4 Limits at Infinity

Definition 4.1. *Let f be a function that is defined at every number in some interval $(a, +\infty)$. The* ***limit of** $f(x)$**, as** x **increases without bound, is L, written***

$$\lim_{x \to +\infty} f(x) = L$$

if for any $\epsilon > 0$ however small, there exists a number $N > 0$ such that if $x > N$ then $|f(x) - L| < \epsilon$.

Definition 4.2. *Let f be a function that is defined at every number in some interval $(-\infty, a)$. The* ***limit of** $f(x)$**, as** x **decreases without bound, is L, written***

$$\lim_{x \to -\infty} f(x) = L$$

if for any $\epsilon > 0$ however small, there exists a number $N < 0$ such that if $x < N$ then $|f(x) - L| < \epsilon$.

4.1 Limit Theorems

1. $\lim_{x \to +\infty} c = c$, *for any constant* c;
2. $\lim_{x \to +\infty}[f(x) \pm g(x)] = \lim_{x \to +\infty} f(x) \pm \lim_{x \to +\infty} g(x)$;
3. $\lim_{x \to +\infty}[f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)] = \lim_{x \to +\infty} f_1(x) \pm \lim_{x \to +\infty} f_2(x) \pm \cdots \pm \lim_{x \to +\infty} f_n(x)$;
4. $\lim_{x \to +\infty}[f(x) \cdot g(x)] = [\lim_{x \to +\infty} f(x)] \cdot [\lim_{x \to +\infty} g(x)]$;
5. $\lim_{x \to +\infty}[f_1(x) \cdot f_2(x) \cdot \cdots \cdot f_n(x)] = [\lim_{x \to +\infty} f_1(x)] \cdot [\lim_{x \to +\infty} f_2(x)] \cdots [\lim_{x \to +\infty} f_n(x)]$;
6. $\lim_{x \to +\infty}[f(x)]^n = [\lim_{x \to +\infty} f(x)]^n, n \in \mathbb{Z}^+$;
7. $\lim_{x \to +\infty} \dfrac{f(x)}{g(x)} = \dfrac{lim_{x \to \infty} f(x)}{\lim_{x \to +\infty} g(x)}$, *provided* $\lim_{x \to +\infty} g(x) \neq 0$;
8. $\lim_{x \to +\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to +\infty} f(x)}$, $\lim_{x \to +\infty} f(x) \geq 0$;
9. $\lim_{x \to +\infty} \dfrac{1}{x^n} = 0, n \in \mathbb{Z}^+$

These theorems are also valid if "$x \to +\infty$" is replaced with "$x \to -\infty$".

Example 4.1. *Evaluate the following.*

1. $\lim_{x\to+\infty} \dfrac{2x+1}{5x-2}$
2. $\lim_{x\to-\infty} \dfrac{6x-4}{3x+1}$
3. $\lim_{x\to+\infty} \dfrac{2x+1}{5x-2}$

Try this. *Find the limit of the following.*

1. $\lim_{x\to+\infty} \dfrac{7x^2-2x+1}{3x^2+8x+5}$
2. $\lim_{y\to-\infty} \dfrac{2y^3-4}{5y+3}$
3. $\lim_{w\to+\infty} \dfrac{\sqrt{w^2-2w+3}}{w+5}$

Definition 4.3. *The line $y = b$ is a* ***horizontal asymptote*** *of the graph of the function $f(x)$ if at least one of the following statements is true:*

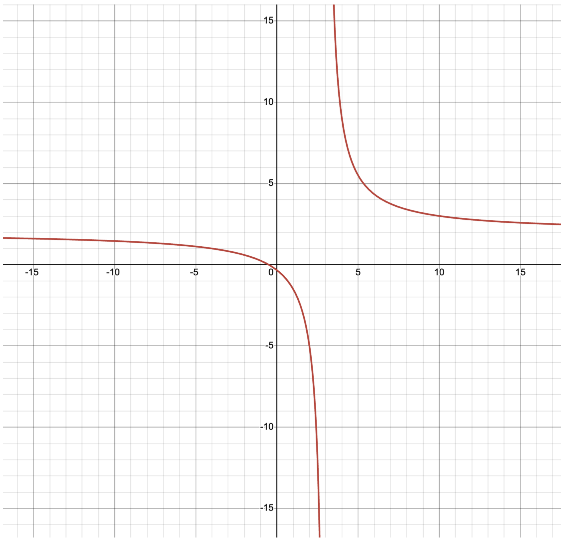
i. $\lim_{x\to+\infty} f(x) = b$ *and for some number N, if $x > N$, then $f(x) \neq b$;*

ii. $\lim_{x\to-\infty} f(x) = b$ *and for some number N, if $x < N$, then $f(x) \neq b$.*

Example 4.2. *Find the horizontal asymptotes of*

$$f(x) = \frac{2x+1}{x-3}.$$

Sketch the graph showing the horizontal asymptotes.



Try this. *Find the vertical asymptotes of*

$$f(x) = \frac{2x+1}{x-3}.$$

Try this. *Find the limits of the following.*

1. $\lim_{x\to+\infty}\left(\frac{6}{\sqrt{x^3}}\right)$
2. $\lim_{x\to4^+}\left(\frac{3}{(4-x)^3}\right)$
3. $\lim_{x\to-\infty}\left(x-x^2\right)$
4. $\lim_{x\to3^-}\left(\frac{2x}{x-3}\right)$
5. $\lim_{x\to+\infty}\left(x^3+x\right)$

Try this. *Find the vertical and horizontal asymptotes of*

$$f(x) = \frac{x^2-1}{x^2-4}.$$