CHA.4 AND 5
HYPOTHESES TESTING AND INFERENCE BASED ON TWO SAMPLE

- The hypotheses to be test is called null hypotheses H_0 and negation is called alternative hypotheses H_a
- A hypotheses is said to be a sample hypotheses when H completely specifies the density of the population $(i.e. \theta = 2,..)$. Otherwise is called composite hypotheses$(i.e. \theta < 2, ...)$

➢ Method of finding test

❖ The likelihood ratio test statistic denoted with (W)

and (C) refers to critical region or just rejection region is the set store the value for reject the null hypotheses H_0 . It is defined by

$$W(x_1, x_2, \dots, x_n) = \frac{\max_{\theta \in \Omega_0} L(\theta, x_1, \dots, x_n)}{\max_{\theta \in \Omega_a} L(\theta, x_1, \dots, x_n)} \; or \; W = \frac{L(\theta_0)}{L(\theta_a)}$$

then $C = \{(x_1, x_2, \dots, x_n) | W(\theta, x_1, x_2, .., x_n) \leq k \}, k \in [0,1]$

❖ We have two type of errors are

1. Reject H_0 when H_0 $is\ true$. It is called type I error

2. Accept H_0 $when$ H_0 is false . It is called type 2 error

❖ The significance level α is the value of probability of type I error

❖ β is the value of probability of type II error

❖ The function or power function is denoted by π and defined by

$$\pi(\theta) = P(type\ I\ error) or\ \pi(\theta) = 1 - P(type\ 2\ error)$$

- The uniformly most powerful (UMP)

if T is UMP than w of the same test of level δ $if\ : \pi_T(\theta) \geq \pi_W(\theta)$

- Neyman-Pearson Lemma is the set of critical region c defined by

$$C = \{(x_1, .., x_n) | \frac{L(\theta_0)}{L(\theta_a)} \leq k\}, where\ L\ is\ the\ likelihood\ function$$

- If $X_1, \ldots, X_n$ $is\ random\ sample\ from\ normal\ distribution\ that\ has\ parameter\ \mu\ and\ \sigma^2$ are known and n <30

- The z test for population mean is defined as the test statistic value is $z = \frac{\bar{x}-\mu_0}{\sigma_0/\sqrt{n}}$

- Three case of hypotheses test when $H_0: \mu = \mu_0\ and\ if$

1. $H_a: \mu > \mu_a\ then\ critical\ region\ C\ or\ test\ is\ C = \{z:\ z = \frac{\bar{x}-\mu_0}{\frac{\sigma_0}{\sqrt{n}}} \geq z_\alpha\}$
2. $H_a: \mu < \mu_a\ then\ C = \{z:\ z \leq -z_\alpha\}$
3. $H_a: \mu \neq \mu_a$then C={z: $|z| \geq z_{\frac{\alpha}{2}}$}

- For large sample $n \geq 30$ and it is from a distribution with know μ , then

$z = \frac{\bar{x}-\mu}{S/\sqrt{n}}, where\ is\ the$ standard deviation of the sample

- We use T-test when it is a normal distribution with unknown parameter μ and σ^2 also n <30

when $H_0: \mu = \mu_0$ $then$ test statistic value for T-test : $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$

- the critical region (C) or rejection region (RR)

1. when $H_a: \mu > \mu_0$, $then\ C = \{t: t \geq t_{\alpha,n-1}\}$
2. when $H_a: \mu < \mu_0$ $then\ C = \{t : t \leq -t_{\alpha,n-1}\}$
3. When $H_a: \mu \neq \mu_0$ then $\mathrm{C} = \{\mathrm{t}: |\mathrm{t}| \geq \mathrm{t}_{\frac{\alpha}{2},n-1}\}$

- Without testing hypotheses by using critical region , we can also use P-value for testing the hypotheses for z test

P-value $= 1 - \phi(z)$ for upper-tailed or right-tailed test $(H_a: \mu_a > \mu_0)$

$= \phi(z)$ for lower-tailed or left –tailed test $(H_a: \mu_a < \mu_0)$

$= 2[1 - \phi(|z|)]$ $for\ two - tailed\ test$ $(H_a: \mu_a \neq \mu_0)$

Decision for P-value

If P-value $\leq \alpha$, then reject H_0

If P-value $> \alpha$ not reject H_0

❑ β *and sample size*

Recall: β *is probability of type II error*

❑ To find the power π , *we have* $\pi(\mu') = 1 - \beta(\mu')$, *where*

1. If $\mathrm{H_a}: \mu > \mu_0$ *then* $\beta(\mu') = \phi\left(z_\alpha + \frac{\mu_0 - \mu'}{\frac{\sigma}{\sqrt{n}}}\right)$

2. If $H_a: \mu < \mu_0$ *then* $\beta(\mu') = 1 - \phi\left(-z_\alpha + \frac{\mu_0 - \mu'}{\frac{\sigma}{\sqrt{n}}}\right)$

3. If $H_a: \mu \neq \mu_0$then $\beta(\mu') = \phi\left(z_{\frac{\alpha}{2}} + \frac{\mu_0 - \mu'}{\frac{\sigma}{\sqrt{n}}}\right) - \phi\left(-z_{\frac{\alpha}{2}} + \frac{\mu_0 - \mu'}{\frac{\sigma}{\sqrt{n}}}\right)$

❑ Size determinant

1. For one tailed test $n = \left[\frac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu\prime}\right]^2$

2. For two-tailed test $n = \left[\frac{\sigma\left(z_{\frac{\alpha}{2}} + z_\beta\right)}{\mu_0 - \mu'}\right]^2$, n is rounded up to the whole number

- Test for a Variance and standard deviation

for random sample from normal distribution of size n with mean and variance are unknown

We use χ^2 test

$Null\ hypotheses\ H_0: \sigma^2 = \sigma_0^2$

And test statistic value $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$

- Determine the rejection region

1. If $H_a: \sigma^2 > \sigma_0^2\ then\ C = \{\chi^2: \chi^2 \geq \chi^2_{\alpha,n-1}\}$
2. If $H_a: \sigma^2 < \sigma_0^2\ then\ C = \{\chi^2: \chi^2 \leq \chi^2_{1-\alpha,n-1}\}$
3. If $H_a : \sigma^2 \neq \sigma_0^2\ then\ C = \left\{\chi^2: \chi^2 \leq \chi^2_{1-\frac{\alpha}{2},n-1}\ or\ \chi^2 \geq \chi^2_{\frac{\alpha}{2},n-1}\right\}$

❑ Test about a population proportion

❖ Assumption $x_1, \dots x_n \sim Bernulli(p)\ sample\ , x_i \in \{0,1\}$

then $X = \sum x_i \sim Ber(n,p)$, where $\hat{p} = \frac{x}{n}$ $(probability\ of\ success)$

We use z –test

- Null Hypotheses : $H_0: p = p_0$

- Test statistic value $z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$

- Critical region

1. For $H_a: p > p_0\ then\ C = \{z: z \geq z_\alpha\}$
2. For $H_a: p < p_0\ then\ C = \{z: z \leq -z_\alpha\}$
3. For $H_a: p \neq p_0\ then\ C = \left\{z: z \leq -z_{\frac{\alpha}{2}}\ or\ z \leq z_{\frac{\alpha}{2}}\right\} = \left\{z: |z| \geq z_{\frac{\alpha}{2}}\right\}$

Remark : test will provided that $np_0 \geq 5\ and\ n(1 - p_0) \geq 5$

Beta and sample size

we have alternative hypotheses $\beta(p'), where$

1. $$H_a: p > p_0 \ then \ \beta(p') = \phi\left[\frac{p_0 - p' + z_\alpha\sqrt{\frac{p_0(1-p_0)}{n}}}{\sqrt{\frac{p'(1-p')}{n}}}\right]$$

2. $$H_a: p < p_0 \ then \ \beta(p') = 1 - \phi\left[\frac{p_0 - p' - z_\alpha\sqrt{\frac{p_0(1-p_0)}{n}}}{\sqrt{\frac{p'(1-p')}{n}}}\right]$$

3. $$H_a: p \neq p_0 \ then \ \beta(p') = \phi\left[\frac{p_0 - p' + z_{\frac{\alpha}{2}}\sqrt{\frac{p_0(1-p_0)}{n}}}{\sqrt{\frac{p'(1-p')}{n}}}\right] - \phi\left[\frac{p_0 - p' - z_{\frac{\alpha}{2}}\sqrt{\frac{p_0(1-p_0)}{n}}}{\sqrt{\frac{p'(1-p')}{n}}}\right]$$

Sample size

For one tailed test: $n = \left[\frac{z_{\alpha}\sqrt{p_0(1-p_0)}+z_{\beta}\sqrt{p'(1-p')}}{p'-p_0}\right]^2$

For two tailed test : $n = \left[\frac{z_{\frac{\alpha}{2}}\sqrt{p_0(1-p_0)}+z_{\beta}\sqrt{p'(1-p')}}{p'-p_0}\right]^2$

CH.5

Test for normal distribution with known the variance

- The null hypotheses $H_0 : \Delta_0 = \mu_1 - \mu_2$
- We use z-test and test statistic value is $z = \frac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}$
- We have the critical region , when

1. $H_a : \mu_1 - \mu_2 > \Delta_0 \; then \; C = \{z : z \geq z_\alpha\}$
2. $H_a : \mu_1 - \mu_2 < \Delta_0 \; then \; C = \{z : z \leq -z_\alpha\}$
3. $H_a : \mu_1 - \mu_2 \neq \Delta_0 \; then \; C = \{z : |z| \geq z_{\frac{\alpha}{2}}\}$

Alternative hypotheses $\beta(\Delta')$

Recall $\beta(\Delta') = P(type\ II\ error), when\ \Delta' = \mu_1 - \mu_2$ is defined by

1. If $H_a: \mu_1 - \mu_2 > \Delta_0\ then\ \beta(\Delta') = \phi\left(z_\alpha - \frac{\Delta' - \Delta_0}{\sigma}\right)$

2. $If\ H_a: \mu_1 - \mu_2 < \Delta_0\ then\ \beta(\Delta') = 1 - \phi\left(-z_\alpha - \frac{\Delta' - \Delta_0}{\sigma}\right)$

3. $If\ H_a: \mu_1 - \mu_2 \neq \Delta_0\ then\ \beta(\Delta') = \phi\left(z_{\frac{\alpha}{2}} - \frac{\Delta' - \Delta_0}{\sigma}\right) - \phi\left(-z_{\frac{\alpha}{2}} - \frac{\Delta' - \Delta_0}{\sigma}\right)$

Where $\sigma = \sigma_{\bar{X}-\bar{Y}} = \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}$

Sample size

We use $\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}=\frac{(\Delta'-\Delta_0)^2}{(z_\alpha+z_\beta)^2}$ for one –tailed test , where m and n is the sample size

We use $\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}=\frac{(\Delta'-\Delta_0)^2}{\left(z_{\frac{\alpha}{2}}+z_\beta\right)^2}$ for two tailed test

Large Sample Tests

- A assumption normal distribution when n $\geq$ 30 and m ≥ 30,

from Central limit theorem

- Test statistic value $z = \frac{\bar{x}-\bar{y}-\Delta_0}{\sqrt{\frac{S_1^2}{m}+\frac{S_2^2}{n}}}$, where $\Delta_0 = \mu_1 - \mu_2$ (null hypotheses)

- Confidence Interval for $\mu_1 - \mu_2$

we get $CI = \left[\bar{x} - \bar{y} \pm z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}\right]$

For Large sample $CI = \left[\bar{x} - \bar{y} \pm z_{\frac{\alpha}{2}}\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}\right]$

T-test for two sample

When distribution are both normal and $\sigma_X^2 \neq \sigma_Y^2$, n and m < 30

- Null hypotheses $H_0\colon \Delta_0 = \mu_1 - \mu_2$

- Test statistic value $T = \frac{\bar{x}-\bar{y}-\Delta_0}{\sqrt{\frac{S_1^2}{m}+\frac{S_2^2}{n}}}$

- The critical region is defined by

1. $H_a\colon \mu_1 - \mu_2 > \Delta_0 \; then \; C = \{t\colon t \geq t_{\alpha,\nu}\}$
2. $H_a\colon \mu_1 - \mu_2 < \Delta_0 \; then \; C = \{t \leq -t_{\alpha}, \nu\}$
3. $H_a\colon \mu_1 - \mu_2 \neq \Delta_0 \; then \; C = \{t\colon |t| \geq t_{\frac{\alpha}{2},\nu}\}$

Where degree of freedom $\nu = \frac{\left(\frac{S_1^2}{m}+\frac{S_2^2}{n}\right)^2}{\frac{\left(\frac{S_1^2}{m}\right)^2}{m-1}+\frac{\left(\frac{S_2^2}{n}\right)^2}{n-1}}$ (rounded it to nearest integer)

- For $100(1-\alpha)\%$ confidence interval is defined by

$$CI = \left[\bar{x} - \bar{y} \pm t_{\frac{\alpha}{2},\nu}\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}\right]$$

- Pooled t procedures

- Alternative to the two sample t procedures assuming not only normal distributed and we don't know the variance for the population but we know that $\sigma_1^2 = \sigma_2^2$

- The standardized variable $T = \frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{S_p\sqrt{\frac{1}{m}+\frac{1}{n}}} \sim t$ distributed with df $\nu = m + n - 2$

Where , S_p *is called* the pooled estimator or standard deviation of σ and defined by

$$S_p = \sqrt{\frac{(m-1)s_1^2 + (n-1)s_2^2}{m+n-2}}$$

- Analysis for Paired Data

- Using one sample t-test

- Null Hypotheses $H_0: \mu_D = \Delta_0$

- D_i is the different d= $before - after = X_i - Y_i$

Test statistic value $t = \frac{\bar{d}-\Delta_0}{\frac{s_D}{\sqrt{n}}}$

- Alternative hypotheses and the critical region

1. When $H_a: \mu_D > \Delta_0$ $then$ $C = \{t: t \geq t_{\alpha,n-1}\}$
2. $H_a: \mu_D < \Delta_0$ $then$ $C = \{t: t \leq -t_{\alpha,n-1}\}$
3. $H_a: \mu_D \neq \Delta_0$ $then$ $C = \left\{t: |t| \geq t_{\frac{\alpha}{2},n-1}\right\}$

- Confidence Interval for the Mean Difference

The paired t CI for μ_D is : $CI = \left[\bar{d} \pm t_{\frac{\alpha}{2},n-1} \times \frac{S_D}{\sqrt{n}}\right]$

- Inferences about two population variance

➢ $X_1, .., X_n$ *be a* random from normal distribution with variance σ_1^2, and let $Y_1, \dots, Y_n$ be a random sample from normal distribution with variance σ_2^2 and let S_1^2 *and* S_2^2 denote the two sample variance .

❖ we use Fisher distribution F that is $F = \frac{\frac{S_1^2}{\sigma_1^2}}{\frac{S_2^2}{\sigma_2^2}}$ with df $\nu_1 = m - 1$ *and* $\nu_2 = n - 1$

The test for equality of Variances

- Use Fisher distribution F
- Null hypotheses $H_0: \sigma_1^2 = \sigma_2^2$

- Test statistic value : $f = \frac{S_1^2}{S_2^2}$
- Alternative Hypotheses and rejection region

1. When $H_a: \sigma_1^2 > \sigma_2^2$ $then\ C\ or\ RR = \{f: f \geq F_{\alpha,m-1,n-1}\}$
2. When $H_a: \sigma_1^2 < \sigma_2^2$ $then\ C = \{f: f \leq F_{1-\alpha,m-1,n-1}\}$
3. When $H_a: \sigma_1^2 \neq \sigma_2^2$ $then\ C = \{f: f \geq F_{\frac{\alpha}{2},m-1,n-1}\ or\ f \leq F_{1-\frac{\alpha}{2},m-1,n-1}\}$

A confidence Interval for σ_1^2/σ_2^2 and σ_1/σ_2

- A $100(1-\alpha)\%$ CI for σ_1^2/σ_2^2 is $CI = \left[\frac{s_1^2}{S_2^2} \cdot \frac{1}{F_{\frac{\alpha}{2},m-1,n-1}}, \frac{s_1^2}{S_2^2} \cdot F_{\frac{\alpha}{2},n-1,m-1}\right]$

- A random $100(1-\alpha)\%$ CI for σ_1/σ_2 is

$$CI = \left[\frac{s_1}{s_2} \cdot \frac{1}{\sqrt{F_{\frac{\alpha}{2},m-1,n-1}}}, \frac{s_1}{s_2}\sqrt{F_{\frac{\alpha}{2},n-1,m-1}}\right]$$

- Inferences about two population proportion

- Test statistic value $Z = \frac{\hat{p}_1 - \hat{p}_2 - (p_1 - p_2)}{\sqrt{\frac{p_1 q_1}{m} + \frac{p_2 q_2}{n}}}$ approximate to standard normal

- For null hypotheses $H_0: p_1 = p_2$, then
- the test Statistic value $z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{m} + \frac{1}{n}\right)}}$, where

$\hat{p}_1 = \frac{x}{m}$ and $\hat{p}_2 = \frac{y}{n}$ $, \hat{p} = \frac{x+y}{m+n}$ $, \hat{q} = 1 - \hat{p}$

For test safely be used as long as $m\hat{p}_1, m\hat{q}_1, n\hat{p}_2, n\hat{q}_2$ are all at least 5 (≥ 5)

- Alternative hypotheses and Rejection region

1. $H_a: p_1 - p_2 > 0$ $then$ $R = \{z: z \geq z_\alpha\}$
2. $H_a: p_1 - p_2 < 0$ $then$ $R = \{z: z \leq -z_\alpha\}$
3. $H_a: p_1 - p_2 \neq 0$ $then$ $R = \left\{z: |z| \geq z_{\frac{\alpha}{2}}\right\}$

❑ Alternative Hypotheses $\beta(p_1, p_2), if$

1. $H_a: p_1 - p_2 > 0 \; then \; \beta(p_1, p_2) = \phi\left[\frac{z_\alpha\sqrt{\bar{p}\bar{q}\left(\frac{1}{m}+\frac{1}{n}\right)}-(p_1-p_2)}{\sigma}\right]$

2. $H_a: p_1 - p_2 < 0 \; then \; \beta(p_1, p_2) = 1 - \phi\left[\frac{-z_\alpha\sqrt{\bar{p}\bar{q}\left(\frac{1}{m}+\frac{1}{n}\right)}-(p_1-p_2)}{\sigma}\right]$

3. $H_a: p_1 - p_2 \neq 0 \; then \; \beta(p_1, p_2) = \phi\left[\frac{z_{\frac{\alpha}{2}}\sqrt{\bar{p}\bar{q}\left(\frac{1}{m}+\frac{1}{n}\right)}-(p_1-p_2)}{\sigma}\right] - \phi\left[\frac{-\frac{z_\alpha}{2}\sqrt{\bar{p}\bar{q}\left(\frac{1}{m}+\frac{1}{n}\right)}-(p_1-p_2)}{\sigma}\right]$

Where $\bar{p} = \frac{(mp_1+np_2)}{(m+n)}, \bar{q} = \frac{mq_1+nq_2}{m+n} \; and \; \sigma = \sigma_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{p_1q_1}{m} + \frac{p_2q_2}{n}}$

- For sample size
- In case $m = n \; and \; d = p_1 - p_2$ then

- Sample size $n = \frac{\left[z_{\alpha}\sqrt{\frac{(p_1+p_2)(q_1+q_2)}{2}}+z_{\beta}\sqrt{p_1q_1+p_2q_2}\right]^2}{d^2}$ $for \; one \; tailed \; test$
- For two-tailed test $n = \frac{\left[z_{\frac{\alpha}{2}}\sqrt{\frac{(p_1+p_2)(q_1+q_2)}{2}}+z_{\beta}\sqrt{p_1q_1+p_2q_2}\right]^2}{d^2}$

- A large sample confidence interval
- A $100(1-\alpha)\%$ CI for $p_1 - p_2$ is

$$CI = \left[\hat{p}_1 - \hat{p}_2 \pm z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}_1\hat{q}_1}{m} + \frac{\hat{p}_2\hat{q}_2}{n}}\right]$$