CHANGE OF BASIS

- **Example 1** Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2 \qquad (1)$$

- Suppose

$$\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2 \qquad (2)$$

- That is, suppose $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{C}}$.

CHANGE OF BASIS

- **Solution** Apply the coordinate mapping determined by $\mathcal{C}$ to **x** in (2). Since the coordinate mapping is a linear transformation,

$$\begin{aligned}[x]_{\mathcal{C}} &= [3\mathrm{b}_1 + \mathrm{b}_2]_{\mathcal{C}} \\ &= [3\mathrm{b}_1]_{\mathcal{C}} + [\mathrm{b}_2]_{\mathcal{C}}\end{aligned}$$

- We can write the vector equation as a matrix equation, using the vectors in the linear combination as the columns of a matrix:

$$[x]_{\mathcal{C}} = [\mathrm{b}_1]_{\mathcal{C}}[\mathrm{b}_2]_{\mathcal{C}}\begin{bmatrix}3\\1\end{bmatrix} \qquad (3)$$

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- This formula gives $[\mathbf{x}]_{\mathcal{C}}$, once we know the columns of the matrix. From (1),

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{and } [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

- Thus, (3) provides the solution:

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

- The $\mathcal{C}$-coordinates of $\mathbf{x}$ match those of the $\mathbf{x}$ in Fig. 1, as seen on the next slide.

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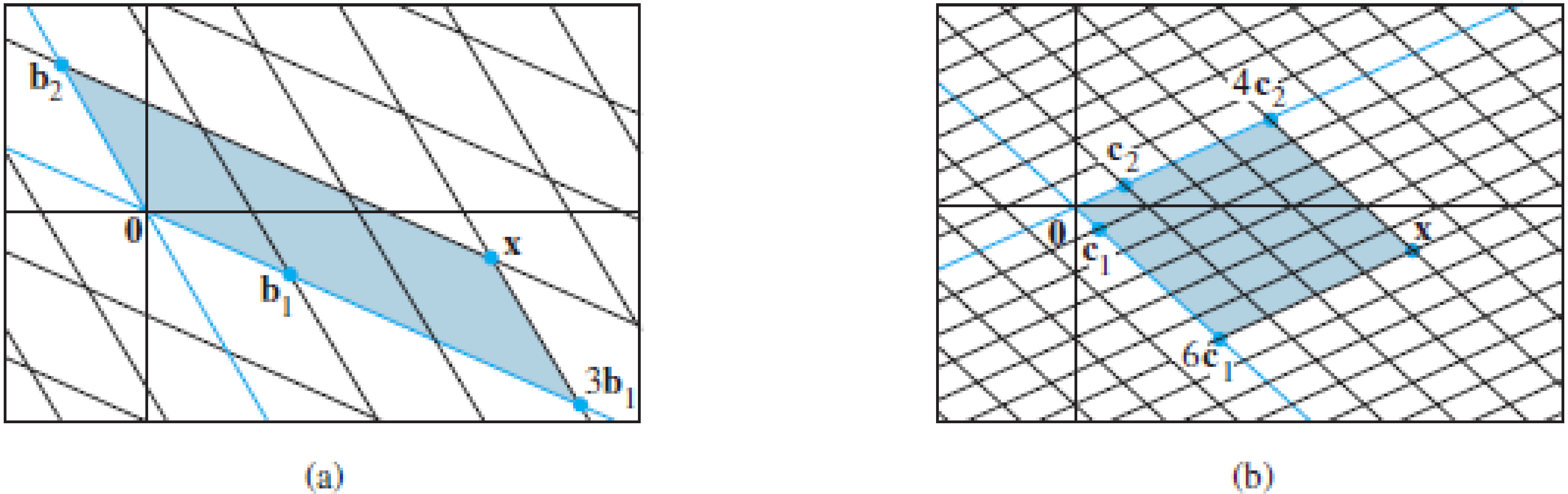

FIGURE 1 Two coordinate systems for the same vector space.

CHANGE OF BASIS

- **Theorem 15**: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ for a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} \tag{4}$$

- The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \tag{5}$$

CHANGE OF BASIS

- The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ in Theorem 15 is called the **change-of-coordinates matrix from β to $\mathcal{C}$**. Multiplication by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ converts β-coordinates into $\mathcal{C}$-coordinates.
- Figure 2 below illustrates the change-of-coordinates equation (4).

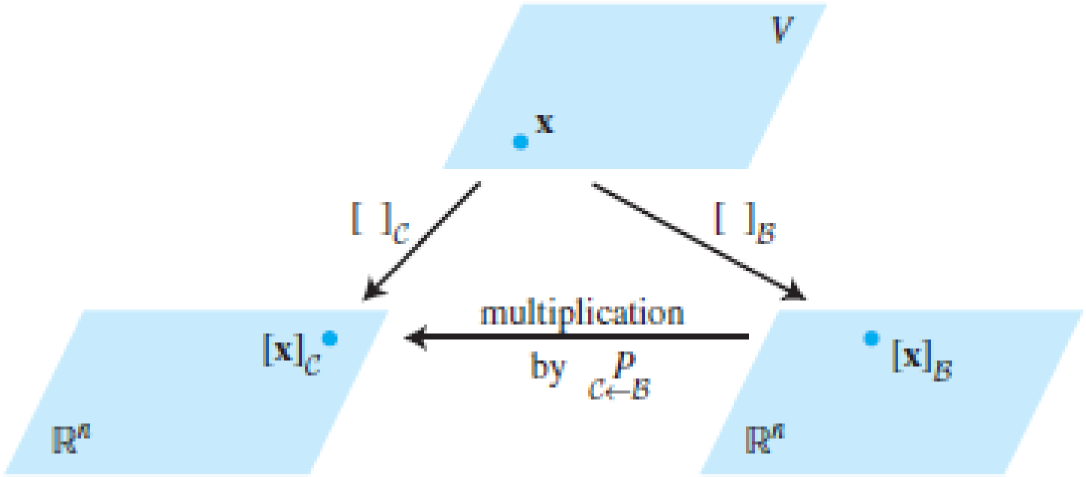

FIGURE 2 Two coordinate systems for V.

CHANGE OF BASIS

- The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are linearly independent because they are the coordinate vectors of the linearly independent set $\mathcal{B}$.

- Since $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is square, it must be invertible, by the Invertible Matrix Theorem. Left-multiplying both sides of equation (4) by $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$ yields

$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}[\mathbf{x}]_{\mathcal{C}}=[\mathbf{x}]_{\mathcal{B}}$$

Thus $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$ is the matrix that converts $\mathcal{C}$-coordinates into $\mathcal{B}$–coordinates. That is, (6)

$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} \qquad P$$

CHANGE OF BASIS IN $\mathbb{R}^n$

- If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{E}$ is the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$, then $[\mathbf{b}_1]_{\mathcal{E}} = \mathbf{b}_1$, and likewise for the other vectors in $\mathcal{B}$. In this case, $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ is the same as the change-of-coordinates matrix $P_{\mathcal{B}}$ introduced in Section 4.4, namely,

$$P_{\mathcal{B}} = [\, \mathbf{b}_1 \quad \mathrm{b}_2 \ldots \mathbf{b}_n]$$

- To change coordinates between two nonstandard bases in $\mathbb{R}^n$, we need Theorem 15. The theorem shows that to solve the change-of-basis problem, we need the coordinate vectors of the old basis relative to the new basis.

CHANGE OF BASIS IN $\mathbb{R}^n$

- **Example 2** Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ and consider the bases for $\mathbb{R}^n$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change-of-coordinates matrix from β to $\mathcal{C}$.

- **Solution** The matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ involves the $\mathcal{C}$-coordinate vectors of $\mathbf{b}_1$ and $\mathbf{b}_2$. Let $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then, by definition,

$$[\mathbf{c}_1 \quad \mathbf{c}_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}_1 \quad \text{and} \quad [\mathbf{c}_1 \quad \mathbf{c}_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{b}_2$$

CHANGE OF BASIS IN $\mathbb{R}^n$

- To solve both systems simultaneously, augment the coefficient matrix with b_1 and b_2, and row reduce:

$$[\mathbf{c}_1 \;\; \mathbf{c}_2 \;\vdots\; \mathbf{b}_1 \;\; \mathbf{b}_2] = \left[\begin{array}{cc:cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right] \quad (7)$$

- Thus

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \text{ and } [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

- The desired change-of-coordinates matrix is therefore

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[[\mathbf{b}_1]_{\mathcal{C}} \;\; [\mathbf{b}_2]_{\mathcal{C}}\right] = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$