10. Suppose that $X_1, X_2, ..., X_n$ are i.i.d. $N(\mu, \sigma^2)$.

A) If μ is known, what the mle of σ?

B) If σ is known, what is the mle of μ?

Solution

A. Find the mle of σ

We have

$X_1, X_2, X_3, \dots, X_n \sim^{iid} N(\mu, \sigma^2)$, then the pdf is $f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

then the likelihood function is $L(x, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{\sum \frac{-(x_i-\mu)^2}{2\sigma^2}}$

then the log – function is $\ln\left(L(\sigma^2|x)\right) = \ln\left(\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{\sum \frac{-(x_i-\mu)^2}{2\sigma^2}}\right)$

$\Rightarrow \ln\left(L(\sigma^2|x)\right) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$

then $\frac{\partial(\ln(L(x|\sigma^2)))}{\partial\sigma^2} = -\frac{n}{2} \times \frac{2\pi}{2\pi\sigma^2} + \frac{2}{4\sigma^4}\sum(x_i-\mu)^2 = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(x_i-\mu)^2$

after that set $\frac{\partial(\ln(L(x|\sigma^2)))}{\partial\sigma^2} = 0 \Leftrightarrow -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(x_i-\mu)^2 = 0$

$\Rightarrow \frac{1}{\sigma^4}\sum(x_i-\mu)^2 = \frac{n}{\sigma^2} \Rightarrow \sigma^2 = \frac{1}{n}\sum(x_i-\mu)^2$

check the second derivative , then

$\frac{\partial^2 \ln\left(L(x|\mu)\right)}{\partial(\sigma^2)^2} = \frac{2n}{4\sigma^4} - \frac{2}{4\sigma^8}\sum(x_i-\mu)^2 = \frac{1}{2\sigma^4}\left(n - \frac{1}{\sigma^2}\sum(x_i-\mu)^2\right) \le 0$

Thus $\hat{\sigma}^2 = \frac{1}{n}\sum(x_i-\mu)^2$, *then the mle of* σ *is* $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$ (*Invariance Principle*)

B. find the mle of μ

We have $\ln\left(L(x|\mu)\right) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$

then $\frac{\partial \ln\left(L(\mu|\, x)\right)}{\partial\mu} = -\frac{1}{2\sigma^2}\sum\left(-2(x_i-\mu)\right) = \frac{1}{\sigma^2}\sum(x_i-\mu)$

set $\frac{\partial \ln\left(L(x|\mu)\right)}{\partial\mu} = 0 \Leftrightarrow \frac{1}{\sigma^2}\sum(x_i-\mu) = 0 \Leftrightarrow \sum x_i - n\mu = 0$

then $\mu = \frac{1}{n}\sum x_i$

Moreover, $\frac{\partial^2 \ln\left(L(x|\mu)\right)}{\partial\mu^2} = \frac{\partial}{\partial\mu}\left(\frac{1}{\sigma^2}\sum x_i - \frac{n}{\sigma^2}\,\mu\right) = -\frac{n}{\sigma^2} < 0$

Hence, $\hat{\mu} = \frac{1}{n}\sum x_i$

11. Consider an i.i.d. sample of random variables with density function

$$f(x|\sigma) = \frac{1}{2\sigma}\exp(-\frac{|x|}{\sigma})$$

A) Obtain the likelihood function.

B) Find the mle of σ.

Solution

A. Obtain the likelihood function

We have X is the random sample that its pdf is defined by

$$f(x|\sigma) = \frac{1}{2\sigma}\exp\left(-\frac{|x|}{\sigma}\right)$$

The llikelihood is $L(x,\sigma) = \prod_{i=1}^{n}\left(\frac{1}{2\sigma}e^{-\frac{|x_i|}{\sigma}}\right) = \left(\frac{1}{2\sigma}\right)^n e^{-\frac{1}{\sigma}\Sigma|x|}$

Next, we define the log − likelihood $\ln\big(L(x,\sigma)\big) = \ln\left(\left(\frac{1}{2\sigma}\right)^n e^{-\frac{1}{\sigma}\Sigma|x|}\right) = -n\ln(2\sigma) - \frac{1}{\sigma}\Sigma|x_i|$

Thus the log likelihood is $\ln(L(x,\sigma)) = -n\ln(2\sigma) - \frac{1}{\sigma}\sum_{i=1}^{n}|x_i|$

B. Find the mle of σ

We have $\ln(L(x,\sigma)) = -n\ln(2\sigma) - \frac{1}{\sigma}\sum_{i=1}^{n}|x_i|$

then, $\frac{\partial(\ln(L(x,\sigma))}{\partial\sigma} = -\frac{2n}{2\sigma} + \frac{1}{\sigma^2}\Sigma|x_i|$

Take $\frac{\partial(\ln(L(x,\sigma))}{\partial\sigma} = 0 \Leftrightarrow -\frac{n}{\sigma} + \frac{1}{\sigma^2}\Sigma|x_i| = 0 \Leftrightarrow \sigma = \frac{1}{n}\Sigma|x_i|$

Therefore $\hat{\sigma} = \frac{1}{n}\Sigma|x_i|$

12. Let $X_1, ..., X_n$ be an i.i.d. sample from a Poisson distribution with probability mass function

$$f(x|\lambda) = \lambda^x \frac{e^{-\lambda}}{x!}$$

A) Obtain the likelihood function.

B) Find mle of λ.

C) Prove that the Fisher information is $I(\lambda) = \frac{1}{\lambda}$

Solution

A. Obtain the likelihood function.

We have $X_1, X_2, \dots, X_n \sim^{iid} \text{Poi}(\lambda)$ and its pmf is $f(x|\lambda) = \lambda^x \frac{e^{-\lambda}}{x!}$

The likelihood function of X is $L(x,\lambda) = \prod_{i=1}^{n} \lambda^{x_i} \frac{e^{-\lambda}}{x_i!} = e^{-n\lambda} \prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!}$

B. Find the mle of λ

We have looklihood function is

$$\ln\big(L(x,\lambda)\big) = -n\lambda + \sum_{i=1}^{n} \ln\left(\frac{\lambda^{x_i}}{x_1!}\right) = -n\lambda + \ln(\lambda) \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \ln(x_i!)$$

$$\Rightarrow \frac{\partial \ln(L(x,\lambda))}{\partial\lambda} = -n + \frac{1}{\lambda}\sum x_i$$

When $\frac{\partial \ln(L(x,\lambda))}{\partial\lambda} = 0 \Leftrightarrow -n + \frac{1}{\lambda}\sum x_i = 0 \Rightarrow \lambda = \frac{1}{n}\sum x_i = \bar{x}$

When $\frac{\partial \ln^2\big(L(x,\lambda)\big)}{\partial\lambda^2} = -\frac{1}{\lambda^2}\sum x_i < 0,$

Therefore $\hat{\lambda} = \bar{x}$

C. Proved that $I(\lambda) = \frac{1}{\lambda}$

The fisher Information has the formula

$$I_n(\lambda) = E\left(\left[\frac{\partial \ln\big(L(x,\lambda)\big)}{\partial\lambda}\right]^2\right) \text{ or } I_n(\lambda) = -E\left(\frac{\partial \ln^2 L(x,\lambda)}{\partial \lambda^2}\right) \text{ or } I_n(\lambda) = V\left(\frac{\partial \ln\big(L(x,\lambda)\big)}{\partial \lambda}\right)$$

FISHER INFORMATION AND INFORMATION CRITERIA

The Fisher Information in a random variable X:

$$\boxed{I[\theta]=E[\lambda'(x;\theta)]^2=V[\lambda'(x;\theta)]=-E[\lambda''(x;\theta)]\geq 0}$$

The Fisher Information in the random sample:

$$I_n[\theta]=nI(\theta)$$

Let's prove the equalities above.

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Knowing that $X \sim \mathrm{Poi}(\lambda)$, then $E(X)=\lambda, V(X)=\lambda, I_n(\lambda)=n\lambda$

First method

$$I_n(\lambda)=E\left[\left(-n+\frac{1}{\lambda}\sum x_i\right)^2\right]=E\left(n^2-2\frac{n}{\lambda}\sum x_i+\frac{1}{\lambda^2}\left(\sum x_i\right)^2\right)$$

$$=n^2-2\frac{n}{\lambda}E\left(\sum x_i\right)+\frac{1}{\lambda^2}E\left(\sum x_i\right)^2=n^2-2\frac{n}{\lambda}\sum E(x_i)+\frac{1}{\lambda^2}E\left[\left(\sum x_i\right)^2\right]$$

$$=n^2-2\frac{n}{\lambda}(n\lambda)+\frac{1}{\lambda^2}\left[V\left(\sum x_i\right)-\left(E\left(\sum x_i\right)\right)^2\right]=-n^2+\frac{1}{\lambda^2}\left[\sum V(x_i)+\left(\sum E(x_i)\right)^2\right]$$

$$=-n^2+\frac{1}{\lambda^2}\left[n\lambda-(n\lambda)^2\right]=-n^2+\frac{n}{\lambda^2}(\lambda-n\lambda^2)=-n^2+n\left(\frac{1}{\lambda}+n\right)=\frac{n}{\lambda}$$

Second method $I_n(\lambda)=V\left(\frac{\partial \ln(L(x,\lambda))}{\partial\lambda}\right)=V\left(-n+\frac{1}{\lambda}\sum x_i\right)=\frac{1}{\lambda^2}V\left(\sum x_i\right)=\frac{1}{\lambda^2}\sum V(x_i)=\frac{1}{\lambda^2}(n\lambda)=\frac{n}{\lambda}$

Third $I_n(\lambda)=-E\left(\frac{\partial^2(\ln(L(x,\lambda)))}{\partial\lambda^2}\right)=-E\left(-\frac{1}{\lambda^2}\sum x_i\right)=\frac{1}{\lambda^2}\sum E(x_i)=\frac{n}{\lambda}$

And we also know that

$$I(\lambda)=-E\left(\frac{\partial^2\ln f(x))}{\partial\lambda^2}\right)$$

Since $X \sim \mathrm{Poi}(\lambda)$ then pmf $f(x,\lambda)=\frac{\lambda^x e^{-\lambda}}{x!}$

then $\ln f(x|\lambda)=x\ln\lambda-\lambda-\ln x!$

$$\frac{d(\ln f(x|\lambda))}{d\lambda}=\frac{x}{\lambda}-1 \text{ and } \frac{d^2\ln(f(x|\lambda)}{d\lambda^2}=-\frac{x}{\lambda^2}$$

We get $I(\lambda)=-E\left(-\frac{x}{\lambda^2}\right)=\frac{1}{\lambda^2}E(x)=\frac{1}{\lambda}$ (becasue $E(x)=\lambda$)

We can also usen $I(\lambda)=V\left(\frac{\partial\ln f(x,\lambda)}{\partial\lambda}\right)=V\left(\frac{x}{\lambda}-1\right)=\frac{1}{\lambda^2}V(x)=\frac{1}{\lambda}$(because $V(x)=\lambda$)

and also this $I(\lambda) = E\left[\left(\frac{\partial \ln f(x,\lambda)}{\partial\lambda}\right)^2\right] = E\left[\left(\frac{x}{\lambda} - 1\right)^2\right] = V\left(\frac{x}{\lambda} - 1\right) + E\left(\frac{x}{\lambda} - 1\right)^2$

$$= \frac{1}{\lambda^2}V(x) + \left(\frac{1}{\lambda}E(x) - 1\right)^2 = \frac{1}{\lambda} + 0 = \frac{1}{\lambda}$$

13. Suppose that $X_1, ..., X_n$ are i.i.d. with density function

$$f(x|\theta) = e^{-(x-\theta)}, \qquad x \geq \theta$$

and $f(x|\theta) = 0$ otherwise.

A) Obtain and plot the likelihood and log-likelihood functions.

B) Find the mle of θ. (Hint: Be careful, and don't differentiate before thinking. For what values of θ is the likelihood positive?)

C) Find the Fisher information.

Solution

We have $X_1, X_2, \dots, X_n \sim^{iid} f(x|\theta) = e^{-(x-\theta)}, x \geq \theta$

A. Obtain the plot the likelihood and log − likelihood function

We have $L(x|\theta) = \prod_{i=1}^{n} f(x|\theta) = \prod_{i=1}^{n} e^{-(x_i-\theta)} = \exp\left(-\sum_{i=1}^{n}(x_i - \theta)\right)$

log − likelihood $\ln L(x|\theta) = -\sum_{i=1}^{n} x_i + n\theta = -n\bar{x} + n\theta$

B. Find the mle of θ

check $\frac{d\ln(L(x|\theta))}{d\theta} = n$ and $\frac{d^2\ln(L(x|\theta))}{d\theta^2} = 0$

We get the log − likelihood function is recreasing functions.

Knowing that for all $x_i \geq \theta$

Thus mle of θ is $\hat{\theta} = \min(X_i)$

C. Find the fisher information

$I_n(\theta) = E\left(\frac{d\ln(x,\theta)}{d\,\theta}\right)^2 = E[(n)^2] = n^2$ or $I(\theta) = n$ (because $I_n(\theta) = nI(\theta)$