Chapter 1

Functions and their Limits

1.1 Introduction

This chapter begins with a short review on functions. Students need to be familiar with functions and their notations and operations as these will be needed to better understand the concepts of Calculus.

After a brief review on functions, limits of a function will then be discussed. At the end of this chapter, students are expected to:

1. understand the intuitive concept of limit of a function;

2. have a grasp of the $\epsilon - \delta$ definition of a limit of a function;

3. demonstrate, describe, and recognize ways in which limits do not exist

4. evaluate limits of a function by applying the various theorems on limits;and

5. identify applications of limits in the real world.

Definition 1.1. *Let A and B be two nonempty sets. The* ***cross product*** *of A and B, denoted by $A \times B$, is given as*

$$A \times B = \{(x, y) : x \in A \wedge y \in B\}.$$

Definition 1.2. *A* ***relation****, say r, from A to B, is any nonempty subset of the cross product $A \times B$. Symbolically,*

$$r \subseteq A \times B \wedge r \neq \emptyset.$$

Definition 1.3. *Suppose r is a relation from set A to set B. Then*

domain of r: $\text{dom} \quad r = \{x : (x, y) \in r\}$

range of r: $\text{rng} \quad r = \{y : (x, y) \in r\}$

field of r: $\text{fld} \quad r = \text{dom}r \cup \text{rng}r$

Definition 1.4. *Let A and B be nonempty sets.*

*A **function** f from A to B is a relation from A to B such that the following conditions hold:*

i. dom f = A

ii. no two ordered pairs in f must have the same first components

*If to each element in A there is assigned a unique element in B, then such assignment is called a **function** on A to B, denoted by $f : A \to B$.*

Definition 1.5. *Let $f : A \to B$ be a function. Then*

*i. A is called the **domain** of f and B is called the **codomain** of f.*

*ii. If $a \in A$ is assigned to $b \in B$, then b is called the **image** of a under f, denoted by $f(a) = b$.*

*The **range** of f, denoted by $f(A)$ or $\text{rng}f$, is the set*

$$f(A) = \{f(x) : x \in A\}.$$

Definition 1.6. *A **function** is a set of ordered pairs of numbers (x, y) in which no two distinct ordered pairs have the same first number.*

Definition 1.7. *Let $f : A \to B$ be a function. Then*

*i. A is called the **domain** of f and B is called the **codomain** of f.*

*ii. If $a \in A$ is assigned to $b \in B$, then b is called the **image** of a under f, denoted by $f(a) = b$.*

*The **range** of f, denoted by $f(A)$ or $\text{rng}f$, is the set*

$$f(A) = \{f(x) : x \in A\}.$$

The set of all admissible values of x is called the **domain** of the function and the set of all resulting values of y is called the **range** of the function.

Example 1.1. *Verify if the following are functions.*

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$

2. $g : \mathbb{R} \to \mathbb{R}, g(x) = 3x^2 - 5x + 2$

3. $h : \mathbb{R} \to \mathbb{R}, h(x) = |x|$

Definition 1.8. *A function $f : A \to B$ is called **one-to-one** (1-1) (or **injective**) if*

$$f(x) = f(y) \Rightarrow x = y;$$

or equivalently,

$$x \neq y \Rightarrow f(x) \neq f(y).$$

*The function f is called **onto** (or **surjective**) if and only if the range of f is B.*

*If the function f is both one-to-one and onto, then f is said to be **bijective**.*

Definition 1.9. *The **identity function** $I_A : A \to A$ on a set A is defined by*

$$I_A(x) = x.$$

Definition 1.10. *Let $f : A \to B$ and $g : B \to C$ be functions. Then f is **equal** to g, written $f = g$, if*

$$f(x) = g(x), \quad \forall x \in A.$$

Definition 1.11. *Let $f : A \to B$ and $b \in B$.*

*The **inverse of b under** $\mathbf{f}$, denoted by $f^{-1}(b)$, is the set*

$$f^{-1}(b) = \{x \in A : f(x) \in D\}.$$

If $D \subseteq B$, then

$$f^{-1}(D) = \{x \in A : f(x) \in D\}.$$

Remark. *Let $f : A \to B$. Then*

i. $f^{-1}(B) = A$

ii. $f^{-1}[f(A)] = A$

Theorem 1.1. *If $f : A \to B$ is 1-1 and onto, then f^{-1} is a function. $f^{-1} : B \to A$ is called the **inverse function** of f.*

1.2 Operations on Functions

Definition 1.12. *(The Four Basic Operations on Functions)*
Let $f : A \to B$ and $g : C \to D$. Then

i. $(f + g)(x) = f(x) + g(x)$

ii. $(f - g)(x) = f(x) - g(x)$

iii. $(fg)(x) = f(x)g(x)$

iv. $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, *provided* $g(x) \neq 0$

The domain of (i)- (iii) is $A \cap C$ while the domain of (iv) is $A \cap C - \{x : g(x) = 0\}$.

Definition 1.13. *Let $f : A \to B$ and $g : B \to C$ be functions. The* ***composition function*** *of f and g, denoted by*

$$g \circ f : A \to C,$$

is defined by

$$(g \circ f)(x) = g[f(x)]$$

where x is any element in the domain of f.

The domain of $g \circ f$ is the set of all values of x in the domain of f such that $f(x)$ is in the domain of g.

Note.

$rng\ f \subseteq B = dom\ g$
$dom(g \circ f) = dom\ f$

Theorem 1.2. *Given that f and g are both functions such that $f : A \to B$ and $g : B \to A$, if one is the other's inverse, then*

$$f \circ g = I_B$$

and

$$g \circ f = I_A,$$

where I_A and I_B are identity functions on sets A and B, respectively.

1.3 Introduction to Limits of a Function

Consider the function defined by $f(x) = \dfrac{2-2x^2}{x-1}$. Note that $f(x)$ is undefined when $x = 1$.

Now, let's pick some values of x and see what happens to the value of $f(x)$ as these values get closer and closer to 1.

x	$f(x)$	x	$f(x)$
2	-6	0	-2
1.5	-5	0.5	-3
1.1	-4.2	0.9	-3.8
1.01	-4.02	0.99	-3.98
1.001	-4.002	0.999	-3.998
1.0001	-4.0002	0.9999	-3.9998

Sketch the graph of the function above.

What do you observe?

Try this. *Let $f(x) = 2x + 2$. Compute $f(x)$ as x takes values closer to 1. Hint: First consider values of x approaching 1 from the left ($x < 1$). Then consider values of x approaching 1 from the right.*

Definition 1.14. *Let f be a function that is defined at every number in some open interval containing a, except possibly at the number a itself. The* ***limit of $f(x)$ as x approaches a is L****, written*

$$\lim_{x \to a} f(x) = L$$

if the following statement is true:

If $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon|$.

Example 1.2. *Show that the limits of the given functions are as shown using the $\epsilon - \delta$ definition.*

1. $\lim_{x\to 4}(2x - 5) = 3$

2. $\lim_{x\to -1}(5x + 8) = 3$