1.4 One-sided Limits

Definition 1.15. *Let $f(x)$ be a function that is defined at every number in some open interval (a, c). Then the* ***limit of*** $f(x)$ ***as*** x ***approaches*** a ***from the right (left) is L****, written*

$$\lim_{x \to a^+} f(x) = L$$

(alternately,

$$\lim_{x \to a^-} f(x) = L$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \epsilon$.

Theorem 1.3. $\lim_{x \to a} f(x)$ *exists and is equal to L if and only if* $\lim_{x \to a^-} f(x)$ *and* $\lim_{x \to a^+} f(x)$ *both exist and both are equal to L.*

Example 1.4. *Find the limits of the following functions.*

1. *Consider the function*

$$f(x) = \begin{cases} 2 & \text{if } x < 1 \\ -1 & \text{if } x = 1 \\ -3 & \text{if } 1 < x \end{cases}$$

   Find

   (a) $\lim_{x \to 1^+} f(x)$
   (b) $\lim_{x \to 1^-} f(x)$
   (c) $\lim_{x \to 1} f(x)$

2. *Consider the function*

$$f(t) = \begin{cases} t + 4 & \text{if } t \leq -4 \\ 4 - t & \text{if } -4 < t \end{cases}$$

   Find

   (a) $\lim_{t \to -4^+} f(t)$
   (b) $\lim_{t \to -4^-} f(t)$
   (c) $\lim_{t \to -4} f(t)$