Limit and Continuity

Mr. CHEA Makara
4th Year Engineering in Majoring Data Science,
Department of Applied Mathematics and Statistics, ITC

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Contents

Definition 1

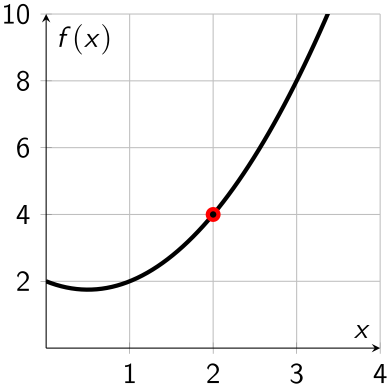
The limit of $f(x)$, as x approaches to a, equals to L, denoted by

$$\lim_{x \to a} f(x) = L \text{ or } f(x) \to L \text{ as } x \to a$$

If the values of $f(x)$ moves arbitrarily close to L as x moves sufficiently close to a (on either side of a) but not equal to a.

Graph and Table

Consider $\lim_{x \to 2} (x^2 - x + 2) = 4$.



$x < 2$	$f(x)$	$x > 2$	$f(x)$
1.0	2.0000	3.0	8.0000
1.5	2.7500	2.5	5.7500
1.9	3.7100	2.1	4.3100
1.99	3.9701	2.01	4.0301
1.995	3.9850	2.005	4.0150
1.999	3.9970	2.001	4.0030

One-sided Limits

Theorem 1

If a function f has limit, then it is unique.

- Left-hand limit of f

$$\lim_{x \to a^-} = L \tag{1}$$

- Right-hand limit of f

$$\lim_{x \to a^+} = L \tag{2}$$

If (1) and (2) is hold, we say that f has limit at point a, or

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \tag{3}$$

Example 1: Does $h(x)$ have the limit at $x = 2$? if $h(x) = \begin{cases} x+1 & , x < 2 \\ (x-2)^2+3 & , x > 2 \end{cases}$

Example 2: Does $f(x)$ has limit at $x = 2$? Given that $f(x) = \begin{cases} x+1 & , x \leq 2 \\ (x-2)^2+1 & , x > 2 \end{cases}$

Limit at a point

We denote I be an interval in $\mathbb{R}$, $c \in I, l \in \mathbb{R}$ and $f : I \to \mathbb{R}$

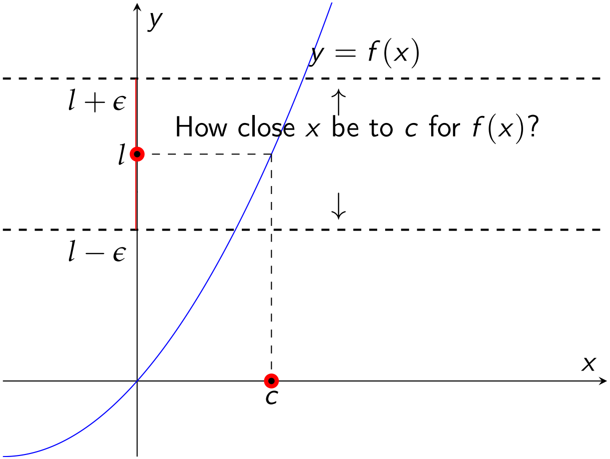
Definition 2

We say that f has limit l as x tends to c iff

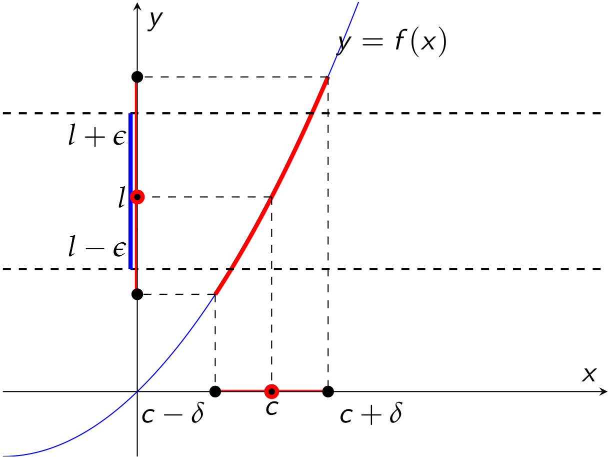
$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, (|x - c| < \delta \Longrightarrow |f(x) - l| < \epsilon)$$

In this case, we write $\lim\limits_{x \to c} f(x) = l$

Limit at a point

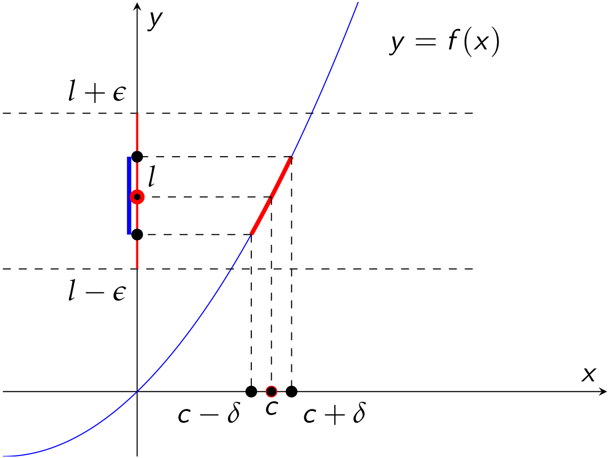


Limit at a point



For some x in this interval $f(x)$ is not between $l - \epsilon$ and $l + \epsilon$. Therefore the δ in this picture is too big for the given ϵ. We need a smaller δ.

Limit at a point



If you choose x in the interval $[c - \delta, c + \delta]$ then $f(x)$ will be between $l - \epsilon$ and $l + \epsilon$. Therefore the δ is small enough for the given ϵ.

Limit at a point

Example: Using the definition of limit, prove that $\lim_{x \to 2} 2x + 1 = 5$

Recall that f has limit l as x approach to c iff

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, (|x - c| < \delta \implies |f(x) - l| < \epsilon)$$

Let $\forall \epsilon > 0$, Consider

$$\begin{aligned} |f(x) - 5| < \epsilon &\iff |2x + 1 - 5| < \epsilon \\ &\iff |2x - 4| < \epsilon \\ &\iff 2|x - 2| < \epsilon \\ &\iff |x - 2| < \frac{\epsilon}{2} \end{aligned}$$

Choose $\delta = \dfrac{\epsilon}{2}$

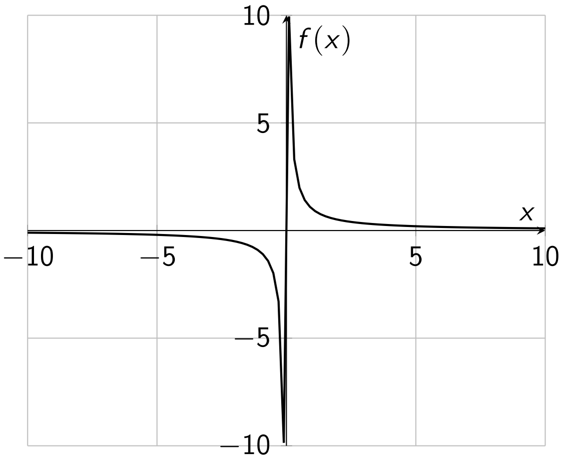
Therefore $\forall \epsilon > 0, \exists \delta = \dfrac{\epsilon}{2} > 0, \forall x \in \mathbb{R}, |x - 2| < \delta \implies |f(x) - 5| < \epsilon$

Limits at Infinity

Definition 3

- Let f be a function defined on some interval (a, ∞). Then
$$\lim_{x\to\infty} f(x) = L$$
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.
- Let f be a function defined on some interval $(-\infty, a)$. Then
$$\lim_{x\to-\infty} f(x) = L$$
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

Example Consider the function $f(x) = \dfrac{1}{x}$



x	10	100	$\to +\infty$
$y = f(x)$	0.1	0.01	$\to 0$
x	-10	-100	$\to -\infty$
$y = f(x)$	-0.1	-0.01	$\to 0$

We get $\lim\limits_{x\to\infty} \dfrac{1}{x} = \lim\limits_{x\to-\infty} \dfrac{1}{x} = 0$

Moreover, $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$ and $\lim\limits_{x\to 0^+} \dfrac{1}{x} = +\infty$

Limits at Infinity

Definition 4

We say that f has limit $+\infty$ as x tends to c iff

$$\forall A > 0, \exists \delta > 0, \forall x \in I, (|x - c| < \delta \Longrightarrow f(x) > A)$$

In this case, we arite $\lim\limits_{x \to c} f(x) = +\infty$

Definition 5

We say that f has limit $-\infty$ as x tends to c iff

$$\forall A < 0, \exists \delta > 0, \forall x \in I, (|x - c| < \delta \Longrightarrow f(x) < A)$$

In thus case, we write $\lim\limits_{x \to c} f(X) = -\infty$

Limits at Infinity

Definition 6

We say that f has limit l as x tends to $+\infty$ iff

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, (x > \delta \Longrightarrow |f(x) - l| < \epsilon)$$

In this case, we write $\lim\limits_{x \to +\infty} f(x) = l$

Definition 7

We say that f has limit $+\infty$ as x tends to $+\infty$ iff

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, (x > \delta \Longrightarrow f(x) > \epsilon)$$

In this case, we write $\lim\limits_{x \to +\infty} f(x) = +\infty$

Directed Substitutions

Theorem 2

Suppose f be any function, $f : I \to \mathbb{R}$ and $a \in I$ then

$$\lim_{x \to a} f(x) = f(a)$$

Changing Variable

Suppose $\lim\limits_{x \to x_0} f(x)$ is exists. we can create the new variable and approach it into zero in the limit such that

Let $u = x - x_0$, when $x \to x_0 \implies u \to 0$. Then

$$\lim_{x \to x_0} f(x) = \lim_{u \to 0} f(u + x_0)$$

Limits Properties

Theorem 3

Suppose that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = l'$ where $l, l', c \in \mathbb{R}$. Then

1. $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x) = cl$
2. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = l \pm l'$
3. $\lim_{x\to a} [f(x).g(x)] = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x) = l \times l'$
4. $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{l}{l'}$, where $l' \neq 0$

Theorem 4

1. $\lim_{x\to a} x = a$
2. $\lim_{x\to a} c = c$, for any constant c
3. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
4. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

Continue

Theorem 6

Let $l \in \mathbb{R}$

1. $\begin{cases} \lim\limits_{x \to a} f(x) = l \\ \lim\limits_{x \to a} g(x) = \infty \end{cases} \implies \lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = 0$
2. $\begin{cases} \lim\limits_{x \to a} f(x) = 0 \\ \lim\limits_{x \to a} g(x) = \infty \end{cases} \implies \lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = 0$
3. $\begin{cases} \lim\limits_{x \to a} f(x) = l \\ \lim\limits_{x \to a} g(x) = 0 \end{cases} \implies \lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \infty$
4. $\begin{cases} \lim\limits_{x \to a} f(x) = \infty \\ \lim\limits_{x \to a} g(x) = 0 \end{cases} \implies \lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \infty$

Continue

Theorem 5 (Squeeze Theorem)

Let $f, g, h : I \to \mathbb{R}$. Suppose that

$$g(x) \leq f(x) \leq h(x), \forall x \in I$$

If $\lim\limits_{x \to a} g(x) = \lim\limits_{x \to a} h(x) = l$, then $\lim\limits_{x \to a} f(x) = l$

Limits of Trigonometric Functions

Remark

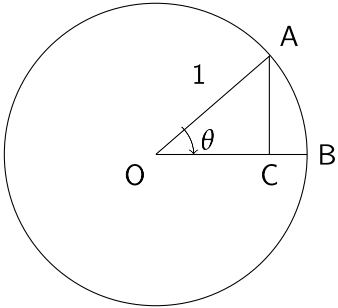
- $\lim_{x \to a} \sin(x) = \sin(a)$
- $\lim_{x \to a} \cos(x) = \cos(a)$
- $\lim_{x \to a} \cot(x) = \cot(a)$
- $\lim_{x \to a} \tan(x) = \tan(a)$

corollary 1

- $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$
- $\lim_{x \to 0} \frac{\tan(x)}{x} = 1$

Limits of Trigonometric Functions

Proof



- $AC = \sin\theta$,arclength $AB = \theta$
- $\dfrac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$
- Principle: Short pieces of curves are nearly straight.

Limits of Trigonometric Functions

Remark

- $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$
- $\lim\limits_{x\to 0} \dfrac{x}{\sin x} = 1$
- $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1$
- $\lim\limits_{x\to 0} \dfrac{x}{\tan x} = 1$
- $\lim\limits_{x\to 0} \dfrac{\sin(nx)}{nx} = 1$
- $\lim\limits_{x\to 0} \dfrac{nx}{\sin(nx)} = 1$
- $\lim\limits_{x\to 0} \dfrac{\tan(nx)}{nx} = 1$
- $\lim\limits_{x\to 0} \dfrac{nx}{\tan(nx)} = 1$

Limits of Exponential and Logarithmic Function

- We have $y = e^x$ is the exponential function, where the value $e \approx 2.7182$

Table of values

x	1	2	3	$\cdots$
y	2.7182	7.3886	20.0837	$\cdots$

We assume that $\lim\limits_{x\to+\infty} e^x = +\infty$ and $\lim\limits_{x\to-\infty} e^x = \lim\limits_{x\to-\infty} \dfrac{1}{e^{-x}} = 0$

- We have $y = \ln(x)$ is the inverse function of $y = e^x$

Table of Values

x	20	40	60	$\cdots$
y	2.995	3.668	4.094	$\cdots$

We assume that $\lim\limits_{x\to+\infty} \ln(x) = +\infty$ and $\lim\limits_{x\to 0^+} \ln(x) = -\infty$

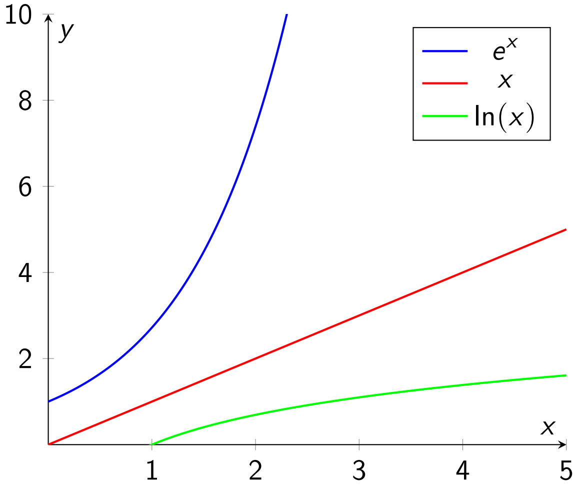
Proof: if $x \to 0^+$ then $\dfrac{1}{x} \to +\infty$. Let $X = \dfrac{1}{x}$ then $x = \dfrac{1}{X}$

we have $\lim\limits_{x\to 0^+} \ln(x) = \lim\limits_{X\to+\infty} \ln(\dfrac{1}{X}) = -\lim\limits_{X\to+\infty} \ln(X) = -\infty$

Limits of Exponential and Logarithmic Function

Corollary 2

- $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x = e$
- $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$
- $\lim\limits_{x\to\infty}\left(1+\frac{1}{nx}\right)^{nx} = e$
- $\lim\limits_{x\to 0}(1+nx)^{\frac{1}{nx}} = e$
- $\lim\limits_{x\to 0}\frac{e^x-1}{x} = 1$
- $\lim\limits_{x\to 0}\frac{x}{e^x-1} = 1$
- $\lim\limits_{x\to 0}\frac{e^{nx}-1}{nx} = 1$
- $\lim\limits_{x\to 0}\frac{nx}{e^{nx}-1} = 1$
- $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x} = 1$
- $\lim\limits_{x\to 0}\frac{x}{\ln(1+x)} = 1$
- $\lim\limits_{x\to 0}\frac{\ln(1+nx)}{nx} = 1$
- $\lim\limits_{x\to 0}\frac{nx}{\ln(1+nx)} = 1$



Corollary 3

- $\displaystyle\lim_{x\to+\infty} \frac{e^x}{x} = +\infty$
- $\displaystyle\lim_{x\to+\infty} \frac{e^x}{x^n} = +\infty, n > 0$
- $\displaystyle\lim_{x\to+\infty} \frac{x^n}{e^x} = 0, n > 0$
- $\displaystyle\lim_{x\to+\infty} \frac{\ln(x)}{x} = 0$
- $\displaystyle\forall(\alpha,\beta) \in (\mathbb{R}_+^*)^2 \lim_{x\to0^+} x^\beta(\ln(x))^\alpha = 0$
- $\displaystyle\forall(\alpha,\beta) \in (\mathbb{R}_+^*)^2 \lim_{x\to+\infty} \frac{(\ln x)^\alpha}{x^\beta} = 0$
- $\displaystyle\lim_{x\to0^+} x^n \ln(x) = 0, n \geq 0$

Indeterminate Form of Limits

Indeterminate Forms of Limits are $\frac{0}{0}, \frac{\infty}{\infty}, +\infty - \infty, 1^{\infty}, 0 \times \infty, \infty^0, 0^0$

Indeterminate form $\frac{0}{0}$

For computing the indeterminate of limit $\frac{0}{0}$ we need to find the common factorization on numerator and denominator, then divide the common factor. After that we can evaluate the new limit.

Frequently Expanded Form of Polynomial

- $a^2 - b^2 = (a+b)(a-b)$
- $a^3 - b^3 = (a-b)(a^2+ab+b^2)$and $a^3 + b^3 = (a+b)(a^2-ab+b^2)$
- $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$
- $a^n + b^n = (a+b)\left(a^{n-1} - a^{n-2}b + \cdots + (-1)^{n-2}ab^{n-2} + (-1)^{n-1}b^{n-1}\right)$

Indeterminate Form $\dfrac{0}{0}$

Example: Evaluate the limit $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

Solution

$$\begin{aligned} \lim_{x \to 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x \to 2} (x+2) \\ &= 2 + 2 = 4 \end{aligned}$$

Corollary 3

Let $P(x) = a_m x^m + \cdots + a_0$ and $Q(x) = b_n x^n + \cdots + b_0$ be the polynomials of degree m and n. respectively, so that $a_m \neq 0$ and $b_n \neq 0$. Then

$$\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)}$$

1. equals zero if $m < n$
2. equals $\dfrac{a_m}{b_n}$ if $m = n$,
3. does not exist if $m > n$. Or equivalently, the limit is $+\infty$ or $-\infty$

Example: Evaluate the limit $\displaystyle\lim_{x \to \infty} \frac{x^2 + 2x + 1}{2x^2 + x - 3} = \lim_{x \to \infty} \frac{x^2}{2x^2} = \frac{1}{2}$

Corollary 4

Suppose that we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

or

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

where a can be any values, positive or negative infinity. In case l'hôpital rule we have,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Indeterminate Form of Limit 1^∞

Let's suppose that $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$, $a \in \mathbb{R} \cup \{\pm\infty\}$,then we have that

$$\lim_{x\to a} f(x)^{g(x)} = 1^\infty$$

Remark

- $\lim\limits_{x\to\infty} \left(1 + \dfrac{1}{p(x)}\right)^{p(x)} = e$, where $\lim\limits_{x\to\infty} p(x) = \infty$
- $\lim\limits_{x\to 0} (1 + q(x))^{\frac{1}{q(x)}} = e$, where $\lim\limits_{x\to 0} q(x) = 0$

Indeterminate Form of Limit 1^∞

Corollary 5

Let's suppose that $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty, a \in \mathbb{R} \cup \{\pm\infty\}$. Then

$$\lim_{x\to a} f(x)^{g(x)} = e^{\alpha}$$

, where

$$\alpha = \lim_{x\to a} [g(x). \ln f(x)]$$

or

$$\alpha = \lim_{x\to a} [(f(x) - 1).g(x)]$$

Indeterminate Form of Limit 1^∞

Proof.

$$\text{We have} \quad \lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln(f(x))^{g(x)}} = \lim_{x\to a} e^{g(x)\ln(f(x))}$$

$$= e^{\lim_{x\to a} g(x)\ln f(x)} = e^{\alpha}$$

$$\Longrightarrow \alpha = \lim_{x\to a} g(x)\ln f(x)$$

$$\text{Moreover,} \quad \lim_{x\to a} g(x)\ln f(x) = \lim_{x\to a} g(x)\ln[1+(f(x)-1)]$$

$$= \lim_{x\to a} g(x).\frac{\ln[1+(f(x)-1)]}{f(x)-1}.[f(x)-1]$$

$$\Longrightarrow \lim_{x\to a} g(x)\ln f(x) = \lim_{x\to a} g(x)\,(f(x)-1)\ (\text{because } \lim_{x\to a}\frac{\ln[f(x)-1]}{f(x)-1} = 1)$$

$$\text{Then} \quad \alpha = \lim_{x\to a} g(x)\,(f(x)-1)$$

□

Example: Evaluate the limit $\lim_{x\to\infty}\left(\frac{x^2+2x+2}{x^2+3}\right)^x$ (1^∞ form)

Solution

$$\text{We have } \lim_{x\to\infty}\left(\frac{x^2+2x+2}{x^2+3}\right)^x = e^\alpha$$

$$\text{where} \quad \alpha = \lim_{x\to\infty} x\left(\frac{x^2+2x+2}{x^2+3}-1\right)$$

$$= \lim_{x\to\infty} x\left(\frac{2x-1}{x^2+3}\right)$$

$$\Longrightarrow \alpha = \lim_{x\to\infty}\frac{2x^2}{x^2} = 2$$

$$\text{Therefore } \lim_{x\to\infty}\left(\frac{x^2+2x+2}{x^2+3}\right)^x = e^2 \quad \square$$

Other Indeterminate Forms of Limit

Let c be any value such that $c \in \mathbb{R} \cup \{\pm\infty\}$. For other indeterminate forms of limits, we can transform them into the forms $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ and then apply L'Hôpital's rule for the following cases:

Form	Conditions	Transform to $\dfrac{0}{0}$	Transform to $\dfrac{\infty}{\infty}$
$0 \cdot \infty$	$\lim\limits_{x\to c} f(x) = 0$, $\lim\limits_{x\to c} g(x) = \infty$	$\lim\limits_{x\to c} f(x)\cdot g(x) = \lim\limits_{x\to c} \dfrac{f(x)}{1/g(x)}$	$= \lim\limits_{x\to c} \dfrac{g(x)}{1/f(x)}$
$\infty - \infty$	$\lim\limits_{x\to c} f(x) = \infty$, $\lim\limits_{x\to c} g(x) = \infty$	$\lim\limits_{x\to c} (f(x) - g(x)) = \lim\limits_{x\to c} \dfrac{1/g(x) - 1/f(x)}{1/(f(x)g(x))}$	$= \lim\limits_{x\to c} \ln \dfrac{e^{f(x)}}{e^{g(x)}}$
0^0	$\lim\limits_{x\to c} f(x) = 0^+$, $\lim\limits_{x\to c} g(x) = 0$	$\lim\limits_{x\to c} f(x)^{g(x)} = \exp \lim\limits_{x\to c} \dfrac{g(x)}{1/\ln(f(x)}$	$= exp \lim\limits_{xtoc} \dfrac{\ln f(x)}{1/g(x)}$
∞^0	$\lim\limits_{x\to c} f(x) = \infty$, $\lim\limits_{x\to c} g(x) = 0$	$\lim\limits_{x\to c} f(x)^{g(x)} = \exp \lim\limits_{x\to c} \dfrac{g(x)}{1/\ln(f(x))}$	$=\exp \lim\limits_{x\to c} \dfrac{\ln f(x)}{1/g(x)}$

Continuity

Definition 8

- Let $a \in I$ and $f : I \to \mathbb{R}$. We say that f is continuous at a iff

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, (|x - a| < \delta \implies |f(x) - f(a)| < \epsilon).$$

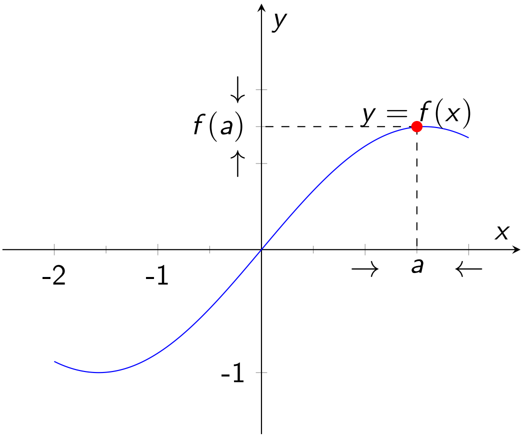
- f is said to be continuous on I if it is continuous at every point in I.

Theorem 6

Let $a \in I$ and $f : I \to \mathbb{R}$. Then f is continuous at a if and only if

$$\lim_{x \to a} = f(a)$$

Continuity-continue



- $f(a)$ is defined (a in the domain of f)
- $\lim_{x\to a} f(x)$ exists.(means that $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$)
- $\lim_{x\to a} f(x) = f(a)$

Theorem 7

If $f, g : I \to \mathbb{R}$, are continuous at $x = a$ and c is a constant, then the following functions are also continuous at a.

- $|f|$
- $f \pm g$
- cf
- fg
- $\dfrac{f}{g}, g(a) \neq 0$

Remark: The following functions are always continuous at every number in their domains.

- Polynomial functions
- Rational functions
- Power and root functions
- Trigonometric functions

Theorem 8

If g is continuous at a and f is continuous at $g(a)$, then $(f \circ g)(x) = f(g(x))$ is continuous at a.

Theorem 9 (Heine's Theorem)

If f is continuous on $[a, b]$ then f is uniformly continuous on $[a, b]$

Theorem 10

Suppose that $f(x)$ is continuous on $[a, b]$ and y_0 is any number between $f(a)$ and $f(b)$. Then, there is at least one number $c \in [a, b]$ for which $f(c) = y_0$

Corollary 6

Suppose that $f(x)$ is continuous on $[a, b]$ with $f(a)$ and $f(b)$ have the opposite sign.[i.e., $f(a)f(b) < 0$]. Then, there is at least one number $c \in (a, b)$ for

$$f(c) = 0$$

Continuity-IMV