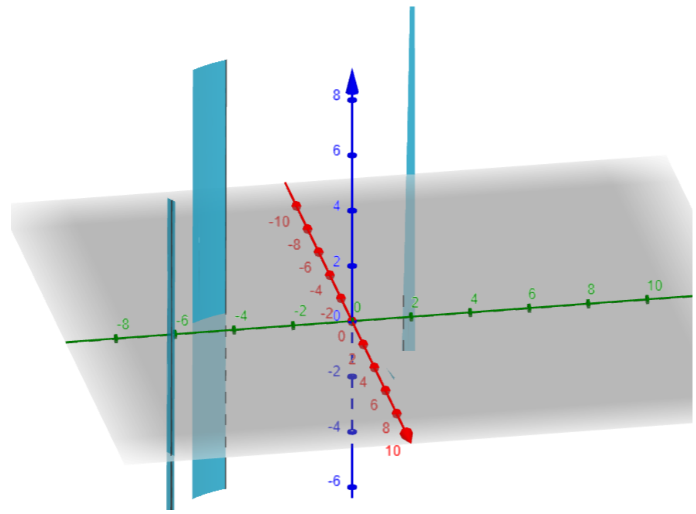
2). $z = 2y^3 - x^3 + 147x - 54y + 12$

Let $z = f(x, y) = 2y^3 - x^3 + 147x - 54y + 12$

$f_x(x, y) = \dfrac{\partial f(x, y)}{\partial x} = -3x^2 + 147x$ and

$f_y(x, y) = \dfrac{\partial f(x, y)}{\partial y} = 6y^2 - 54$



set $f_x(x, y) = 0 \ \Leftrightarrow \ -3x^2 + 147x = 0 \ \Rightarrow x = 0 \, , x = 49$

then set $f_y(x, y) = 0 \ \Leftrightarrow 6y^2 - 54 = 0 \ \Rightarrow y = \pm 3$

we obtain $a_1(x, y) = a_1 \ (0, -3)$and $a_2(49,3)$

find Hessian Matrix $H_f(x, y) = \begin{pmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{pmatrix}$

where $f_{xx}(x, y) = \dfrac{\partial^2 f(x, y)}{\partial x^2} = -6 \, , f_{yy}(x, y) = 12$

$f_{xy}(x, y) = f_{yx}(x, y) = 0$

then we get $H_f(x, y) = \begin{pmatrix} -6 & 0 \\ 0 & 12 \end{pmatrix}$

Since $D_1 = |-6| = -6 < 0$ (determinant) and $D_2 = \begin{vmatrix} -6 & 0 \\ 0 & 12 \end{vmatrix} = -12 < 0$

Thus $a_1(0, -3)$and $a_2(49,3)$ is not neither the positive definite nor negative definite

Hence , $a_1(0, -3)$ and $a_2(49,3)$ is the saddle point

Remark : if D_i are positive for all $i = 1,2,3$ ....

Thus H_f (Hessian matrix) is the positive definite , that is function f(x,y) has maximum at (x_0, y_0)

If $D_i = (-1)^k, k = 1,2, ..$ mean D1 negative than D2 positive , D3 negative ,… then H_f is negative definite that is function f(x,y) has minimum at (x_0, y_0)

None of all above the function f(x,y) is saddle point

5). $u = x^{0.5}y^{0.3}$ under the condition $10x + 3y = 140$

Let $u \leftrightarrow z$

$\Rightarrow z = x^{0.5}y^{0.3}$ with constrain $10x + 3y = 140$

Using Lagrange multiplier

we obtain $L(\lambda, x, y) = f(x, y) - \lambda(g(x) - c)$, where

$f(x, y) = x^{0.5}y^{0.3}$ and $g(x) = 10x + 3y$ and $c = 140$

We obtain $L(\lambda, x, y) = x^{0.5}y^{0.3} - \lambda(10x + 3y - 140)$

Consider $\nabla L(\lambda, x, y) = \left(\frac{\partial L(\lambda, x, y)}{\partial \lambda}, \frac{\partial L(\lambda, x, y)}{\partial x}, \frac{\partial L(\lambda, x, y)}{\partial y}\right)$

$$\text{set } \nabla L(\lambda, x, y) = 0 \Leftrightarrow \begin{cases} \frac{\partial L(\lambda, x, y)}{\partial \lambda} \\ \frac{\partial L(\lambda, x, y)}{\partial x} \\ \frac{\partial L(\lambda, x, y)}{\partial y} \end{cases} \Rightarrow \begin{cases} -(10x + 3y - 140) = 0 & (1) \\ 0.5x^{-0.5}y^{0.3} - 10\lambda = 0 & (2) \\ 0.3x^{0.5}y^{-0.7} - 3\lambda = 0 & (3) \end{cases}$$

from (2)and (3):

$$\begin{cases} 0.5x^{-0.5}y^{0.3} - 10\lambda = 0 \ (\times 3) \\ 0.3x^{0.5}y^{-0.7} - 3\lambda = 0 \ \ (\times 10) \end{cases} \Leftrightarrow \begin{cases} 1.5x^{-0.5}y^{0.3} - 30\lambda = 0 \\ 3x^{0.5}\,y^{-0.7} - 30\lambda = 0 \end{cases} (-) \Leftrightarrow 1.5x^{-0.5}y^{0.3} - 3x^{0.5}y^{-0.7} = 0$$

$$\Leftrightarrow \frac{1.5y^{0.3}}{x^{0.5}} - \frac{3x^{0.5}}{y^{0.7}} = 0 \Leftrightarrow 1.5y - 3x = 0 \ \ (4)$$

from (1) and (4):

$$\begin{cases} 10x + 3y = 140 \\ -3x + 1.5y = 0 \quad (\times 2) \end{cases} \Rightarrow \begin{cases} 10x + 3y = 140 \\ -6x + 3y = 0 \end{cases} (-) \Rightarrow 16x = 140 \Rightarrow x = \frac{140}{16} = \frac{70}{8} = \frac{35}{4}$$

$$\text{then from (4)}: -3x + 1.5y = 0 \Leftrightarrow y = \frac{3}{1.5}x = \frac{3}{1.5} \times \frac{35}{4} = \frac{175}{6}$$

$$\text{from (3)}: 0.3x^{0.5}y^{-0.7} - 3\lambda = 0 \Rightarrow \lambda = \frac{0.3x^{0.5}}{3y^{0.7}} = \frac{0.1\sqrt{x}}{y^{0.7}} = \frac{0.1\sqrt{\frac{35}{4}}}{\left(\frac{175}{6}\right)^{0.7}} = \frac{0.29}{10.6} = 0.027$$

so $a(x, y) = a\left(\frac{35}{4}, \frac{175}{6}\right)$ with $\lambda = 0.027$

$$\text{The Hessian matrix } H_f(\lambda, x, y) = \begin{pmatrix} \frac{\partial^2 L}{\partial \lambda^2} & \frac{\partial^2 L}{\partial \lambda\, \partial x} & \frac{\partial^2 L}{\partial \lambda\, \partial y} \\ \frac{\partial^2 L}{\partial x\, \partial \lambda} & \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x\, \partial y} \\ \frac{\partial^2 L}{\partial y\, \partial \lambda} & \frac{\partial^2 L}{\partial y\, \partial x} & \frac{\partial^2 L}{\partial y^2} \end{pmatrix}$$

we have $L(\lambda, x, y) = x^{0.5}y^{0.3} - \lambda(10x + 3y - 140)$

at the point $a\left(\frac{35}{4}, \frac{175}{6}\right)$ with $\lambda = 0.027$

then $\frac{\partial^2 L}{\partial \lambda^2} = 0$, $\quad \frac{\partial^2 L}{\partial \lambda\, \partial x} = \frac{\partial^2 L}{\partial x\, \partial \lambda} = -10, \quad \frac{\partial^2 L}{\partial \lambda\, \partial y} = \frac{\partial^2 L}{\partial y\, \partial \lambda} = -3,$

$$\frac{\partial^2 L}{\partial x\, \partial y} = \frac{\partial^2 L}{\partial y\, \partial x} = \frac{0.15}{x^{0.5} y^{0.7}} = \frac{0.15}{\left(\frac{35}{4}\right)^{0.5} \left(\frac{175}{6}\right)^{0.7}} = \frac{0.15}{2.95 \times 10.6} = 0.0047$$

$$\frac{\partial^2 L}{\partial x^2} = -0.25 x^{-1.5} y^{0.3} = -0.25 \left(\frac{35}{4}\right)^{-1.5} \left(\frac{175}{6}\right)^{0.3} = -\frac{0.68}{25.88} = -0.026\ ,$$

$$\frac{\partial^2 L}{\partial y^2} = -0.21 x^{0.5} y^{-1.7} = -\frac{0.62}{309.24} = -0.002$$

so $H_f(\lambda, x, y) = \begin{pmatrix} 0 & -10 & -3 \\ -10 & -0.026 & 0.0047 \\ -3 & 0.0047 & -0.026 \end{pmatrix}$

Conside the sequence (s): $(-1)^k d_{2k+1}, (-1)^k d_{2k+2}, \dots, (-1)^k d_{k+n}$

we have $k = 1$ (number of constrian), $n = 2$ (number of variable)

so (s): $-d_3, \dots, -d_3$, hence we just compute $-d_3$, where (d_3 means determinant of order 3)

we get $-d_3 = \begin{vmatrix} 0 & -10 & -3 \\ -10 & -0.026 & 0.0047 \\ -3 & 0.0047 & -0.026 \end{vmatrix} = 3.116 > 0$

Thus $a\left(\frac{35}{4}, \frac{175}{6}\right)$ is a minimum point of $f(x, y)$ subject to the constrain

$g(x, y) = 10x + 3y = 140$

Remark: we have (s): $(-1)^k d_{2k+1}, (-1)^k d_{2k+2}, \dots, (-1)^k d_{k+n}$

- If the sequence (s) consists entirely of positive numbers, then f(x,y) is a local minimum subject to $g_i(x, y) = c_i, i = 1,2, \dots k$
- If the sequence (s) begin with a negative number and thereafter alternatives in sign, then f(x,y) is a local maximum subject to constrain $g_i(x, y) = c_i, i = 1,2, \dots, k$
- If neither case 1 nor 2 above , then f is the saddle point at a(x,y)
- If $\det(H_f) = 0$, then the test is fail