One-dimensional Quantum Integer CDF(2,2) Wavelet Transform

Freddy Alejandro Chaurra-Gutierrrez*, Gustavo Rodriguez-Gomez*, Claudia Feregrino-Uribe* and Julio Cesar Perez-Sansalvador*,**

Abstract. Quantum transforms have shown the power and capabilities of quantum computing, being the basis for developing powerful algorithms. Particularly, quantum Wavelet transforms plays a fundamental role in information processing and data hiding applications, decreasing computational complexity but limiting the application capabilities due to the reduced set of quantum transforms and the constraints of quantum computing. Therefore, we propose a new class of one-dimensional quantum integer wavelet transform based on the lifting scheme called CDF(2,2), extending the quantum transform set. We design the quantum circuits for the transform, avoiding the nonlinearities for some operations and providing a polynomial complexity. Also, we present the unitary definition and algorithmic description of the decomposition process. Furthermore, we perform a comparative analysis, showing a complexity decrease compared to the classical version. Finally, the simulation results show the feasibility and applicability of the CDF(2,2) transformation for signal decomposition.

Keywords. Quantum Transforms, Quantum Wavelet, Integer Wavelet, CDF Transform, Lifting Scheme.

1. Introduction

Quantum information processing (QIP) enables storing and manipulating information in new ways using quantum interference, entanglement, and superposition properties not available in classical processing, decreasing computational costs and memory requirements [1–3]. In addition, application areas such as information hiding, signal analysis, feature extraction, and quantum formats have shown the efficiency of QIP [4–7].

Additionally, research on quantum transforms definition has shown the power and capability of quantum computing in information processing. For

* Instituo Nacional de Astrofísica, Óptica y Electrónica (INAOE), Computer Science Department

** Consejo Nacional de Humanidades, Ciencias y Tecnologías (CONAHCYT)

instance, the Quantum Fourier Transform (QFT) is the basis for developing powerful algorithms such as Shor's, phase estimation, and Simon's algorithm [2, 5]. Also, quantum versions of the Cosine (QCT) and Wavelet Transform (QWT) have increased the quantum toolkit for information processing [8, 9]. These are used in encryption, watermarking, denoising, and compression applications [4, 5, 10, 11].

Mainly, QWTs are useful in quantum information hiding, compression, and coding [11–13]. However, only the Haar and Daubechies-4 QWTs focusing on real-valued transformation are developed [14–18], decreasing the computational complexity but limiting the application capabilities compared to the classical set of Wavelet transforms. Thus, recent work has presented a new quantum wavelet version, developing the first quantum integer-to-integer wavelet transform (QIWT) based on the lifting scheme and the Haar kernel [19], extending the set of quantum transforms and application areas [20–22].

Integer transforms are fundamental in lossless applications where the information content cannot be modified, and preservation of the original data is critical, such as military tasks, medical signal analysis, and banking transactions [23–25]. Therefore, we develop a new class of one-dimensional Biorthogonal QIWT called CDF (2,2), based on the vanishing moments, or (5/3), based on the filter sizes. This transform uses a linear approximation as the decomposition basis and is the core for the standard lossless JPEG2000 [21, 22]. Thus, we design the quantum circuits for the operations on the transform. Also, we give the unitary definition and algorithmic description for the decomposition process. In addition, we analyze the quantum complexity, giving a polynomial complexity. Finally, simulation results show the feasibility and applicability of the transformation.

This paper is organized as follows. Section 2 introduces the Integer-to-Integer CDF(2,2) wavelet transform and describes the proposed quantum version. Section 3 presents the experimental analysis and results. Finally, section 4 draws the conclusions.

2. Integer-to-Integer Transform

The integer Haar wavelet transform (S-Transform) assumes constant signal values to define the operations in the lifting scheme, involving prediction and update steps for the decomposition process. However, a constant signal assumption is rigid for most signals. Therefore, we can define different lifting operations to support other signal assumptions.

2.1. Classical CDF Transform

The next step from the Haar transform is to assume piecewise linear signals, decomposing them through Linear Prediction and Update steps in the lifting scheme using the CDF(2,2) wavelet transform, which is biorthogonal (symmetric and linear phase) and preserves the first moment of the signal [26]. Thus, this transform assumes that all the signal values, S, lie on a straight

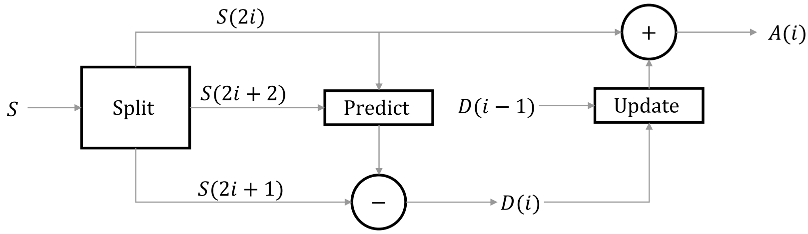

FIGURE 1. Prediction and update scheme of the one-level CDF(2,2) Transform.

line, such that a new value, D, is predicted by the two closest elements using a Prediction operator $P(\cdot)$:

$$P(S) = \frac{1}{2}\left[S(2i) + S(2i+2)\right] \tag{2.1}$$

Thus,

$$\begin{aligned} D(i) &= S(2i+1) - \lfloor P(S) \rfloor \\ &= S(2i+1) - \left\lfloor \frac{1}{2}\left[S(2i) + S(2i+2)\right] \right\rfloor \end{aligned} \tag{2.2}$$

Finally, the Update step, W, is determined by the previous two D values, given a new element, A:

$$\begin{aligned} W(D) &= \frac{1}{4}\left[D(i-1) + D(i)\right] \\ A(i) &= S(2i) + \lfloor W(D) \rfloor \end{aligned} \tag{2.3}$$

The element $A(i)$ and $D(i)$ correspond to the Approximation and Detail coefficients of the decomposition, respectively. The floor function is to guarantee the integer and lossless transformation [24, 26]. Fig. 1 shows the prediction and update scheme over a signal, S.

Additionally, the inverse transform is given by

$$\begin{aligned} S'(2i) &= A(i) - \lfloor W(D) \rfloor \\ S'(2i+1) &= D(i) + \left\lfloor \frac{1}{2}\left[S'(2i) + S'(2i+2)\right] \right\rfloor \end{aligned} \tag{2.4}$$

where the recovered signal S' is equal to the original signal S because the transformation is reversible.

2.2. Quantum CDF Transform

Given the previous definition, we propose a quantum version of the CDF (2,2) transform by designing the quantum circuits for addition, subtraction, halving, and rounding operations. The quantum complexity of each operation has polynomial growth, and we guarantee the linearity and reversibility of the decomposition process by using quantum operators. We describe each quantum operation below.

2.2.1. Quantum Full Addition. Full addition of two binary numbers a and b is defined by

$$U_a\ |a,b\rangle|0\ldots0\rangle = |a,b\rangle|a_0+b_0, a_1+b_1,\ldots, carry\rangle \tag{2.5}$$

where U_a is the addition operator on qubits $|a\rangle$ and $|b\rangle$, and $|0\ldots0\rangle$ are the auxiliary qubits to store the binary addition and carry bit. This operation is based on the $\sqrt{X}$ and $\sqrt{X^\dagger}$ gates [27]. Fig. 2 shows the quantum circuit to add two binary numbers.

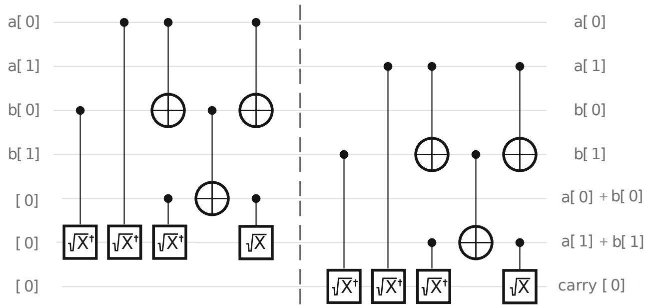

FIGURE 2. Quantum Circuit-gate of the full addition operation for two binary numbers.

2.2.2. Quantum Full Subtraction. Full subtraction is performed as

$$U_s\,|a\rangle|b,0\ldots0\rangle = |a\rangle|b_0-a_0, b_1-a_1,\ldots, borrow\rangle \tag{2.6}$$

where U_s is the subtraction operator, and the register $|b\rangle|0\ldots0\rangle$ gives the binary subtraction, including the borrow bit [28]. The subtraction of q-bits numbers requires $|0\ldots0\rangle = |0\rangle^{\otimes q}$ auxiliary qubits. Fig. 3 presents the quantum operation. The segment A of the circuit must be cascaded to subtract more than two bits.

2.2.3. Quantum Halving. Quantum halving operation on $|a\rangle = |a_0, a_1, \ldots, a_{n-1}\rangle$ is defined by binary shifting,

$$U_H|a\rangle|0\rangle = |a_0\rangle\,|a_1,a_2,\ldots,a_{n-1}\rangle\,|0\rangle \tag{2.7}$$

where the element $|a_0\rangle$ is discarded, and $|a/2\rangle = |a_1, a_2, \ldots, a_{n-1}\rangle\,|0\rangle$. Fig. 4 shows the halving circuit using only identity gates.

2.2.4. Quantum Rounding. Quantum rounding uses halving properties on integers to avoid nonlinearities. For example, if $a = 2m$ with m integer, the rounding operation is expressed as the integer number m,

$$\left\lfloor \frac{a}{2} \right\rfloor = \left\lfloor \frac{2m}{2} \right\rfloor = \lfloor m \rfloor = m \tag{2.8}$$

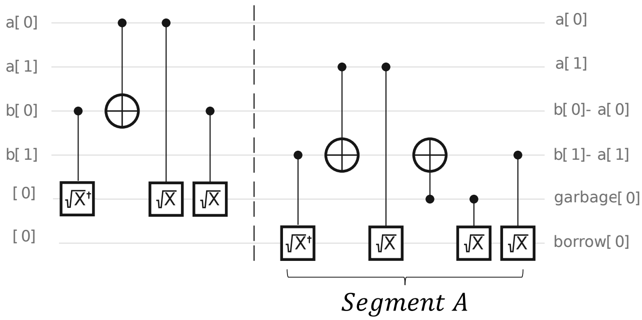

FIGURE 3. Quantum Circuit-gate of the full subtraction operation of two binary numbers

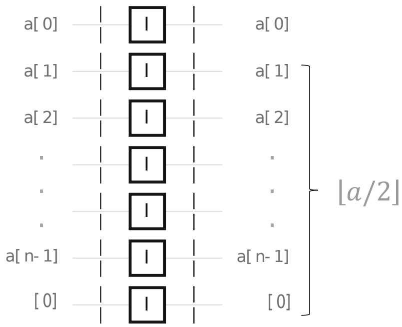

FIGURE 4. Quantum halving operation.

On the other hand, if $a = 2m + 1$, the rounding operation could be reduced by subtracting 1 before the halving operation,

$$\left\lfloor \frac{a-1}{2} \right\rfloor = \left\lfloor \frac{2m}{2} \right\rfloor = \lfloor m \rfloor = m \tag{2.9}$$

This is equivalent to perform the conventional process,

$$\left\lfloor \frac{2m+1}{2} \right\rfloor = \left\lfloor \frac{2m}{2} + \frac{1}{2} \right\rfloor = \lfloor m \rfloor = m \tag{2.10}$$

because the additional term, 1/2, does not contribute to the floor rounding operation. Therefore, a quantum rounding operations can be defined by

$$U_R \left| a \right\rangle = \left| \lfloor a \rfloor \right\rangle \ \leftrightarrow a \in \mathbb{Z} \tag{2.11}$$

where this quantum rounding is implemented by the halving operation in (2.7).

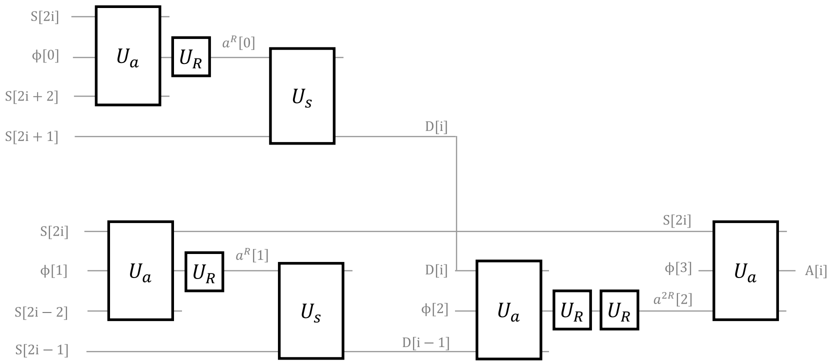

FIGURE 5. Block Circuit-gate of the quantum CDF(2,2) transform.

2.3. Quantum Circuit Representation

The quantum CDF(2,2) transform uses the odd/even signal elements to obtain the decomposition coefficients, $A(i)$ and $D(i)$. Initially, the CDF transform uses the $S(2i)$, $S(2i+1)$ and $S(2i+2)$ components in the prediction and update steps. However, this transform also requires the $D(i-1)$ term in (2.3) to compute the approximation coefficients, but computing $D(-1)$ requires the $S(-1)$ and $S(-2)$, which are not defined. Our proposed solution uses zero padding to obtain the value at that index [24]. There are other methods that could be explored, however, the analysis is beyond the scope of this research.

Finally, the CDF quantum transform is defined by

$$\begin{aligned}
&U_{s(24)}U_{R(2)}U_{a(123)}|S(2i),\phi(0),S(2i+2),S(2i+1)\rangle \rightarrow |D(i)\rangle \\
&U_{s(24)}U_{R(2)}U_{a(123)}|S(2i),\phi(1),S(2i-2),S(2i-1)\rangle \rightarrow |D(i-1)\rangle \\
&U_{a(124)}U^2_{R(4)}U_{a(345)}\ |S(2i),\phi(3),D(i),\phi(2),D(i-1)\rangle \rightarrow |A(i)\rangle
\end{aligned} \tag{2.12}$$

where $S\,(2i-1) = S\,(2i-2) = 0$ for $i = 0$. Also, the right side on (2.12) is simplified, showing only the qubits of the decomposition coefficients. Fig. 5 shows the general quantum circuit of the transform for one decomposition level, where $\phi(i)$ are auxiliary qubits $|0\ldots0\rangle$, and $a^R(i)$ are the addition results after rounding operations. Each rounding operation is equivalent to performing the halving operation ($|a/2\rangle = |\lfloor a/2\rfloor\rangle$).

3. Experimental Analysis and Results

This section provides the algorithmic description of the CDF(2,2) transform. Then, the simulation experiments present the decomposition results using Qiskit by IBM [29]. Next, a complexity analysis shows the polynomial growth of the transform. Finally, a comparative analysis depicts the characteristics of the classical and quantum versions.

3.1. Quantum CDF(2,2) Transform Algorithm

The decomposition process requires a one-dimensional signal, $|S\rangle$, of 2^n+1 elements. Thus, we operate on a first block of qubits $|S(2i), \phi(0), S(2i+2), S(2i+1)\rangle$, applying the addition operator on the three first qubits followed by the rounding operator on the second qubit, and the subtraction operator with qubits two and qubit four (algorithm 1 on line 8 to line 14). Next, we select the second block of qubits $|S(2i), \phi(1), S(2i-2), S(2i-1)\rangle$, and perform the addition, rounding, and subtraction operations on them (algorithm 1 on line 15 to line 21). Then, we operate on the last set of qubits $|S(2i), \phi(3), D(i), \phi(2), D(i-1)\rangle$ using the addition operator on the three last qubits, the rounding operator applied two times on qubit four, and again the addition operator on qubits one, two, and four (algorithm 1 on line 23 to line 29). Finally, we extract the coefficients $A(i)$ and $D(i)$ from the individual qubits (algorithm 1 on line 31).

Algorithm 1 : Quantum CDF Transform - Decomposition Process

1: ▷ ***Input***: One-dimensional signal $|S\rangle$ of 2^n+1.
2: ▷ ***Output***: Decomposition coefficients $|A(i)\rangle$ and $|D(i)\rangle$.
3:
4: **Step 1:** State initialization
5: $|0\dots0\rangle = |S(2i), S(2i+1), S(2i+2), S(2i-1), S(2i-2)\rangle$
6: $|0\dots0\rangle = |\phi(0), \phi(1), \phi(2), \phi(3)\rangle$ {Auxiliary qubits}
7:
8: **Step 2:** Select the first block of elements
9: $|S(2i), \phi(0), S(2i+2), S(2i+1)\rangle$
10: ▷ **Step 2.1:** Apply the addition operator $U_{a(123)}$
11: ▷ **Step 2.2:** Apply the rounding operator on qubit two $U_{R(2)}$
12: ▷ **Step 2.3:** Apply the subtraction operator $U_{s(24)}$
13: These operations produce the state,
14: $|S(2i), a_0^R, S(2i+2), D(i)\rangle$ $\{a_i^R$ are the addition results after rounding$\}$

3.2. Decomposition Results

Given a one-dimensional integer[1] signal of 2^6+1 elements, we apply the quantum and classical CDF(2,2) transform for three decomposition levels; one approximation coefficient and three detail coefficients (Fig. 6). The figure shows the decomposition levels of the quantum and classical CDF transform, where the coefficients are identical, and the error between them is zero, verifying the practical feasibility of the quantum transform.

3.3. Complexity Analysis

Based on the quantum operations in the CDF(2,2) transform, we perform a quantum complexity analysis given by the number of quantum gates (QG), auxiliary (A) and garbage (G) qubits on each quantum operation. Table 1

[1] All the signal elements are in $\{\mathbb{Z}\}$ to guarantee the integer-to-integer mapping.

Step 3: Select the next block of elements
$|S(2i), \phi(1), S(2i-2), S(2i-1)\rangle$
▷ **Step 3.1:** Apply addition operator $U_{a(123)}$
▷ **Step 3.2:** Apply rounding operator on qubit two $U_{R(2)}$
▷ **Step 3.3:** Apply the subtraction operator $U_{s(24)}$
This gives,
$|S(2i), a_1^R, S(2i-2), D(i-1)\rangle$

Step 4: Select the last block of qubits
$|S(2i), \phi(3), D(i), \phi(2), D(i-1)\rangle$
▷ **Step 4.1:** Apply the addition operator $U_{a(345)}$
▷ **Step 4.2:** Apply the rounding operator two-times in qubit four $U_{R(4)}$
▷ **Step 4.3:** Apply the subtraction operator $U_{a(124)}$
This produces the final state,
$|S(2i), A(i), D(i), a_2^{2R}, D(i-1)\rangle$

Step 5: Extract the decomposition coefficients $A(i)$ and $D(i)$
▷ Repeat n-times on $A(i)$ to get n decomposition levels.

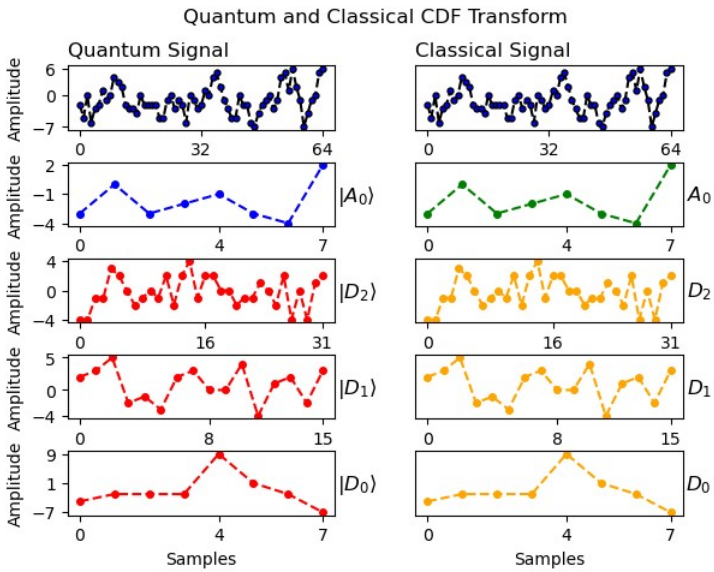

FIGURE 6. Quantum and classical decomposition by the CDF(2,2) transform on a one-dimensional integer signal. The dot points are the integer values of the signals.

presents the quantum complexity[2], where q is the bit precision of the signal

[2]Table 1 does not show the complexity of the halving operation, as it is part of the rounding operation.

elements. Thus, the quantum complexity of the CDF(2,2) transform is $O(qn)$ for the maximum decomposition level n.

TABLE 1. Complexity analysis of the quantum CDF(2,2) Transform.

Operation	**QG**	**A**	**G**
Add	$4 \times (7q)$	$4 \times (q+1)$	0
Sub	$2 \times (6q-2)$	$2 \times (q)$	$2 \times (q-1)$
Rounding	$4 \times (q+1)$	$4 \times (1)$	0

3.4. Comparative Analysis

We present a comparative analysis of the quantum and classical CDF(2,2) and Haar transforms. Table 2 shows some characteristics of the wavelet transforms, including the type of transformation, the scheme for the decomposition, the domain of decomposition, the class of wavelet transform, and complexity, where the quantum integer versions significantly decrease the computational complexity. Moreover, to our knowledge, the proposed CDF(2,2) transform is the first developed biorthogonal quantum wavelet.

TABLE 2. Comparative analysis of quantum and classical CDF(2,2) and Haar transforms, where 2^n is the signal length and q the bit precision.

	Quantum			**Classical**	
Wavelet	**CDF(2,2) (Ours)**	**Haar [19]**	**Haar [14–18]**	**CDF(2,2)**	**Haar [30]**
Type	Integer	Integer	Real-valued	Integer	Integer
Scheme	Lifting	Lifting	Filter	Lifting	Lifting
Domain	Spatial	Spatial	Spatial	Spatial	Spatial
Class	Biorthogonal	Orthogonal	Orthogonal	Biorthogonal	Orthogonal
Complexity	$O(qn)$	$O(qn)$	$O(n^3)$	$O(n2^n)$	$O(n2^n)$

Table 2 describes two definitions of Wavelet transforms, quantum and classical. The integer type is based on the Lifting scheme, useful in lossless applications, and the real-valued ones use the traditional filter bank decomposition. In addition, the signal decomposition analysis is performed in the spatial domain instead of the frequency domain. Furthermore, the proposed transform belongs to the class of biorthogonal wavelets, being the first quantum wavelet of this class. Finally, the CDF(2,2) quantum transform significantly decreases the complexity compared to the classical version, providing an exponential speedup.

4. Conclusions

In this research, we presented a new class of quantum Integer-to-Integer wavelet transform called CDF(2,2), using a linear approximation as the decomposition basis and being the first proposed biorthogonal quantum wavelet. Also, we designed the quantum circuits for the one-dimensional version, achieving a polynomial complexity, $O(qn)$, for the quantum decomposition process.

We avoided the nonlinearities of the rounding operation using the features of the CDF transform, where the halving operation implements the quantum rounding. In addition, we provided the unitary definition and algorithmic description of the transformation, facilitating the mathematical and practical manipulation.

Also, we showed the simulation results of the decomposition process in a one-dimensional integer signal using the quantum and classical CDF(2,2) transform, reaching three decomposition levels, and a zero error between the coefficient values, verifying the practical feasibility and applicability of the proposed quantum transform.

Furthermore, we performed a comparative analysis between quantum and classical versions of the CDF(2,2) and Haar wavelet transforms, showing the type of transformation, the decomposition scheme, the decomposition domain, the class of wavelet transform, and its complexity. Thus, we revealed that the proposed quantum CDF(2,2) transform, and the quantum integer Haar transform, significantly decrease the computational complexity compared to the quantum and the classical counterparts.

Finally, our proposed quantum wavelet transform extends the set of quantum transforms and their application areas, allowing the development of lossless quantum versions of standards such as JPEG, proving their feasibility and applicability, as well as the exponential reduction of complexity for the decomposition process.

Author Contributions

Freddy Alejandro Chaurra-Gutierrez: Writing - Original Draft, Conceptualization, Methodology, Software, Formal analysis, Investigation, Validation. **Gustavo Rodriguez-Gomez:** Writing - Review & Editing, Supervision, Formal analysis, Conceptualization, Methodology, Investigation. **Claudia Feregrino-Uribe:** Writing - Review & Editing, Supervision, Funding acquisition, Conceptualization, Methodology, Investigation. **Julio Cesar Perez-Sansalvador:** Writing - Review & Editing, Supervision, Software, Conceptualization, Methodology, Investigation.

Acknowledgment

This research was supported by the Consejo Nacional de Ciencia y Tecnología (CONACYT) with the CVU 937653.

References

[1] F. de Lima Marquezino, R. Portugal, and C. Lavor, *A primer on quantum computing*. Springer, 2019.

[2] J. D. Hidary and J. D. Hidary, *Quantum computing: an applied approach*, vol. 1. Springer, 2019.

[3] R. K. Bera, *The amazing world of quantum computing*. Springer, 2020.

[4] G. Luo, R.-G. Zhou, J. Luo, W. Hu, Y. Zhou, and H. Ian, "Adaptive lsb quantum watermarking method using tri-way pixel value differencing," *Quantum Information Processing*, vol. 18, pp. 1–20, 2019.

[5] X. Liu, D. Xiao, W. Huang, and C. Liu, "Quantum block image encryption based on arnold transform and sine chaotification model," *Ieee Access*, vol. 7, pp. 57188–57199, 2019.

[6] Z. Wang, M. Xu, and Y. Zhang, "Review of quantum image processing," *Archives of Computational Methods in Engineering*, vol. 29, no. 2, pp. 737–761, 2022.

[7] F. Yan, A. M. Iliyasu, and P. Q. Le, "Quantum image processing: a review of advances in its security technologies," *International Journal of Quantum Information*, vol. 15, no. 03, p. 1730001, 2017.

[8] C.-Y. Pang, R.-G. Zhou, B.-Q. Hu, W. Hu, and A. El-Rafei, "Signal and image compression using quantum discrete cosine transform," *Information Sciences*, vol. 473, pp. 121–141, 2019.

[9] C. P. Williams, *Explorations in Quantum Computing*. Springer Science & Business Media, 2010.

[10] X.-Z. Li, W.-W. Chen, and Y.-Q. Wang, "Quantum image compression-encryption scheme based on quantum discrete cosine transform," *International Journal of Theoretical Physics*, vol. 57, pp. 2904–2919, 2018.

[11] S. Chakraborty, S. H. Shaikh, A. Chakrabarti, and R. Ghosh, "An image denoising technique using quantum wavelet transform," *International Journal of Theoretical Physics*, vol. 59, pp. 3348–3371, 2020.

[12] W.-W. Hu, R.-G. Zhou, J. Luo, S.-X. Jiang, and G.-F. Luo, "Quantum image encryption algorithm based on arnold scrambling and wavelet transforms," *Quantum Information Processing*, vol. 19, pp. 1–29, 2020.

[13] D. Ghai, H. K. Gianey, A. Jain, and R. S. Uppal, "Quantum and dual-tree complex wavelet transform-based image watermarking," *International Journal of Modern Physics B*, vol. 34, no. 04, p. 2050009, 2020.

[14] H.-S. Li, P. Fan, H. Peng, S. Song, and G.-L. Long, "Multilevel 2-d quantum wavelet transforms," *IEEE Transactions on Cybernetics*, 2021.

[15] A. Fijany and C. P. Williams, "Quantum wavelet transforms: Fast algorithms and complete circuits," in *NASA international conference on quantum computing and quantum communications*, pp. 10–33, Springer, 1998.

[16] H.-S. Li, P. Fan, H.-y. Xia, S. Song, and X. He, "The multi-level and multi-dimensional quantum wavelet packet transforms," *Scientific reports*, vol. 8, no. 1, pp. 1–23, 2018.

[17] H.-S. Li, P. Fan, H.-y. Xia, and S. Song, "Quantum multi-level wavelet transforms," *Information Sciences*, vol. 504, pp. 113–135, 2019.

[18] A. Klappenecker, “Wavelets and wavelet packets on quantum computers,” in *Wavelet Applications in Signal and Image Processing VII*, vol. 3813, pp. 703–713, International Society for Optics and Photonics, 1999.

[19] F. A. Chaurra-Gutierrez, C. Feregrino-Uribe, J. C. Perez-Sansalvador, and G. Rodriguez-Gomez, “Qist: One-dimensional quantum integer wavelet s-transform,” *Information Sciences*, vol. 622, pp. 999–1013, 2023.

[20] F. Zhang and D. Bull, *Intelligent image and video compression: communicating pictures*. Academic Press, 2021.

[21] K. Sayood, *Lossless compression handbook*. Elsevier, 2002.

[22] Z.-M. Lu and S.-Z. Guo, *Lossless information hiding in images*. Syngress, 2016.

[23] K. Sayood, *Introduction to data compression*. Morgan Kaufmann, 2017.

[24] A. Jensen and A. la Cour-Harbo, *Ripples in Mathematics: The Discrete Wavelet Transform*. Springer Science & Business Media, 2011.

[25] R. A. Alotaibi and L. A. Elrefaei, “Text-image watermarking based on integer wavelet transform (iwt) and discrete cosine transform (dct),” *Applied Computing and Informatics*, vol. 15, no. 2, pp. 191–202, 2019.

[26] A. la Cour-Harbo and A. Jensen, “Wavelets and the lifting scheme,” 2007.

[27] W. N. Hung, X. Song, G. Yang, J. Yang, and M. Perkowski, “Optimal synthesis of multiple output boolean functions using a set of quantum gates by symbolic reachability analysis,” *IEEE transactions on Computer-Aided Design of integrated circuits and Systems*, vol. 25, no. 9, pp. 1652–1663, 2006.

[28] H. Thapliyal, “Mapping of subtractor and adder-subtractor circuits on reversible quantum gates,” in *Transactions on Computational Science XXVII*, pp. 10–34, Springer, 2016.

[29] A. tA v, M. S. ANIS, Abby-Mitchell, H. Abraham, AduOffei, R. Agarwal, G. Agliardi, M. Aharoni, V. Ajith, I. Y. Akhalwaya, G. Aleksandrowicz, T. Alexander, M. Amy, S. Anagolum, André, Anthony-Gandon, I. F. Araujo, E. Arbel, E. Arellano, A. Asfaw, I. E. Ashimine, A. Athalye, A. Avkhadiev, C. Azaustre, P. BHOLE, V. Bajpe, A. Banerjee, S. Banerjee, W. Bang, A. Bansal, P. Barkoutsos, A. Barnawal, G. Barron, G. S. Barron, L. Bello, Y. Ben-Haim, M. C. Bennett, D. Bevenius, D. Bhatnagar, P. Bhatnagar, A. Bhobe, P. Bianchini, L. S. Bishop, C. Blank, S. Bolos, S. Bopardikar, S. Bosch, S. Brandhofer, Brandon, S. Bravyi, Bryce-Fuller, D. Bucher, L. Burgholzer, A. Burov, F. Cabrera, P. Calpin, L. Capelluto, J. Carballo, G. Carrascal, A. Carriker, I. Carvalho, R. Chakrabarti, A. Chen, C.-F. Chen, E. Chen, J. C. Chen, R. Chen, F. Chevallier, K. Chinda, R. Cholarajan, J. M. Chow, S. Churchill, CisterMoke, C. Claus, C. Clauss, C. Clothier, R. Cocking, R. Cocuzzo, J. Connor, F. Correa, Z. Crockett, A. J. Cross, A. W. Cross, S. Cross, J. Cruz-Benito, C. Culver, A. D. Córcoles-Gonzales, N. D, S. Dague, T. E. Dandachi, A. N. Dangwal, J. Daniel, DanielAja, M. Daniels, M. Dartiailh, A. R. Davila, F. Debouni, A. Dekusar, A. Deshmukh, M. Deshpande, D. Ding, J. Doi, E. M. Dow, P. Downing, E. Drechsler, M. S. Drudis, E. Dumitrescu, K. Dumon, I. Duran, K. EL-Safty, E. Eastman, G. Eberle, A. Ebrahimi, P. Eendebak, D. Egger, EgrettaThula, ElePT, I. Elsayed, Emilio, A. Espiricueta, M. Everitt, D. Facoetti, Farida, P. M. Fernández, S. Ferracin, D. Ferrari, A. H. Ferrera, R. Fouilland, A. Frisch, A. Fuhrer, B. Fuller, M. GEORGE, J. Gacon, B. G. Gago, C. Gambella, J. M. Gambetta, A. Gammanpila, L. Garcia, T. Garg, S. Garion, J. R. Garrison, J. Garrison, T. Gates,

N. Gavrielov, G. Gentinetta, H. Georgiev, L. Gil, A. Gilliam, A. Giridharan, Glen, J. Gomez-Mosquera, Gonzalo, S. de la Puente González, J. Gorzinski, I. Gould, D. Greenberg, D. Grinko, A. Großardt, W. Guan, D. Guijo, Guillermo-Mijares-Vilarino, J. A. Gunnels, H. Gupta, N. Gupta, J. M. Günther, M. Haglund, I. Haide, I. Hamamura, O. C. Hamido, F. Harkins, K. Hartman, A. Hasan, V. Havlicek, J. Hellmers, Ł. Herok, R. Hill, S. Hillmich, I. Hincks, C. Hong, H. Horii, C. Howington, S. Hu, W. Hu, C.-H. Huang, J. Huang, R. Huisman, H. Imai, T. Imamichi, K. Ishizaki, Ishwor, R. Iten, T. Itoko, A. Ivrii, A. Jalali, A. Javadi, A. Javadi-Abhari, W. Javed, Q. Jianhua, M. Jivrajani, K. Johns, S. Johnstun, Jonathan-Shoemaker, JosDenmark, JoshDumo, J. Judge, T. Kachmann, A. Kale, N. Kanazawa, J. Kane, Kang-Bae, A. Kapila, A. Karazeev, P. Kassebaum, T. Kato, T. Kehrer, J. Kelso, S. Kelso, H. van Kemenade, V. Khanderao, S. King, Y. Kobayashi, Kovi11Day, A. Kovyrshin, J. Krishna, R. Krishnakumar, P. Krishnamurthy, V. Krishnan, K. Krsulich, P. Kumkar, G. Kus, LNoorl, R. LaRose, E. Lacal, R. Lambert, H. Landa, J. Lapeyre, D. Lasecki, J. Latone, S. Lawrence, C. Lee, G. Li, T. J. Liang, J. Lishman, D. Liu, P. Liu, Lolcroc, A. K. M, L. Madden, Y. Maeng, S. Maheshkar, K. Majmudar, A. Malyshev, M. E. Mandouh, J. Manela, Manjula, J. Marecek, M. Marques, K. Marwaha, D. Maslov, P. Maszota, D. Mathews, A. Matsuo, F. Mazhandu, D. McClure, M. McElaney, J. McElroy, C. McGarry, D. McKay, D. McPherson, S. Meesala, D. Meirom, C. Mendell, T. Metcalfe, M. Mevissen, A. Meyer, A. Mezzacapo, R. Midha, A. Miessen, D. Millar, D. Miller, H. Miller, Z. Minev, A. Mitchell, A. Mohammad, N. Moll, A. Montanez, G. Monteiro, M. D. Mooring, R. Morales, N. Moran, D. Morcuende, S. Mostafa, M. Motta, R. Moyard, P. Murali, D. Murata, J. Müggenburg, T. NEMOZ, D. Nadlinger, K. Nakanishi, G. Nannicini, P. Nation, E. Navarro, Y. Naveh, S. W. Neagle, P. Neuweiler, A. Ngoueya, T. Nguyen, J. Nicander, Nick-Singstock, P. Niroula, H. Norlen, NuoWenLei, L. J. O'Riordan, O. Ogunbayo, P. Ollitrault, T. Onodera, R. Otaolea, S. Oud, D. Padilha, H. Paik, S. Pal, Y. Pang, A. Panigrahi, V. R. Pascuzzi, S. Perriello, E. Peterson, A. Phan, K. Pilch, F. Piro, M. Pistoia, C. Piveteau, J. Plewa, P. Pocreau, C. Possel, A. Pozas-Kerstjens, R. Pracht, M. Prokop, V. Prutyanov, S. Puri, D. Puzzuoli, Pythonix, J. Pérez, Quant02, Quintiii, R. I. Rahman, A. Raja, R. Rajeev, I. Rajput, N. Ramagiri, A. Rao, R. Raymond, O. Reardon-Smith, R. M.-C. Redondo, M. Reuter, J. Rice, M. Riedemann, Rietesh, D. Risinger, P. Rivero, M. L. Rocca, D. M. Rodríguez, RohithKarur, B. Rosand, M. Rossmannek, M. Ryu, T. SAPV, N. R. C. Sa, A. Saha, A. Ash-Saki, A. Salman, S. Sanand, M. Sandberg, H. Sandesara, R. Sapra, H. Sargsyan, A. Sarkar, N. Sathaye, N. Savola, B. Schmitt, C. Schnabel, Z. Schoenfeld, T. L. Scholten, E. Schoute, J. Schuhmacher, M. Schulterbrandt, J. Schwarm, P. Schweigert, J. Seaward, Sergi, D. E. Serrano, I. F. Sertage, K. Setia, F. Shah, P. A. Shah, N. Shammah, W. Shanks, R. Sharma, P. Shaw, Y. Shi, J. Shoemaker, A. Silva, A. Simonetto, D. Singh, D. Singh, P. Singh, P. Singkanipa, Y. Siraichi, Siri, J. Sistos, J. Sistos, I. Sitdikov, S. Sivarajah, Slavikmew, M. B. Sletfjerding, J. A. Smolin, M. Soeken, I. O. Sokolov, I. Sokolov, V. P. Soloviev, SooluThomas, Starfish, D. Steenken, M. Stypulkoski, A. Suau, S. Sun, K. J. Sung, M. Suwama, O. Słowik, R. Taeja, H. Takahashi, T. Takawale, I. Tavernelli, C. Taylor, P. Taylour, S. Thomas, K. Tian, M. Tillet, M. Tod, M. Tomasik, C. Tornow, E. de la Torre, J. L. S. Toural, K. Trabing, M. Treinish, D. Trenev, TrishaPe, F. Truger, TsafrirA, G. Tsilimigkounakis, K. Tsuoka, D. Tulsi, D. Tuna, W. Turner, K. Ueda,

Y. Vaknin, C. R. Valcarce, F. Varchon, A. Vartak, A. C. Vazquez, P. Vijaywargiya, V. Villar, B. Vishnu, D. Vogt-Lee, C. Vuillot, WQ, J. Weaver, J. Weidenfeller, R. Wieczorek, J. A. Wildstrom, J. Wilson, E. Winston, WinterSoldier, J. J. Woehr, S. Woerner, R. Woo, C. J. Wood, R. Wood, S. Wood, J. Wootton, M. Wright, L. Xing, J. YU, Yaiza, B. Yang, U. Yang, J. Yao, D. Yeralin, R. Yonekura, D. Yonge-Mallo, R. Yoshida, R. Young, J. Yu, L. Yu, Yuma-Nakamura, C. Zachow, L. Zdanski, H. Zhang, E. Zheltonozhskii, I. Zidaru, B. Zimmermann, B. Zindorf, C. Zoufal, a matsuo, aeddins ibm, alexzhang13, b63, bartek bartlomiej, bcamorrison, brandhsn, nick bronn, chetmurthy, choerst ibm, comet, dalin27, deeplokhande, dekel.meirom, derwind, dime10, ehchen, ewinston, fanizzamarco, fs1132429, gadial, galeinston, georgezhou20, georgios ts, gruu, hhorii, hhyap, hykavitha, itoko, jeppevinkel, jessica angel7, jezerjojo14, jliu45, johannesgreiner, jscott2, kUmezawa, klinvill, krutik2966, luciacuervovalor, ma5x, merav aharoni, michelle4654, msuwama, nico lgrs, nrhawkins, ntgiwsvp, ordmoj, sagar pahwa, pritamsinha2304, rickyzcode, rithikaadiga, ryancocuzzo, saktar unr, saswati qiskit, sebastian mair, septembrr, sethmerkel, sg495, shaashwat, smturro2, sternparky, strickroman, tigerjack, tsura crisaldo, upsideon, vadebayo49, welien, willhbang, wmurphy collabstar, yang.luh, yuri@FreeBSD, and M. Čepulkovskis, "Qiskit: An open-source framework for quantum computing," 2021.

[30] R. A. Brown, M. L. Lauzon, and R. Frayne, "A general description of linear time-frequency transforms and formulation of a fast, invertible transform that samples the continuous s-transform spectrum nonredundantly," *IEEE Transactions on Signal Processing*, vol. 58, no. 1, pp. 281–290, 2009.

Freddy Alejandro Chaurra-Gutierrrez
Instituto Nacional de Astrofísica, Óptica y Electrónica, Computer Science Department, Santa María Tonantzintla, Puebla 72840, Mexico
e-mail: `chaura@inaoep.mx`

Gustavo Rodriguez-Gomez
Instituto Nacional de Astrofísica, Óptica y Electrónica, Computer Science Department, Santa María Tonantzintla, Puebla 72840, Mexico
e-mail: `grodrig@inaoep.mx`

Claudia Feregrino-Uribe
Instituto Nacional de Astrofísica, Óptica y Electrónica, Computer Science Department, Santa María Tonantzintla, Puebla 72840, Mexico
e-mail: `cferegrino@inaoep.mx`

Julio Cesar Perez-Sansalvador
Instituto Nacional de Astrofísica, Óptica y Electrónica, Computer Science Department, Santa María Tonantzintla, Puebla 72840, Mexico
Consejo Nacional de Ciencia y Tecnología, Mexico City 03940, Mexico
e-mail: `jcp.sansalvador@inaoep.mx`