f''(x)$
   - If $f''(x)$ changes sign both sides at x_0 then the graph has the reflection point at $I(x_0, f(x_0))$
   - If $f''(x)$ does not change sign then the graph have not reflection.



Example: Study the convex and concave of function f , that is $f(x) = x^3 - 3x^2 + 2$
Example: Given $f(x) = x^3 - 6x^2 + 5$. Does the graph $f(x)$ has the reflection ? If yes, find the reflection.

Relative Extreme and Second Derivative

Proposition

Suppose $y = f(x)$ be the function that have the second differential at x_0, then

- *f has local maximum at x_0 if* $\begin{cases} f'(x_0) & = 0 \\ f''(x_0) & < 0 \end{cases}$
- *f has local minimum at x_0 if* $\begin{cases} f'(x_0) & = 0 \\ f''(x_0) & > 0 \end{cases}$

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Tangent Line to the Curve at a Point x_0

The line D given the slop a and go through the point (x_0, y_0) is given by

$$(D) : y = a(x - x_0) + y_0$$

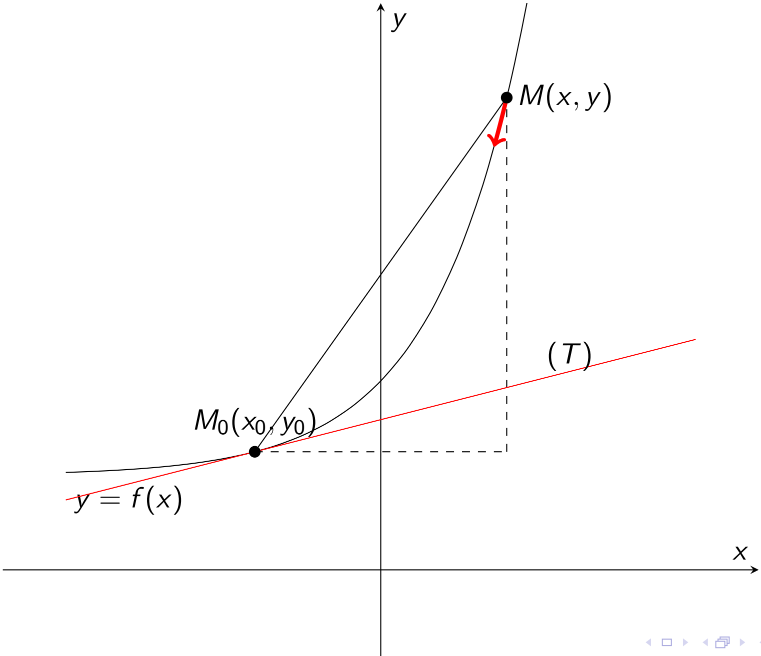
where $a = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x}$ if we know two points that graph have through.
Take a look for the definition of derivative at x_0

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

when the point x approach closely to x_0 then the point (x, y) and (x_0, y_0) is approximately on the same coordinate. That's why the slop of tangent to the graph at x_0 is $f'(x_0)$.

Proof.

Therefore the form of tangent line to the curve is $(T) : y = f'(x_0)(x - x_0) + y_0$ □



y
$M(x, y)$
(T)
$M_0(x_0, y_0)$
$y = f(x)$
x

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Odd-Even Functions

Definition

Suppose f be the function on $\mathbb{D}_f$

- We said f is the odd function iff $\forall x \in \mathbb{D}_f, -x \in \mathbb{D}_f : f(-x) = -f(x)$
- We said f is the even function iff $\forall x \in \mathbb{D}_f, -x \in \mathbb{D}_f : f(-x) = f(x)$

Example: Given function f

1. $f(x) = x^4 + 2x^2$. Show that $f(x)$ is the even function.
2. Show that $f(x) = x^7 + 3x^3 - x$ is the odd function

Graph of Odd-even Functions

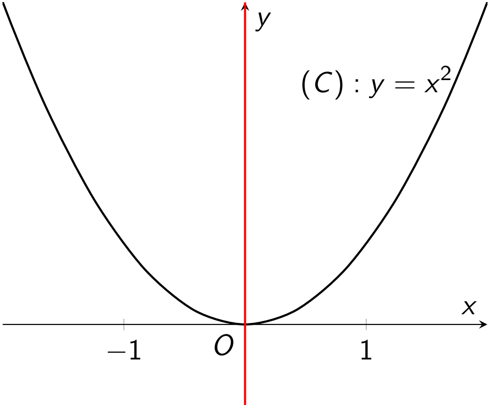


Figure: Even-function

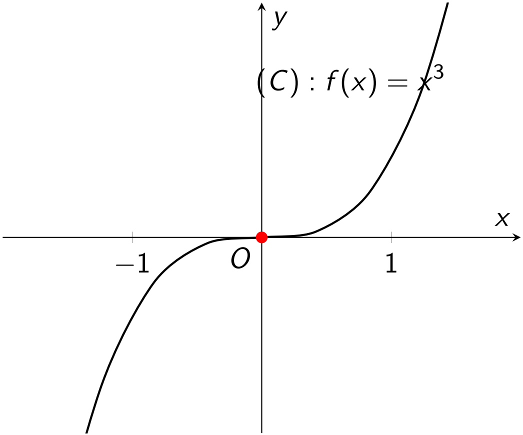


Figure: odd-function

Odd-even Function Properties

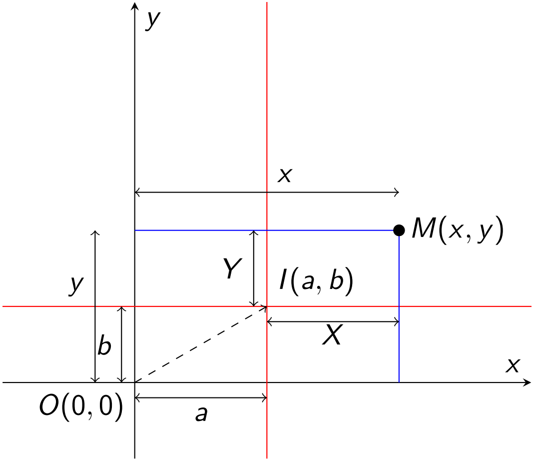
Proposition

- *Graph of even function is symmetry with respect to the* **y-axis**. *We call it* **Axis of symmetry (Mirror Line)**.
- *Graph of odd function is symmetry with respect to the* **origin**. *We call it* **Center(Point) of Reflection**

Translation

Sometimes the axes that we want to show as **center of symmetry** or **axis of symmetry** is not **y-axis** or **origin**. In this case we can not use the odd-even function properties directly. We need to change the origin (**translation**) to the new origin. We want to translation the origin $O(0,0)$ to $I(a,b)$. Take (x,y) is the coordinate M in $O(0,0)$ and (X,Y) be the coordinate M in $I(a,b)$
From Graphic we have

$$\begin{cases} x &= a + X \\ y &= b + Y \end{cases}$$



Example Given function f such that $f(x) = \dfrac{x^2+x-3}{x-1}$ that has the graph (C)

1. Find the domain of f
2. (D) and (L) are the vertical asymptote and oblique asymptote respectively. Find the equation of (D) and (L)
3. Find the coordinate intersection point I of (D) and (L)
4. Show that I is the point of reflection of(C)

Center of Symmetry and Line of Symmetry

Theorem

If $I(a, b)$ is the center of symmetric to the graph $(c) : y = f(x)$ then

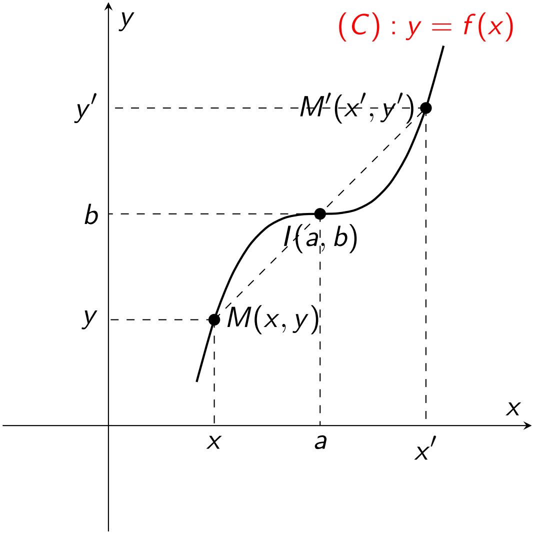
$$f(2a - x) + f(x) = 2b$$

Theorem

If $x = a$ is the symmetric line (axis of symmetry)to the curve $(C) : y = f(x)$ then

$$f(2a - x) - f(x) = 0$$

Proof



Let $M(x, y)$ and $M'(x', y')$ are the points lie on the graph (C) and symmetry on $I(a, b)$
Then we get I is the middle point on the line $[MM']$

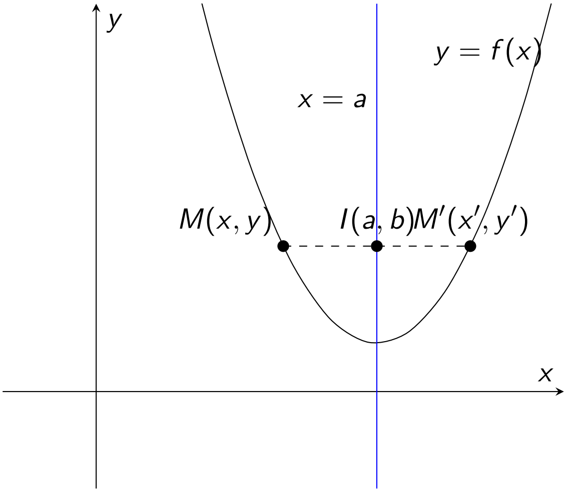
$$\Longrightarrow \begin{cases} x_I &= \dfrac{x + x'}{2} \\ y_I &= \dfrac{y + y'}{2} \end{cases} \Longleftrightarrow \begin{cases} x + x' &= 2a \\ y + y' &= 2b \end{cases}$$

we get $x' = 2a - x$

Consider $y + y' = 2b \Longleftrightarrow f(x) + f(x') = 2b$

Therefore $\boxed{f(2a - x) + f(x) = 2b}$

Proof



Let $y = f(x)$ has the domain $\mathbb{D}$ and $M(x, y)$ and $M'(x', y') \in \mathbb{D}$. The point $I(a, b)$ is the middle on the line $[MM']$ then

$$\begin{cases} x_I &= \dfrac{x + x'}{2} \\ y_I &= \dfrac{y + y'}{2} \end{cases}$$

we have $x + x' = 2a$ or $x' = 2a - x$

$y = y' = y_I = b$, which means $f(x') = f(x)$

Therefore

$$\boxed{f(2a - x) = f(x) \text{ or } f(2a - x) - f(x) = 0}$$

yield $x = a$ is the symmetry line.

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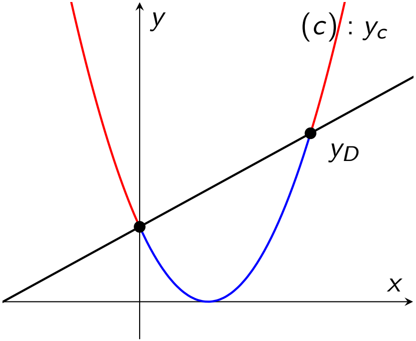
Relative Position between Line and Curve

Proposition

Suppose $y_c = f(x)$ *be the function which have the graph representation* (c) *and* $y_D = ax + b$ *be the relative line. then*

1. *If* $y_c - y_D > 0$ *we say graph*(c) *is above the line* y_D
2. *If* $y_c - y_D < 0$ *we say graph* (c) *is below the line* y_D
3. *If* $y_c - y_D = 0$ *we say graph* (c) *and line* y_D *have intersection.*

Relative Position



Example Given $f(x) = \dfrac{x^2}{x-1}$ has the graph (C).(D) is the oblique asymptote to the graph.

1. Find the equation of the line (D)
2. Study the relative positive between (C) and (D)

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Proposition (Solving Equation by Graphing)

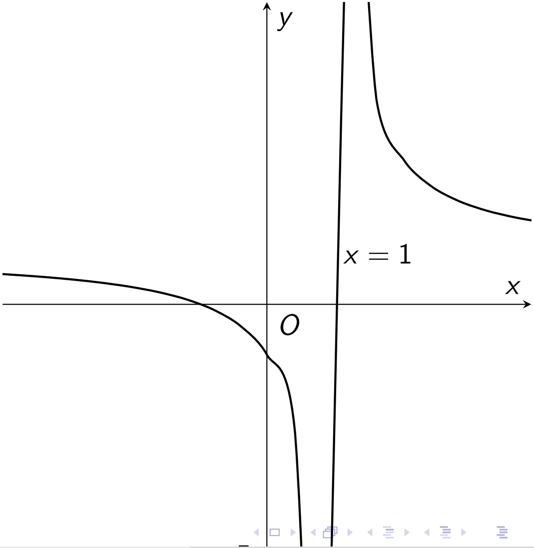
Suppose $y = f(x)$ and $y = k$ in domain D. The equation $f(x) = k$ is number of point-intersection between the horizontal line $y = k$ and the curve $y = f(x)$

Proposition (Solving Inequality by Graphing)

The solution $f(x) < g(x)$ is the set of value x that $(C_1) : y = f(x)$ is below the curve $(C_2) : y = g(x)$

Solving Equation by Graphing

Given $f(x)=\dfrac{x+1}{x-1}$ represent by graph (c) as below. Discuss the roots of equation $\dfrac{x+1}{x-1}=k$ base on the value of parameter k.



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Study the Variation and Tracing the Curve

Study the variation and the step for tracing the curve:

1. Domain
2. Direct Variation
   - Limit and Asymptote
   - First derivative and its sign
   - Ralative Extrema
   - Varaition Table
3. Sketching the Graph
   - Intersection between graph and axes (if possible)
   - axis and center of reflection (if they ask)
   - Table of value for some lines or graph

Example: Study the variation and plot the curve $(c) : y = \dfrac{(x+2)^2}{2x+1}$