MATH1021

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1 W1

1. Terminology (Numbers and intervals). A number $r \in \mathbb{R}$ is called rational if there are integers $p, q \in \mathbb{Z}$ with $q \neq 0$ such that $r = p/q$. If it is not rational, it is called irrational. Interval notation if $a \leq b$:
   - $(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$ open interval
   - $[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$ closed interval
   - $[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}$ half open (or half closed) interval
   - $(a, \infty) := \{x \in \mathbb{R} \mid a < x\}$ open half line
   - $(-\infty, a] := \{x \in \mathbb{R} \mid x \leq a\}$ closed half line
2. Terminology (Complex numbers). Complex numbers are numbers of the form $z = x + iy$ with x and y real numbers and imaginary unit i having the property that $i^2 = -1$. Any complex number represents a point on the plane with coordinates (x, y). With that identification we obtain the complex plane or Argand diagram. We call $x + iy$ the Cartesian form of z. We have
   - $\underline{\operatorname{Re} z := x \text{ is called the real part of } z}$
   - $\underline{\operatorname{Im} z := y \text{ is called the imaginary part of } z}$
   - $\underline{\bar{z} := x - iy \text{ is called the complex conjugate of } z = x + iy}$
   - $|z| := \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$ is calle the modulus of z (distance of z from origin)
   - $\frac{z}{w} = \frac{z\bar{w}}{|w|^2}$ to make the denominator real
3. Computations work exactly the same as for real numbers, taking into account that $i^2 = -1$.

   Terminology (Sets). If A, B are subsets of a larger set X we define
   - the union of A and B : $\quad A \cup B := \{x \in X \mid x \in A \text{ or } x \in B\}$;
   - the intersection of A and B : $A \cap B := \{x \in X \mid x \in A \text{ and } x \in B\}$;
   - the complement of A : $\quad A^c := \{x \in X \mid x \notin A\}$;
   - the complement of B in A : $A \backslash B := A \cap B^c = \{x \in A \mid x \notin B\}$.

2 W2

1. Definition (Polar form, complex exponential function). A complex number z with modulus $|z| = r$ and argument $\arg(z) = \theta$ can be written in standard polar form as

$$z = r(\cos(\theta) + i\sin(\theta))$$

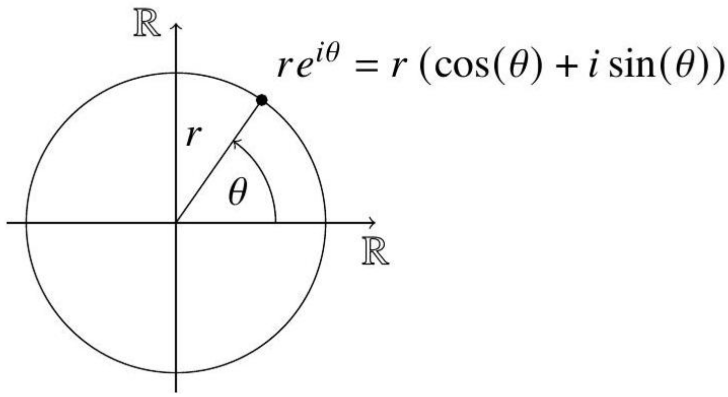
or shorter in exponential polar form

$$z = re^{i\theta}$$

where by definition $e^{i\theta} := \cos(\theta) + i\sin(\theta)$ for all $\theta \in \mathbb{R}$ measured in radians. Note that $\theta \mapsto e^{i\theta}$ is 2π-periodic, that is, $e^{i(\theta+2\pi k)} = e^{i\theta}$ for all $k \in \mathbb{Z}$. More generally we define the complex exponential function

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y).$$

for all $z = x + iy$ with $x, y \in \mathbb{R}$.



Note : The complex exponential function has the same properties as the usual exponential function. For $z, w \in \mathbb{C}$ we have

$$e^0 = 1, \quad e^z e^w = e^{z+w}, \quad e^{-z} = \frac{1}{e^z}.$$

W2-Exercises-Q8

2. Theorem (De Moivre's Theorem). For any $n \in \mathbb{Z}$,

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$$

or in exponential polar form (more intuitive and natural)

$$\left(e^{i\theta}\right)^n = e^{in\theta}$$

corresponding to the usual index laws for powers.

W2-Exercises-Q7(f) W2-Exercises-Q9 W2-Exercises-Q10 W2-TUT-Q4

3. Additional typical problem *W2-Exercises-Q12 W2-Exercises-Q13 W2-Exercises-Q14 W2-TUT-Q6 W2-TUT-Q7*

3 W3

1. Terminology (Functions). Let $A, B \subseteq \mathbb{R}$ be sets. A function $f : A \to B$ is a rule which assigns exactly one element of B to each element of A. We call A the domain of f and B the codomain of f. The graph of f is the set $\{(x, f(x)) \mid x \in A\}$. For a function we require that every vertical line through $x \in A$ meets the graph at exactly one point.

2. If $f(x)$ is given by a formula, the natural domain is the set of $x \in \mathbb{R}$ such that the formula makes sense, for instance $\log(x)$ makes sense for $x > 0$, so $A = (0, \infty)$ is the natural domain.

   W3-Exercises-Q2 W3-Exercises-Q6 W3-TUT-Q3

3. The range of f is the set $\{f(x) \mid x \in A\}$. We call the function f is surjective or onto if the range is B(*Range* = *Codomain*). For f to be surjective, every horizontal line through $y \in B$ meets the graph at least once.
   W3-Exercises-Q7 W3-Exercises-Q8

4. The function f is injective or one-to-one if every point in the image comes from exactly one element in the domain. For f to by injective every horizontal line through $y \in B$ meets the graph at most once, that is, exactly once or not at all.
   W3-Exercises-Q10

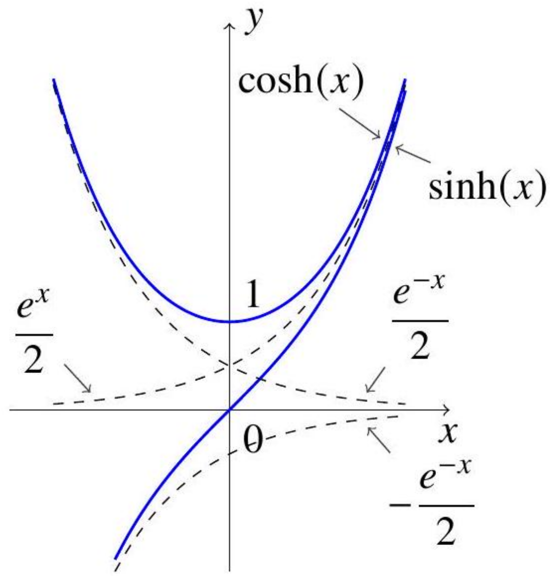
5. The function f is bijective or invertibleif it is both injective and surjective. Then the equation $y = f(x)$ has a unique solution $x \in A$ for each $y \in B$. The inverse function $f^{-1} : B \to A$ recovers the value of $x \in A$ from the value of $y \in B$. To find f^{-1}, solve $y = f(x)$ for $x \in A$, then swap the names of the variables. The graph of $f^{-1} : B \to A$ is obtained by reflecting the graph of f at the diagonal $x = y$.

   W3-Exercises-Q1 W3-Exercises-Q11 W3-TUT-Q2 W3-TUT-Q6 W3-TUT-Q7

6. Definition (hyperbolic sine and cosine functions.). The hyperbolic cosine and hyperbolic sine functions are defined by

$$\cosh(x) := \frac{e^x + e^{-x}}{2}$$
$$\sinh(x) := \frac{e^x - e^{-x}}{2}$$

for all $x \in \mathbb{R}$. They share many properties with the cosine and sine functions.



$$\cosh^2 x - \sinh^2 x = 1$$

$$\frac{d}{dx}\sinh x = \cosh x$$

$$\frac{d}{dx}\cosh x = \sinh x$$

$$\cosh A \cosh B + \sinh A \sinh B = \cosh(A + B)$$

$$2(\cosh A)^2 - 1 = \cosh(2A)$$

W3-TUT-Q5

7. Composite function

- Know that the composite $g \circ f$ of the function $f: A \to B$ and the function $g: B \to C$ is the function from A to C given by the rule $(g \circ f)(x) = g(f(x))$ for all $x \in A$, and be able to determine the range of $g \circ f$ in simple cases.

W3-Exercises-Q6 *W3-TUT-Q4*

8. Additional typical problem:

W3-Exercises-Q5

4 W4

1. One side limit:

Two sided limits vs one sided limits

- Fact: We have

$$\boxed{\lim_{x\to a} f(x) = \ell \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \ell.}$$

- Critiria for "$\lim_{x\to a} f(x)$ DNE":

  If one of the following holds: e.g. step function
  - $\lim_{x\to a^-} f(x) = \ell \neq m = \lim_{x\to a^+} f(x)$, or
  - $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ DNE ← e.g. vertical asymptote

  then $\lim_{x\to a} f(x)$ DNE.

2. DNE problem:

Two sided limits vs one sided limits

- Fact: We have

$$\boxed{\lim_{x\to a} f(x) = \ell \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \ell.}$$

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  - $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ DNE ← e.g. vertical asymptote

  then $\lim_{x\to a} f(x)$ DNE.

3. *Note*:
Limits are usually computed by using some elementary limits and the limit and squeeze laws.

4. Limit Laws. If $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$, then

   - $\lim_{x\to a}(kf(x)) = k\ell$ for all $k \in \mathbb{R}$.
   - $\lim_{x\to a}(f(x)g(x)) = \ell m$
   - $\lim_{x\to a}(f(x) + g(x)) = \ell + m$
   - $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\ell}{m}$ provided $m \neq 0$

5. Squeeze Law. Suppose that $f(x) \le g(x) \le h(x)$ for all $x \neq a$ near a. If $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} h(x) = \ell$, then $\lim_{x\to a} g(x) = \ell$.
   W4-Exercises-Q5 W4-TUT-Q4

6. Elementary limits:
   Frequently used elementary limits are $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Roots $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ $(a > 0)$ and other elementary functions allow the computation of limits by substitution (they are continuous, see below).
   W4-Exercises-Q7

7. Limits with fractions:
   Limits of the form $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ are often computed by dividing numerator and denomintor by the fastest growing term, often the highest power of x.
   W4-Exercises-Q3 W4-Exercises-Q4 W4-TUT-Q2 W4-TUT-Q3

8. Definition of continutity:
A function f is <u>continuous</u> at a point a if a is in the domain of f and $\lim_{x\to a} f(x) = f(a)$, that is, the limit exists and equals the function value at a. We often make use of the fact that elementary functions such as polynomials, roots, exponentials and the trigonometric functions are continuous.

- **continuity at a point**

Continuity

- **Definition (continuity at a point):** $f(x)$ is continuous at $x = a$ if

$$\boxed{\lim_{x\to a} f(x) = f(a).}$$

- Remarks: The condition $\lim\limits_{x\to a} f(x) = f(a)$ means

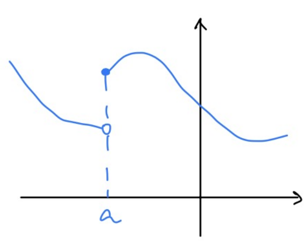
1. $\lim\limits_{x\to a} f(x)$ exists, so $\lim\limits_{x\to a} f(x) = \ell$ for some $\ell \in \mathbb{R}$;
2. $f(x)$ is defined at $x = a$, so we have the functional value $f(a) \in \mathbb{R}$;
3. $\ell = f(a)$, i.e. $\lim\limits_{x\to a} f(x) = f(a)$.

- **Variations of the theme of continuity**

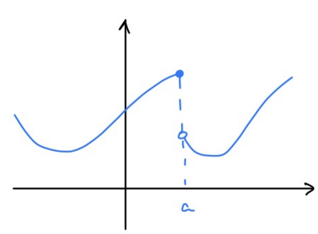
Variations of the theme of continuity

(1) $f(x)$ is left continuous at $x = a$ if $\lim\limits_{x\to a^-} f(x) = f(a)$.

(2) $f(x)$ is right continuous at $x = a$ if $\lim\limits_{x\to a^+} f(x) = f(a)$.



right - cts



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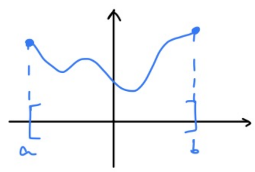
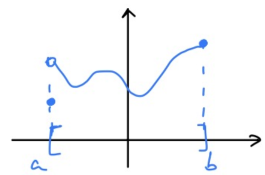
Variations of the theme of continuity

(3) $f(x)$ is continuous on the interval $[a,b]$ if

  (a) $f(x)$ is continuous at each point $x = c$ with $c \in (a,b)$, and
  (b) $f(x)$ is right continuous at $x = a$, and
  (c) $f(x)$ is left continuous at $x = b$.



Cts on [a,b]



not Cts on [a,b]

(4) $f(x)$ is continuous on the interval $(a,b]$ if

  (a) $f(x)$ is continuous at each point $x = c$ with $c \in (a,b)$, and
  (b) $f(x)$ is left continuous at $x = b$.

Remark: Similar definitions for intervals $[a,b)$, (a,b).

- **Limits of continuous functions**
  - If $f(x)$ is continuous at $x = a$, then $\lim_{x \to a} f(x) = f(a)$ by direct substitution of $x = a$ into $f(x)$.
  - Many familiar functions are continuous on their natural domains, including:
    - polynomials
    - trig functions
    - inverse trig functions
    - exponential functions
    - logarithmic functions
    - hyperbolic trig functions and their inverses
    - $\sqrt{x}$, $x^{1/n}$

      $\vdots$

W4-TUT-Q6 W4-TUT-Q7 W4-TUT-Q8

- **Composition Law**
  - **Theorem (Composition Law):** If

$$f(x) \text{ is continuous at } x = a, \quad \text{and} \quad g(x) \text{ is continuous at } x = f(a),$$

then $(g \circ f)(x) = g(f(x))$ is continuous at $x = a$. That is,

$$\lim_{x \to a} g(f(x)) = g(f(a)).$$

5 W5

1. **Definition (Derivative):**
The derivative of a function f at x_0 is the slope of the tangent to the graph of f at the point $(x_0, f(x_0))$. If it exists, it is the limit of the difference quotient
$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
Derivatives also represent rates of change.

2. **Note (Rules of Differentiation):**
If $f(x)$ and $g(x)$ are differentiable, then the following functions are also differentiable, with derivatives as stated:
(1) $(\alpha f)' = \alpha f'$ for $\alpha \in \mathbb{R}$
(2) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ (if $g'(x) \neq 0$, quotient rule)
(3) $(f + g)'(x) = f' + g'$
(4) $(fg)' = f'g + fg'$(product rule)
(5) $(f \circ g)'(x) = g'(x) f'(g(x))$ (chain rule)

3. **Note (Implicit differentiation):**
Functions can be given implicitly by an equation such as $f(x, y) = c$, where c is a constant. For example $\cos(y) + xy = 3$. Assuming that y is a function of x we use the rules of differentiation to take the derivative. Since the derivative of a constant is zero differentiation with respect to x is given by $\sin(y)y' + x = 0$. Solve for y' to get the derivative.

6 W6

1. **Note:**
   Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable with a continuous derivative.

   - If $f'(x) > 0[< 0]$ on some interval (c, d), then f is strictly increasing [decreasing] on (c, d).
   - If $f''(x) > 0[< 0]$ on some interval (c, d), then f is concave up [down] on (c, d).
   - If f has a (local) maximum or minimum at some $c \in (a, b)$, then $f'(c) = 0$.
   - If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$. Likewise, if $f'(c) = 0$ and $f''(c) > 0$, then f has a minimum at $x = c$. (In both cases f'' needs to be continuous)
   - If f has a point of inflection at $x = c$, then $f''(c) = 0$.
   - If $f''(c) = 0$ and the sign of f'' changes at $x = c$, then f has a point of inflection at $x = c$.
   - The function f has a global maximum [minimum] at some $c \in [a, b]$ with $f'(c) = 0$ if $c \in (a, b)$.

2. **Method (L'Hôpital's rule):**
   L'Hôpital's rule is a method to compute limits of fractions that lead to indeterminate expressions of the form $0/0$ or $\pm\infty/\pm\infty$. If f, g are such that $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ and

   $$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$$

   exists, then

   $$\lim_{x \to a} \frac{f(x)}{g(x)} = L$$

   A similar statement holds if $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$. It is also valid if $a = \pm\infty$ or $L \pm \infty$.

7 W7

1. **Note (Taylor polynomial):**
   The n-th order Taylor polynomial T_n of a n times differentiable function f at a point a is the polynomial of degree at most n providing the best possible approximation of f near a. It is given by

$$\begin{aligned}T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \\ + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n\end{aligned}$$

   Note that $f^{(k)}(a) = T_n^{(k)}(a)$ for $k = 0, \ldots, n$. We call T_n the n-th order Taylor polynomial of f about $x = a$.

   Substitution method:
   If T_n is the n-th order Taylor polynomial of f about $a = 0$ and $m \in \mathbb{N}$, then $T_n(x^m)$ is the Taylor polynomial of order mn of $f(x^m)$ about 0 .

2. **Note (Lagrange form of remainder):**
   If T_n is the n-th order Taylor polynomial of f centred at a we call

$$R_n(x) = f(x) - T_n(x)$$

   the n-th order remainder. It satsifies the limit

$$\lim_{x \to a} \frac{R_n(x)}{(x-a)^n} = 0$$

   If f has $n+1$ derivatives, then $R_n(x)$ can be represented in the Lagrange form: There exists c strictly between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

   It is almost like the $(n+1)$-st term, but with $f^{(n+1)}(a)$ replaced by $f^{(n+1)}(c)$.

8 W8

1. **Note (Taylor series expansion):**
   The Taylor series expansion of a function f at a point a is the limit of Taylor the polynomials. It represents the function f if

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{n!}(x-a)^k = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{n!}(x-a)^k.$$

   If $T_n(x)$ is the n-th order polynomial and $R_n(x) = f(x) - T_n(x)$, then the Taylor series represents f if $R_n(x) \to 0$ as $n \to \infty$. <u>To show this we use the Lagrange form of remainder given by</u>

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

   for some c between x and a. Often, $|R_n(x)|$ is maximised over c between a and x to show that $|R_n(x)| \to 0$.

2. **Note (Geometric series):**<u>If $x \neq 1$, then for every $n \in \mathbb{N}$ we have</u>

$$1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

   and if $|x| < 1$, then

$$\frac{1}{1-x} = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{\infty} x^k$$

9 W9

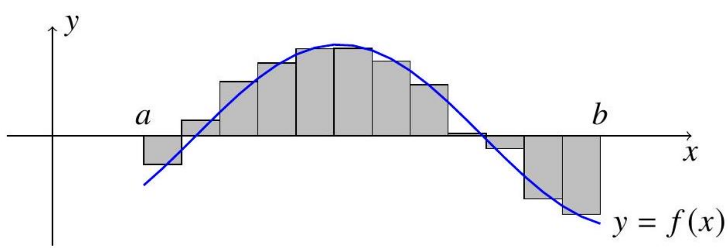
1. Definition (Riemann sums). Let f be a continuous function on an interval $[a, b]$ and let $a = x_0 < x_1 < x_2 < \cdots < x_N = b$ be a partition (subdivision) of $[a, b]$ into N intervals $[x_{k-1}, x_k]$ of equal length (for simplicity). Then

$$\Delta x = \frac{b-a}{N}$$

is the length of these intervals and $x_k = a + k\Delta x$ for $k = 0, \ldots, N$. Choose points a point x_k^* from each of the intervals $[x_{k-1}, x_k]$. The sum

$$\sum_{k=1}^{N} f\left(x_k^*\right) \Delta x$$

is called a Riemann sum. It approximates the area of the region under the graph $y = f(x)$ given by sums of the area of rectangles as shown below.



2. The limit of these sums as $N \to \infty$ is the definite integral of f over $[a, b]$, denoted by

$$\lim_{N\to\infty} \sum_{k=1}^{N} f\left(x_k^*\right) \Delta x = \int_a^b f(x)dx.$$

3. **Note (Upper and lower Riemann sums):**
If we choose x_k^* such that $f\left(x_k^*\right)$ is the maximum (or minimum) of f on $[x_{k-1}, x_k]$, then the corresponding Riemann sum is called the upper (or lower) Riemann sum. We denote them by U_N and L_N, respectively. We have

$$L_N \leq \int_a^b f(x)dx \leq U_N$$

If the function is increasing, then for x^*k choose the left endpoint of $[x_{k-1}, x_k]$ for the lower sum and the right endpoint for the upper sum. For a decreasing function it is the other way.

10 W10

1. Theorem (Fundamental Theorem of Calculus.). The fundamental theorem of calculus makes a connection between integration and differentiation. It comes in two parts.

   Part I If $f : [a, b] \to \mathbb{R}$ is continuous, then

   $$\frac{d}{dx}\int_c^x f(t)dt = f(x) \quad \text{for all } x \in [a, b]$$

   W10-Exercises-Q3 W10-Exercises-Q4 W10-TUT-Q3 W10-TUT-Q4 W10-TUT-Q5

   Part II If $F : [a, b] \to \mathbb{R}$ is differentiable and $F' : [a, b] \to \mathbb{R}$ is continuous, then

   $$F(b) - F(a) = \int_a^b F'(t)dt$$

2. **Note:**
   As a consequence of Part I we also have

   $$\frac{d}{dx}\int_x^c f(t)dt = -f(x) \quad \text{for all } x \in [a, b]$$

   and <u>if g is a differentiable function</u>, then the chain rule implies that

   $$\frac{d}{dx}\int_c^{g(x)} f(t)dt = f(g(x))g'(x) \quad \text{for all } x \in [a, b]$$

   and <u>f a and b is a differentiable function</u>, then the chain rule implies that

   $$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x))b'(x) - f(a(x))a'(x) \quad \text{for all } x \in [a, b]$$

11 W11

The following methods of integration are frequently used:

1. Change of variable formula (for definite integrals)
   Integration by substitution Setting $s = u(t)$ we have $ds = u'(s)dt$ and

$$\int_{u(a)}^{u(b)} f(s)ds = \int_a^b f(u(t))u'(t)dt$$

$$\int_0^{\frac{\pi}{2}} \cos(x)\sin^7(x)\,dx$$

Let $u = \sin(x) \Rightarrow du = \cos(x)\,dx \Rightarrow dx = \frac{du}{\cos(x)}$

$x = 0;\ u = 0$

$x = \frac{\pi}{2}\ \ u = 1$

$[*] = \int_0^1 \cos(x)\, u^7 \frac{du}{\cos(x)} = \int_0^1 u^7\,du = \left[\frac{u^8}{8}\right]_0^1 = \frac{1}{8}$

Change of variable formula (for indefinite integrals)

$$\int f(u(x))u'(x)\,dx = \int f(u)\,du$$

$$\int x\cos(x^2)\,dx$$

Let $x^2 = u \Rightarrow 2x\,dx = du \Rightarrow dx = \frac{du}{2x}$

$[*] = \int x\cos(u)\cdot\frac{du}{2x} = \frac{1}{2}\int \cos(u)\,du$

$= \frac{1}{2}\sin(u) + C$

$= \frac{1}{2}\sin(x^2) + C$

appendix:
Trig substitution
The trigonometric substitution method is a technique used in calculus to simplify integrals involving radicals. It involves substituting trigonometric functions for the variables in the integral. The most common trigonometric substitutions are:

- For $\sqrt{a^2 - x^2}$, where $a > 0$, we use the substitution $x = a\sin\theta$ $(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$.
- For $\sqrt{a^2 + x^2}$, we use the substitution $x = a\tan\theta$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$.
- For $\sqrt{x^2 - a^2}$, where $a > 0$, we use the substitution $x = a\sec\theta$ $(0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2})$.

These substitutions help in simplifying the integrals by expressing the radical term in terms of trigonometric functions, which often leads to easier calculations and solution of the integral.

Example:

$\int_0^1 \sqrt{1-x^2}\, dx$

Let $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$\begin{bmatrix} x=0, & \theta=0 \\ x=1, & \theta=\frac{\pi}{2} \end{bmatrix}$

$[*] = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\, d\theta \cos\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}\, d\theta$

$= \left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

2. Integration by parts We have

$$\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}v dx$$

3. Integration by partial fractions Assume that f can be written in the form

$$f(x) = \frac{a + bx}{(x-\lambda)(x-\mu)}$$

with $\lambda \neq \mu$ by possibly factorising the numerator. Write

$$f(x) = \frac{A}{x-\lambda} + \frac{B}{x-\mu} = \frac{A(x-\mu) + B(x-\lambda)}{(x-\lambda)(x-\mu)}$$

and determine A, B by equating $A(x-\mu)+B(x-\lambda) = ax+b$. The easiest way is to choose $x = \lambda$ and $x = \mu$ to determine A and B in terms of a, b, μ, λ.

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1. **Note (Area between graphs):**
   Let $f, g : [a, b] \to \mathbb{R}$ be continuous funcitons with $f(x) \geq g(x)$ for all $x \in [a, b]$. Then the region between the graphs has surface area given by

   $$\int_a^b f(x) - g(x) dx$$

   If the graphs cross over, one has to compute sum of the surface areas between the cross-over points to make sure it has the correct positive sign.

   W12-Exercises-Q1 W12-Exercises-Q2

2. **Note (Length of a graph)**:
   If $f : [a, b] \to \mathbb{R}$ is differentiable, then the length of the curve given by the graph is given by

   $$\int_a^b \sqrt{1 + [f'(x)]^2} dx$$

   The formula is obtained by approximating the length by the length of a polygon and passing to the limit with a Riemann sum.

   Note (Volumes of revolution).
   Disc method: The volume of the solid obtained by revolving the region between the graph $y = f(x)$, the x-axis, $x = a$ and $x = b$ about the x-axis is

   $$V = \int_a^b \pi f(x)^2 dx$$

   The formula is obtained by slicing the solid into thin disks of radius $f(x)$ with volume $\pi f(x)^2 \Delta x$, letting $\Delta X \to 0$.
   W12-Exercises-Q4

Shell method: The volume of the solid obtained by revolving the region between the graph $y = f(x)$, the x-axis, $x = a$ and $x = b$ about the y-axis is

$$V = 2\pi \int_a^b xf(x)dx$$

The formula is obtained by slicing the solid into thin cylindrical shells of radius x and height $f(x)$ with approximate volume $2\pi xf(x)\Delta x$, then letting $\Delta X \to 0$.
W12-Exercises-Q5