Instructor Version

Overview

Complex patterns often arise from repeatedly applying a simpler set of rules. In this activity, we explore how two sets of rules generate a well-known fractal.

Work together as a group to complete the activities. Write the answers to the questions on the large sheet of paper, and be sure to write large enough so that other groups can see your answers.

Activity A (18 minutes)

1. Grab a single sheet labeled "Rule 90". Tape it to a window or pin it to a board, so that all members in your group can see and draw on it.
2. Beginning with the first row, fill out the first 6 rows by applying the cellular automaton's rules. Make sure everyone understands how to apply the rules.
3. Using a "divide and conquer" approach, quickly fill out the rest of the grid. This means that every one in the group should pick up a marker and try to apply the rules at the same time.

   Q1: What feature of the rules makes it possible to cooperatively fill out the rest of the grid?

   A1: The rules are applied locally, and only require knowledge of the 3 cells directly above it. This means, that students can apply the rules in a parallel fashion…it is embarrassingly parallel.

   Q2: Could one person work on filling out the top half of the grid, while another person worked on filling out the bottom half? Why or why not?

   A2: No. The rules require the previous three cells. So you must always work down from the top.

   Q3: Why is color of the middle square in each triplet is not important? Find a simpler description of the rules that does not rely on the middle square. Describe it in words.

   A3: The rule set is symmetric. For example, 010 and 000 both map to the same output. The simpler rule set is just to look at the L and R cells above. An ideal description might be something like: The next square is black if just one of the L or R squares is black. It is an exclusive or (xor) function.

Activity B (12 minutes)

1. Grab a calculator and a single sheet labeled "Pascal's Triangle". Tape the sheet to a window or pin it to a board, so that all members in your group can see and draw on it.
2. Begin at the top of the triangle and fill out the first 8 rows of Pascal's triangle, using the addition rule specified on the sheet. Make sure everyone understands how to apply the rule.
3. Color all the circles with an even number inside them.

   Q1: Using only the colors of the circles, find a new set of rules that will allow you to finish coloring the rest of Pascal's triangle, without having to do addition.

   A1: New set of rules is just like the cellular automata. Next circle is colored iff only one of the previous circles is colored.

   Q2: How does the rule on colors relate to the properties of even and oddness?

   A2: odd + even = odd, odd + odd = even, etc.

4. Finish coloring Pascal's triangle. [It is not necessary to put numbers inside the circles.]

Q3: What pattern do you see? Compare and contrast it to the pattern we saw for "Rule 90".

A3: It is the same: Sierpinski's triangle. Some differences are the grid. We could also use a hexagonal grid.

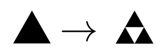
Q4: How does the coloring rule set compare to the simplified rule set that we found for "Rule 90"?

A3: It is the same, but with different spacing. Generally, there were two different rule sets, applied on different lattices, that ended up being similar and generating the same pattern.

Bonus Questions

1. Where does the "90" in "Rule 90" come from? Hint: $90 = 2^6 + 2^4 + 2^3 + 2^1$.

   Answer:
   90 is the decimal equivalent of the rule written as the binary number 01011010_2.
   Recall, the number 425 has a decimal expansion of $425 = 4 \cdot 10^2 + 2 \cdot 10^1 + 5 \cdot 10^0$.
   Similarly, $01011010_2 = 0 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$.

2. The Sierpinski triangle can also be generated using the following rule: 

   Begin with an equilateral triangle, having side length equal to 1. Then, apply the rule repeatedly. What are the formulas for the perimeter and area of the pattern as a function of the number of iterations? What is the perimeter and area as the number of iterations tends to infinity?

   Answer:

$$P(n) = 3\left(\frac{3}{2}\right)^n \qquad \lim_{n\to\infty} P(n) = \infty$$

$$A(n) = \frac{\sqrt{3}}{4}\left(\frac{3}{4}\right)^n \qquad \lim_{n\to\infty} A(n) = 0$$

3. For Pascal's triangle, x and y correspond to $\binom{n}{k}$ and $\binom{n}{k+1}$. The "$x + y$" circle corresponds to $\binom{n+1}{k+1}$. Show that generation rule takes the following form: $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.

   Answer:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n}{k} + \frac{n-k}{k+1}\binom{n}{k} = \frac{n+1}{k+1}\binom{n}{k} = \binom{n+1}{k+1}$$

Resources

If you want to learn more, try searching for some of the following keywords:

fractals, cellular automata, iterated function systems, chaos game,
Sierpinski triangle, Turing complete, game of life, Mandelbrot set

Here are some fun webpages and a nice YouTube video:

http://ecademy.agnesscott.edu/~lriddle/ifs/ifs.htm
http://www.kevs3d.co.uk/dev/lsystems
https://www.youtube.com/watch?v=5plLxMnbtAw