RMSC 4007

Study on Multi-Asset Income Notes

Group Six

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Abstract

The major target of this project is to price a structured product and implement risk contorl, using various models and techniques, including Heston Model, Duan GARCH Model, and hedging methods. We may also discuss the risk embedded in this product.

1 Introduction

1.1 Product Decription

The product is a 8 Percent Income Notes linked to the FTSE 100 Index and the S&P 500 Index, due July 25, 2017, issued by Goldman Sachs Group, Inc.. The product pays yearly coupon and final pay-off on maturity date, both of which are jointly determined by both underlying indices. The market quote from Sep to Oct is within 1.09 and 1.11 and on Oct. 11 the price is 1.10.

Pay-off Structure
Interest
On each Interest Payment Date, investors will receive GBP 0.08 for each certificate if both indeces are greater or equal than respective coupon level; if on any preceeding valuation date both indices are greater or equal than respective lock-in level, investors will receive all following coupon payments regardless of index level.
Settlement Amount at Maturity
On Maturity Date, investors will receive GBP 1.00 for each certificate, if level of both indices are greater than respective Barrier Level on final valuation date; Otherwise, investors will receive GBP 1.00 * Minimum Index Performance (MIP) for each certificate, where MIP is defined as the worse of two index performance (Final Level / Initial Level)

1.2 Feature of the Product

The product provides exposure to both British and US stock market, and is suitable for investors who expect relatively low downward volatility in both markets during contract period. As shown in the graph, the profit from this product is capped, while the loss can be as much as the total amount invested.

1.3 Product Structure

Let C be the nominal amount of coupon per payment, and $\{t_i\},\ i = 0, 1, ..., n$ be the date for payment. Define $r_i(t),\ i = 1, 2,\ 0 \leq t \leq T$ be the risk-free interest rate and exchange rate at time t, and $I_i(0),\ I_i(T),\ i = 1, 2$ be the index levels at time=0 and maturity T respectively, where $I_1(0) = 1316.14$ and $I_2(0) = 5843.66$. Let $\xi(t_k) = min\{\frac{I_1(t_k)}{I_1(0)}, \frac{I_2(t_k)}{I_2(0)}\}$ for $k = 1, 2, ..., 6$. Then the barrier events could be defined as

- $\mathcal{R}_B(t_k) = \{j < k \mid \xi(t_j) \geq 1.2\}$
- $\mathcal{R}_A(t_k) = \{\xi(t_k) \geq 0.6\}$
- $\mathcal{B} = \{\xi(T) \leq 0.6\}$

The total payoff at T is

$$\Pi(T) = C\Sigma_{k=1}^{n}\{\mathbf{1}_{\mathcal{R}_A(t_k)} + \mathbf{1}_{\mathcal{R}_A(t_k)^c \cap \mathcal{R}_B(t_k)}\}exp\int_{t_k}^{T} r(t)dt + \{\xi(T)\mathbf{1}_{\mathcal{B}} + \mathbf{1}_{\mathcal{B}^c}\}$$

2 Black Schole Model

In this section, we investigate BS model on the two indeces and exchange rate for constant r_d and r_f. Under domestic $\mathcal{Q}$ measure, we have

$$\frac{dF(t)}{F(t)} = (r_d - r_f)dt + \sigma_E dW^d_{E,t} \quad (1)$$

$$\frac{dI_1(t)}{I_1(t)} = r_d dt + \sigma_1 dW^d_{1,t} \quad (2)$$

$$\frac{dI_2(t)}{I_2(t)} = (r_f - \rho_{E,3}\sigma_E\sigma_2)dt + \sigma_2 dW^d_{2,t} \quad (3)$$

for convenience, we define $\hat{I}_i(t) = \frac{I_i(t)}{I_i(0)}$ and $X_i = ln(\frac{I_i(t)}{I_i(0)})$ for i=1,2. The procedure to change I_2 to demostic measure is attached in appendix.

For Black-Schole Model, the product is replicated by three products:

- Coupons consist of digital call option which would be paid if $\xi(t_k) > 0.6$
- Coupons consist of up-and-in put option which would be paid if $\xi(t_k) < 0.6$ with lock-in event $\xi(\tau) > 1.2$ for $\tau = t_1, t_2, ...$
- Final settlement.

Thus the payoff function could be shown as

$$\Pi(T) = \Pi_1(T) + \Pi_2(T) + \Pi_3(T)$$

where

- $\Pi_1(T) = C\Sigma_{k=1}^n\{\ e^{-r(t_k-t)}\mathbf{1}_{\mathcal{R}_A(t_k)}\}$
- $\Pi_2(T) = C\Sigma_{k=1}^n\{e^{-r(t_k-t)}\mathbf{1}_{\mathcal{R}_A(t_k)^c\cap\mathcal{R}_B(t_k)}\}$
- $\Pi_3(T) = e^{-r(T-t)}(\xi(T)\mathbf{1}_{\mathcal{B}} + \mathbf{1}_{\mathcal{B}^c})$

The correctness of the treatment of the coupon in original product is given in appendix.

2.1 Multi-Dimensional Normal Distribution and Monte Carlo Simulation

The close form for the payoff of this product involves a 12-dimensional multivariate normal distribution at time t_0 for 6 times of coupon and 6-dimension for time t for 3 times of coupon, involving 14 events. The form is sophisticated and lacks universality, and the computation time will grow fast from 10 sec for 6-dim to 3 min for 12-dim if more dimensions are involved, compared with few seconds using Monte Carlo simulation.

	Coupon	Total	Error	Time
6-D	0.2130445	1.092	$o(tolerance) * o(2^n)$	$O(compute) * O(2^n)$
MC (10^4)	0.2134255	1.092381	$o(\sqrt{\frac{S^2(m)}{m}})$	$O(mn)$

2.2 Multi-dimensional Random Walk Tree

Both the close form and MC are not satisfying since multi dimension costs a long time while MC result fluctuates greater than close form (for multi normal, the calculation for probability itself has an error term). Thus we hope to use multi-dimensional random walk tree to find a value which converges to close form solution.

To build a RW tree, we need to guarantee weak convergence to the continuous provess, where

$$\mu_i(N) := \frac{1}{\delta t}[E_{i(N)}(ln(\frac{S_{k,i}}{S_{k-1,i}})|S_{k-1,i})] \to r_i - \frac{1}{2}\sigma_i^2 \tag{4}$$

$$\sigma_i^2(N) := \frac{1}{\delta t}[Var_{i(N)}(ln(\frac{S_{k,i}}{S_{k-1,i}})|S_{k-1,i})] \to \sigma_i^2 \tag{5}$$

$$c_{i,j}(N) := \frac{1}{\delta t}[Cov_N(ln(\frac{S_{k,i}}{S_{k-1,i}}), ln(\frac{S_{k,j}}{S_{k-1,j}})|S_{k-1,i}, S_{k-1,j})] \to \sigma_i\sigma_j\rho_{i,j} \tag{6}$$

Thus we applied multi-dimensional RB tree and the four events are defined as

	probability	amount A	amount B
uu	$\frac{1}{4}(1+\rho)$	$exp\{(r_1 - \sigma_1^2/2)\delta t + \sigma_1\sqrt{\delta t}\}$	$exp\{(r_2 - \sigma_2^2/2)\delta t + \sigma_2\sqrt{\delta t}\}$
ud	$\frac{1}{4}(1-\rho)$	$exp\{(r_1 - \sigma_1^2/2)\delta t + \sigma_1\sqrt{\delta t}\}$	$exp\{(r_2 - \sigma_2^2/2)\delta t - \sigma_2\sqrt{\delta t}\}$
du	$\frac{1}{4}(1-\rho)$	$exp\{(r_1 - \sigma_1^2/2)\delta t - \sigma_1\sqrt{\delta t}\}$	$exp\{(r_2 - \sigma_2^2/2)\delta t + \sigma_2\sqrt{\delta t}\}$
dd	$\frac{1}{4}(1+\rho)$	$exp\{(r_1 - \sigma_1^2/2)\delta t - \sigma_1\sqrt{\delta t}\}$	$exp\{(r_2 - \sigma_2^2/2)\delta t - \sigma_2\sqrt{\delta t}\}$

Hence the mean and variance processes converge to continuous process if δt tends to 0. We used totally 100 time points to compute the probability and the result is

	Coupon	Time
RW tree	0.2137536	$O(n^2)$

2.3 Lower Bound and Upper Bound Using Digital Option

Although MC and RW tree converge fast and cost little time, we hope to find some theoretical results rather than technical methods to study the product.

Consider

$$\Pi_1(t) = C\Sigma_{k=1}^n \mathbf{1}_{\mathcal{R}_B(t_k)} e^{-r(t_k-t)} + e^{-r(T-t)}\{\xi(T)\mathbf{1}_{\mathcal{B}} + \mathbf{1}_{\mathcal{B}^c}\} \tag{7}$$

$$\Pi_2(t) = C\Sigma_{k=1}^n e^{-r(t_k-t)} + e^{-r(T-t)}\{\xi(T)\mathbf{1}_{\mathcal{B}} + \mathbf{1}_{\mathcal{B}^c}\} \tag{8}$$

For $\Pi_1(T)$ only focuses on the performence at the current time point, giving the lower bound of the original one. For $\Pi_2(T)$, this product just gives coupon at every check point, thus it gives an upper bound. The close form solution of the payoff of the two products above are given in appendix.

	Final	Coupon	Sum
Upper	0.8789555	0.235782	1.114737
Lower	0.8789555	0.2126706	1.091626

2.4 An Alternative Upper Bound Using Continuous-Time Barrier

It is difficult to compute the price if the stopping event involves two Wienner process, thus we hope to use stopping event only concerning one index to provide an alternative upper bound. The replicate coupon contians two parts:

- A up-and-in digital option. If return of index one has ever been greater than 1.2 and at t_k index return is lower than 0.6, then 0.08 will be paid.
- A up-and-in rainbow option. If return of index one has ever been greater than 1.2 and at t_k index two has return lower than 0.6, then 0.08 will be paid.

2.4.1 One asset continous-time look-back option

We first considered a barrier option with index 1 as underlying asset.
For $\tau = inf\{X_\tau > ln(1.2)\}$

$$\Pi_3(k) = \begin{cases} 0.08 & if\ \tau < t_k,\ X_{t_k} < ln(0.6) \\ 0 & otherwise \end{cases} \tag{9}$$

The expected payoff for one coupon has close form solution using relection principle

$$E^*(\Pi_3(k)) = e^{-r(t_k-t)}(\frac{1.2}{\hat{I}_1})^{2(\frac{r-\frac{\sigma^2}{2}}{\sigma^2})}\Psi(-\frac{2ln(1.2)-ln(0.6)-X_1+(r-\frac{\sigma^2}{2})(t_k-t)}{\sigma\sqrt{t_k-t}})$$

For detailed computation, please refer to appendix.

2.4.2 Adjustment to discrete monitored barrier one-asset case

It is possible to adjust coupon price to discrete monitored barrier case. Broadie et al(1997) proposed a continuity correction for this type of option. Let $V(H)$ be the price of continuous barrier option, and $V_m(H)$ be the option with m monitoring points. Then

$$V_m(H) = V(He^{\beta\sigma\frac{T}{m}}) + o(\frac{1}{\sqrt{m}})$$

This result could be reached from renewal theory. For details, please refer to appendix.

Now define overshoot $R_{t_k} = X_\tau - ln(1.2)$, using reflection principle

$$P(X_{t_k} < x, \tau \le t_k) \quad = \quad P(X_{t_k} > 2ln(1.2) + 2R_{t_k} - x) + o(\frac{1}{\sqrt{m_k}}) \tag{10}$$

Replace R_{t_k} by $E(R_{t_k})$, which could be computed by renewal theory. The result of this $E(R_{t_k})$, which could be represented by a function involved zeta function.

$$E(R_{t_k}) = -\frac{\sigma t_k}{m_k}\frac{\zeta(1/2)}{\sqrt{2\pi}}$$

Here $\zeta(.)$ is zeta function. Thus the pricing formula for $\Pi_3(i)$ is

$$E_t^{**}(\Pi_3(k)) \quad = \quad e^{-r(t_k-t)}((\frac{1.2}{\hat{I}_1})e^{E(R_{t_k})})^{2(\frac{r-\frac{\sigma^2}{2}}{\sigma^2})} \tag{11}$$

$$*\Psi(-\frac{2ln(1.2)-ln(0.6)-X_1+2E(R_{t_k})+(r-\frac{\sigma^2}{2})(t_k-t)}{\sigma\sqrt{t_k-t}}) + o(\frac{1}{\sqrt{m_k}}) \tag{12}$$

Notice that there is an error term in the formula. For our product, since the coupon pays at most 0.08, this term is around $0.08 * o(\frac{1}{\sqrt{m}})$. This result provide us with an idea that if we could solve the problem of 2-D Wiener process, we may adjust it to discrete case using results provided by Cao and Kou(2007).

2.4.3 Digital rainbow option

Rainbow option involves two dimentional Wiener process with barrier on one asset. This kind of option has close form solution using Fourier tranform to solve partial differential equation. Since we have already computed the barrier digital option on index 1, if we could find the up-and-in option giving 0.08 if return of index 2 is lower than 0.6, then we would have an alternative upper bound.

The probability for payment is

$$
\begin{aligned}
E_t^{***}(\Pi_3(k)) &= e^{-r_1(t_k-t)}P(\tau < t_k) - e^{-r_1(t_k-t)}P(X_2 > ln(0.6)) + \frac{1.2^{1-a-b}\hat{I_1}^a\hat{I_2}^b e^{c(t_k-t)}}{2\pi\sigma_1\sigma_2(t_k-t)\sqrt{1-\rho^2}} && (13)\\
&\quad * \int_{-\infty}^{0} d\xi_1 \int_{ln(0.5)}^{\infty} [e^{-\frac{(-ln(1.2)+X_1-\xi_1)^2}{2\sigma_1^2\eta}} + e^{-\frac{(-ln(1.2)+X_1+\xi_1)^2}{2\sigma_1^2 t_k}}]e^{-a\xi_1-b\xi_2} * e^B d\xi_2 && (14)\\
B &= -\frac{[-ln(1.2)+X_2-\xi_2-\frac{\rho\sigma_2(-ln(1.2)+X_1-\xi_1)}{\sigma_1}]^2}{2\sigma_2^2(1-\rho^2)(t_k-t)} && (15)\\
P(\tau < t_k) &= \Psi(-\frac{ln(1.2)-X_1-(r-\frac{\sigma^2}{2})(t_k-t)}{\sigma\sqrt{t_k-t}}) && (16)\\
&\quad +(\frac{1.2}{\hat{I}_1})^{2(\frac{r-\frac{\sigma^2}{2}}{\sigma^2})}\Psi(-\frac{ln(1.2)-X_1+(r-\frac{\sigma^2}{2})(t_k-t)}{\sigma\sqrt{t_k-t}}) && (17)
\end{aligned}
$$

The original pricing formula is not for digital option. We got this based on the derivation provided by WANG (2007) and got this formula (for detailed steps, please refer to appendix), then combine the payoff in 2.4.1 and 2.4.3 together and adjust to discrete case (the computation formula is in appendix, we made some modification). Since there has been an error term if adjusted to discrete case, we plug in R_{t_k} term without prove its correctness. However, $R^*_{t_k}$ should be changed since the second process involved W_1. For this issue, further study is needed.

The coupon would be paid if index 1 hitted 1.2 and at t_k anyone of the index has return lower than 0.6. If both are lower than 0.6, the payment would be twice. The probability of index 1 hit 1.2 is greater than the minimum one hits, while overlap exists for payment, thus this provide an alternative upper bound.

lower	MC	E***	upper
0.2123647	0.2134255	0.2172053	0.235782

For lock-in event focusing on only one index, the pricing formula has already included a double integration, which is not convenient for ue to study on its properties. If we tighten up the lock-in event condition from "index one has ever been greater than 1.2" to "the minimum performance has ever been greater than 1.2", we need to solve the PDE for coupon.

2.5 Approximation using two indeces

Two assets barrier digital option for the last part of coupon using heating equation to solve 2D PDE. We tried to use heating equation to solve the 2D PDE refering to Rossella's result, where the original problem is defined in one quadrant for X_1 and X_2. When trying to solve this PDE, we found problems in initial conditions, which leading to failure when using the orthogonal property of sine series, thus we loose the condition to

- If both returns reach 1.2, then the ith coupon will be paid.
- If minimum of return reaches 0.6 while any one has never reached 1.2, then coupon paid.

to make sure the domain is within π. Then for the second part,

$$\frac{\partial V}{\partial t} + (r_1 - \frac{\sigma_1^2}{2})\frac{\partial V}{\partial X_1} + (r_2 - \frac{\sigma_2^2}{2})\frac{\partial V}{\partial X_2} + \frac{\sigma_1^2}{2}\frac{\partial^2 V}{\partial X_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial X_2^2} + \rho\sigma_1\sigma_2\frac{\partial^2 V}{\partial X_1 \partial X_2} - r_1 V = 0$$

where

$$X_1 = ln(\hat{I_1}) - ln(1.2) \qquad X_2 = ln(\hat{I_2}) - ln(1.2) \qquad V(X_1 \geq 0 \; or \; X_2 \geq 0) = 0 \tag{18}$$

$$V(t=T) = 1\{\{X_1(T) < ln(0.5)\} \cup \{X_2(T) < ln(0.5)\}\} \tag{19}$$

Here $ln(0.5) = ln(0.6) - ln(1.2)$. After transformation, the initial conditions are

$$S_2 = Ve^{r_1\eta + aX_1 + bX_2} \qquad D_2 = \{\pi \leq \theta \leq \pi + \psi_0\} \tag{20}$$

$$S_2(\tau = 0) = V(\tau = 0)e^{prcos\theta + qrsin\theta} := \omega(r, \theta) \tag{21}$$

and the solution is

$$S_2(\eta, r_0, \theta_0) = \Sigma_{k=1}^{+\infty} e^{-\lambda_k^2 \eta} D_k J_{\nu_k}(\lambda r_0) \int_0^{+\infty} \int_{\pi}^{\pi+\psi_0} J_{\nu_k}(\lambda r) sin(\nu_k \theta_0) sin(\nu_k \theta) \omega(r, \theta) r d\theta dr$$

for some η, D_k, λ_k, ν_k. And (r_0, θ_0) are computed from (X_1, X_2) and θ_0 is in D_2.

For the firsrt part,

$$S_1(\eta, r_0, \theta_0) = 1 - \Sigma_{k=1}^{+\infty} e^{-\lambda_k^2 \eta} D_k J_{\nu_k}(\lambda r_0) \int_0^{+\infty} \int_0^{\psi_0} J_{\nu_k}(\lambda r) sin(\nu_k \theta_0) sin(\nu_k \theta) r d\theta dr$$

For detailed computation, please refer to appendix. However, the computation of some terms is too complicated thus we failed to find a numerical solution, so as to verify its correctness. In addition, even if we found a numerical solution, it would cost about $O(n^4)$ time, compared with $O(mn)$ of BS and $O(n^2)$ of RW tree since the solution contains four integrations.

3 Multi-Asset Heston Model

Since the simple Black Schole model assumes constant volatility, which could not explain the volatility-smile problem, we introduce two models, Heston model and Duan GARCH model, to explain the changing volatility.
In Heston model (1993), both index itself and its volatility follow stochastic process.

$$\frac{dF}{F} = (r1 - r2)dt + \sigma_E dW_E \tag{22}$$

$$\frac{dI_1}{I_1} = r_1 dt + \sqrt{\nu_1} dW_1 \tag{23}$$

$$d\nu_1 = \kappa_1(\theta - \nu_1)dt + \eta_1\sqrt{\nu_1} d\dot{W}_1 \tag{24}$$

$$\frac{dI_2}{I_2} = r_2 dt - \sigma_E \rho_{E,I_2} \sqrt{\nu_2} dt + \sqrt{\nu_2} dW_2 \tag{25}$$

$$d\nu_2 = \kappa_2(\theta - \nu_2)dt - \eta_2 \sigma_E \rho_{E,\nu_2} \sqrt{\nu_2} dt + \eta_2 \sqrt{\nu_2} d\dot{W}_2 \tag{26}$$

where $W_i(t)$ and $\dot{W}_i(t)$ Brownian process under measure $\mathbf{Q}$ and μ_i has been adjusted to domestic measure for foreign index. The method for changing foreign to domestic measure is similar with the one for BS model, thus we skip the proof.

3.1 Calibration for Single-Asset Heston Model

For Single-asset Heston Model, the close form solution of European call option is available and calibration can be conducted with efficiency with 5 parameters to be calibrated. However, it is sophisticated to calibrate on multi-asset cross-currency framework, where avaliable products are not activated and correlation parameters are hard to calibrated. The parameters could be calibrated by minimizing the loss function

$$L(\Theta|I, r) = \Sigma_{i=1}^{N} w_i \{C_M(i) - C_H(K_i, \tau_i, I_i, r_i, \Theta)\}^2$$

where Θ is the collection of parameters in the ith index model, N is the number of European call option prices available, $C_M(i)$ is the market price for the ith option, $C_H(.)$ is the theoretical price under Heston model, K_i,τ_i, I_i, r_i are the strike price,futures price, maturity, current index level, and risk-free rate for each call option.

$$V(I,\nu,\tau;\Theta) = e^{-r\tau}(IP_1(I,\nu,\tau;\Theta) - KP_2(I,\nu,\tau;\Theta)) \tag{27}$$

$$P_k(I,\nu,\tau;\Theta) = [\frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Re\{\frac{e^{-i\psi log(K)}\Phi_k(I,\nu,\tau;\Theta)}{i\psi}\}d\psi]e^{r\tau 1\{k=1\}} \tag{28}$$

$$\hat{\Theta} = argmin\{\Theta : L(\Theta|I,r)\} \tag{29}$$

The result of calibration is as follow:

	κ	θ	ν_0	ϵ	ρ
FTSE	0.945468	0.101905	0.011370	0.438973	-0.999999
SP500	0.941560	0.148626	0.007087	0.529037	-0.999999

We noticed that correlation of index level and respective volatility is essentially -1 and it seems unnatural. Besides, we observed that market price for some call options significantly deviated from their theoretical price under Heston model. Therefore there exists probability that the market does not match with Heston model.

3.2 Valuation Under Single Heston Model

Using the result in 4.1.1 and Full Truncated Discretization Scheme, we made simultaion to price the product. Since the whole model consists of 4 brownian motions and is sophisticated to compute the close form solution, we only apply simulation on this model.

	Coupon	Total	Time
Heston	0.217118	1.075623	30 sec
BS	0.2134255	1.092381	10 sec

The expectation of total payoff for Heston model is lower than the one of BS model, while the payoff for coupon is much higher, there are some possible reasons:

- Effect on Full Truncation: We notice that the simulation result for log return itself does not follow normal distribution. Instead, it follows a left-skewed distribution. This may be caused by full truncation scheme. Although the stock price converges to theoretical expectation, the distribution may not be the same, leading to a different probability under 0.6 and above 1.2. As a result, for coupon, the lock-in event occurs with larger probability and the expectation for this part is higher.
- Model Misspecification: When searching FTSE option data, we found the option price fluctuates in a strange pattern: in some specific time points, the available strike prices are much lower than other perioulds and the bid prices are much higher than others. The mechanism of the real world is comlicated than a simple model.

4 Multi-Asset Duan-GARCH Model

Although Heston's model intends to capture the stochastic volatility structure, it may not be practical in real use due to the complirrelation structure. Hence we consider a Duan-GARCH Model in multi-asset framewrok with qunto properties as a substitute. It could well explain the volatility clustering and leptokurtosis issues, while utilizing the locally risk-neutral valuation relationship and providing a risk-neutral solution to price structured products.

Under physical measure $\mathbb{P}$, the model for two index is defined as:

$$R_t^{(j)} := ln(\frac{I_t^{(j)}}{I_{t-1}^{(j)}}) = r^{(j)}\Delta t + \lambda^{(j)}\sqrt{h_t^{(j)}} - \frac{1}{2}h_t^{(j)} + \varepsilon_t^{(j)} \tag{30}$$

$$h_t^{(j)} = w^{(j)} + a^{(j)}(\varepsilon_{t-1}^{(j)})^2 + \beta^{(j)}h_{t-1}^{(j)} \tag{31}$$

where $\varepsilon_t^{(j)}|_{\mathcal{F}_{t-1}} \sim N^{\mathbb{P}}(0, h_t^{(j)})$ for $j = 1,2$ with $E^{\mathbb{P}}(\varepsilon_t^{(1)}\varepsilon_t^{(2)}) = \rho^{1,2}\sqrt{h_t^{(1)}h_t^{(2)}}$

Exchange rate is assumed to follow standard Blacl Schole Model, the discretized version is formulated as following.

$$\Delta F_t := F_t - F_{t-1} = F_{t-1}(\mu_F \Delta t + \sigma_F \varepsilon_t^{(3)}) \tag{32}$$

For simplicity, define the correlation matrix among underlying indeces and the exchange rate as

$$Corr^P\{(\varepsilon^{(1)}, \varepsilon^{(2)}, \varepsilon^{(3)})^T\} =: [\rho^{i,j}]_{i,j=1}^3 \; in \; R^{3\times 3}$$

4.1 Estimation Methodology and Result

Standard Maximum Likihood Estimation Method was adopted to estimate the parameters in this Multi-asset Duan GARCH Model under measure $\mathbb{P}$ since there is no anaytical formula for European call price under GARCH to facilitate calibration. Marginal MLE for two indices will be obtained first, and serve as initial guess for the optimization of joint likihood function to accelerate the convergence. Historical data series is of length 1600 days.

The detailed procedure is stated below:

- STEP 1: Obtain the marginal MLE for each index given by

$$\hat{\Theta}^{(j)} = argmax\{\Theta^{(j)} : L(\Theta^j|\mathcal{F}_0)\}$$

for $j = 1, 2$, with

$$\Theta^j = \{h_0^{(j)}, \lambda^{(j)}, w^{(j)}, a^{(j)}, \beta^{(j)}\}$$

$$L^j(\Theta|\mathcal{F}_0) = \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi(h_{i_t}^{(j)})}} exp(\frac{-(Ln\frac{I_t^{(j)}}{I_{t-1}^{(j)}} - r^{(j)} - \lambda^{(j)}\sqrt{h_t^{(j)}} + \frac{1}{2}(h_t^{(j)}))^2}{2(h_t^{(j)})}) \tag{33}$$

- STEP 2: Obtain the joint MLE from $\hat{\Theta} = argmax\{\Theta : L(\Theta|\mathcal{F}_0)\}$

$$\Theta = \{h_0^{(1)}m, \lambda^{(1)}, w^{(1)}, a^{(1)}, \beta^{(1)}, h_0^{(2)}, \lambda^{(2)}, w^{(2)}, a^{(2)}, \beta^{(2)}, \rho^{(1,2)}\}$$

with

$$L(\Theta|\mathcal{F}_0) = \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi det(\Sigma_t)}} exp(-\frac{1}{2}(R_t - \mu_t)^T \Sigma^{-1}(R_t - \mu_t) \tag{34}$$

$$\mu_t = \begin{bmatrix} r^{(1)} - \frac{h_t^{(1)}}{2} + \lambda^{(1)}\sqrt{h_t^{(1)}} \\ r^{(2)} - \frac{(h_t^{(2)})}{2} + \lambda^{(2)}\sqrt{h_t^{(2)}} \end{bmatrix} \Sigma_t = \begin{bmatrix} h_t^{(1)} & \rho^{1,2}\sqrt{h_t^{(1)}h_t^{(2)}} \\ \rho^{1,2}\sqrt{h_t^{(1)}h_t^{(2)}} & h_t^{(2)} \end{bmatrix} \tag{35}$$

- STEP 3: Volatility of exchange rate and correlation between index 2 and exchange rate $\sigma_F, \rho^{2,3}$ will be directly estimated using sample data.

Please refer to appendix for the resulting marginal and joint MLE.

4.2 Pricing under Duan-GARCH Model

Using change of measure, equation 30 and 31 can be transformed into domestic risk neutral measure $\mathbb{Q}^{(1)}$.

$$R_t^{(1)} := ln(\frac{I_t^{(1)}}{I_{t-1}^{(1)}}) = r^{(1)}\Delta t - \frac{1}{2}h_t^{(1)} + \widetilde{\varepsilon^{(1)}}_t \tag{36}$$

$$h_t^{(1)} = w^{(1)} + a^{(1)}(\widetilde{\varepsilon^{(1)}}_{t-1} - \lambda^{(1)}\sqrt{h_{t-1}^{(1)}})^2 + \beta^{(1)}h_{t-1}^{(1)} \tag{37}$$

$$R_t^{(2)} := ln(\frac{I_t^{(2)}}{I_{t-1}^{(2)}}) = (r^{(2)} - \sigma_F\rho^{2,3}\sqrt{h_t^{(2)}})\Delta t - \frac{1}{2}h_t^{(2)} + \widetilde{\varepsilon^{(2)}}_t \tag{38}$$

$$h_t^{(2)} = w^{(2)} + a^{(2)}(\widetilde{\varepsilon^{(2)}}_{t-1} - \sigma_F\rho^{2,3}\sqrt{h_{t-1}^{(2)}}\Delta t - \lambda^{(2)}\sqrt{h_{t-1}^{(2)}})^2 + \beta^{(2)}h_{t-1}^{(2)} \tag{39}$$

where

$$\widetilde{\varepsilon_t^{(j)}}|_{\mathcal{F}_{t-1}} \sim N^{\mathbb{Q}^{(1)}}(0, h_t^{(j)})$$

for $j = 1, 2$
The product price was calculated using Monte Carlo simulation for 20000 sample paths under above Duan-GARCH Model in risk-neutral measure.

	Mean	SD
Estimated Price	1.17606	0.1356577

By averaging, we obtain a risk-free price around 1.17606 on Oct 1, 2014, which is higher than market quote and Heston Model. The possible reason is that the both indices experience stable growth during the past 6 years. Unlike the Heston which implements the calibiration, the MLE method based on the historical data assums the stationarity of the log return series, thus it cannot capture the market expection and news. So the resulting simulated index paths are tend to go over the barrier level and overestimate the product price.

5 Risk Analysis

The return of this note depends on the worse performance between the two indices. In the case of barrier event, investors may lose part or all of their initial investment.

5.1 Risk Indicators

We calculated Value at Risk (VaR), Expected Shortfall (ES) and Probability of Barrier Event (PBE) under all the 3 models. The results are showed in table in section 8.4. At 95% confidence level, the worst loss can be around 50% of initial investment, and in the case of the worst 5% outcome, average loss is around 60%. Thus this note is considerably risky and may cause significant loss.

5.2 Greek Letters

By finite difference method, we calculated Delta and Gamma with respect to the two underlying indices. From graph in section 8.4 we can see under both Duan-GARCH and Heston model, Delta is significantly high around respective barrier level. In other words, price of the security is very sensitive to index fluctuation when indices are close to its barrier level, and less sensitive when indices are significantly higher than barrier level. Gamma is almost constant and is close to zero, indicating relatively small curvature of the price curve against index level. This is also consistent with the price surface under Duan-GARCH model in graph in section 8.4.

5.3 Replication and Hedging

As discussed in previous sections, the security can be decomposed into 3 parts. Nevertheless, such specific products barely exist in the market. Therefore we consider hedging strategy by maintaining Delta-neutral position.

We investigate both static and dynamic Delta hedging strategy for this security. In static Delta hedging, we hold short position of futures on the two underlying indices with amount Delta1 and Delta2. As for dynamic Delta hedging, it would be computationally intensive if we compute Delta at each time point, thus we adopt Simple Least-Square method proposed by Longstaff and Schwartz (2001). We first regress simulated final pay-off over the index levels at previous time point, and take the discounted fitted value as price at previous time:

$$f(t) = \beta_{0,t-1} + \beta_{1,t-1} I_1(t-1) + \beta_{2,t-1} I_2(t-1) \tag{40}$$

$$\hat{f}(t-1) = e^{-r_d \Delta_t}(\hat{\beta}_{0,t-1} + \hat{\beta}_{1,t-1} I_1(t-1) + \hat{\beta}_{2,t-1} I_2(t-1)) \tag{41}$$

The coefficients of indices, on the other hand, are estimators of deltas. By repeating the regression backwards, we can estimate delta at each time point and construct dynamic delta hedging strategy accordingly.

As shown in graphs in section 8.4, under all the 3 models, dynamic hedging provides more stable return. However in practice, frequent rebalancing will incur high transaction cost, which must be taken into consideration.

5.4 Other Risk

The issuer has Baa1 rating by Moodys and A- by S&P, thus issuer credit risk is slight yet unnegligible. Moreover, as issue size is small and secondary market is not likely to develop for exotic certificates, there exists significant liquidity risk.

Besides, investors of this security also face exchange rate risk, as the two underlying indices are based in two different markets and are denoted in two different currencies. Although the note itself is settled with single currency, variance of exchange rate will affect the risk-neutral valuation of this Quanto product. In each market, various factors, such as economic, regulatory and political events, jointly influence the performance of indices as well as exchange rate.

6 Conclusion

This project aims at derivative pricing and risk analysis for multi-asset income notes. For derivative pricing, we used Black-Schole, Heston and Duan-GARCH model to model underlying assets, and provided several methods to compute or estimate the price. For risk analysis, we analyzed the risk of the product and provided some strategies for hedging.

For Black-Schole model, we used techniques including look-back option and rainbow option to derive upper and lower bound of our product. These results might be helpful for further study and risk analysis.

For Heston model, the full truncation scheme leads to a different distribution for log return of indeces. With the suggestion of group 7, we tried their scheme, but the result is around 1.050. It is possible that the schemes are not good for barrier option and look-back option, thus for further study, we may try to find a suitable scheme.

For Duan-GARCH model, it overestimate the price, probably because of the long time span of the product. Before issuance, the economy was depressed and the return for both underlying assets were negative. The return of them had a three-year up-and-down cyclical pattern. But for recent years, the two assets kept rising, thus using short-term option and data for only recent years may not be accurate for predicting the price after three years.

For risk analysis, we do not have enough time to correct the mistake in our project, where we used Q measure from the beginning to the end of the holding period in calculating VaR. Thus for further study, we need to correct this mistake first. In addition, in the future, we could use empirical distribution for underlying assets to analyze the risk and provide results combined with theoretical results from BS model.

7 Appendix

7.1 Group Member Information

Name	SID	Part of Work
CHEN Wenxin	1155014455	Risk Analysis, Calibration
XING Yue	1155014329	BS Model, Simulation for Heston Model
ZHOU Zhihao	1155014412	Duan-GARCH Model

7.2 Proof for Replication of Coupon

Let $\gamma = C\Sigma_{k=1}^{n}\{\mathbf{1}_{\mathcal{R}_A(t_k)} + \mathbf{1}_{\mathcal{R}_A(t_k)^c\cap\mathcal{R}_B(t_k)}\}exp\int_{t_k}^{T} r(t)dt$, then the scenarios are

Scenario at time t_k		Coupon 1	Coupon 2	Total
$\xi(t_k) < 0.6$	$\xi(t_j) < 1.2\ \forall\ j < k$	0	0	0
$0.6 \leq \xi(t_k)$	$\xi(t_j) < 1.2\ \forall\ j < k$	0.08	0	0.08
$\xi(t_k) < 0.6$	$\exists\ j < k,\ \xi(t_j) \geq 1.2$	0	0.08	0.08
$0.6 \leq \xi(t_k)$	$\exists\ j < k,\ \xi(t_j) \geq 1.2$	0.08	0	0.08

Thus the replicating coupons has the same payoff structure with the original coupon.

7.3 Changing Foreign Index to Domestic Measure

The correlation matrix for (W_E, W_1, W_2) is given by

$$\Sigma = \begin{bmatrix} 1 & \rho_{E,1} & \rho_{E,2} \\ \rho_{E,1} & 1 & \rho_{1,2} \\ \rho_{E,2} & \rho_{1,2} & 1 \end{bmatrix}$$

the dynamics could be writtern as

$$\begin{aligned} dln(F) &= (r_d - r_f - \frac{\sigma_E^2}{2})dt + \sigma_E dW_E^d && (42)\\ dX_1 &= (r_d - \frac{\sigma_1^2}{2})dt + \sigma_1 dW_1^d && (43)\\ &= (r_d - \frac{\sigma_1^2}{2})dt + \sigma_1(\rho_{E,1}dW_E^d + \sqrt{1-\rho_{E,1}^2}dB_1) && (44)\\ dX_2 &= (r_f - \frac{\sigma_2^2}{2})dt + \sigma_2 dW_2^f && (45)\\ &= (r_f - \frac{\sigma_2^2}{2})dt + \sigma_2(\rho_{E,2}dW_E^f + \sqrt{1-\rho_{E,2}^2}dB_2) && (46) \end{aligned}$$

now consider changing dW_E^d to foreign measure,

$$dW_E^f = dW_E^d - \sigma_E dt$$

thus

$$\begin{aligned} dX_2 &= (r_f - \frac{\sigma_2^2}{2})dt + \sigma_2(\rho_{E,2}dW_E^f + \sqrt{1-\rho_{E,2}^2}dB_2) && (47)\\ &= (r_f - \frac{\sigma_2^2}{2})dt + \sigma_2(\rho_{E,2}dW_E^d - \sigma_E\rho_{E,2}dt + \sqrt{1-\rho_{E,2}^2}dB_2) && (48)\\ &= (r_f - \frac{\sigma_2^2}{2} - \sigma_E\sigma_2\rho_{E,2})dt + \sigma_2(\rho_{E,2}dW_E^d + \sqrt{1-\rho_{E,2}^2}dB_2) && (49) \end{aligned}$$

consider $\Sigma = H'DH$ be the LU factorization and $H = \{h_{i,j}\}_{E,1,2}$, then $E(dB_1 dB_2) = h_{1,1}h_{1,2}dt$ thus we have

$$dX_2 = (r_f - \frac{\sigma_2^2}{2} - \sigma_E\sigma_2\rho_{E,2})dt + \sigma_2 dW_2^d$$

7.4 Lower and Upper Bound Using No-Barrier Singal Option

$$E_t(\Pi_1) = E_t(\Pi_{final}) + E_t(\Pi_{1,coupon})$$

$$E_t(\Pi_2) = E_t(\Pi_{final}) + C\Sigma_{k=1}^{n} e^{r(T-t_k)}$$

$$E_t(\Pi_{final}) = e^{X_1} P_s(A_1, B_1, \rho = \rho_1) + e^{X_2+(r_2-r_1)(T-t)} P_s(A_2, B_2, \rho = \rho_2) + e^{-r(T-t)} P_l(A(T-t), B(T-t), \rho = \rho)$$

$$E_t(\Pi_{1,coupon}) = C\Sigma_{k=1}^{n} e^{-r(t_k-t)} P_l(A(t_k-t), B(t_k-t), \rho = \rho)$$

where

$$P_s(x, y, \rho) = P(z_1 < x, z_2 < y, \rho = \rho)$$

$$P_l(x, y, \rho) = P(z_1 > x, z_2 > y, \rho = \rho)$$

$$A(t) = \frac{ln(0.6) - r_1 t - X_1 + \frac{\sigma_1^2 t}{2}}{\sigma_1\sqrt{t}} \qquad B(t) = \frac{ln(0.6) - r_2 t - X_2 + \frac{\sigma_2^2 t}{2}}{\sigma_2\sqrt{t}}$$

$$A_1 = \frac{(r_2 - r_1)(T-t) + (X_2 - X_1) - \frac{\hat{\sigma}^2(T-t)}{2}}{\hat{\sigma}\sqrt{(T-t)}} \qquad A_2 = \frac{(r_1 - r_2)(T-t) + (X_1 - X_2) - \frac{\hat{\sigma}^2(T-t)}{2}}{\hat{\sigma}\sqrt{(T-t)}}$$

$$B_1 = \frac{ln(0.6) - r_1(T-t) - X_1 - \frac{\sigma_1^2(T-t)}{2}}{\sigma_1\sqrt{(T-t)}} \qquad B_2 = \frac{ln(0.6) - r_2(T-t) - X_2 - \frac{\sigma_2^2(T-t)}{2}}{\sigma_2\sqrt{(T-t)}}$$

$$\hat{\sigma}^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 \qquad \rho_1 = \frac{\sigma_1 - \rho\sigma_2}{\hat{\sigma}} \qquad \rho_2 = \frac{\sigma_2 - \rho\sigma_1}{\hat{\sigma}}$$

7.5 Details on Computing Digital Barrier Option

Lemma $***$: for $W\ N(0,1),\ a > 0$

$$E(e^{mW}\mathbf{1}\{aW < b\}) = E(e^{-mW}\mathbf{1}\{aW > -b\})$$

Proof:

$$\begin{aligned}
E(e^{mW}\mathbf{1}\{aW < b\}) &= \int_{-\infty}^{+\infty} e^{mw}\mathbf{1}\{aw < b\} f(w)dw && (50)\\
&= \int_{-\infty}^{\frac{b}{a}} e^{mw} f(w)dw && (51)\\
&= \int_{+\infty}^{-\frac{b}{a}} e^{-mw^*} f(-w^*)d(-w^*)\frac{dw^*}{dw^*} && (52)\\
&= \int_{-\frac{b}{a}}^{+\infty} e^{-mw^*} f(-w^*)dw^* && (53)\\
&= \int_{-\frac{b}{a}}^{+\infty} e^{-mw^*} f(w^*)dw^* && (54)\\
&= E(e^{-mW}\mathbf{1}\{aW > -b\}) && (55)
\end{aligned}$$

where $f(.)$ is the pdf for N(0,1).

The price for the digital barrier option is

$$\begin{aligned}
E^*(\Pi_3(k)) &= e^{-r(t_k-t)} E(1\{\sup_{t\le\tau\le t_k} e^{(r-\frac{\sigma^2}{2})(\tau-t)+\sigma(W_\tau - W_t)>} 1.2, && (56)\\
&\quad e^{(r-\frac{\sigma^2}{2})t_k+\sigma W_{t_k}} < 0.6\}) && (57)\\
&= e^{-r(t_k-t)} E(1\{\sup_{0\le\tau\le t_k-t} e^{(r-\frac{\sigma^2}{2})(t_k-t-\tau)+\sigma B_\tau + X_1>} 1.2, && (58)\\
&\quad e^{(r-\frac{\sigma^2}{2})(t_k-t)+\sigma B_{t_k-t}+X_1} < 0.6\}) && (59)
\end{aligned}$$

Consider changing measure

$$B_t^* = B_t + \frac{\frac{\sigma^2}{2} - r}{\sigma} t$$

Then

$$\begin{aligned}
E_t^*(\Pi_3(k)) &= e^{-r(t_k-t)} E_t(e^{(\frac{r-\frac{\sigma^2}{2}}{\sigma})B^*_{t_k-t} - \frac{1}{2}(\frac{r-\frac{\sigma^2}{2}}{\sigma})^2(t_k-t)} && (60)\\
&\quad \mathbf{1}\{\sup_{0\le\tau\le t_k-t} \sigma B^*_\tau > ln(1.2) - X_1, \sigma B^*_{t_k-t} < ln(0.6) - X_1\}) && (61)\\
&= e^{-r(t_k-t)} E_t(e^{m(B^*_\tau - B^*_\tau + B^*_{t_k-t}) - \frac{1}{2}m^2(t_k-t)} && (62)\\
&\quad \mathbf{1}\{\sup_{0\le\tau\le t_k-t} \sigma B^*_\tau > ln(1.2) - X_1, \sigma(B^*_{t_k-t} - B^*_\tau + B^*_\tau) < ln(0.6) - X_1\}) && (63)\\
&= e^{-r(t_k-t)} E_t(E(e^{m(B^*_\tau - B^*_\tau + B^*_{t_k-t}) - \frac{1}{2}m^2(t_k-t)} && (64)\\
&\quad \mathbf{1}\{\sigma(B^*_{t_k-t} - B^*_\tau + B^*_\tau) < ln(0.6) - X_1\}|\tau)) && (65)\\
&= e^{-r(t_k-t)} E_t(e^{mB^*_\tau - \frac{1}{2}m^2(t_k-t)} E(e^{m(B^*_{t_k-t} - B^*_\tau)} && (66)\\
&\quad \mathbf{1}\{\sigma(B^*_{t_k-t} - B^*_\tau) < ln(0.6) - X_1 - \sigma B^*_\tau\}|\tau)) && (67)\\
&= e^{-r(t_k-t)} E_t(e^{mB^*_\tau - \frac{1}{2}m^2(t_k-t)} E(e^{m(-B^*_{t_k-t} + B^*_\tau)} && (68)\\
&\quad \mathbf{1}\{\sigma(B^*_{t_k-t} - B^*_\tau) > -ln(0.6) + X_1 + \sigma B^*_\tau\}|\tau)) \qquad *** && (69)\\
&= e^{-r(t_k-t)} E_t(e^{m(2B^*_\tau - B^*_{t_k-t}) - \frac{1}{2}m^2(t_k-t)} \mathbf{1}\{\sigma B^*_{t_k-t} > 2ln(1.2) - ln(0.6) - X_1\}) && (70)\\
&= e^{-r(t_k-t)} (\frac{1.2}{\hat{I}_1})^{\frac{2m}{\sigma}} E_t(1\{\sigma B^{**}_{t_k-t} - \sigma m(t_k - t) > ln(2.4) - X_1\}) && (71)\\
&\qquad\qquad (changing\ measure, \quad e^{B_\tau} = (\frac{1.2}{\hat{I}_1})^{\frac{1}{\sigma}}) && (72)\\
&= e^{-r(t_k-t)} (\frac{1.2}{\hat{I}_1})^{2(\frac{r-\frac{\sigma^2}{2}}{\sigma^2})} \Psi(-\frac{ln(2.4) - X_1 + (r - \frac{\sigma^2}{2})(t_k - t)}{\sigma\sqrt{t_k - t}}) && (73)
\end{aligned}$$

7.6 Discrete Adjustment Using Renewal Process and Reflection Principle

Define $U(t) := vt + B(t)$, $Un := \Sigma_{i=1}^n (Z_i + \frac{v}{\sqrt{m}})$. U_n is a random walk with a small drift ($m \to \infty$), then let $\tau := \inf\{t \ge 0 : U(t) \ge b\}$, for any constants $b \ge y$, as $m \to \infty$,

$$P(U_m < y\sqrt{m}, \tau(b, U) \le m) = P(U(1) \le y, \tau(b + \frac{\beta}{\sqrt{m}}) \le 1) + o(\frac{1}{\sqrt{m}}) \tag{74}$$

Here β is the limiting expectation of overshoot.
Define the overshoot process $A_n := \Sigma_{i=1}^n Z_i$ and $N := \min\{n \ge 0 : A_n > 0\}$. $f(.)$ is the average length of overshoot given total length of δ. $L(.)$ is the total length of the first $A_{N,t}$ overshoots for $t = 1, 2, ..., k$. Then

$$\begin{aligned}
\beta &= \int_0^\infty f(\delta)d\delta = \frac{\Sigma_{t=1}^k A^2_{N,t}}{2L(k)} + \int_{L(k)}^\infty f(\delta)d\delta && (75)\\
&= \lim_{k\to\infty} \frac{\Sigma_{t=1}^k A^2_{N,t}}{2L(k)} = \lim_{k\to\infty} \frac{\Sigma_{t=1}^k A^2_{N,t}}{2L(k)} \frac{k}{k} = \frac{E(A_N^2)}{2E(A_N)} && (76)
\end{aligned}$$

For Wiener process, β could be calculated. The computation of $\frac{E(A_N^2)}{2E(A_N)}$ is complicated thus we used the result without derivation.

$$\beta = -\frac{\zeta(1/2)}{\sqrt{2\pi}}$$

Thus for a Wiener process,

$$P(\sigma Z_1 \ge k) \approx P(\sigma Z_\tau \ge (k + \sigma\beta))$$

7.7 Digital Rainbow Option

7.7.1 Up-and-Out Digital Call Option

$$\begin{aligned}\frac{\partial V}{\partial \eta}+r_1V &= (r_1-\frac{\sigma_1^2}{2})\frac{\partial V}{\partial X_1}+(r_2-\frac{\sigma_2^2}{2})\frac{\partial V}{\partial X_2}+\frac{\sigma_1^2}{2}\frac{\partial^2 V}{\partial X_1^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial X_2^2}+\rho\sigma_1\sigma_2\frac{\partial^2 V}{\partial X_1\partial X_2} \quad (77)\\ V(X_1,X_2,0) &= \mathbf{1}_{\{X_2>ln(0.5)\}} \quad (78)\\ V(0,X_2,0) &= 0 \quad (79)\\ D &= \{(X_1,X_2,t)|X_1<0,-\infty<X_2<\infty,0\le t\le t_k\} \quad (80)\end{aligned}$$

Substitute $\eta = T-t$ and let $U = Ve^{r_1\eta}$, we have

$$\frac{\partial U}{\partial \eta}=(r_1-\frac{\sigma_1^2}{2})\frac{\partial U}{\partial X_1}+(r_2-\frac{\sigma_2^2}{2})\frac{\partial U}{\partial X_2}+\frac{\sigma_1^2}{2}\frac{\partial^2 U}{\partial X_1^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 U}{\partial X_2^2}+\rho\sigma_1\sigma_2\frac{\partial^2 U}{\partial X_1\partial X_2}$$

Consider $S = Ue^{aX_1+bX_2}$,

$$\begin{aligned}\frac{\partial S}{\partial X_1} &= aS+S\frac{\partial U}{\partial X_1} \quad (81)\\ \frac{\partial S}{\partial X_2} &= bS+S\frac{\partial U}{\partial X_2} \quad (82)\\ \frac{\partial^2 S}{\partial X_1^2} &= a^2S+2aS\frac{\partial U}{\partial X_1}+S\frac{\partial^2 U}{\partial X_1^2} \quad (83)\\ \frac{\partial^2 S}{\partial X_2^2} &= b^2S+2bS\frac{\partial U}{\partial X_2}+S\frac{\partial^2 U}{\partial X_2^2} \quad (84)\\ \frac{\partial^2 S}{\partial X_1\partial X_2} &= abS+aS\frac{\partial U}{\partial X_2}+bS\frac{\partial U}{\partial X_1}+S\frac{\partial^2 U}{\partial X_1\partial X_2} \quad (85)\end{aligned}$$

to eliminate the terms $\frac{\partial U}{\partial X_1}$ and $\frac{\partial U}{\partial X_2}$, we have

$$(r_1-\frac{\sigma_1^2}{2})+2a\frac{\sigma_1^2}{2}+b\rho\sigma_1\sigma_2=0 \quad (86)$$

$$(r_2-\frac{\sigma_2^2}{2})+2b\frac{\sigma_2^2}{2}+a\rho\sigma_1\sigma_2=0 \quad (87)$$

thus

$$\begin{bmatrix}a\\ b\end{bmatrix}=-\begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}^{-1}\begin{bmatrix}r_1-\frac{\sigma_1^2}{2}\\ r_2-\frac{\sigma_2^2}{2}\end{bmatrix}$$

at the same time, the terms without partial differentiations are eliminated, so

$$\begin{aligned}\frac{\partial S}{\partial \eta} &= \frac{\sigma_1^2}{2}\frac{\partial^2 S}{\partial X_1^2}+\frac{\sigma_2^2}{2}\frac{\partial^2 S}{\partial X_2^2}+\rho\sigma_1\sigma_2\frac{\partial^2 S}{\partial X_1\partial X_2} \quad (88)\\ S(X_1,X_2,0) &= \mathbf{1}_{\{X_2>ln(0.5)\}} \quad (89)\\ S(0,X_2,0) &= 0 \quad (90)\\ V &= e^{aX_1+bX_2+c\tau}S \quad (91)\end{aligned}$$

Then WONG solved this $S(X_1,X_2,\eta)$, where

$$\begin{aligned}S &= \frac{1}{2\pi\sigma_1\sigma_2(t_k-t)\sqrt{1-\rho^2}} \quad (92)\\ & \int_{-\infty}^{0}d\xi_1\int_{-\infty}^{\infty}[e^{-\frac{(-ln(1.2)+X_1-\xi_1)^2}{2\sigma_1^2\eta}}+e^{-\frac{(-ln(1.2)+X_1+\xi_1)^2}{2\sigma_1^2t_k}}]e^{-a\xi_1-b\xi_2}e^{B}d\xi_2 \quad (93)\\ B &= -\frac{[-ln(1.2)+X_2-\xi_2-\frac{\rho\sigma_2(-ln(1.2)+X_1-\xi_1)}{\sigma_1}]^2}{2\sigma_2^2(1-\rho^2)(t_k-t)} \quad (94)\end{aligned}$$

Thus the solution for up-and-out digital call option is

$$V_{up-and-out} = e^{-r_1(t_k-t)} \frac{1.2^{1-a-b}\hat{I_1}^a \hat{I_2}^b e^{c(t_k-t)}}{2\pi\sigma_1\sigma_2(t_k-t)\sqrt{1-\rho^2}} \quad (95)$$

$$* \int_{-\infty}^{0} d\xi_1 \int_{ln(0.5)}^{\infty} [e^{-\frac{(-ln(1.2)+X_1-\xi_1)^2}{2\sigma_1^2\eta}} + e^{-\frac{(-ln(1.2)+X_1+\xi_1)^2}{2\sigma_1^2 t_k}}] e^{-a\xi_1-b\xi_2} e^B d\xi_2 \quad (96)$$

7.7.2 Deriving Other Digital Call Option

Define the terms

- C_1: up-and-out call
- C_2: up-and-in call
- P_2: up-and-in put

then it is clear that

$$C_1 + C_2 = e^{-r_1(t_k-t)} P(X_2 > ln(0.6)) \quad (97)$$

$$C_2 + P_2 = e^{-r_1(t_k-t)} P(\tau < t_k) \quad (98)$$

Using steps similar with 7.5, we get

$$P(\tau < t_k) = \Psi(-\frac{ln(1.2) - X_1 - (r - \frac{\sigma^2}{2})(t_k-t)}{\sigma\sqrt{t_k-t}}) \quad (99)$$

$$+(\frac{1.2}{\hat{I_1}})^{2(\frac{r-\frac{\sigma^2}{2}}{\sigma^2})} \Psi(-\frac{ln(1.2) - X_1 + (r - \frac{\sigma^2}{2})(t_k-t)}{\sigma\sqrt{t_k-t}}) \quad (100)$$

thus the price for up-and-in put option is

$$E_t^{***}(\Pi_3(k)) = e^{-r_1(t_k-t)} P(\tau < t_k) - e^{-r_1(t_k-t)} P(X_2 > ln(0.6)) + \frac{1.2^{1-a-b}\hat{I_1}^a \hat{I_2}^b e^{c(t_k-t)}}{2\pi\sigma_1\sigma_2(t_k-t)\sqrt{1-\rho^2}} \quad (101)$$

$$* \int_{-\infty}^{0} d\xi_1 \int_{ln(0.5)}^{\infty} [e^{-\frac{(-ln(1.2)+X_1-\xi_1)^2}{2\sigma_1^2\eta}} + e^{-\frac{(-ln(1.2)+X_1+\xi_1)^2}{2\sigma_1^2 t_k}}] e^{-a\xi_1-b\xi_2} e^B d\xi_2 \quad (102)$$

In fact, when summing up the three parts of coupon, some terms can be cancelled. For easy computation, we modified a little.

$$E_t^{***}(\Pi_3(k)) = e^{-r_1(t_k-t)} P(X_1 > ln(0.6), X_2 > ln(0.6) \quad (103)$$

$$+e^{-r(t_k-t)}(\frac{1.2}{\hat{I_1}})^{2(\frac{r-\frac{\sigma^2}{2}}{\sigma^2})} \Psi(-\frac{ln(2.4) + 2E(R_m) - X_1 + (r - \frac{\sigma^2}{2})(t_k-t)}{\sigma\sqrt{t_k-t}}) \quad (104)$$

$$+\Psi(-\frac{ln(1.2) + E(R_m) - X_1 - (r - \frac{\sigma^2}{2})(t_k-t)}{\sigma\sqrt{t_k-t}}) + V^*_{up-and-out} \quad (105)$$

$$+e^{-r_1(t_k-t)}(\frac{1.2}{\hat{I_1}})^{2(\frac{r-\frac{\sigma^2}{2}}{\sigma^2})} \Psi(-\frac{ln(1.2) + E(R_m) - X_1 + (r - \frac{\sigma^2}{2})(t_k-t)}{\sigma\sqrt{t_k-t}}) \quad (106)$$

$$-e^{-r_1(t_k-t)} P(X_2 > ln(0.6)) \quad (107)$$

$$\leq e^{-r(t_k-t)}(\frac{1.2}{\hat{I_1}})^{2(\frac{r-\frac{\sigma^2}{2}}{\sigma^2})} \Psi(-\frac{ln(2.4) + 2E(R_m) - X_1 + (r - \frac{\sigma^2}{2})(t_k-t)}{\sigma\sqrt{t_k-t}}) \quad (108)$$

$$+\Psi(-\frac{ln(1.2) + E(R_m) - X_1 - (r - \frac{\sigma^2}{2})(t_k-t)}{\sigma\sqrt{t_k-t}}) + V^*_{up-and-out} \quad (109)$$

$$+e^{-r_1(t_k-t)}(\frac{1.2}{\hat{I_1}})^{2(\frac{r-\frac{\sigma^2}{2}}{\sigma^2})} \Psi(-\frac{ln(1.2) + E(R_m) - X_1 + (r - \frac{\sigma^2}{2})(t_k-t)}{\sigma\sqrt{t_k-t}}) \quad (110)$$

where $V^*_{up-and-out}$ adjust $ln(0.5)$ to $[ln(0.5) + E(R_m)]$

7.8 Two-Asset Continous-Time Barrier Option

Set V be the part of coupon: anyone of the indeces reaches 1.2 and at t_k the minimum performance is smaller than 0.6. The PDE is

$$\frac{\partial V}{\partial t} + (r_1 - \frac{\sigma_1^2}{2})\frac{\partial V}{\partial X_1} + (r_2 - \frac{\sigma_2^2}{2})\frac{\partial V}{\partial X_2} + \frac{\sigma_1^2}{2}\frac{\partial^2 V}{\partial X_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial X_2^2} + \rho\sigma_1\sigma_2\frac{\partial^2 V}{\partial X_1 \partial X_2} - r_1 V = 0$$

where

$$X_1 = ln(\hat{I}_1) - ln(1.2) \qquad X_2 = ln(\hat{I}_2) - ln(1.2) \qquad V(X_1 \geq 0 \; or \; X_2 \geq 0) = 0 \tag{111}$$

$$V(t = T) = \mathbf{1}\{\{X_1(T) < ln(0.5)\} \cup \{X_2(T) < ln(0.5)\}\} \tag{112}$$

Here $ln(0.5) = ln(0.6) - ln(1.2)$. We tried to use heating equation to solve this 2D PDE and transform this PDE to a form like

$$\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial a^2} + \frac{\partial^2 F}{\partial b^2}$$

Using steps similar with 7.7, we get

$$\frac{\partial S}{\partial \eta} = \frac{\sigma_1^2}{2}\frac{\partial^2 S}{\partial X_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 S}{\partial X_2^2} + \rho\sigma_1\sigma_2\frac{\partial^2 S}{\partial X_1 \partial X_2}$$

reduce cross term, consider Z

$$S = Ve^{r_1\eta + aX_1 + bX_2} = Ve^{r_1\eta + pZ_1 + qZ_2} \tag{113}$$

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}^{-1} \begin{bmatrix} \sqrt{\frac{\sigma_1^2}{2}}X_1 \\ \sqrt{\frac{\sigma_2^2}{2}}X_2 \end{bmatrix} \tag{114}$$

thus

$$\frac{\partial S}{\partial \eta} = \frac{\partial^2 S}{\partial Z_1^2} + \frac{\partial^2 S}{\partial Z_2^2}$$

represent Z_1 and Z_2 as $Z_1 = rsin\theta$ and $Z_2 = rcos\theta$, then

$$\frac{\partial S}{\partial \eta} = \frac{\partial^2 S}{\partial r^2} + \frac{1}{r}\frac{\partial S}{\partial r} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2} \tag{115}$$

$$S(\eta, r, \theta) = T(\eta)R(r)\Theta(\theta) \tag{116}$$

$$\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} := -\lambda^2 \tag{117}$$

$$T = Ae^{-\lambda^2\eta} \tag{118}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \lambda^2\frac{R}{r^2} = -\frac{\Theta''}{\Theta} := \nu^2 \tag{119}$$

then for the second part of the coupon,

$$r^2R'' + rR' - (\lambda_k^2 r^2 + \nu_k^2)R = 0 \qquad R = B_k J_{\nu_k}(\lambda_k r) \tag{120}$$

$$\Theta = C_k sin(\nu_k\theta) \qquad \nu_k = \frac{k\pi}{\psi_0} \qquad \lambda_k = \frac{r_0}{2\eta} \tag{121}$$

here $J_V(.)$ is Modified Bessel function of type I.thus

$$S(\eta, r, \theta) = T_k(\eta)R_k(r)\Theta_k(\theta) \tag{122}$$

$$\omega(r, \theta) = M_k J_{\nu_k}(\lambda_k r) sin(\nu_k\theta) \tag{123}$$

$$M_k = A_k B_k C_k \tag{124}$$

using sine transform, note that we changed the original condition in order to use this method

$$\int_{\pi}^{\pi+\psi_0} sin(\nu_k\theta)\omega(r,\theta)d\theta = M_k J_{\nu_k}(\lambda_k r)\int_{\pi}^{\pi+\psi_0} sin^2(\nu_k\theta)d\theta \tag{125}$$

$$= M_k J_{\nu_k}(\lambda_k r)\frac{\psi_0}{2} \tag{126}$$

$$= \frac{\psi_0}{2} M_k J_{\nu_k}(\lambda_k r) \tag{127}$$

Then set $t = \frac{1}{\eta}$, multiply $e^{-\lambda_k^2 \eta} t$ on both side, then use inverse Hankel transform, integrate t on both side

$$\int_{\pi}^{\pi+\psi_0} sin(\nu_k\theta)\omega(r,\theta)d\theta \int_0^{+\infty} e^{-\frac{\lambda_k^2}{t}} tdt = \int_0^{+\infty} e^{-\lambda_k^2\eta}\frac{\psi_0}{2} M_k J_{\nu_k}(\lambda_k r)tdt \tag{128}$$

$$= \int_0^{+\infty} \frac{\psi_0}{2} e^{-\lambda_k^2\eta} M_k J_{\nu_k}(\frac{trr_0}{2})tdt \tag{129}$$

$$\frac{\psi_0}{2}e^{-\lambda_k^2\eta}M_k = \int_0^{+\infty} e^{-\frac{\lambda_k^2}{t}} tdt \int_0^{+\infty}\int_{\pi}^{\pi+\psi_0} sin(\nu_k\theta)\omega(r,\theta)d\theta \frac{rr_0}{2}\frac{d\frac{rr_0}{2}}{dr}dr \tag{130}$$

$$= \frac{r_0^2}{4}P_k \int_0^{+\infty}\int_{\pi}^{\pi+\psi_0} sin(\nu_k\theta)\omega(r,\theta)rd\theta dr \tag{131}$$

$$e^{-\lambda_k^2\eta}M_k = \frac{r_0^2}{2\psi_0}P_k \int_0^{+\infty}\int_{\pi}^{\pi+\psi_0} sin(\nu_k\theta)\omega(r,\theta)rd\theta dr \tag{132}$$

$$= D_k \int_0^{+\infty}\int_{\pi}^{\pi+\psi_0} sin(\nu_k\theta)\omega(r,\theta)rd\theta dr \tag{133}$$

$$D_k = \frac{r_0^2}{2\psi_0}\int_0^{+\infty} e^{-\frac{r_0^2 t}{4}} tdt \tag{134}$$

Thus using superposition,

$$S(\eta, r_0, \theta_0) = \Sigma_{k=1}^{+\infty} e^{-\frac{r_0^2}{\eta}} D_k J_{\frac{k\pi}{\psi_0}}(\frac{r_0^2}{2\eta}) \int_0^{+\infty}\int_{\pi}^{\pi+\psi_0} J_{\nu_k}(\lambda r) sin(\nu_k\theta_0) sin(\nu_k\theta)\omega(r,\theta)rd\theta dr \tag{135}$$

now the initial conditions are converted to

$$D = \{\pi \le \theta \le 2\pi\} \tag{136}$$

$$S(\tau = 0) = V(\tau = 0)e^{prsin\theta + qrcos\theta} := \omega(r,\theta) \tag{137}$$

and (r_0, θ_0) are computed from (X_1, X_2), i.e., current log returns.

7.9 Correlation Matrix for Multi-Asset Heston Model

$$\Sigma = \begin{bmatrix} 1 & \rho_{E,I_1} & \rho_{E,I_1}\rho_1 & \rho_{E,I_2} & \rho_{E,I_2}\rho_2 \\ \rho_{E,I_1} & 1 & \rho_1 & \rho_{1,2} & \rho_{1,2}\rho_2 \\ \rho_{E,I_1}\rho_1 & \rho_1 & 1 & \rho_{1,2}\rho_1 & \rho_{1,2}\rho_1\rho_2 \\ \rho_{E,I_2} & \rho_{1,2} & \rho_{1,2}\rho_1 & 1 & \rho_2 \\ \rho_{E,I_2}\rho_2 & \rho_{1,2}\rho_2 & \rho_{1,2}\rho_1\rho_2 & \rho_2 & 1 \end{bmatrix}$$

7.10 Single-Asset Call Option under Heston Model

$$V(I,\nu,\tau;\Theta) = e^{-r\tau}(IP_1(I,\nu,\tau;\Theta) - KP_2(I,\nu,\tau;\Theta)) \tag{138}$$

$$P_k(I,\nu,\tau;\Theta) = [\frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Re\{\frac{e^{-i\psi log(K)}\Phi_k(I,\nu,\tau;\Theta)}{i\psi}\}d\psi]e^{r\tau 1\{k=1\}} \tag{139}$$

$$\Phi_k(I,\nu,\tau;\Theta) = exp\{C_k(\tau;\Theta) + D_k(\tau;\Theta)\nu + i\psi I\} \tag{140}$$

$$C_k(\tau;\Theta) = r\psi i\tau + \frac{a}{\epsilon^2}\{(b_k - \rho\epsilon\psi i + d_k)\tau - 2ln(\frac{1-g_k e^{d_k\tau}}{1-g_k})\} \tag{141}$$

$$D_k(\tau;\Theta) = \frac{b_k - \rho\epsilon\psi i + d_k}{\epsilon^2}\frac{1-e^{d_k\tau}}{1-g_k e^{d_k\tau}} \tag{142}$$

$$g_k = \frac{b_k - \rho\epsilon\psi i + d_k}{b_k - \rho\epsilon\psi i - d_k} \tag{143}$$

$$d_k = \sqrt{(\rho\epsilon\psi i - b_k)^2 - \epsilon(2u_k\psi i - \psi^2)} \tag{144}$$

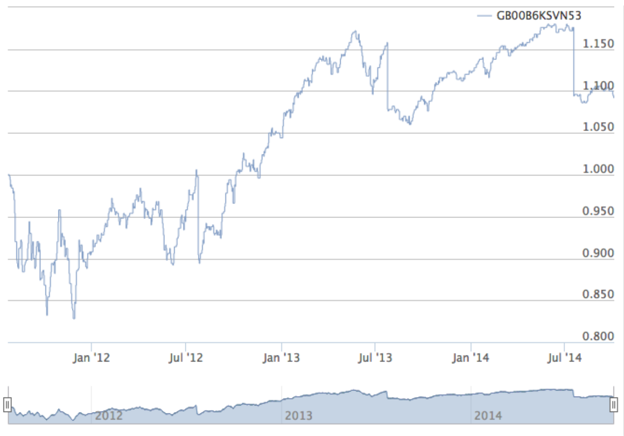
$$u_k = \frac{3}{2} - k, \qquad a = \kappa\theta \qquad , b_k = \kappa - \rho\epsilon 1_{k=1} \tag{145}$$

8 Graphs and Charts

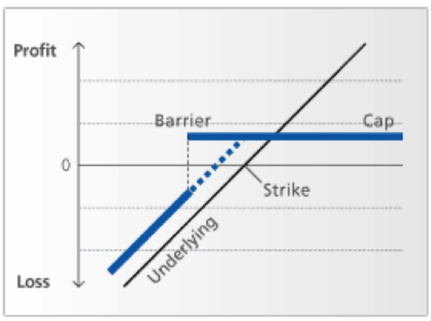
8.1 Product Information

Issue Date	Jul 22, 2011	**Notional Price**	GBP 1.00 / Certificate
Maturity Date	Jul 25, 2017	**Settlement Currency**	GBP
Issue Price	GBP 1.00 / Certificate	**Interest Amount**	GBP 0.08 per Annum
Valuation Date	Jul 16, 2012, Jul 16, 2013 Jul 16, 2014, Jul 16, 2015 Jul 18, 2016, Jul 18, 2017	**Interest Payment Date**	Jul 23, 2012, Jul 23, 2013 Jul 23, 2014, Jul 23, 2015 Jul 25, 2016, Jul 25, 2017

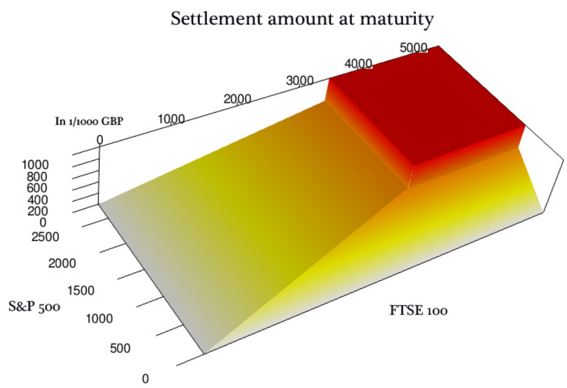
Underlying Asset	Currency	Initial Level	Barrier /Coupon level	Lock-in Level
S&P 500 (SPX)	USD	USD 1316.14	USD 789.684	USD 1579.368
FTSE 100 (UKX)	GBP	GBP 5843.66	GBP 3506.196	7012.392



Historical Price and Volume



Total profit and loss



Final settlement amount at maturity

8.2 Simulation Results for BS and Heston Model

	BS	Heston
path	5000	5000
time	50	50
mean	1.092381	1.075623
sd	0.004745	0.0036056

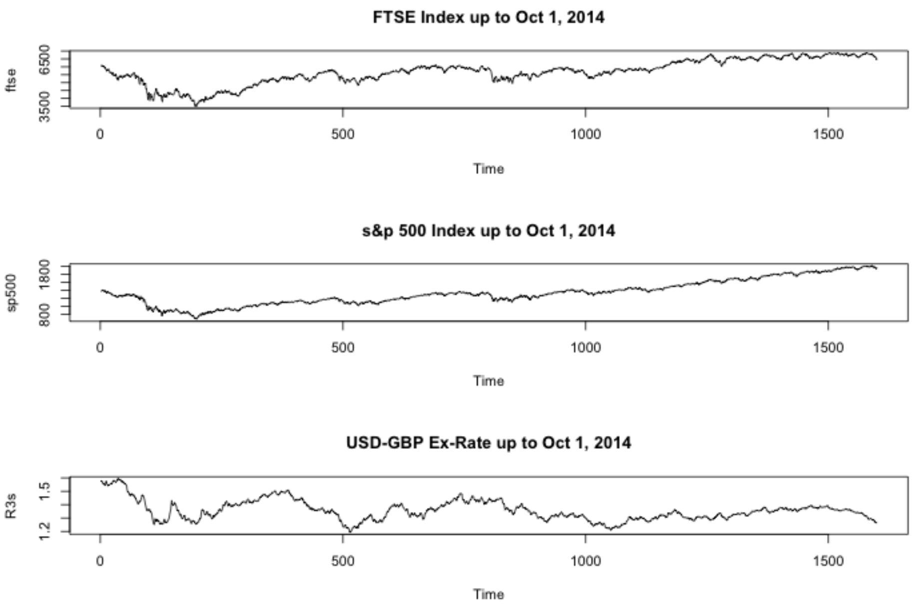
8.3 Duan-GARCH MLE Result

Parameters	Value
$h_0^{(1)}$	0.0001167749
$\lambda^{(1)}$	0.05663344
$w^{(1)}$	1.261072e-06
$a^{(1)}$	0.08807004
$\beta^{(1)}$	0.904558
$h_0^{(2)}$	9.432635e-06
$\lambda^{(2)}$	0.1997192
$w^{(2)}$	2.982831e-06
$a^{(2)}$	0.09937672
$\beta^{(2)}$	0.8692852

Summary Statistics for Marginal MLE

Parameters	Value
$h_0^{(1)}$	0.0001505208
$\lambda^{(1)}$	0.08783096
$w^{(1)}$	1.770349e-06
$a^{(1)}$	0.08433492
$\beta^{(1)}$	0.9019514
$h_0^{(2)}$	0.0003124885
$\lambda^{(2)}$	0.1185714
$w^{(2)}$	2.482179e-06
$a^{(2)}$	0.1115622
$\beta^{(2)}$	0.8710128
$\rho^{1,2}$	0.6492056
$\rho^{2,3}$	0.02220879
σ_F	0.004773923

Summary Statistics for Joint MLE

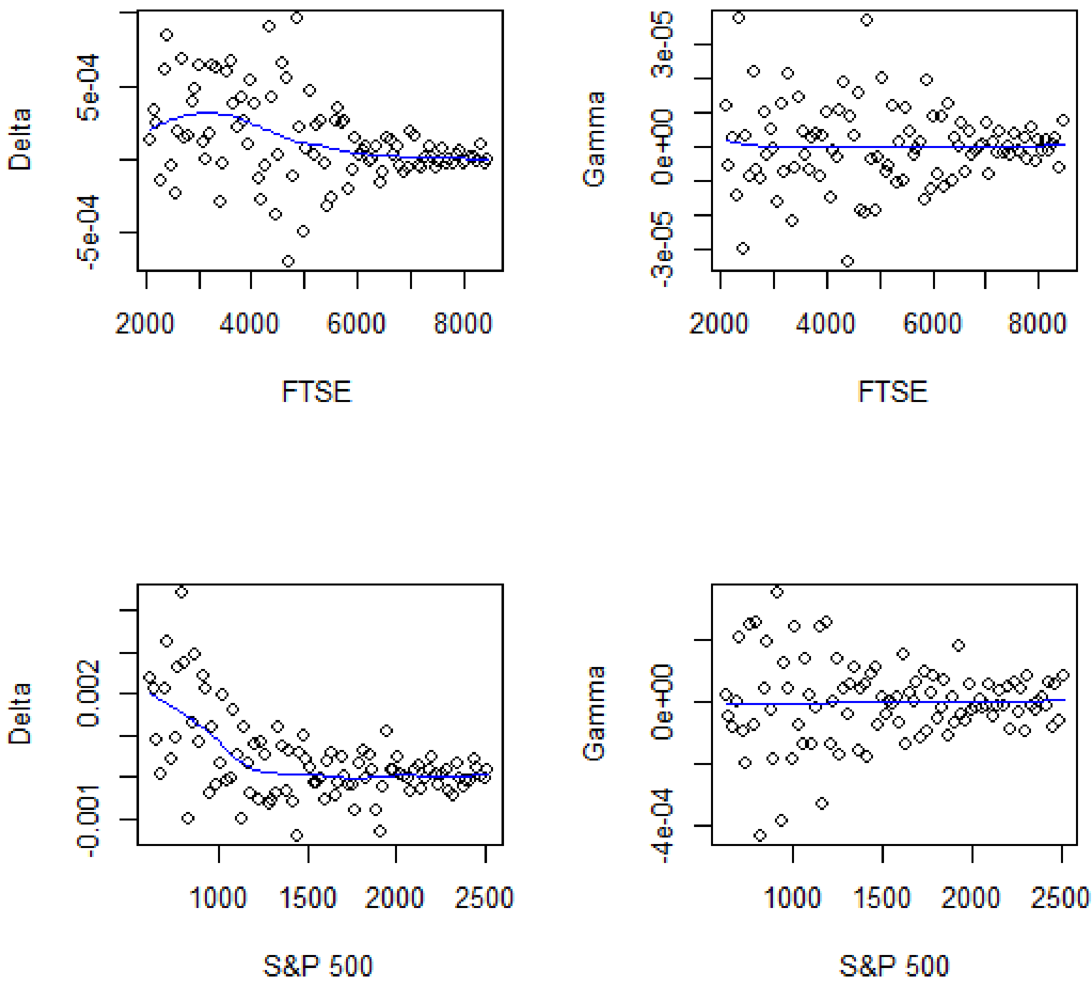


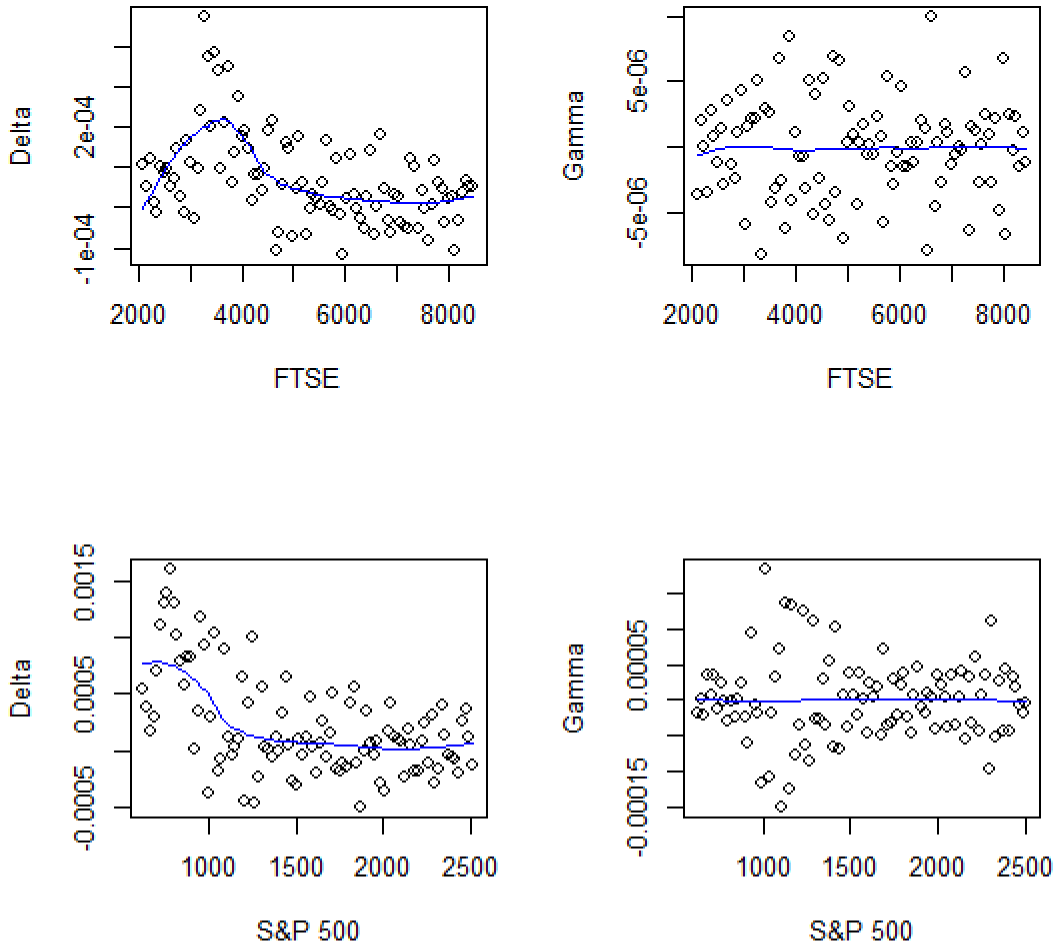
Time series plot of 1600 days of historical data

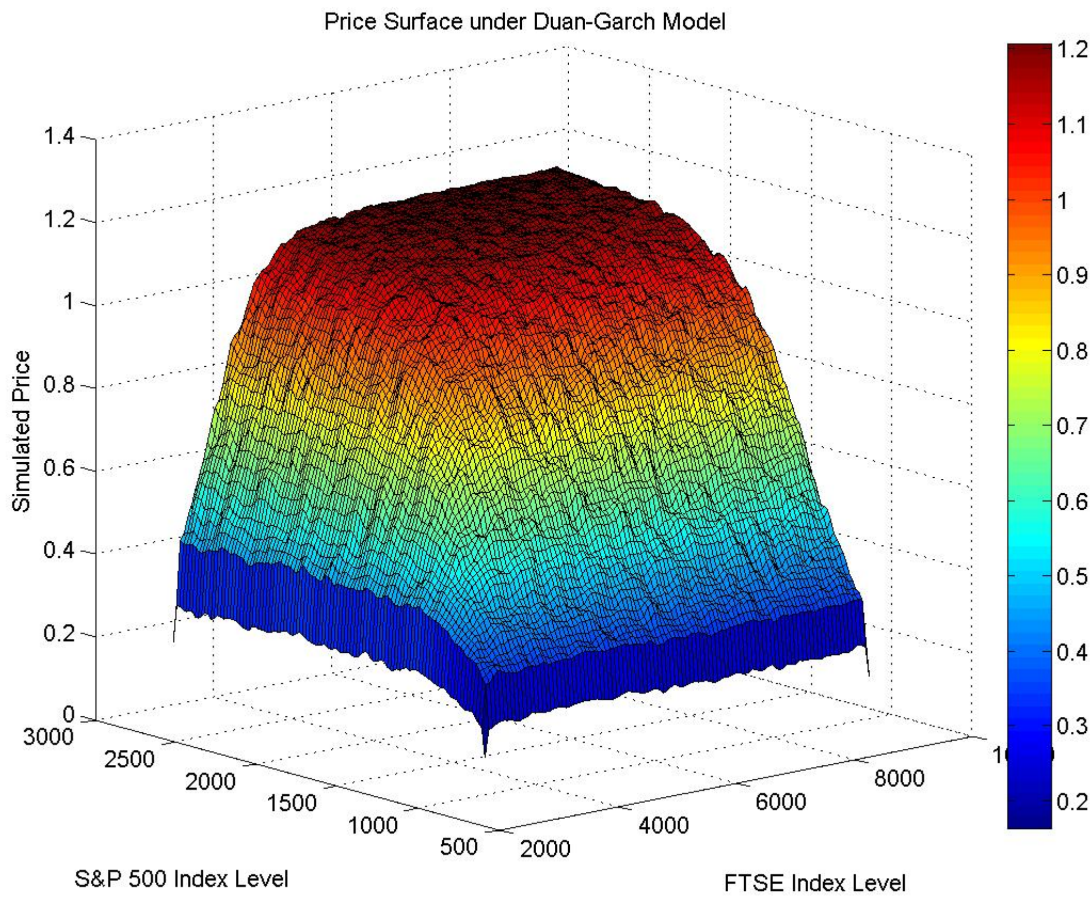
8.4 Risk Analysis

Indicator / Model	Black-Scholes	Heston	Duan-GARCH
95% VaR	52.23%	63.60%	43.96 %
95% ES	61.21%	75.13%	56.56%
PBE	17.57%	18.71%	2.93%

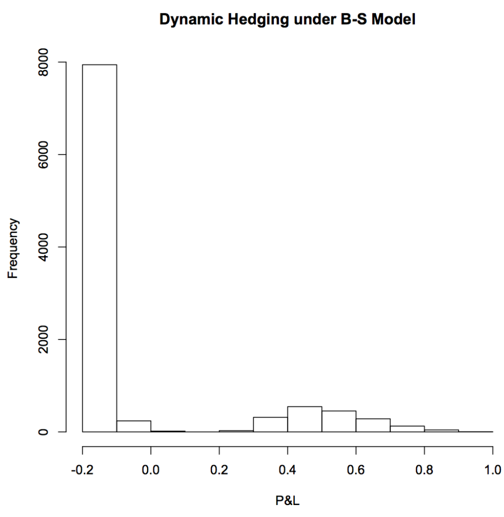
*PBE is defined as the probability that at maturity, the worse index is lower than its barrier level.

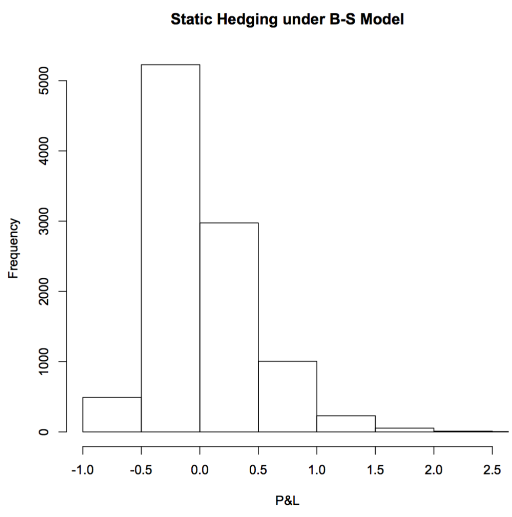

Delta
5e-04
-5e-04
2000 4000 6000 8000
FTSE
Gamma
3e-05
0e+00
-3e-05
2000 4000 6000 8000
FTSE
Delta
0.002
-0.001
1000 1500 2000 2500
S&P 500
Gamma
0e+00
-4e-04
1000 1500 2000 2500
S&P 500

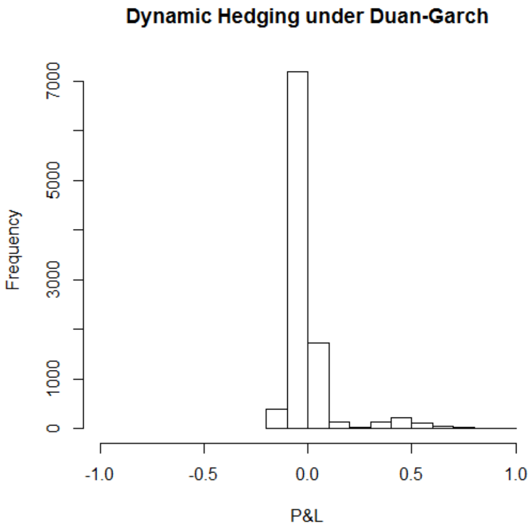

Delta
2e-04
-1e-04
2000
4000
6000
8000
FTSE
Gamma
5e-06
-5e-06
2000
4000
6000
8000
FTSE
Delta
0.0015
0.0005
-0.0005
1000
1500
2000
2500
S&P 500
Gamma
0.00005
-0.00015
1000
1500
2000
2500
S&P 500

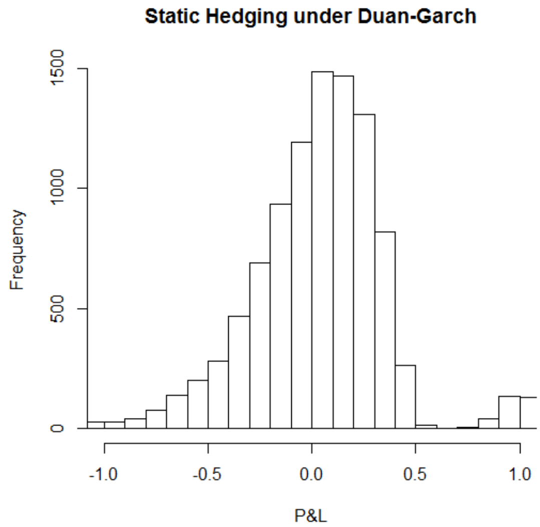


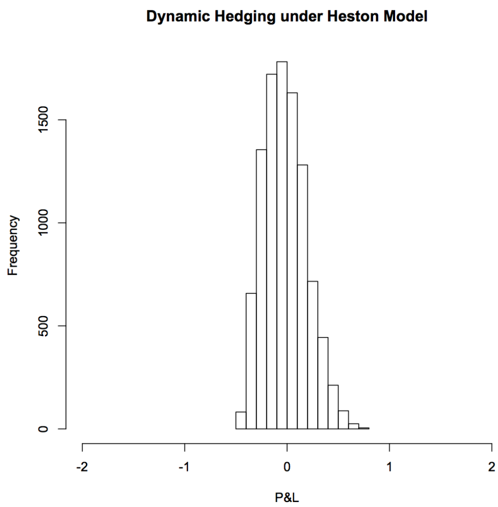
Price Surface under Duan-GARCH by one set of parameter

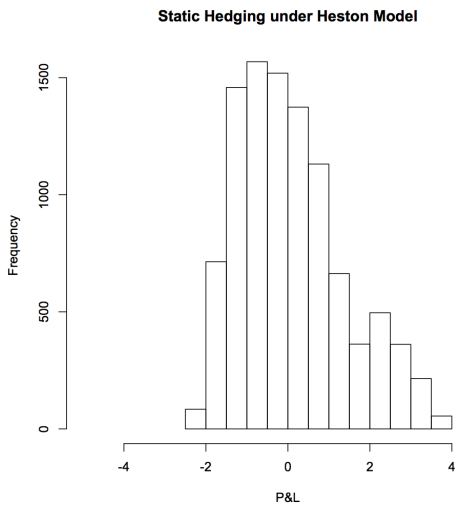





Dynamic Hedging under Duan-Garch
Frequency
0 1000 3000 5000 7000
-1.0 -0.5 0.0 0.5 1.0
P&L


Static Hedging under Duan-Garch
Frequency
0 500 1000 1500
-1.0 -0.5 0.0 0.5 1.0
P&L


Dynamic Hedging under Heston Model
Frequency
0 500 1000 1500
-2 -1 0 1 2
P&L


Static Hedging under Heston Model
Frequency
0 500 1000 1500
-4 -2 0 2 4
P&L

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