

$$\text{Cov}(\widehat{\beta}_1^0, \widehat{\beta}_1^0 - \widehat{\beta}_1) = 0$$

$$\text{cov}(\sum \alpha_i x_i, \sum \beta_j y_j) = \sum \sum x_i \beta_j \dots$$

$$\begin{aligned} & \text{Cov}\left(\frac{\sum xy}{\sum x^2}, \frac{\sum xy}{\sum x^2} - \frac{\sum (x-\bar{x})y}{\sum (x-\bar{x})^2}\right) = \\ &= \frac{1}{(\sum x^2)^2} \cdot \sum x^2 \text{Var } y - \frac{1}{\sum x^2} \cdot \frac{1}{\sum (x-\bar{x})^2} \cdot \sum x(x-\bar{x}) \text{Var } y = \\ &= \sigma^2 \left[\frac{1}{\sum x^2} - \frac{1}{\sum x^2} \cdot \frac{1}{\sum (x-\bar{x})^2} \cdot \sum (x-\bar{x})^2 \right] = 0. \end{aligned}$$

$$\begin{aligned} \text{Var}(\widehat{\beta}_1^0 - \widehat{\beta}_1) &= \text{Var} \widehat{\beta}_1^0 + \text{Var} \widehat{\beta}_1 - 2 \text{Cov}(\widehat{\beta}_1^0, \widehat{\beta}_1) = \\ &= \frac{\sigma^2}{\sum x^2} + \frac{\sigma^2}{\sum (x-\bar{x})^2} - 2 \cdot \text{Cov}(\widehat{\beta}_1^0 - \widehat{\beta}_1^0, \widehat{\beta}_1^0) - 2 \underbrace{\text{Cov}(\widehat{\beta}_1^0, \widehat{\beta}_1^0)}_{=0} = \\ &= \sigma^2 \left(\frac{1}{S_{xx}} - \frac{1}{S_{xx}^0} \right). \end{aligned}$$