

MATH1023/MATH1062 Calculas

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1 Week1

1.1 Differential Equation

1. **Differential Equation(DE)**: A differential equation (DE) is a mathematical equation that relates some function with its derivatives
2. **Order**: The order of a differential equation equals to a highest derivative occuring in it.
 - $\frac{dy}{dx} = -ky$ has order 1
 - $\frac{dy}{dx} = y^{18} + \frac{d^5y}{dx^2}y + x^2$ has order 5
3. **Standard Form**: The standard form of a first-order differential equation is

$$\frac{dy}{dx} = f(x, y)$$

4. **General Solution**: A general solution is a solution incoporating all constants of integration.
5. **Initial Condition**: An initial condition is a pair (x_0, y_0) such that $y(x_0) = y_0$

2 Week2

2.1 Direction Field

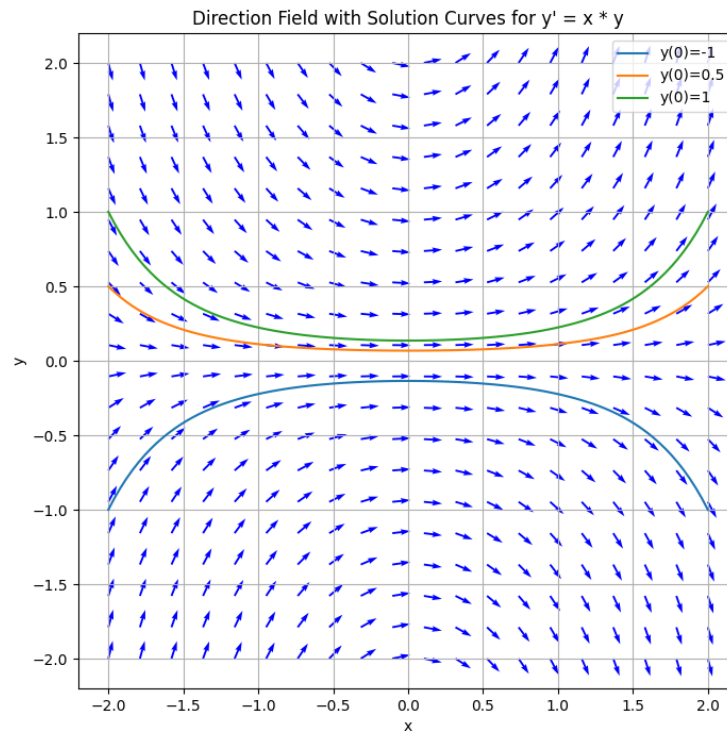
1. **Definition**: A direction field of a DE

$$y' = f(x, y)$$

consists of a grid of short line segments with slope $f(a, b)$ drawn at points (a, b) . So the line segment at (a, b) is tangent to any solution passing through (a, b)

2. **Example:** Draw some solution curves on the given direction field for the DE:

$$y' = xy$$



2.2 Separable equations

1. **Definition:** A first-order DE $y' = f(x, y)$ is called **separable** if there are functions $g(x)$ and $h(y)$ such that $f(x, y) = g(x)h(y)$, so a separable DE can be written

$$y' = g(x)h(y)$$

2. **Goal:** We want to find a method for solving separable DEs
3. **Method:** We can solve a separable DE:

$$\frac{dy}{dx} = g(x)h(y)$$

by separating variables.

Dividing both sides by $h(y)$ gives

$$\frac{1}{h(y)} \frac{dy}{dx} = g(x)$$

Integrating both sides gives:

$$\int \frac{1}{h(y)} dy = \int g(x) dx$$

If we can find antiderivatives $H(y)$ for $\frac{1}{h(y)}$ and $G(x)$ for $g(x)$, then we have

$$H(y) = G(x) + C$$

3 Week3

3.1 Modelling Population Growth

1. **Constant Growth:** This occurs when the population x increases at a constant rate. The DE is

$$\frac{dx}{dt} = k$$

where k is constant

2. **Exponential Growth:** The exponential growth model assumes the growth rate is proportional to the size of the population.

The general form of a DE modelling exponential growth is

$$\frac{dx}{dt} = kx$$

where k is constant

3. **Logistic Growth:** Exponential growth is **not** a realistic growth model for all values of t . **A small animal population** with unlimited resources of food and space **may show exponential growth initially**

As the population gets larger there will be food shortages, overcrowding, and other factors that **slow down the growth rate**.

The growth rate k should decrease as the population x increases.

Since k is no longer constant, we write $k = g(x)$, so the DE becomes

$$\frac{dx}{dt} = g(x)x$$

A small population can grow exponentially, so we want $g(x) \approx k$ when $x \approx 0$. But as x increases $g(x)$ should decrease.

The simplest formula with this behaviour is

$$g(x) = k - ax$$

So the DE becomes

$$\frac{dx}{dt} = (k - ax)x$$

We introduce a new constant $b = \frac{k}{a}$ so

$$(k - ax)x = ax\left(\frac{k}{a} - x\right) = ax(b - x)$$

Let $\frac{b}{a} = b$, the logistic DE is then given by

$$\frac{dx}{dt} = ax(b - x)$$

4 week4

4.1 First-order linear DEs

1. **First-order linear differential equation:** A first-order linear differential equation is a DE of the form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

$\frac{dy}{dx}$ and y occur only linearly

2. **How to solve first-order linear DEs ?:** The idea is multiplying the DE by a function $r(x)$ give:

$$r(x)\frac{dy}{dx} + r(x)p(x)y = r(x)q(x)$$

If we can find $r(x)$ such that:

$$r(x)\frac{dy}{dx} + r(x)p(x)y = \frac{d}{dx}(r(x)y(x))$$

then the DE becomes:

$$\frac{d}{dx}(r(x)y(x)) = r(x)q(x)$$

Integrating with respect to x gives:

$$\begin{aligned} \int \frac{d}{dx}(r(x)y(x))dx &= \int r(x)q(x)dx \\ &\rightarrow \\ r(x)y(x) &= \int r(x)q(x)dx + C \end{aligned}$$

so the general solution is

$$y = \frac{1}{r(x)} \left[\int r(x)q(x)dx + C \right]$$

3. **Integrating factor:** The function

$$r(x) = e^{\int p(x)dx}$$

is an integrating factor for the first-order linear DE

$$\frac{dy}{dx} + p(x)y = q(x)$$

4. **General Solution** the general solution of the DE is

$$y = \frac{1}{r(x)} \left[\int r(x)q(x)dx + C \right]$$

5 week5

5.1 Higher order differential equations

Higher order DEs involve higher order derivatives. For example, the DE:

$$\frac{d^2y}{dx^2} + f(x, y)\frac{dy}{dx} = g(x, y)$$

is a second-order differential equation.

1. Solving higher-order DEs is harder.
2. The general solution of a second-order DE has 2 degrees of freedom, so needs two initial conditions.
3. The general solution of an nth-order DE has n degrees of freedom, so needs n initial conditions.

5.2 Second-order linear DEs with constant coefficients

1. **Definition** A second-order linear differential equation is a DE that can be expressed in the form:

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_0(x)y = g(x)$$

The DE is linear in y and its derivatives.

2. homogeneous/inhomogeneous

- The DE is homogeneous if $g(x) = 0$
- The DE is inhomogeneous if $g(x) \neq 0$

If $g(x) = 0, f_0(x) = a, f_1(x) = b$ for $a, b \in \mathbb{R}$, then we have a homogeneous second-order linear differential equation with constant coefficient:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

3. Solve the above DE :

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

- **Observation 1:** y is a linear combination of its first two derivatives, so we try:

$$y(x) = e^{mx}$$

We have

$$\frac{dy}{dx} = me^{mx}, \frac{d^2y}{dx^2} = m^2e^{mx}$$

- **Observation 2:** Find m such that $y = Ce^{mx}$ satisfies the DE

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

substituting y and its derivatives we get:

$$\begin{aligned} Cm^2e^{mx} + aCme^{mx} + bCe^{mx} &= 0 \\ \Rightarrow Ce^{mx}(m^2 + am + b) &= 0 \\ \Rightarrow m &= \frac{-a \pm \sqrt{a^2 - 4b}}{2} \end{aligned}$$

So we have 2 solutions

$$m_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, m_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

- **Observation 3:** Show that if $m = m_1, m_2$ are solutions of $m^2 + am + b = 0$, then $y = C_1e^{m_1x} + C_2e^{m_2x}$, satisfies the DE

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

we have

$$\begin{aligned} y &= C_1e^{m_1x} + C_2e^{m_2x} \\ \Rightarrow \frac{dy}{dx} &= m_1C_1e^{m_1x} + m_2C_2e^{m_2x} \\ \Rightarrow \frac{d^2y}{dx^2} &= m_1^2C_1e^{m_1x} + m_2^2C_2e^{m_2x} \end{aligned}$$

substituting into the DE we get

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= m_1C_1e^{m_1x} + m_2C_2e^{m_2x} \\ \Rightarrow \frac{d^2y}{dx^2} &= m_1^2C_1e^{m_1x} + m_2^2C_2e^{m_2x} \end{aligned}$$

substituting into the DE we get

$$\begin{aligned}\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by &= m_1^2 C_1 e^{m_1 x} + m_2^2 C_2 e^{m_2 x} + a(m_1 C_1 e^{m_1 x} + m_2 C_2 e^{m_2 x}) + b(C_1 e^{m_1 x} + C_2 e^{m_2 x}) \\ &= C_1 e^{m_1 x} (m_1^2 + am_1 + b) + C_2 e^{m_2 x} (m_2^2 + am_2 + b) \\ &= 0\end{aligned}$$

- **formal solution:** We now have a good candidate for a general solution of the DE:

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

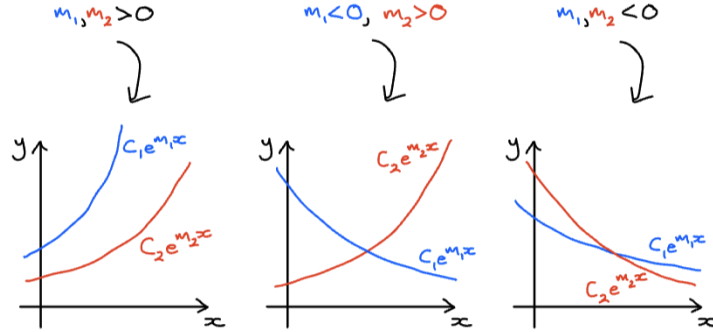
Where $m_1 = \frac{-a + \sqrt{a^2 - 4ab}}{2}, m_2 = \frac{-a - \sqrt{a^2 - 4ab}}{2}$ are solutions of $m^2 + am + b = 0$. We have 3 cases to consider:

- **Case 1:** For $a^2 > 4b$ we have 2 distinct real solutions

$$m_1 \neq m_2, m_1, m_2 \in \mathbb{R}$$

The general solution is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$



- **Case 2:** For $a^2 > 4b$ we have 2 distinct complex solutions:

$$m_1, m_2 = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \frac{-a \pm 2ik}{2} = -\frac{a}{2} \pm ik$$

$$\text{where } k = \frac{1}{2}\sqrt{4b - a^2} > 0$$

Using Euler's formula:

$$e^{ikx} = \cos(kx) + i\sin(kx)$$

We have

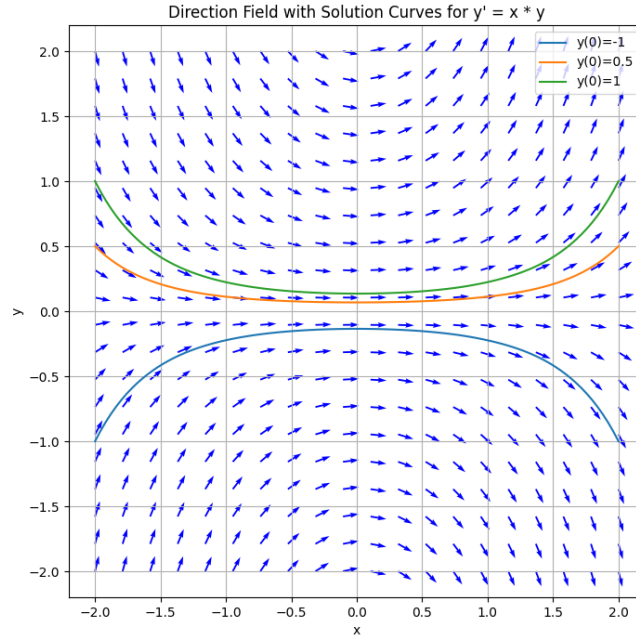
$$\begin{aligned}y &= C_1 e^{m_1 x} + C_2 e^{m_2 x} \\ &= C_1 e^{(-\frac{a}{2} + ik)x} + C_2 e^{(-\frac{a}{2} - ik)x} \\ &= e^{-\frac{a}{2}x} (C_1 e^{ikx} + C_2 e^{-ikx})\end{aligned}$$

$$= e^{-\frac{a}{2}x} (C_1 (\cos(kx) + i \sin(kx)) + C_2 (\cos(kx) - i \sin(kx)))$$

$$= e^{-\frac{a}{2}x} ((C_1 + C_2) \cos(kx) + i (C_1 - C_2) \sin(kx))$$

$$= e^{-\frac{a}{2}x} (D_1 \cos(kx) + D_2 \sin(kx))$$

So the general solution is: $y = e^{-\frac{a}{2}x} (D_1 \cos(kx) + D_2 \sin(kx))$



– **Case 3:** For $a^2 = 4b$ we have 1 real solution:

$$m_1 = m_2 = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = -\frac{a}{2}$$

Our solution becomes

$$y = C_1 e^{-\frac{a}{2}x} + C_2 e^{-\frac{a}{2}x}$$

$$= (C_1 + C_2) e^{-\frac{a}{2}x}$$

$$= D e^{-\frac{a}{2}x}$$

Here, D is a constant ($D = C_1 + C_2$), which means we only have 1 degree of freedom, so this is not a general solution.

We look for a general solution of the form

$$y = f(x)e^{-\frac{a}{2}x}$$

Substituting y and its derivatives into the differential equation (DE)

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

gives

$$e^{-\frac{a}{2}x} \left(f''(x) + \frac{1}{4}(4b - a^2)f(x) \right) = 0 \quad (\text{exercise})$$

Since $e^{-\frac{a}{2}x} \neq 0$,

$$f''(x) = \frac{1}{4}(a^2 - 4b)f(x) = 0$$

which implies

$$f'(x) = C_2$$

$$f(x) = C_2x + C_1$$

Hence, the general solution is

$$y = (C_1 + C_2x)e^{-\frac{a}{2}x}$$