Continuous Random Variable

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Summarise

1. Probability distribution or probability density function (pdf):
   $P(a \le X \le b) = \int_a^b f(x)dx$

* Property

- $f(x) \ge 0$ for all x
- $\int_{-\infty}^{\infty} f(x)dx = 1$

2. Cumulative distribution function (cdf) : $F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)dy$

- $P(X > a) = 1 - F(a)$
- $P(a \le X \le b) = F(b) - F(a)$

3. The $(100p)th$ percentile denoted $\eta(p)$ defined by
   $p = F(\eta(p)) = \int_{-\infty}^{\eta(p)} f(y)dy, 0 \le p \le 1$

4. The median $\tilde{\mu}$ is the $50th$ percentile ,satisfies $\tilde{\mu} = 0.5$. That is $F(\tilde{\mu}) = \int_{-\infty}^{\tilde{\mu}} f(y)dy = \frac{1}{2}$

5. .

- The expected or mean value of X is $\mu_X = E(x) = \int_{-\infty}^{\infty} xf(x)dx$
- The variance of X is $V(X) = E[(X-\mu)^2] = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)dx$
- The standard deviation of X is $\sigma_X = \sqrt{V(X)}$
- The moment-generating function(mgf) of a continuous rv
  X :$M(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f(x)dx$
- $h(X)$ is any function of X ,then $E[h(X)] = \int_{-\infty}^{\infty} h(x).f(x)dx$

* Property : let X is a crv . Then

- $E(aX+b) = aE(X) + b$
- $V(x) = E(X^2) - E^2(X)$
- $V(aX+b) = a^2V(X)$
- $\sigma_{aX+b} = |a|\sigma_X$
- $M^{(n)}(t) = E[X^n e^{tX}]$

- $E(X) = M'(0)$
- $V(X) = M''(0) - [M'(0)]'$

6. The uniform distribution on interval$[a, b]$ denoted by
$X \sim U(a, b) : f(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x \leq b, \\ 0, & otherwise \end{cases}$
For $X \sim U(a, b)$, then
   - $E(X) = \dfrac{a+b}{2}$
   - $V(x) = \dfrac{(b-a)^2}{12}$
   - $M(t) = \begin{cases} \dfrac{e^{tb} - e^{ta}}{t(b-a)}, & t \neq 0 \\ 1, & t = 0 \end{cases}$
7. The normal distribution with parameter μ and σ, where $-\infty < \mu < \infty$ and $0 < \sigma$ denoted by $X \sim N(\mu, \sigma^2)$, then $f(x) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}, -\infty < x < \infty$
8. $X \sim N(\mu, \sigma^2)$,then $E(X) = \mu$, $V(X) = \sigma^2$ and $M(t) = exp(\mu t + \dfrac{\sigma^2 t^2}{2})$
9. Standard normal rv denoted by Z. Then pdf $Z \sim N(0, 1)$ is $f(z) = \dfrac{1}{\sqrt{2\pi}} e^{-z^2/2}$, $-\infty < z < \infty$. The graph of $f(z)$ is called the standard normal (or z) curve. It's inflection are at 1 and -1.
10. The cdf of Z is $\Phi(Z) = P(Z \leq z) = \int_{-\infty}^{z} f(y)dy$
11. $X \sim N(\mu, \sigma^2)$. If $Z = \dfrac{X-\mu}{\sigma}$ then $Z \sim N(0, 1)$.
Thus
$$\begin{aligned} P(a \leq X \leq b) &= P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \\ P(X \leq a) &= \Phi\left(\frac{a-\mu}{\sigma}\right) \\ P(X \geq b) &= 1 - \Phi\left(\frac{b-\mu}{\sigma}\right) \end{aligned}$$
12. Approximating the Binomial Distribution , Let X be a binomial rv based on n trials with p success. X has approximately a normal distribution with $\mu = np$ and $\sigma = \sqrt{npq}$. In particular , for $x = a$ possible values of X.
$P(X \leq x) = B(x; n, p) \approx$(area of the normal curve to the left of x+0.5)$=\Phi(\dfrac{x + 0.5 - np}{\sqrt{npq}})$
. In practice, the approximation is adequate that both $np > 10$ and $nq > 10$.

13. The exponential distribution denoted by $X \sim Exp(\lambda)$, $\lambda > 0$, then pdf of X:
$f(x) = \begin{cases} \frac{1}{\lambda}e^{\frac{-x}{\lambda}} & , x \geq 0 \\ 0 & , otherwise \end{cases}$

- $X \sim Exp(\lambda)$,then
  - $F(X) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-x/\lambda} & , x \geq 0. \end{cases}$
  - $E(X) = \lambda$, $V(X) = \lambda^2$ and $M(t) = \frac{1}{1-\lambda t}, t < \frac{1}{\lambda}$

14. The gamma Function and incomplete gamma function are defined respectively by $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt,$
$\alpha > 0$ and $\Gamma(x, \alpha) = \int_0^{x} t^{\alpha-1}e^{-t}dt, \alpha > 0, x > 0.$

15. $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1), \alpha > 1$
$\Gamma(n) = (n - 1)!, n \in \mathbb{N}$
$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

16. The Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ denoted by $X \sim Gam(\alpha, \beta)$if the pdf of X is $f(x) = \begin{cases} \frac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}e^{-\frac{x}{\beta}}, x \geq 0 \\ 0, otherwise \end{cases}$,
if$\beta = 1$ we call standard gamma distribution.

17. $X \sim Gam(\alpha, \beta)$, then $E(X) = \alpha\beta$, $V(X) = \alpha\beta^2$and $M(t) = \frac{1}{(1-\beta t)^{\alpha}}, \alpha < \frac{1}{\beta}$.

18. $X \sim Gam(\alpha, \beta)$. Then for any $X > 0$ the cdf of X is given by
$P(X \leq x) = F(x) = \Gamma(\frac{x}{\beta}, \alpha)$.

19. The chi-square distribution with parameters ν denoted by $X \sim \chi^2(\nu)$, if the pdf X is the gamma density with $\alpha = \frac{\nu}{2}$and $\beta = 2$, That is $f(x) = \begin{cases} \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}x^{\frac{\nu}{2}-1}e^{-\frac{x}{2}}, x \geq 0, \\ 0, x < 0. \end{cases}$
The parameter ν is called the number of degree of freedom of X.

20. $X \sim \chi^2(\nu)$, then $E(X) = \nu$, $V(X) = 2\nu$ and $M(t) = (1 - 2t)^{-\frac{\nu}{2}}, t < \frac{1}{2}$

21. If the random variable $X \sim N(\mu, \sigma^2), \sigma^2 > 0$, then the random variable
$V = \frac{(X-\mu)^2}{\sigma^2} = Z^2 \sim \chi^2(1)$

22. The beta function defined by $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \alpha, \beta > 0.$
where Γ is the gamma function.

23. if $X \sim Bet(\alpha, \beta)$with parameters $\alpha > 0$ and $\beta > 0$. if its density function is given by
$f(x) = \begin{cases} \dfrac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1 \\ 0, otherwise \end{cases}$

24. If $X \sim B(\alpha, \beta)$, then $E(X) = \dfrac{\alpha}{\alpha+\beta}$ and $V(X) = \dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

25. X has a log-normal distribution if rv $Y = \ln(X)$ has a normal distribution with mean μ and standard deviation σ. we denote $X \sim Log(\mu, \sigma)$ then
$f(x; \mu, \sigma) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{1}{2\sigma^2}[\ln(x)-\mu^2]}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

26. $X \sim Log(\mu, \sigma)$, then $E(X) = e^{\mu+\sigma^2/2}$and $V(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$

27. X has a Weibull distribution with parameter α and β ,if its density function is given by
$f(x; \alpha, \beta) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}, x > 0, \alpha > 0, \beta > 0 \\ 0, otherwise \end{cases}$ we write $X \sim Wei(\alpha, \beta)$

28. If $X \sim Wei(\alpha, \beta)$ then $E(X) = \alpha^{-\frac{1}{\beta}}\Gamma(1 + \frac{1}{\beta})$, $V(X) = \alpha^{-\frac{2}{\beta}}\left\{\Gamma(a + \frac{2}{\beta}) - \left[\Gamma(1 + \frac{1}{\beta})\right]^2\right\}$