3 Infinite Limits

Consider the function

$$f(x) = \frac{1}{x}.$$

What happens to $f(x)$ when we take values of x to the left of 0?
How about when we take values of x to the right of 0?
Look at the following table of values.

x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10000	0.0001	10000

The graph of the function $f(x) = \frac{1}{x}$ is shown in Figure 1.

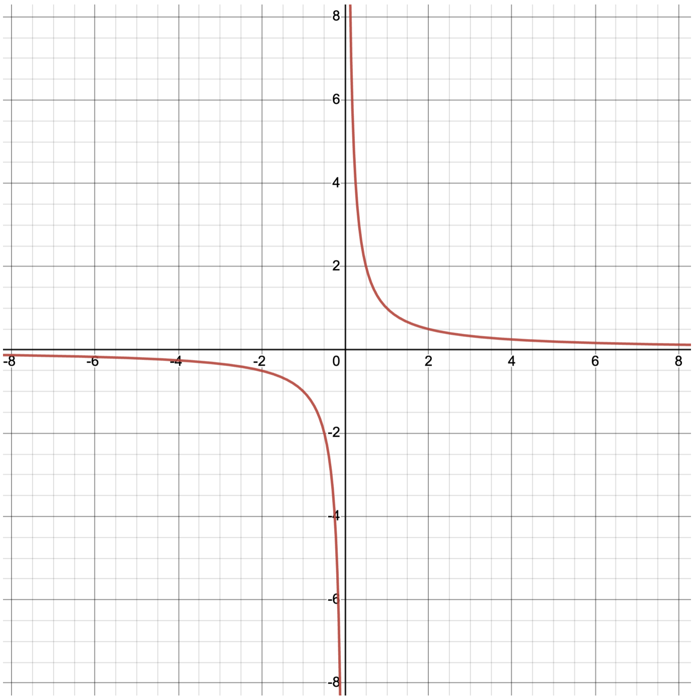

Figure 1: Graph of $f(x) = 1/x$.

Definition 3.1. *Let $f(x)$ be a function that is that is defined at every number in some open interval containing a, except possibly at a itself. As x approaches a, $f(x)$ increases* *(alternately, decreases) without bound, written*

$$\lim_{x\to a} f(x) = +\infty \ \ (-\infty)$$

if for any number $N > 0$ $(N < 0)$, $\exists\delta > 0$ *such that if* $0 < |x-a| < \delta$, *then*

$$f(x) > N \ \ (f(x) < N)$$

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3.1 Limit Theorems

1. *If* r *is any positive integer, then*

   (a)
   $$\lim_{x\to 0+} \frac{1}{x^r} = +\infty$$

   (b)
   $$\lim_{x\to 0-} \frac{1}{x^r} == \begin{cases} -\infty & \textit{if } r \textit{ is odd} \\ +\infty & \textit{if } r \textit{ is even} \end{cases}$$

2. *If* a *is any real number, and if* $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = c$, *where* c *is a constant not equal to 0, then*

   (a) *if* $c > 0$ *and if* $f(x) \to 0$ *through positive values of* $f(x)$,
   $$\lim_{x\to a} \frac{g(x)}{f(x)} = +\infty$$

   (b) *if* $c > 0$ *and if* $f(x) \to 0$ *through negative values of* $f(x)$,
   $$\lim_{x\to a} \frac{g(x)}{f(x)} = -\infty$$

   (c) *if* $c < 0$ *and if* $f(x) \to 0$ *through positive values of* $f(x)$,
   $$\lim_{x\to a} \frac{g(x)}{f(x)} = -\infty$$

(d) *if* $c < 0$ *and if* $f(x) \to 0$ *through negative values of* $f(x)$,

$$\lim_{x \to a} \frac{g(x)}{f(x)} = +\infty$$

3. (a) *if* $\lim_{x\to a} f(x) = +\infty$ *and* $\lim_{x\to a} g(x) = c$, *where* c *is any constant, then*

$$\lim_{x\to a}[f(x) \pm g(x)] = +\infty$$

(b) *if* $\lim_{x\to a} f(x) = -\infty$ *and* $\lim_{x\to a} g(x) = c$, *where* c *is any constant, then*

$$\lim_{x\to a}[f(x) \pm g(x)] = -\infty$$

4. *If* $\lim_{x\to a} f(x) = +\infty$ *and* $\lim_{x\to a} g(x) = c$, *where* c *is any constant except 0, then*

   (a) *if* $c > 0$, $\lim_{x\to a}[f(x) \cdot g(x)] = +\infty$;

   (b) *if* $c < 0$, $\lim_{x\to a}[f(x) \cdot g(x)] = -\infty$

5. *If* $\lim_{x\to a} f(x) = -\infty$ *and* $\lim_{x\to a} g(x) = c$, *where* c *is any constant except 0, then*

   (a) *if* $c > 0$, $\lim_{x\to a}[f(x) \cdot g(x)] = -\infty$;

   (b) *if* $c < 0$, $\lim_{x\to a}[f(x) \cdot g(x)] = +\infty$

Note. The above theorems are also valid if "$x \to a$*" is replaced by "*$x \to a^+$*" or "*$x \to a^-$*".*

Example 3.1. *Find the limits of the following. State which theorem is used to determine the limits.*

a. $\lim_{x\to -2^+} \dfrac{-4}{x+2} = -\infty$

b. $\lim_{x\to -2^-} \dfrac{-4}{x+2} = +\infty$

c. $\lim_{x\to -2} \dfrac{-4}{x+2}$ *DOES NOT EXIST*

Example 3.2. *Find the limits of the following. State which theorem is used to determine the limits.*

a. $\lim_{x\to 0^+} \dfrac{6}{x^2} = +\infty$

b. $\lim_{x\to 0^-} \dfrac{6}{x^2} = -\infty$

c. $\lim_{x\to 0} \dfrac{6}{x^2}$ *DOES NOT EXIST*

Try this. *Find the limits of the following. State which theorem is used to determine the limits.*

a. $\lim_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$

b. $\lim_{x\to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$

c. $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$

Try this. *Find the limits of the following. State which theorem is used to determine the limits.*

a. $\lim_{x\to 3^+} \left(\dfrac{\sqrt{x^2-9}}{x-3}\right)$

b. $\lim_{x\to 3^-} \left(\dfrac{\sqrt{x^2-9}}{x-3}\right)$

c. $\lim_{x\to 3} \left(\dfrac{\sqrt{x^2-9}}{x-3}\right)$

Definition 3.2. *The line $x = a$ is said to be a **vertical asymptote** of the graph of the function f if **at least one** of the following statements is true:*

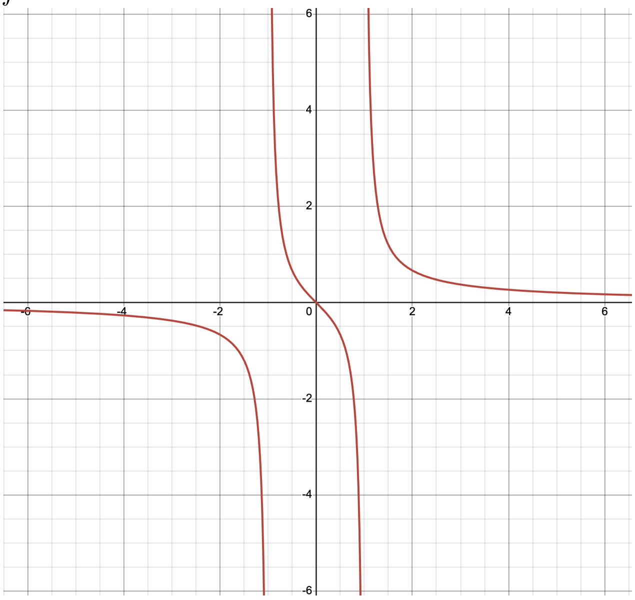
i. $\lim_{x\to a^+} f(x) = +\infty$

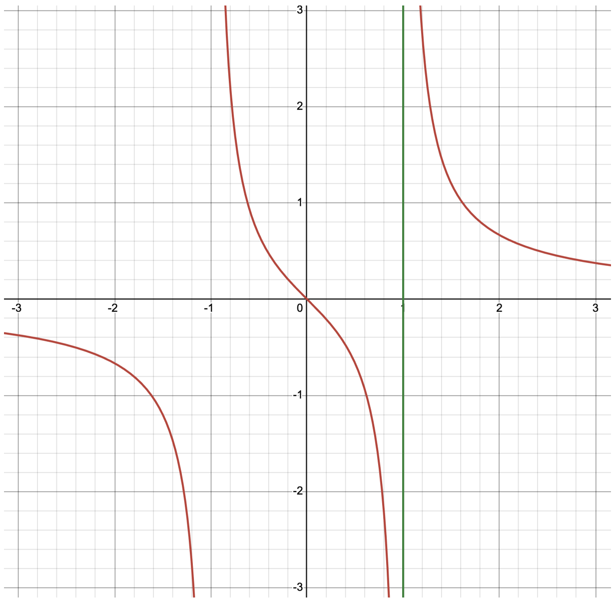
ii. $\lim_{x\to a^+} f(x) = -\infty$

iii. $\lim_{x\to a^-} f(x) = +\infty$

iv. $\lim_{x\to a^-} f(x) = -\infty$

Example 3.3. *Find a vertical asymptote of the function* $f(x) = \dfrac{x}{x^2-1}$. *A graph of this function is shown below.*





Note that the function does not cross the line $x = 1$.
Now, finding $\lim_{x\to 1^+} \dfrac{x}{x^2-1} = +\infty$. *Hence, the line* $x = 1$ *is a vertical asymptote of* f.
Another vertical asymptotes of f *is the line* $x = -1$.