Discrete Random Variable and Probability Distribution

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0.1 Random variable

Let S be a sample space of random experiment. A function $X : S \longrightarrow \mathbb{R}$ is called a random variable (rv).
$D = \{x = X(s) : s \in S\}$ is called the range of random variable X. It is the set of all possible value of X.

- If D is countable set then X is called discrete random variable(drv).
- If D is uncountable set then X is called continuous random variable (crv).

0.2 Probability Distribution or Probability mass function

Definition1: The probability distribution or probability mass function(pmf) of a discrete random variable is denoted $p_X(x)$ and defined by

$$p_X(x) = P(X = x) = P\left(s \in S : X(s) = x\right) \tag{1}$$

Properties
The pmf $p_X(x)$ must satisfy the following properties:

- $p_X(x) \geq 0, \forall x \in D$
- $\sum\limits_{x \in D} p_X(x) = 1$

0.3 The Cumulative Distribution Function

Definition2: For a drv X, the cumulative distribution function (cdf) denoted by F_X , is defined by

$$F_X(x) = P(X \leq x) = \sum_{y \leq x} p_x(y) = \sum_{y \leq x} P(X \leq y) \tag{2}$$

Theorem1: Let $F_X(x)$ be a cdf of drv . Then,

1. If $a < b$, then $F_X(a) \leq F_X(b)$, (F is a non-decreasing function).
2. $\lim\limits_{x \to -\infty} (F_X(x)) = 0$, (the lower limit of F_X is 0).
3. $\lim\limits_{x \to +\infty} (F_X(x)) = 1$, (the upper limit of F_X is 1).
4. $\lim\limits_{x \to x_0^+} (F_X(x)) = F_X(x_0)$, ($F_X$ is right continuous).

Theorem2: Let F_X be a cdf of drv X . Then,

1. $P(a < X \leq b) = F_X(b) - F_X(a)$.
2. $P(X = x) = F_X(x) - F_X(x^-)$, where $F_X(x^-) = \lim_{z \to x^-}(F_X(z))$

0.4 Expected Value

Definition3: The expected value or mean value of X denoted by $E(x)$ or μ_x or μ is

$$E(X) = \sum_{x \in D} x.p_X(x) \tag{3}$$

Theorem 3: The expected value of any function $h(X)$, denoted by $E[h(X)]$, is given by

$$[Eh(X)] = \sum_{x \in D} (h(x).p_X(x))$$

. In particular, if a and b are constants

1. $E(b) = b$
2. $E(aX + b) = aE(X) + b$

0.5 The Variance of X

Definition 4
The variance of drv X, denoted by $V(X)$ or σ_X^2, or just σ^2, is defined by

$$V(X) = E[(X - \mu)^2] = \sum_{x \in D} (x - \mu)^2.p_X(x) \tag{4}$$

The standard deviation (SD) of X is

$$\sigma_X = \sqrt{V(X)} \tag{5}$$

Theorem 4

1. $V(X) = E(X^2) - [E(X)]^2$
2. $V(aX + b) = a^2V(X)$
3. $\sigma_{aX+b} = |a|\sigma_X$.

0.6 The Moment-Generating Function of X

Definition5
The moment-generating function (mgf) of X, denoted by $M(t)$, is defined by

$$M(t) = E(e^{tx}) \tag{6}$$

Theorem 5

1. $M^{(n)}(t) = E[X^n e^{tX}]$
2. $E[X] = M'(0)$
3. $V(X) = M''(0) - [M'(0)]^2$

0.7 The Binomial Probability Distribution

Definition 6
A binomial experiment is an experiment satisfying the following properties:

1. It consists of a sequence of n smaller experiments called **trials** , where n is a (non-random) constant.
2. Each trails can result in one of the same two possible outcomes (dichotomous trails), which we generically denoted by success and the failure.
3. The trials are independent
4. The probability of success is constant from trial to trial, we denote this probability by$p$.

Note that when$n = 1$. it is called Bernoulli experiment.
Definition 7:
The binomial random variable X associated with a binomial experiment consisting of n trials is defined by

$$X = \text{the number of success among the n trials}$$

we write $X \sim Bin(n, p)$ to indicate that X is a binomial rv based on n trials with success probability p.
Theorem 6
if $X \sim Bin(n, p)$, then the pmf of the binomial rv X is given by

$$p_X(x) = \begin{cases} C_n^x \left(p^x(1-p)^{n-x}\right) & , x = 0, 1, 2, .., n \\ 0 & , \text{otherwise} \end{cases}$$

Theorem 7:If $X \sim Bin(n, p)$, then

1. $E(x) = np$
2. $V(X) = npq$
3. $\sigma_X = \sqrt{npq}$
4. $M(t) = (q + pe^t)^n$

where $q = 1 - p$.

0.8 The Hyper geometric Distribution

Definition: The hyper-geometric experiment is an experiment satisfying the following properties:

1.

2. The population or set to be sampled consists of N individuals, objects, or elements (a finite population).

3. Each individual can be characterized as a success (S) or a failure (F), and there are M successes in the population.

4. A sample of n individuals is selected without replacement in such a way that each subject of size n is equally likely to be chosen.

Definition 9: The hyper-geometric random variable X associated with hyper-geometric experiment is defined by

$$X = \text{the number of success in the sample.}$$

$X \sim Hp(n, M, N)$ denotes the parametric rv of sample size n drawn from a population of size N consisting of M success.
Theorem 8
If $X \in Hp(n, M, N)$, then the pmf of the hyper-geometric rv X is given by

$$p_X(x) = \frac{C_M^x . C_{N-M}^{n-x}}{C_N^n} \tag{7}$$

for all integers x satisfying $\max(0, n - N + M) \leq x \leq \min(n, M)$.
Theorem 9: If $X \sim Hp(n, M, N)$, then

1. $E(X) = \dfrac{n.M}{N}$

2. $V(X) = \left(\dfrac{N-n}{N-1}\right) n \dfrac{M}{N}\left(1 - \dfrac{M}{N}\right)$.

. The relationship between Hyper-geometric and Binomial Distributions

A binomial distribution can be used to approximate the hyper-geometric distribution when n is small compared to N. In fact, as a rule thumb, the approximation is good when $\dfrac{n}{N} \leq 0.05$.

0.9 The Negative Binomial Distribution

Definition 10 : The negative binomial experiment is an experiment satisfying the following conditions:

1. The experiment consists of a sequence of independent trials
2. Each trails can result in either a success (S) or a failure (F).
3. The probability of success in constant from trail to trial , so $P(S \text{ on trial } i) = p$ for $i = 1, 2, 3, ..$
4. The experiment continues (trials are performed) unit a total of r success have been observed , where r is a specified positive integers.

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Definition 11: The negative binomial rv X associated with the negative binomial experiment is defined by

$$X = \text{ the number of failures that precede the rth success}$$

$X \sim Nb(r, p)$ denotes the negative binomial rv X with parameters $r = \text{the number of success and and} P(\text{success}) = p$.

Theorem 10

If $X \sim Nb(r, p)$, then the pmf of the negative binomial rv X is given by

$$P_X(x) = C_{x+r-1}^{r-1} p^r (1-p)^x \tag{8}$$

where $x = 0, 1, 2...$

Remark: if $r = 1$ then X is a geometric random variable with pmf

$$p_X(x) = p(1-p)^x, x = 0, 1, 2, ... \tag{9}$$

Theorem11 If $X \sim Nb(r, p)$, then

1. $E(X) = \dfrac{r.(1-p)}{p}$
2. $V(X) = \dfrac{r(1-p)}{p^2}$
3. $M(t) = \left(\dfrac{p}{1 - e^t + pe^t}\right)^r$

0.10 Poisson Probability Distribution

Definition 12

A drv X is called Poisson distribution or Poisson rv with parameter $\lambda(\lambda > 0)$ if the pmf of X is

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \tag{10}$$

$x = 0, 1, 2...$

We write $X \sim Po(\lambda)$.

Theorem 12: Suppose $X \sim Bin(n, p)$ When $n \to \infty, p \to 0$,and $np \to \lambda$ remains constant , then $X \sim Po(\lambda)$

Theorem 13 : If $X \sim Po(\lambda)$, Then

- $E(X) = \lambda$
- $V(x) = \lambda$
- $M(t) = e^{(\lambda)(e^t - 1)}$.