Time Series Analysis

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1. For each of the following ARIMA(p, d, q) model, what are the values of p, d, and q ? Furthermore, state whether the model is stationary and/or invertible. Explain your answer briefly. (It suffices to verify the conditions for stationarity and invertibility for the models).

   Note: $\{e_t\}$ are i.i.d white noise with mean equal to 0 and variance equal to σ_e^2.

   (a) $Y_t = 5 + e_t + 1.5e_{t-1} + 0.5e_{t-2}$

   (b) $Y_t - \frac{5}{6}Y_{t-1} + \frac{1}{6}Y_{t-2} = 1 + e_t + 2e_{t-1}$

   (c) $Y_t - 0.5Y_{t-1} - 0.5Y_{t-2} = e_t + 0.5e_{t-1}$

Proof. . a).

$Y_t = 5 + e_t + 1.5e_{t-1} + 0.5e_{t-2}$
Corresponding to $Y_t = \theta_0 + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$ (moving average process with drift)
$\Longrightarrow p = 0, d = 0, \& \quad q = 2$
This model can be written as $\Phi(B)Y_t = \theta_0 + \Theta(B)e_t$
where $\Phi(B) = 1\& \quad \Theta(B) = (1 + 1.5x + 0.5x^2); (B \leftrightarrow x)$
Stationary case; since $\Phi(B) = 1$ then this process is stationary
Invertibility case, set $\Theta(B) = 0 \Longrightarrow (1 + 1.5x + 0.5x^2) = 0;$
$\Longrightarrow x_1 = -1; x_2 = -2$
since $|x_1| = 1; |x_2| > 1$ consist a root where is not lie outside the unite circle
Thus, this process in not invertible

b). $Y_t - \frac{5}{6}Y_{t-1} + \frac{1}{6}Y_{t-2} = 1 + e_t + 2e_{t-1}$
$(1 - \frac{5}{6}x + \frac{1}{6}x^2)Y_t = 1 + (1 + 2x)e_t \Longrightarrow p = 2; d = 0; q = 1$
Stationary case, set $\Phi(x) = 0 \Longleftrightarrow (1 - \frac{5}{6}x + \frac{1}{6}x^2) = 0$
$\Longrightarrow x^2 - 5x + 6 = 0$ then $x_1 = 2, x_2 = 3;$
since $|x_1| > 1\& \quad |x_2| > 1$ (all roots are lie out the unit circle)
Hence, the model is stationary
Invertibility cases, set $\Theta(x) = 0 \Longrightarrow 1 + 2x = 0$
then $x = -\frac{1}{2}$, and $|x| = \frac{1}{2} < 1$.
Therefore, this model is not invertible

c). $Y_t - 0.5Y_{t-1} - 0.5Y_{t-2} = e_t + 0.5e_{t-1}$
$(1 - 0.5x - 0.5x^2)Y_t = (1 + 0.5x)e_t, \Longleftrightarrow p = 2, d = 0, q = 1$
set $\Phi(x) = 0 \Longrightarrow 1 - 0.5x - 0.5x^2 = 0$
$\Longrightarrow x^2 + x - 2 = 0 \Longrightarrow x_1 = 1; x_2 = -2$
since $|x_1| = 1\&|x_2| = 2$ then the process is not stationary
set $\Theta(x) = 0 \Longleftrightarrow 1 + 0.5x = 0$
then we get $x = -2 \Longrightarrow |x| > 1$
Thus, the process is Invertibility □

2. Consider the following stationary ARMA(2,1) process:

$$Y_t + Y_{t-1} + 0.6Y_{t-2} = e_t - 0.5e_{t-1}$$

(a) Show that its autocorrelation function (ACF) satisfies the equation:

$$\rho_k + \rho_{k-1} + 0.6\rho_{k-2} = 0, \quad k \geq 2$$

(b) Verify that the ACF display a damped sine wave behavior. Find the frequency and period of the ACF.

(c) Given a realization of size 200 from the preceding ARMA(2,1) model, how many "cycles" do you expect to see in the series ? Explain your answer.

Proof. a). Show that the autocorrelation function is satisfies.

$$\begin{aligned}
\text{Consider } Cov(Y_t, Y_{t-k}) &= Cov(Y_t + Y_{t-1} + 0.6Y_{t-2}, Y_{t-k}) \\
&= Cov(Y_t, Y_{t-k}) + Cov(Y_{t-1}, Y_{t-k}) + 0.6Cov(Y_{t-2}, Y_{t-k}); \quad k \geq 2 \\
&= \gamma_k + \gamma_{k-1} + 0.6\gamma_{k-2} \qquad (1) \\
\text{Alternative } Cov(Y_t + Y_{t-1} + 0.6Y_{t-2}, Y_{t-k}) &= Cov(e_t - 0.5e_{t-1}, Y_{t-k}) \\
&= Cov(e_t, Y_{t-k}) - 0.5Cov(e_{t-1}, Y_{t-k}), \quad k \geq 2 \\
&= 0 \qquad (2)
\end{aligned}$$

From (1)&(2)

$\Longrightarrow \gamma_k + \gamma_{k-1} + 0.6\gamma_{k-2} = 0$

divided both side by γ_0.

$\Longrightarrow \rho_k + \rho_{k-1} + 0.6\rho_{k-2} = 0 \quad$ (Recall that $\rho_k = \gamma_k/\gamma_0$)

$\boxed{\text{Therefore } \rho_k + \rho_{k-1} + 0.6\rho_{k-2} = 0, \text{ for } k \geq 2.}$

b). Verify that the ACF display a damped.

$$\begin{aligned}
\text{we have } \rho_k + \rho_{k-1} + 0.6\rho_{k-2} &= 0 \\
\text{Find the homogeneous solutions} & \\
\text{The characteristics equa. is } x^2 + x + 0.6 &= 0 \\
\Longrightarrow x_1 = -0.5 + 0.6i; \& \quad x_2 &= -0.5 - 0.6i \\
\Longrightarrow \alpha = -0.5\& \quad \beta &= 0.6 \text{ respectively,} \\
\text{Module } r = \sqrt{(-0.5)^2 + (0.6)^2} &= 0.78 \\
\text{Argument } \cos(\theta) = -0.5/0.78 &\Longrightarrow \theta = \arccos(-0.64) \\
&\Longrightarrow \theta = 2.24 \text{ rad} \\
\text{The general solution is defined by } \rho_k &= r^k(A\cos(\theta k) + B\sin(\theta k)). \\
\rho_k &= (0.78)^k(A\cos(2.24k) + B\sin(2.24k)) \\
\text{when } k \to \infty; \quad \rho_k &\to 0. \\
\text{Period } T = \frac{2\pi}{2.24} &= 0.89\pi \\
\text{Frequency } F = \frac{2\pi}{T} = \frac{2\pi}{0.89\pi} &= 2.24(\text{Hz})
\end{aligned}$$

□

3. Consider two models:

   (M1): $(1 + 1.3B + 0.65B^2 + 0.325B^3 + 0.1625B^4 + 0.05B^5)Z_t = e_t$

   (M2): $(1 + 0.8B)Z_t = (1 - 0.5B)e_t$

   (a) Find the π weights for the two models.
   (b) Are these two models similar ? Explain your answer.

   Proof. a). Find π-weight

$$\begin{aligned}(M1) : e_t &= (1 + 1.3x + 0.65x^2 + 0.325x^3 + 0.1625x^4 + 0.05x^5)Y_t, \quad (B \leftrightarrow x)\\ \Longleftrightarrow e_t &= Y_t + 1.3Y_{t-1} + 0.65Y_{t-2} + 0.325Y_{t-3} + 0.1625Y_{t-4} + 0.05Y_{t-5}\end{aligned}$$

   we know that $e_t = \sum_{j=0}^{\infty} \pi_j Y_{t-j}$

   So, $\pi_0 = 1;\pi_1 = 1.3; \pi_2 = 0.65; \pi_3 = 0.325, \pi_4 = 0.1625, \pi_5 = 0.05$

$$\begin{aligned}(M2) : e_t &= (\frac{1 + 0.8x}{1 - 0.5x})Z_t = (\frac{1}{1 - 0.5x})(1 + 0.8x)Z_t\\ &= (1 + 0.5x + 0.25x^2 + 0.125x^3 + 0.0625x^4 + 0.03125x^5 + o(x^5))(1 + 0.8x)Z_t\\ &= (1 + 1.3x + 0.65x^2 + 0.325x^3 + 0.1625x^4 + 0.008125x^5)Z_t\end{aligned}$$

   So, $\pi_0 = 1;\pi_1 = 1.3; \pi_2 = 0.65; \pi_3 = 0.325, \pi_4 = 0.1625, \pi_5 = 0.008125$

   b). This models is similar, it is difference for $\pi_5 = 0.008125$ in model (M2) whereas the $\pi_5 = 0.05$ is model (M1). □

4. From a series of length 100, we have computed $r_1 = 0.8, r_2 = 0.5, r_3 = 0.4, \bar{Y} = 2$. and a sample variance of 5. If we assume that AR(2) model with a constant term is appropriate, how can we (simple) estimate of ϕ_1, ϕ_2, θ_0, and σ_e^2.

   Proof. Estimate of ϕ_1, ϕ_2, θ_0 and σ_e^2
   Using Method of Moments,
   Consider the AR(2) model with drift

$$Y_t = \theta_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t, \quad \text{where } e_t \sim iid(\mu = 0, \sigma_e^2)$$

   Mean: $E[Y_t] = \theta_0 + \phi_1 E[Y_{t-1}] + \phi_2 E[Y_{t-2}] + 0$

$$\Longrightarrow \theta_0 = (1 - \phi_1 - \phi_2)\mu_{Y_t} \tag{1}$$

   Variance function: $Var(Y_t) = \phi_1^2 Var(Y_{t-1}) + \phi_2^2 Var(Y_{t-2}) + 2\phi_1\phi_2 Cov(Y_{t-1}, Y_{t-2}) + \sigma_e^2$
   $\Longleftrightarrow \gamma_0 = \phi_1^2\gamma_0 + \phi_2^2\gamma_0 + 2\phi_1\phi_2\gamma_1 + \sigma_e^2$

$$\Longrightarrow \sigma_e^2 = (1 - \phi_1^2 - \phi_2^2 - 2\phi_1\phi_2\rho_1)\gamma_0 \tag{2}$$

   Covariance function: $Cov(Y_t, Y_{t-k}) = Cov(\theta_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t, Y_{t-k})$
   $= 0 + \phi_1 Cov(Y_{t-1}, Y_{t-k}) + \phi_2 Cov(Y_{t-2}, Y_{t-k}) + Cov(e_t, Y_{t-k})$ for $k \geq 2$

$\Longrightarrow \gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} \Longleftrightarrow \rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}$ (divided both side by γ_0)

- For k =2

$$\rho_2 = \phi_1\rho_1 + \phi_2 \quad (\rho_0 = 1) \tag{3}$$

- For k =3

$$\rho_3 = \phi_1\rho_2 + \phi_2\rho_1 \tag{4}$$

From (1), (2), (3), (4); with replace $\begin{cases} \rho_1 = r_1 = 0.8 \\ \rho_2 = r_2 = 0.5 \\ \rho_3 = r_3 = 0.4 \\ \mu_{Y_t} = \bar{Y} = 2 \\ \gamma_0 = s^2 = 5 \end{cases}$

$$\Longrightarrow \begin{cases} \theta_0 = 2(1-\phi_1-\phi_2) \\ \sigma_e^2 = (1-\phi_1^2-\phi_2^2-1.6\phi_1\phi_2)5 \\ 0.5 = 0.8\phi_1+\phi_2 \\ 0.4 = 0.5\phi_1 + 0.8\phi_2 \end{cases}$$

Using (3) & (4): $\phi_1 = 0; \phi_2 = 0.5$

(2): $\sigma_e^2 = (1-(0.5)^2)5 = (1-0.25)5 = (0.75)5 = 3.75.$

(1): $\theta_0 = 2(1-(0.5)) = 1$

$\boxed{\text{Therefore } \hat{\theta}_0 = 1; \hat{\sigma}_e^2 = 3.75; \hat{\phi}_1 = 0; \hat{\phi}_2 = 0.5}$ □

5. Below, $\{e_t\}$ are i.i.d white noise process with mean 0 and variance σ_e^2.
   Let $Z_1, Z_2, \ldots, Z_{100}$ be an ARMA (1,1) process: $(1-0.5B)Z_t = (1+0.5B)e_t$.
   Listed below are the last two observed and fitted values:

	$t = 99$	$t = 100$
Z_t	3	1
$\hat{Z}_{t-1}(1)$	1	0

   (a) Predict $Z_{101}, Z_{102}, Z_{103}$
   (b) Find the variance of the prediction errors in part (a). (Assume the noise variance to be 1).
   (c) Compute the 95% prediction interval for $Z_{101}, Z_{102}, Z_{103}$.

6. Consider the following model: $Y_t = \phi_0 + \phi_1 Y_{t-4} + e_t - \theta e_{t-1}$.
   (a) Determine thee conditions on the parameters ϕ_0, ϕ_1, θ under which the precedented model is stationary and invertible.
   (b) Suppose that the conditions of invertibility hold and that we have the data $Y_1, Y_2, \ldots, Y_n$. Show that fpr $\ell > 4$, the ℓ step forecasts $\hat{Y}_n(\ell)$ satisfies the equation $\hat{Y}_n(\ell) = \phi_0 + \phi_1\hat{Y}_n(\ell-4)$
   (c) Consider the seasonal model with $\phi_0 = 0, \phi_1 = 0.8, \theta = 0.8$. Suppose that the next five values are predicted to be $1, -3, 5, 6$ and 0.8. Construct an approximate 95% prediction interval for the 100-step ahead prediction. i.e., $\hat{Y}_n(100)$. You may assume that the noise variance to be 1, and the model is stationary.

7. The question concerns the empirical identification of ARIMA models.

   (a) The sample ACFs for a series and its first difference are given below ($n = 100$): based on this information, which ARIMA model(s) would you consider for the

	lag						
ACF	1	2	3	4	5	6	7
Z_t	0.55	0.58	0.53	0.53	0.52	0.53	0.43
ΔZ_t	-0.55	0.063	0.026	0.041	0.073	0.15	-0.18

   series. Explain your reasoning for credits.

   (b) The time series plot of 100 observations of a Z,and its ACF and PACF plots are shown in Fig. 1. Also shown in the figure are similar plots for the log-transformed data. Based on this information, state whether the data should be analyzed on the original scale or the logarithmic scale, and which ARIMA model(s) would you consider for the series. Explain your answer for the credits.

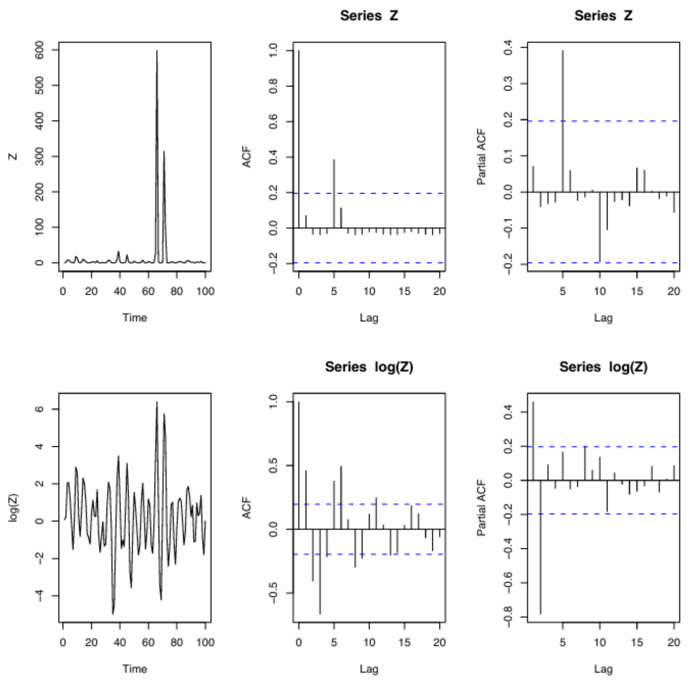

Figure 1:

8. Suppose that annual sales (in millions of dollars) of the Acme Corporation follow the AR(2) model: $Y_t = 5 + 1.1Y_{t-1} - 0.5Y_{t-2} + e_t$ with $\sigma_e^2 = 2$.

   (a) If sales for 2005, 2006, and 2007 were \$11 million, and \$10 million, respectively, forecast sales for 2008 ad 2009.

   (b) Show that $\psi = 1.1$ for this model.

   (c) Calculate 95% prediction limits for your forecast in part (a) for 2006.

   (d) If sales in 2006 turn out to be \$12 million, update your forecast for 2007.

2005	2006	2007
9 M	11 M	10 M

Proof. a). Forecast sales for 2008 and 2009
Recall that Forecast for 2008,
Consider

$$\begin{aligned} Y_t &= 5 + 1.1Y_{t-1} - 0.5Y_{t-2} + e_t \\ \hat{Y}_t(1) &= 5 + 1.1Y_t - 0.5Y_{t-1} \\ \text{and } \hat{Y}_t(2) &= 5 + 1.1\hat{Y}_t(1) - 0.5Y_t \end{aligned}$$

Hence,

$$\begin{cases} \text{For 2008,} & \hat{Y}_t(1) = 5 + 1.1(10) - 0.5(11) = 10.5\text{M } \$ \\ \text{For 2009,} & \hat{Y}_t(2) = 5 + 1.1(10.5) - 0.5(10) = 11.55\text{M } \$ \end{cases}$$

b). Show that $\psi_1 = 1.1$
We know that

$$\begin{aligned} Y_t &= \sum_{j=0}^{\infty} \psi_j e_{t-j} = \psi_0 e_t + \psi_1 e_{t-1} + \dots \\ e_t &= \sum_{j=0}^{\infty} \pi_j Y_{t-j} \end{aligned}$$

and

$$Y_t = 5 + 1.1Y_{t-1} - 0.5Y_{t-2} + e_t$$

in condition $\psi_0 = 1$, then $\psi_1 = \phi_1 = 1.1$
c). 95% Forecasting

$$(1-\alpha)100\%CI = \left[\hat{Y}_t(1) \pm Z_{\alpha/2}\sqrt{Var(e_t(\ell)}\right]$$

For $95\%CI \Longrightarrow Z_{\alpha/2} = Z_{0.025} = 1.96$
and $Var(e_t(\ell)) = \sigma_e^2 \sum_{j=0}^{\ell-1} \psi_j^2 \approx \sigma_e^2(\psi_0^2 + \psi_1^2 + \psi_2^2) = 2^2(1 + 1.1^2 + 0.5^2) = 4.92$
then $\sqrt{Var(e_t(\ell))} = \sqrt{4.92} = 2.22$
So,

$$95\%CI = [10.5 \pm 1.96(2.22)] = [6.15, 14.85]$$

□

9. An AR model has AR characteristic polynomial $(1 - 1.5x + 0.4x^2)(1 - 0.7x^{12})$.
   (a) Is the model stationary ?
   (b) Identify the model as a certain seasonal ARIMA model.

Proof. a). State the stationary
We have the characteristics polynomial for AR

$$\Phi(x) = \underbrace{(1 - 1.5x + 0.4x^2)}_{p}\underbrace{(1 - 0.7x^{12})}_{P}$$

Consider $\Phi(x) = 0 \iff (1 - 1.5x + 0.4x^2)(1 - 0.7x^{12}) = 0$
$\implies 1 - 1.5x + 0.4x^2 = 0$ then $x_1 \approx 0.86$ and $x_2 \approx 2.88$
So, $|x_2| > 1$,but $|x_1| < 1$
$\implies 1 - 0.7x^{12} = 0 \implies x^{12} \approx 1.22$ then $|x|^{12} > 1$.
$\boxed{\text{Overall, the process is not stationary}}$

b). ARIMA model is defined by ARIMA$(2, 0, 0) \times (1, 0, 0)_{12}$ □

10. Consider an intervention analysis model with the trend process m_t following an AR(1): $m_t = \delta m_{t-1} + \omega S_t^{(T)}$. If $\delta = 0.95$, determine the half-life of the intervention effect.

Proof. determine the haft-life of the intervention effect.
In AR(1): model $m_t = \delta m_{t-1} + \omega S_t^{(T)}$

then $m_t = \begin{cases} 0 \text{ if } t \leq T \\ \omega(\dfrac{1-\delta^{t-T}}{1-\delta}) \text{ if } t > T; \text{ where } 0 < \delta < 1 \end{cases}$ For $t \to \infty; \delta^{t-T} \to 0$ then

$m_t = \begin{cases} 0 \text{ if } t \leq T \\ \omega \text{ if } t > T \end{cases}$ Haft-life

$m_t = 0.5(\dfrac{\omega}{1-\delta})$

$\iff \omega(\dfrac{1-\delta^{t-T}}{1-\delta}) = 0.5\dfrac{\omega}{1-\delta)} = \log_\delta(0.5) = \log_{0.95}(0.5) = 12.5134$

$\implies t = T + 13.5134 \implies t - T = \dfrac{\log(0.5)}{\log(\delta}$ □

11. Describe the differences between MA(q), ARCH (q) and GARCH (p, q).

Proof. (a) MA(q): Formula

$$Y_t = \theta_0 + e_t + \sum_{i=1}^{q} \theta_i e_{t-i}$$

The conditional mean of the time series depends on past white noise terms, but the variance is constant.

(b) ARCH(q) Formula

$$\epsilon_t = \sigma_t Z_t, \quad Z_t \sim \mathcal{N}(0, 1)$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2$$

where:

- ϵ_t is the error term,

- σ_t^2 is the conditional variance,
- $\alpha_0 > 0$ and $\alpha_i \geq 0$ for stability,
- Z_t is a standard normal white noise process.

The variance of the series depends on past squared errors, meaning that periods of high and low volatility cluster together.

(c) GARCH(p, q): Formula

$$\epsilon_t = \sigma_t Z_t, \quad Z_t \sim \mathcal{N}(0, 1)$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

where:

- $\beta_j \geq 0$ ensures the non-negativity of variance,
- $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$ ensures stationarity.

GARCH captures both short-term and long-term volatility persistence by modeling conditional variance with both past squared errors and past variances. □