

COMMONWEALTH OF AUSTRALIA

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COMP2123

Data structures and Algorithms

Lecture 10: Divide and Conquer

[GT 3.1 and 8]

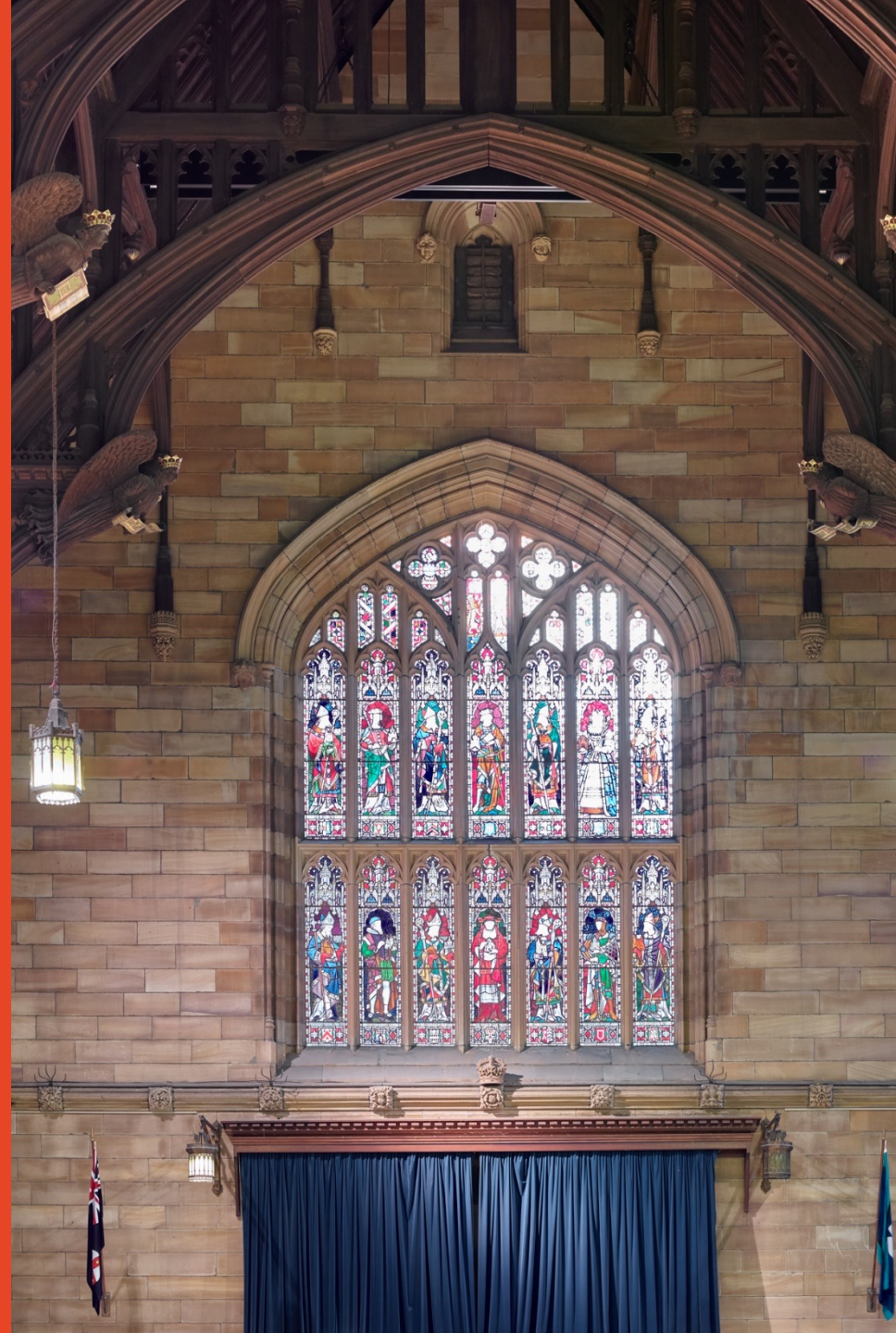
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Divide and Conquer

Divide and Conquer algorithms can normally be broken into these three parts:

1. **Divide** If it is a base case, solve directly, otherwise break up the problem into several parts.
2. **Recur/Delegate** Recursively solve each part [each sub-problem].
3. **Conquer** Combine the solutions of each part into the overall solution.

Divide and Conquer

1. **Divide** If it is a base case, solve directly, otherwise break up the problem into several parts.

Typical base case:

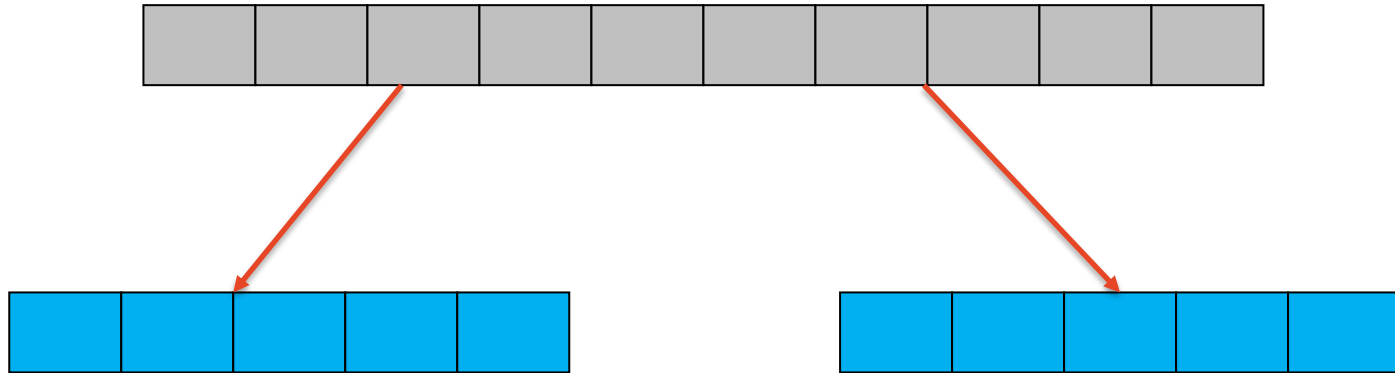
Subproblem of constant size (usually 0 or 1 elements) for which you can compute the solution explicitly



easy to compute solution

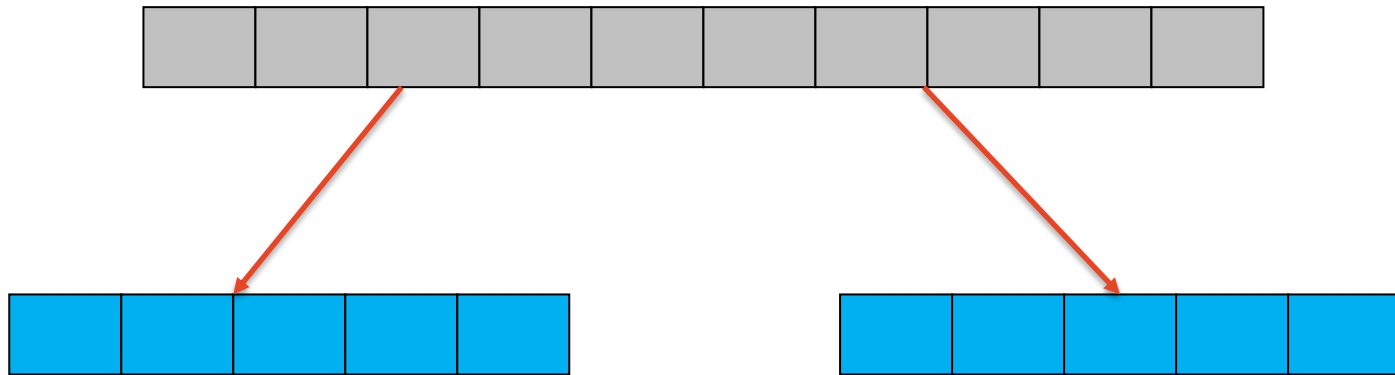
Divide and Conquer

1. **Divide** If it is a base case, solve directly, otherwise break up the problem into several parts.



Divide and Conquer

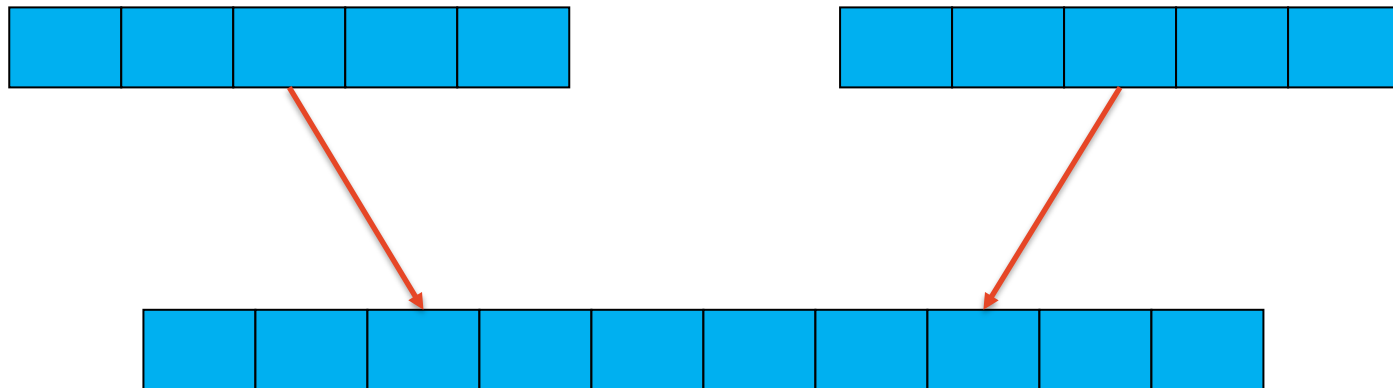
2. **Recur/Delegate** Recursively solve each part [each sub-problem].



The sub-problems are solved by the Recursion Fairy (similar to induction hypothesis), so we don't have to worry about them.

Divide and Conquer

3. **Conquer** Combine the solutions of each part into the overall solution.



Searching Sorted Array

Given A sorted sequence S of n numbers a_0, a_1, \dots, a_{n-1} stored in an array $A[0, 1, \dots, n - 1]$.

Problem Given a number x , is x in S ?

0	2	5	7	12	22	25	37	39	50	55	80
---	---	---	---	----	----	----	----	----	----	----	----

Searching: Naïve Approach

Problem Given a number x , is x in S ?

Idea Check every element in turn to see if it is equal to x .

```
for e in S do
  if e equals x then
    return "Yes"
return "No"
```

Found an element equal to x in S

There was no element equal to x in S

0	2	5	7	12	22	25	37	39	50	55	80
---	---	---	---	----	----	----	----	----	----	----	----

Running Time $O(n)$

Binary Search in sorted $A[0 \text{ to } n-1]$

1. If the array is empty, then return “No”
2. Compare x to the middle element, namely $A[\lfloor n/2 \rfloor]$
3. If this middle element is x , then return “Yes”
4. When the middle element is not x : if $A[\lfloor n/2 \rfloor] > x$, then recursively search $A[0 \text{ to } \lfloor n/2 \rfloor - 1]$
5. if $A[\lfloor n/2 \rfloor] < x$, then recursively search $A[\lfloor n/2 \rfloor + 1 \text{ to } n-1]$

0	2	5	7	12	22	25	37	39	50	55	80
---	---	---	---	----	----	----	----	----	----	----	----

Binary Search Pseudocode

```
def binary_search(A, left, right, x):  
    # A is sorted and left <= right  
    # looking for x in A[left:right]  
  
    if left = right then  
        return “unsuccessful”  
  
    mid = floor((left + right) / 2)  
    if A[mid] < x then  
        return binary_search(A, mid + 1, right, x)  
    else if A[mid] > x then  
        return binary_search(A, left, mid, x)  
    else  
        return mid
```

Heads up: pseudocode textbook uses indexing from 1 to n, not 0 to n-1

Binary Search

- Example, search for $x=5$

0	2	5	7	12	22	25	37	39	50	55	80
---	---	---	---	----	----	----	----	----	----	----	----

Binary Search

- Example, search for $x=5$

0	2	5	7	12	22	25	37	39	50	55	80
---	---	---	---	----	----	----	----	----	----	----	----

A[6]

Binary Search

- Example, search for $x=5$

0	2	5	7	12	22	25	37	39	50	55	80
---	---	---	---	----	----	----	----	----	----	----	----

$$A[6] = 25 > 5 = x$$

Binary Search

- Example, search for $x=5$

0	2	5	7	12	22
---	---	---	---	----	----

A[3]

25	37	39	50	55	80
---------------	----	----	----	----	----

Binary Search

- Example, search for $x=5$

0	2	5	7	12	22
---	---	---	---	----	----

25	37	39	50	55	80
---------------	----	----	----	----	----

$$A[3] = 7 > 5 = x$$

Binary Search

- Example, search for $x=5$

0	2	5
---	---	---

A[1]

7	12	22	25	37	39	50	55	80
--------------	----	----	---------------	----	----	----	----	----

Binary Search

- Example, search for $x=5$

0	2	5	7	12	22	25	37	39	50	55	80
---	---	---	--------------	----	----	---------------	----	----	----	----	----

$$A[1] = 2 < 5 = x$$

Binary Search

- Example, search for $x=5$



A[2]

Binary search correctness

Invariant: If x is in A before the divide step, then x is in A after the divide step

- if $A[\lfloor n/2 \rfloor] > x$, then x must be in $A[0 \text{ to } \lfloor n/2 \rfloor - 1]$
- if $A[\lfloor n/2 \rfloor] < x$, then x must be in $A[\lfloor n/2 \rfloor + 1 \text{ to } n - 1]$

Every divide step leads to a smaller array.

Thus, if x is in A , we will eventually inspect its position due to the invariant and return “Yes”.

Thus, if x is not in A , then eventually we reach the empty array and return “No”.

Recurrence formula

An easy way to analyze the time complexity of a divide-and-conquer algorithm is to define and solve a recurrence

Let $T(n)$ be the running time of the algorithm, we need to find out:

- Divide step cost in terms of n
- Recur step(s) cost in terms of $T(\text{smaller values})$
- Conquer step cost in terms of n

Together with information about the base case, we can set up a recurrence for $T(n)$ and then solve it.

$$T(n) = \begin{cases} \text{“Recur”} + \text{“Divide and Conquer”} & \text{for } n > 1 \\ \text{“Base case” (typically } O(1)) & \text{for } n = 1 \end{cases}$$

Binary search on an array complexity analysis

Divide step (find middle and compare to x) takes $O(1)$

Recur step (solve left or right subproblem) takes $T(n/2)$

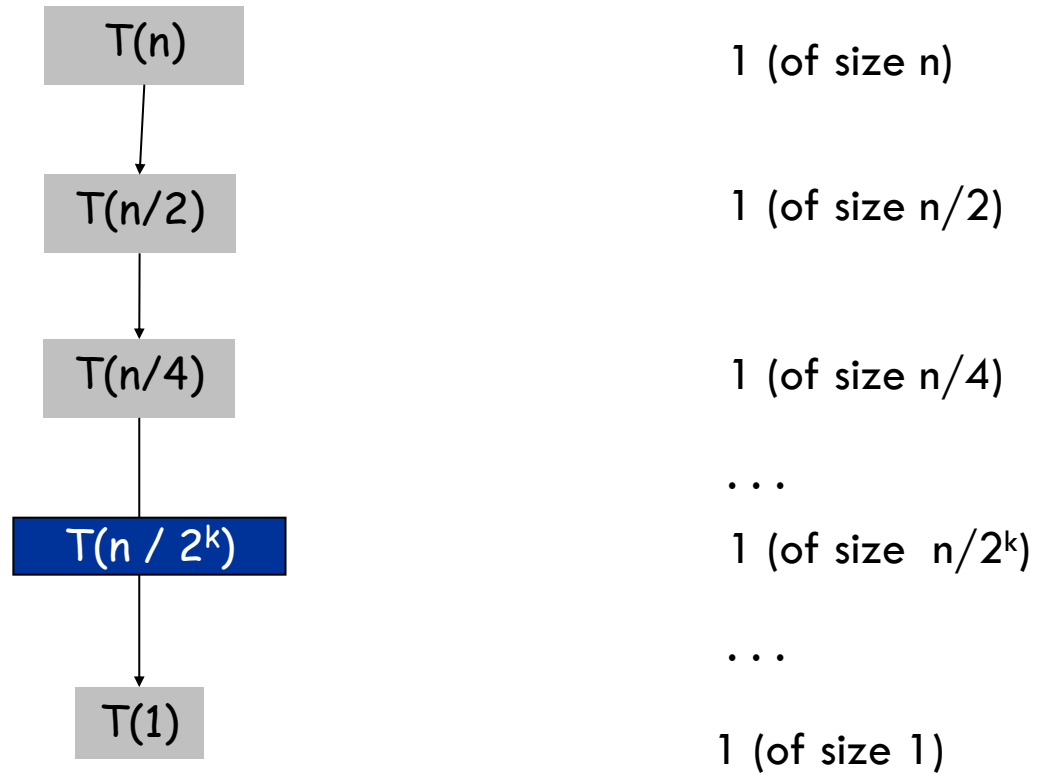
Conquer step (return answer from recursion) takes $O(1)$

Now we can set up the recurrence for $T(n)$:

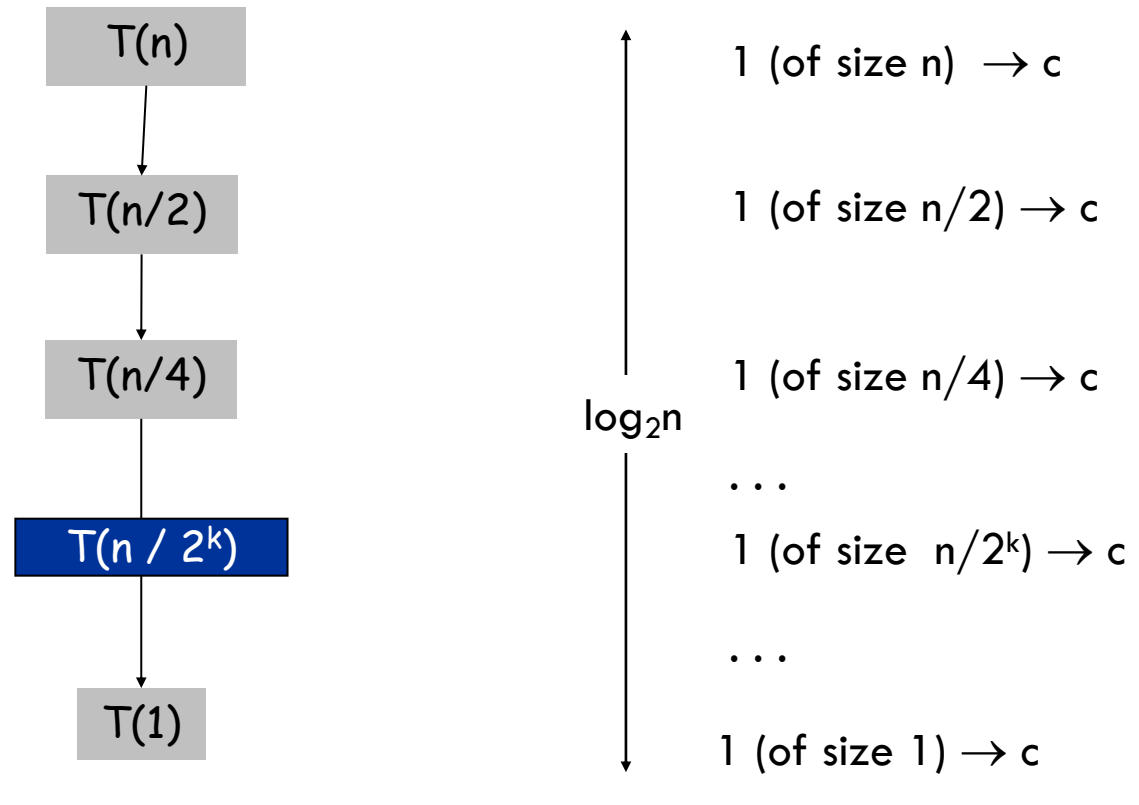
$$T(n) = \begin{cases} T(n/2) + O(1) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

This solves to $T(n) = O(\log n)$, since we can only halve the input $O(\log n)$ times before reaching a base case

Proof by unrolling: $T(n) = T(n/2) + O(1)$



Proof by unrolling: $T(n) = T(n/2) + O(1)$



Binary search on a linked list complexity analysis

Divide step (find middle and compare to x) takes $O(n)$

Recur step (solve left or right subproblem) takes $T(n/2)$

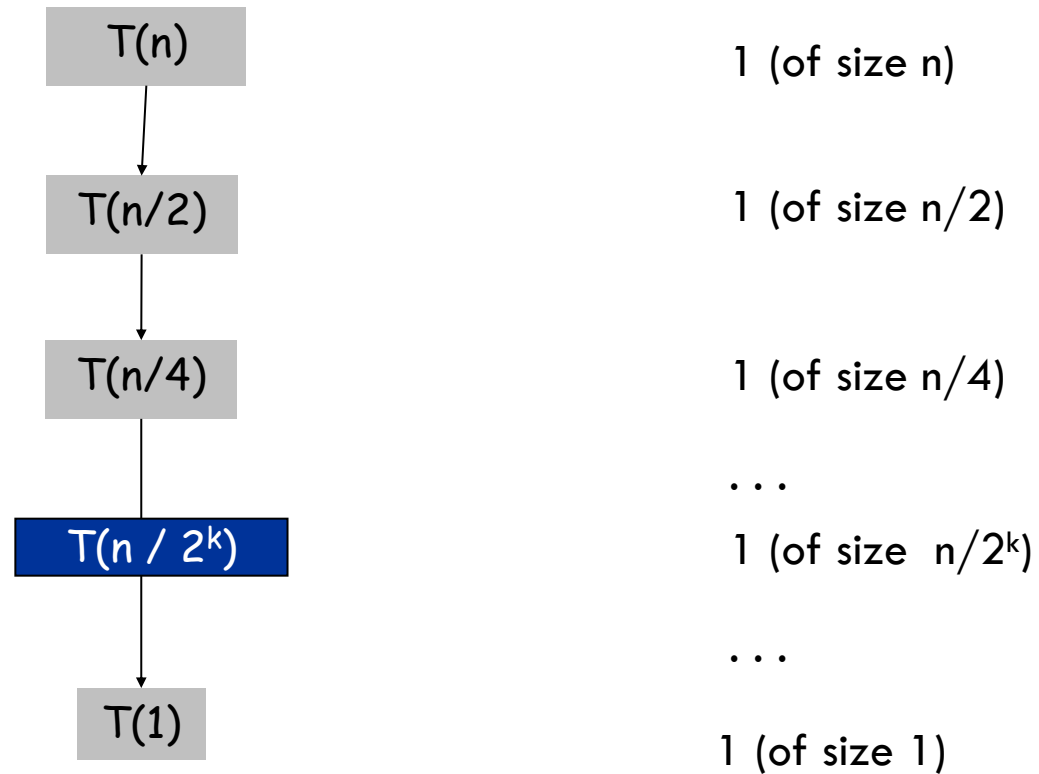
Conquer step (return answer from recursion) takes $O(1)$

Now we can set up the recurrence for $T(n)$:

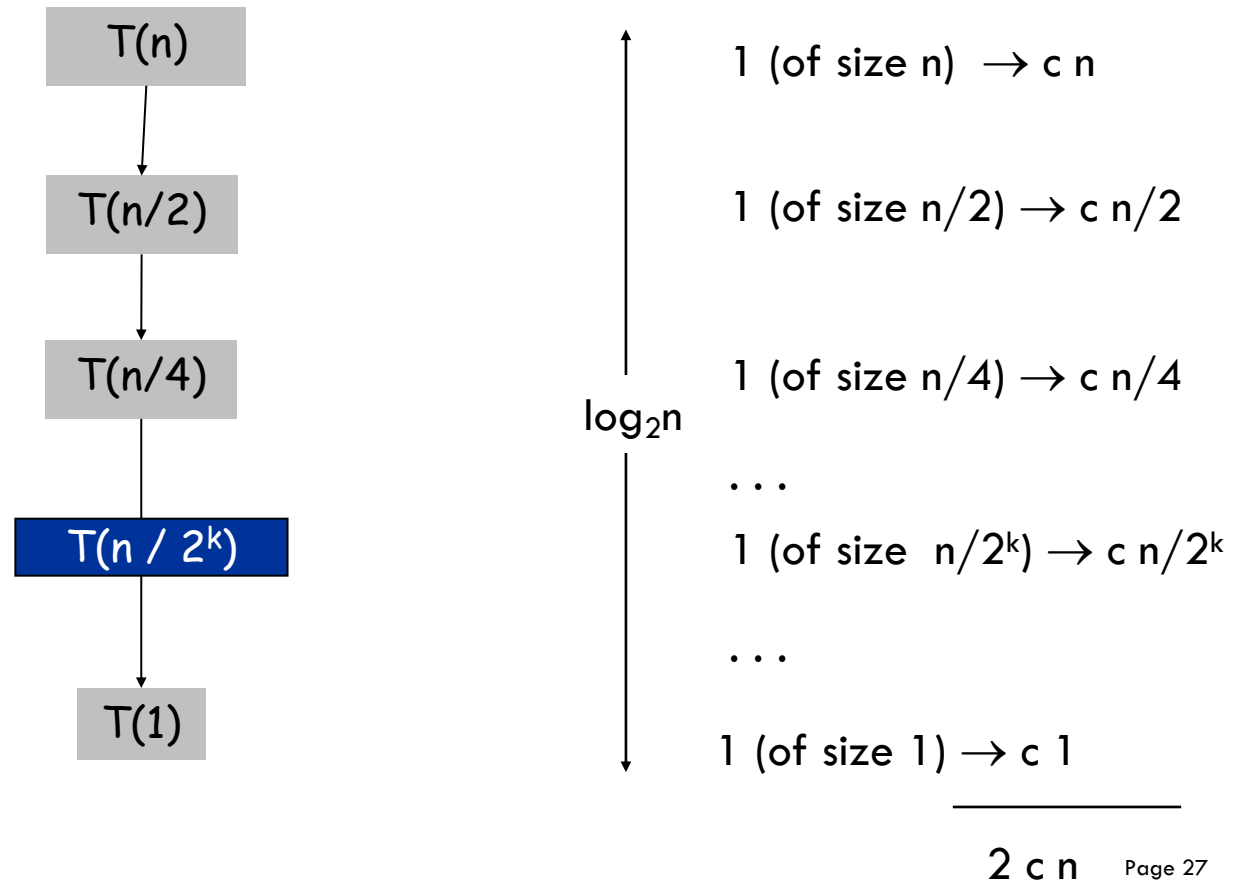
$$T(n) = \begin{cases} T(n/2) + O(n) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

This solves to $T(n) = O(n)$, since to access the next index we end up with $n/2 + n/4 + n/8 + \dots$

Proof by unrolling: $T(n) = T(n/2) + O(n)$



Proof by unrolling: $T(n) = T(n/2) + O(n)$



Merge-Sort

1. **Divide** the array into two halves.
2. **Recur** recursively sort each half.
3. **Conquer** two sorted halves to make a single sorted array.

1	12	5	16	19	7	23	6	13	20
---	----	---	----	----	---	----	---	----	----

1	12	5	16	19
---	----	---	----	----

7	23	6	13	20
---	----	---	----	----

Divide

1	5	12	16	19
---	---	----	----	----

6	7	13	20	23
---	---	----	----	----

Recur

1	5	6	7	12	13	16	19	20	23
---	---	---	---	----	----	----	----	----	----

Conquer

Merge-Sort pseudocode

```
def merge_sort(S):  
    # base case  
    if |S| < 2 then  
        return S  
  
    # divide  
    mid ← ⌊|S|/2⌋  
    left ← S[:mid]      # doesn't include S[mid]  
    right ← S[mid:]     # includes S[mid]  
  
    # recur  
    sorted_left ← merge_sort(left)  
    sorted_right ← merge_sort(right)  
  
    # conquer  
    return merge(sorted_left, sorted_right)
```

How?

Merge

Input Two sorted lists.

Output A new merged sorted list.

To merge, we use:

- $O(n)$ comparisons.
- An array to store our results.



Result:

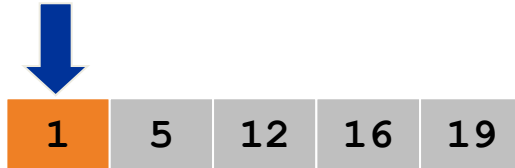


Merge

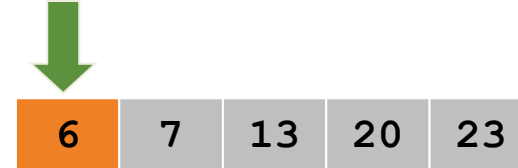
Merge Algorithm

- Keep track of smallest element in each sorted half.
- Insert smallest of two elements into the resultant array.
- Repeat until done.

smallest



smallest



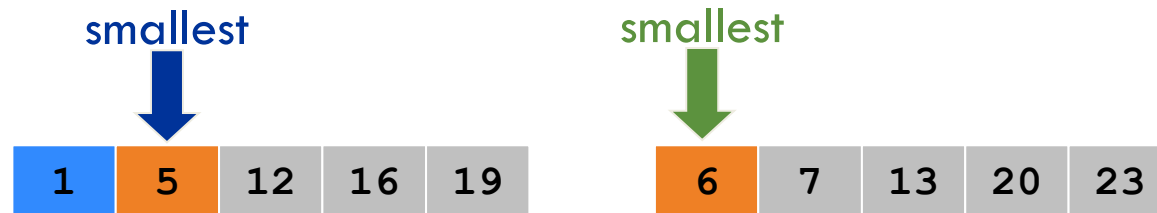
Result:



Merge

Merge Algorithm

- Keep track of smallest element in each sorted half.
- Insert smallest of two elements into the resultant array.
- Repeat until done.



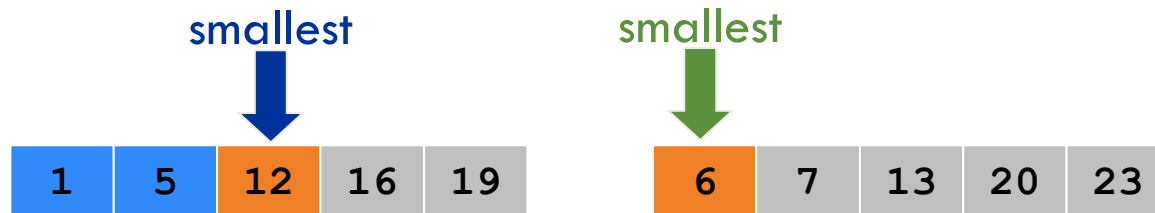
Result:



Merge

Merge Algorithm

- Keep track of smallest element in each sorted half.
- Insert smallest of two elements into the resultant array.
- Repeat until done.



Result:



Merge

Merge Algorithm

- Keep track of smallest element in each sorted half.
- Insert smallest of two elements into the resultant array.
- Repeat until done.



Result:



Merge

Merge Algorithm

- Keep track of smallest element in each sorted half.
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- Repeat until done.



Result:



Merge

Merge Algorithm

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Result:



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- Insert smallest of two elements into the resultant array.
- Repeat until done.



Result:



Merge

Merge Algorithm

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- Insert smallest of two elements into the resultant array.
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Result:



Merge

Merge Algorithm

- Keep track of smallest element in each sorted half.
- Insert smallest of two elements into the resultant array.
- Repeat until done.



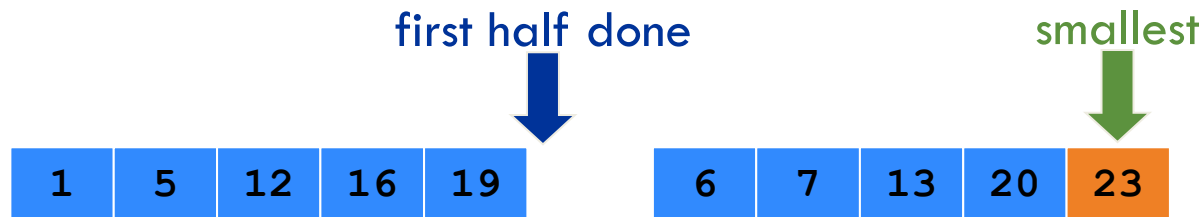
Result:



Merge

Merge Algorithm

- Keep track of smallest element in each sorted half.
- Insert smallest of two elements into the resultant array.
- Repeat until done.



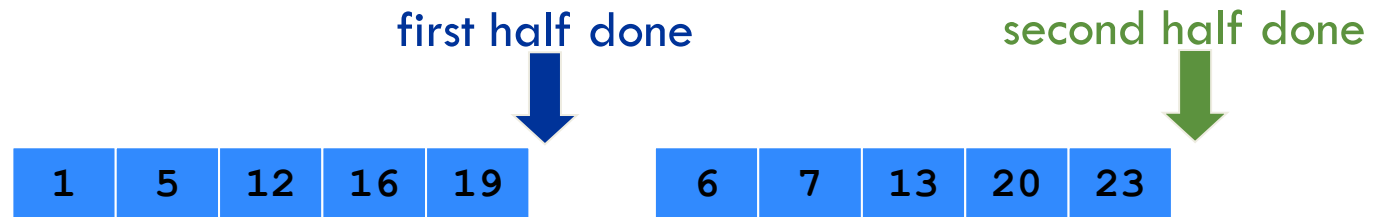
Result:



Merge

Merge Algorithm

- Keep track of smallest element in each sorted half.
- Insert smallest of two elements into the resultant array.
- Repeat until done.



Result:



Merge: Implementation

```
def merge(L, R):  
    result ← array of length (|L| + |R|)  
    l, r ← 0, 0  
    while l + r < |result| do  
        index ← l + r  
        if r ≥ |R| or (l < |L| and L[l] < R[r]) then  
            result[index] ← L[l]  
            l ← l + 1  
        else  
            result[index] ← R[r]  
            r ← r + 1  
    return result
```

Merge: Correctness

Induction hypothesis:

- After the i -th iteration, our result contains the i smallest elements in sorted order

Base case:

- After 0 iterations, our result is empty, so it contains the 0 smallest elements in sorted order

Induction:

- Assume IH after iteration k , to prove it after iteration $k+1$
- Since both halves are sorted and we add the smallest element not already in result, result now contains the $k+1$ smallest elements
- Sorted order follows from the fact that both halves are sorted, thus adding the smallest element implies sorted order of result

Merge-Sort

1. **Divide** array into two halves.
2. **Recur** Recursively sort each half.
3. **Conquer** Merge two sorted halves to make a sorted whole.

1	12	5	16	19	7	23	6	13	20
---	----	---	----	----	---	----	---	----	----

1	12	5	16	19	7	23	6	13	20	divide
---	----	---	----	----	---	----	---	----	----	--------

1	5	12	16	19	6	7	13	20	23	recur
---	---	----	----	----	---	---	----	----	----	-------

1	5	6	7	12	13	16	19	20	23	conquer
---	---	---	---	----	----	----	----	----	----	---------

Merge-Sort: Correctness

Induction hypothesis:

- Merge-Sort correctly sorts an array of size i

Base case:

- If our array has size 0 or 1, it's already sorted

Induction:

- Assume IH for all arrays up to size k , to prove it for array of size $k+1$
- Splitting the array in half gives us two array of size at most k , so by IH those are sorted correctly
- We proved that given two sorted arrays, Merge returns a correctly sorted array containing the elements of both arrays
- Hence, by running Merge on the two sorted halves, we sort the original array

Merge sort complexity analysis

Divide step (find middle and split) takes $O(n)$

Recur step (solve left and right subproblem) takes $2 T(n/2)$

Conquer step (merge subarrays) takes $O(n)$

Now we can set up the recurrence for $T(n)$:

$$T(n) = \begin{cases} 2 T(n/2) + O(n) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

This solves to $T(n) = O(n \log n)$

Solving recurrences by unrolling

General strategy:

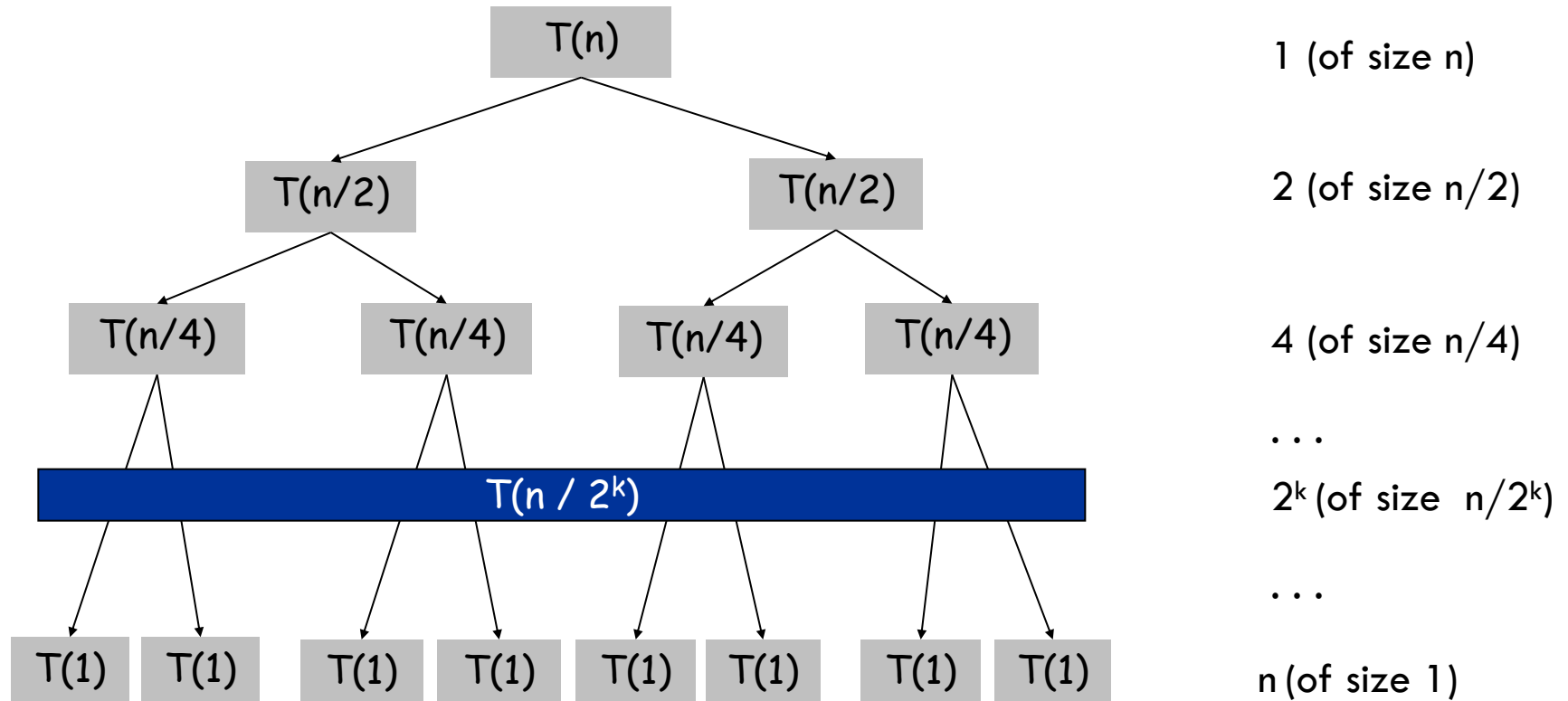
- Analyze first few levels
- Identify the pattern for a generic level
- Sum up over all levels

To verify the solution, we can substitute guess into the recurrence and prove it formally using induction

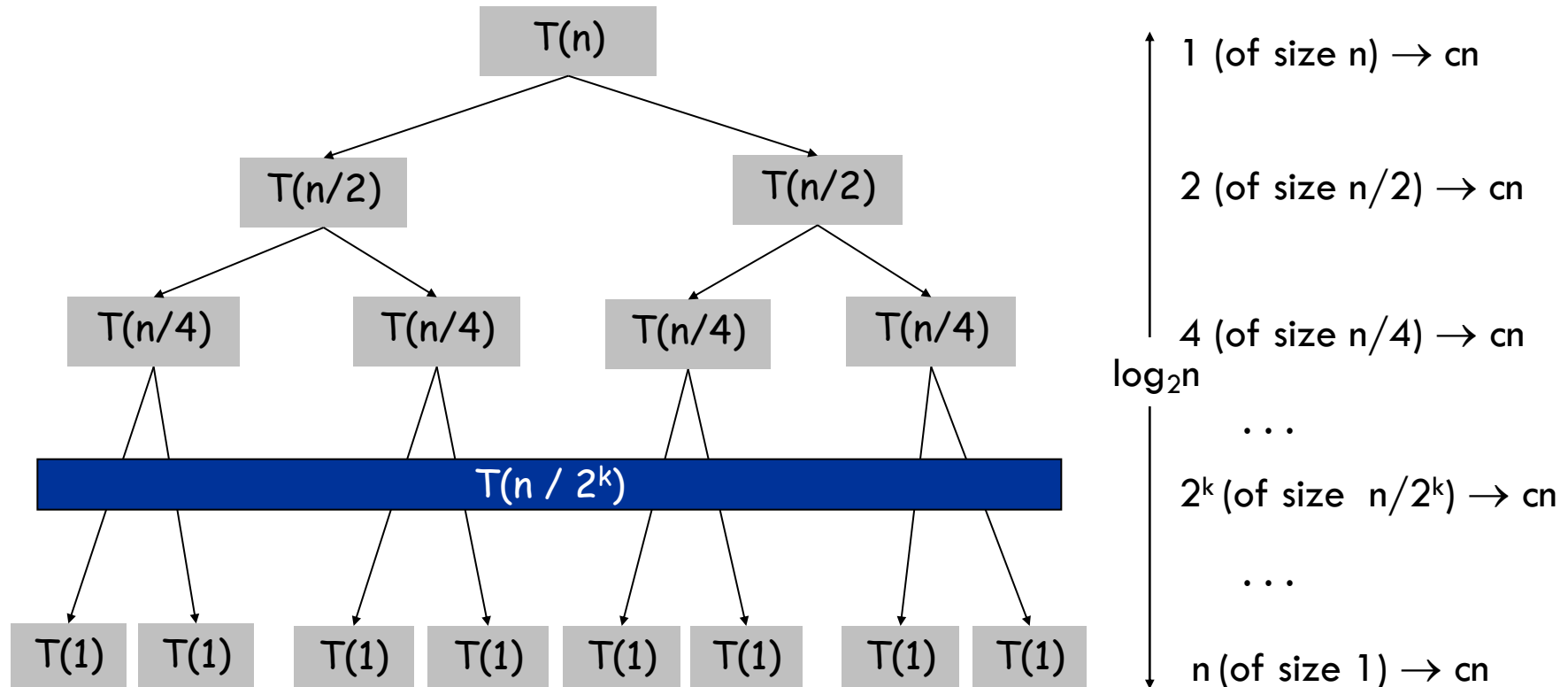
For Merge sort this method yields $T(n) = O(n \log n)$

There is a “Master theorem” (see textbook) that can handle most recurrences of interest, but unrolling is enough for our purposes

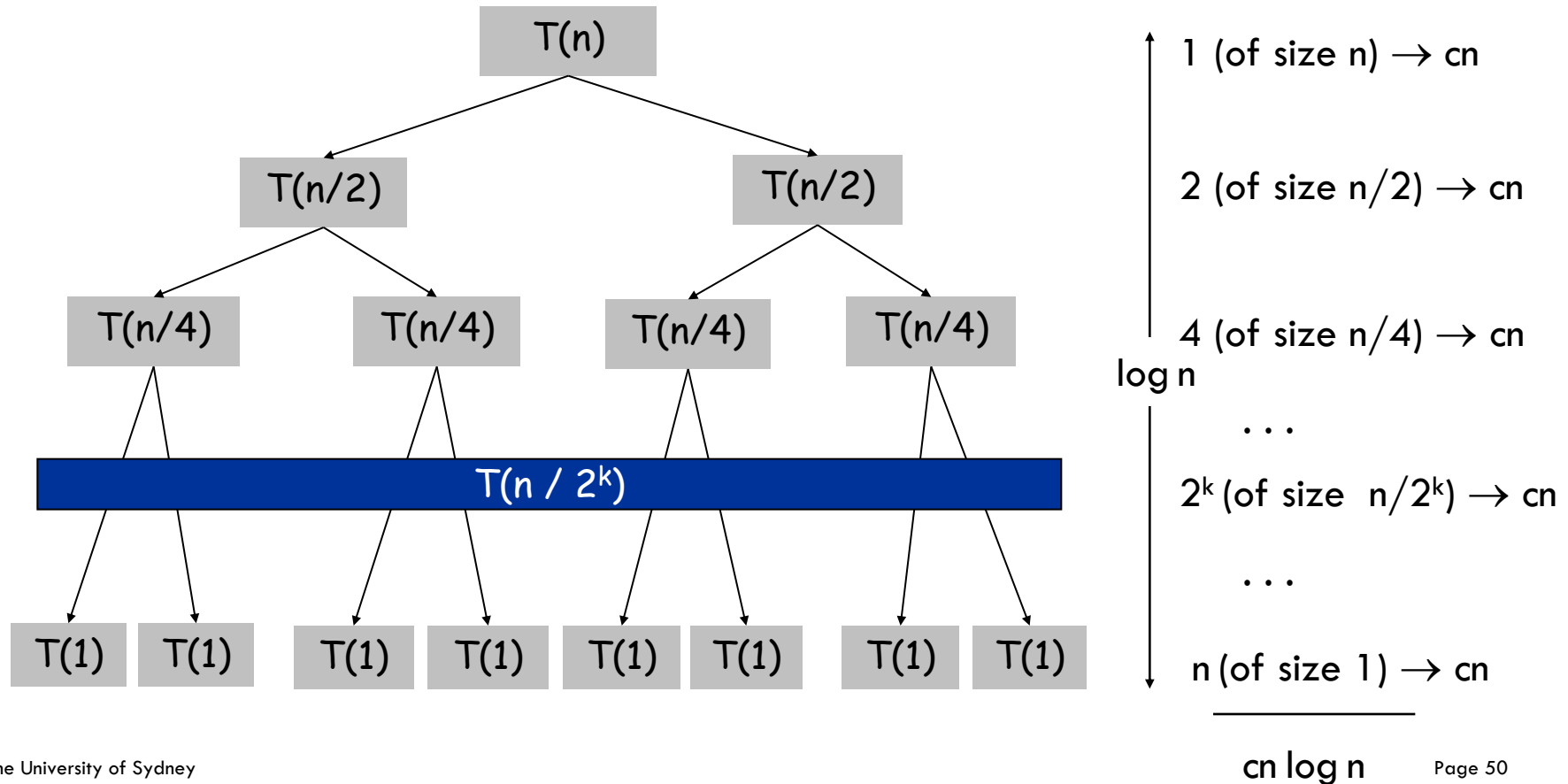
Proof by unrolling: $T(n) = 2 T(n/2) + O(n)$



Proof by unrolling: $T(n) = 2 T(n/2) + O(n)$



Proof by unrolling: $T(n) = 2 T(n/2) + O(n)$

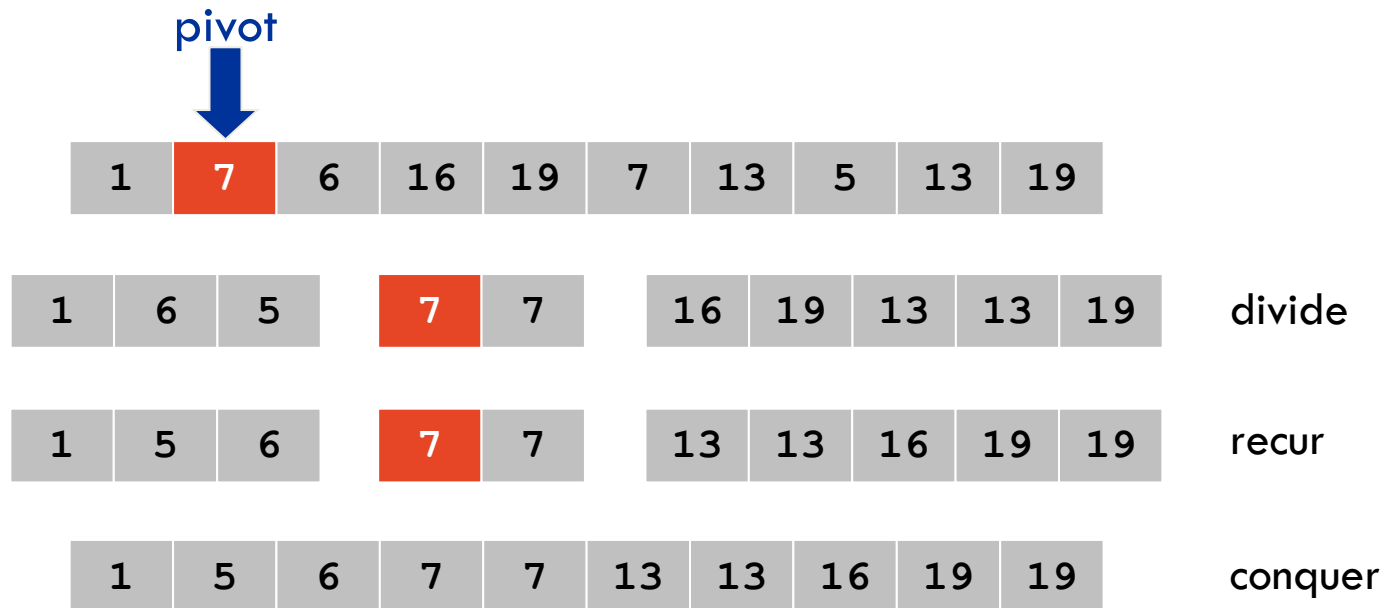


Some recurrence formulas with solutions

Recurrence	Solution
$T(n) = 2 T(n/2) + O(n)$	$T(n) = O(n \log n)$
$T(n) = 2 T(n/2) + O(\log n)$	$T(n) = O(n)$
$T(n) = 2 T(n/2) + O(1)$	$T(n) = O(n)$
$T(n) = T(n/2) + O(n)$	$T(n) = O(n)$
$T(n) = T(n/2) + O(1)$	$T(n) = O(\log n)$
$T(n) = T(n-1) + O(n)$	$T(n) = O(n^2)$
$T(n) = T(n-1) + O(1)$	$T(n) = O(n)$

Quick sort

1. **Divide** Choose a random element from the list as the **pivot**
Partition the elements into 3 lists:
(i) less than, (ii) equal to and (iii) greater than the **pivot**
2. **Recur** Recursively sort the **less than** and **greater than** lists
3. **Conquer** Join the sorted 3 lists together



Quick sort complexity analysis

Divide step (pick pivot and split) takes $O(n)$

Recur step (solve left and right subproblem) takes $T(n_L) + T(n_R)$

Conquer step (merge subarrays) takes $O(n)$

Now we can set up the recurrence for $T(n)$:

$$E[T(n)] = \begin{cases} E[T(n_L) + T(n_R)] + O(n) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

This solves to $E[T(n)] = O(n \log n)$ expected time
(details available on the textbook but not examinable)

Interlude: Comparison sorting lower bound

So far we've seen many sorting algorithms. Some run in $O(n^2)$ time while others run in $O(n \log n)$ time.

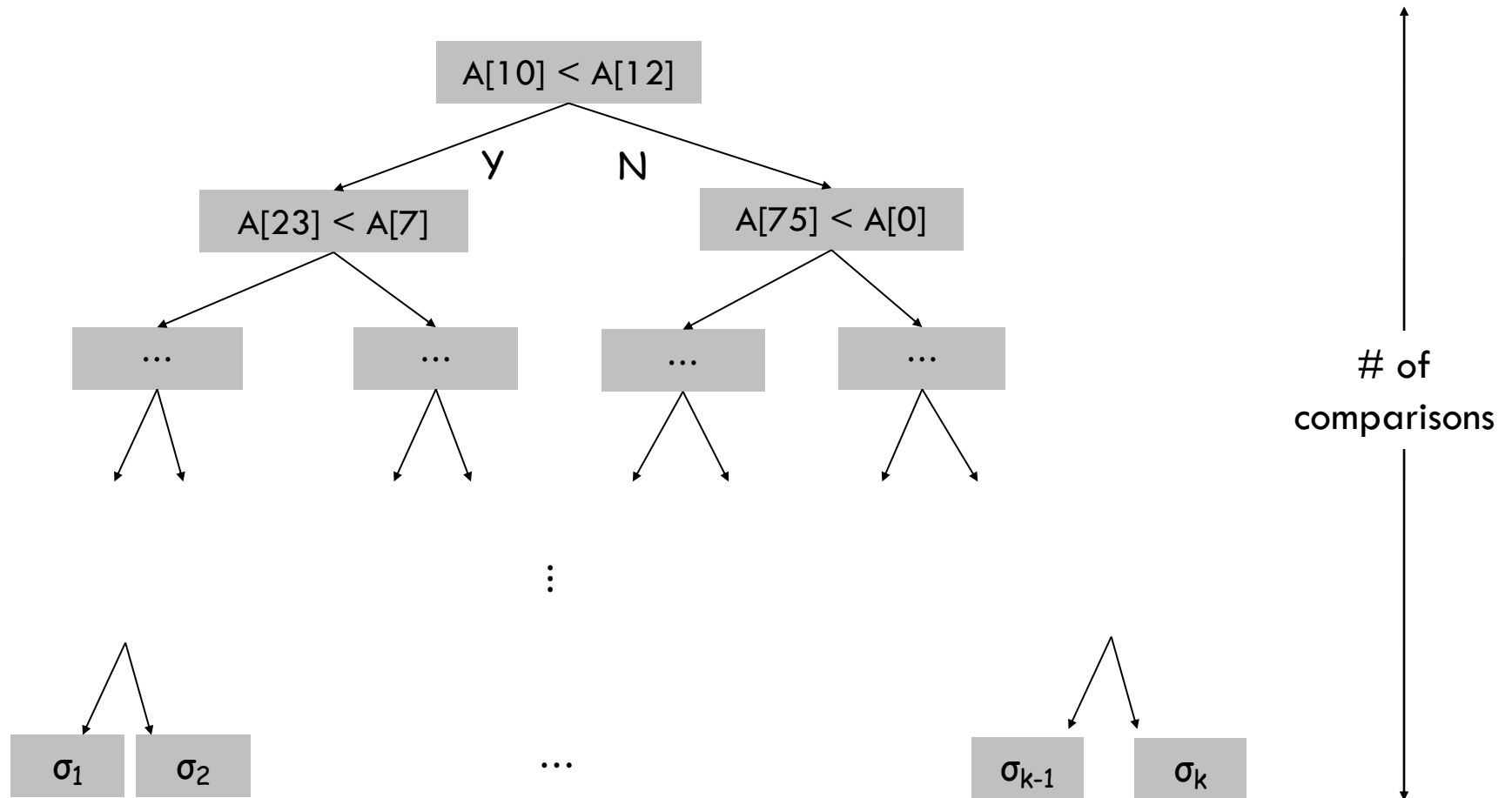
These algorithms work by performing pair-wise comparisons between elements of the sequence we are trying to sort

Such algorithms can be viewed as a decision tree where:

- each internal node compares two indices of the input array
- each external node corresponds to a permutation of $\{1, \dots, n\}$

The height of the decision tree is a lower bound on the running time of the algorithm, since it only counts number of comparisons

Decision tree



The output of a leaf is $A[\sigma(1)], A[\sigma(2)], \dots, A[\sigma(n)]$

Interlude: Comparison sorting lower bound

Fact: Comparison-based sorting algorithms take $\Omega(n \log n)$ time

Proof:

The decision tree associated with a comparison-based sorting algorithm is binary and has $n!$ external nodes. Thus the height is $\log n!$ which is $\Omega(n \log n)$

$$\begin{aligned}\log n! &= \log (n * (n-1) * \dots * 1) \\ &= \log n + \log(n-1) + \dots + \log 1 \\ &> n/2 * (\log n/2) \\ &= \Omega(n \log n)\end{aligned}$$

Remember

Important:

Simply using Merge-Sort in your algorithm doesn't make your algorithm a divide and conquer algorithm.

Example:

A greedy algorithm first sorts the input in some way and then processes the items one by one in that order. Using Merge-Sort for the sorting step doesn't change the fact that the algorithm computes the solution in a greedy way.