

MATRIX FACTORIZATIONS

- A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices.
- Whereas matrix multiplication involves a *synthesis* of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an *analysis* of data.

Examples?

Diagonalize matrix

$$A = PDP^{-1}$$

Eigen Decomposition.

QR FACTORIZATION OF MATRICES

- **Theorem: The QR Factorization**
- If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for $\text{Col} A$ and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.
- **Proof** The columns of A form a basis $\{x_1, \dots, x_n\}$ for $\text{Col } A$. Construct an orthonormal basis $\{u_1, \dots, u_n\}$ for $W = \text{Col } A$ with property (1) in Theorem. This basis may be constructed by the Gram-Schmidt process or some other means.

QR FACTORIZATION OF MATRICES

- Let

$$Q = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_n]$$

- For $k = 1, \dots, k$, \mathbf{x}_k is in $\text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\} = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$. So there are constants, r_{1k}, \dots, r_{kk} , such that

$$\mathbf{x}_k = r_{1k}\mathbf{u}_1 + \dots + r_{kk}\mathbf{u}_k + 0 \cdot \mathbf{u}_{k+1} + \dots + 0 \cdot \mathbf{u}_n$$

- We may assume that $r_{kk} \geq 0$. This shows that \mathbf{x}_k is a linear combination of the columns of Q using as weights the entries in the vector

QR FACTORIZATION OF MATRICES

$$r_k = \begin{bmatrix} r_{1k} \\ \cdot \\ \cdot \\ \cdot \\ r_{kk} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

- That is, $x_k = Qr_k$ for $k = 1, \dots, n$. Let $R = [r_1 \dots r_n]$. Then
$$A = [x_1 \dots x_n] = [Qr_1 \dots Qr_n] = QR$$
- The fact that R is invertible follows easily from the fact that the columns of A are linearly independent. Since R is clearly upper triangular, its nonnegative diagonal entries must be positive.

QR FACTORIZATION OF MATRICES

$$A = QR$$

The diagram illustrates the QR factorization $A = QR$. It shows three matrices represented by boxes: a blue box labeled $m \times n$ for matrix A , a blue box labeled "Orthogonal" for matrix Q , and a white box with a blue upper triangle labeled "Upper triangular" for matrix R . These are connected by an equals sign.

To find QR decomposition:

- 1.) Q : Use Gram-Schmidt to find orthonormal basis for column space of A
- 2.) Let $R = Q^T A$

QR FACTORIZATION OF MATRICES

$$A = QR$$

$$Q^{-1}A = Q^{-1}QR$$

$$Q^{-1}A = R$$

Q has orthonormal columns:

$$\text{Thus } Q^{-1} = Q^T$$

$$\text{Thus } R = Q^{-1}A = Q^T A$$

THE GRAM-SCHMIDT PROCESS

- **The Gram-Schmidt Process**
- Given a basis $\{x_1, \dots, x_p\}$ for a nonzero subspace W of \mathbb{R}^n , define

$$\begin{aligned}v_1 &= x_1 \\v_2 &= x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\v_3 &= x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 \\&\vdots \\v_p &= x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \dots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}\end{aligned}$$

- Then $\{v_1, \dots, v_p\}$ is an orthogonal basis for W . In addition
$$\text{Span}\{v_1, \dots, v_k\} = \text{Span}\{x_1, \dots, x_k\} \quad \text{for } 1 \leq k \leq p \quad (1)$$

QR FACTORIZATION OF MATRICES

- **Example:** Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
- **Solution** The columns of A are the vectors x_1 , x_2 , and x_3 . An orthogonal basis for $\text{Col } A = \text{Span}\{x_1, x_2, x_3\}$ was found in that example:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

QR FACTORIZATION OF MATRICES

- To simplify the arithmetic that follows, scale v_3 by letting $v_3 = 3v_3$. Then normalize the three vectors to obtain u_1 , u_2 , and u_3 , and use these vectors as the columns of Q :

$$Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}.$$

- By construction, the first k columns of Q are an orthonormal basis of $\text{Span}\{x_1, \dots, x_k\}$.

QR FACTORIZATION OF MATRICES

- From the proof of Theorem, $A = QR$ for some R . To find R , observe that $Q^T Q = I$, because the columns of Q are orthonormal. Hence

$$Q^T A = Q^T (QR) = IR = R$$

- and

$$R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 3/\sqrt{12} \end{bmatrix}$$

Find the QR decomposition of

$$A = \begin{bmatrix} 1 & -1 & 6 \\ 1 & 2 & 0 \\ 1 & 3 & -1 \\ 1 & 4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{10}} \\ \frac{1}{2} & 0 & \frac{-2}{\sqrt{10}} \\ \frac{1}{2} & \frac{1}{\sqrt{14}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{2} & \frac{2}{\sqrt{14}} & \frac{2}{\sqrt{10}} \end{bmatrix} = Q$$

$$\begin{bmatrix} 1 & -3 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-3}{\sqrt{14}} & 0 & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{10}} & \frac{-2}{\sqrt{10}} & \frac{-1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 1 & -1 & 6 \\ 1 & 2 & 0 \\ 1 & 3 & -1 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & \frac{14}{\sqrt{14}} & \frac{-2}{\sqrt{14}} \\ 0 & 0 & \frac{5}{\sqrt{10}} \end{bmatrix}$$

Matrix A:

$$\begin{pmatrix} 1.0 & -1.0 & 4.0 \\ 1.0 & 4.0 & -2.0 \\ 1.0 & 4.0 & 2.0 \\ 1.0 & -1.0 & 0.0 \end{pmatrix}$$

Matrix with orthonormal basis:

$$\begin{pmatrix} -0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & -0.5 \\ -0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & 0.5 \end{pmatrix}$$

Upper diagonal matrix:

$$\begin{pmatrix} -2.0 & -3.0 & -2.0 \\ 0.0 & -5.0 & 2.0 \\ 0.0 & 0.0 & -4.0 \\ 0.0 & -0.0 & -0.0 \end{pmatrix}$$

Find the QR decomposition of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

1.) Use Gram-Schmidt to find orthogonal basis for column space of A

$$\text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix} \right\}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}$$

QR Decomposition-Based Algorithm for Background Subtraction



Figure 1. Sample frames of a traffic movie¹.

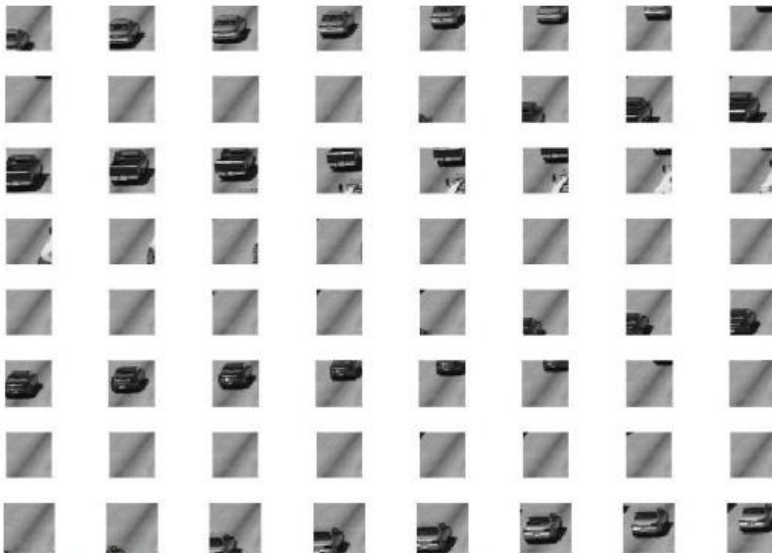


Figure 2. 64 Consequence frames of an instance block in its original order.

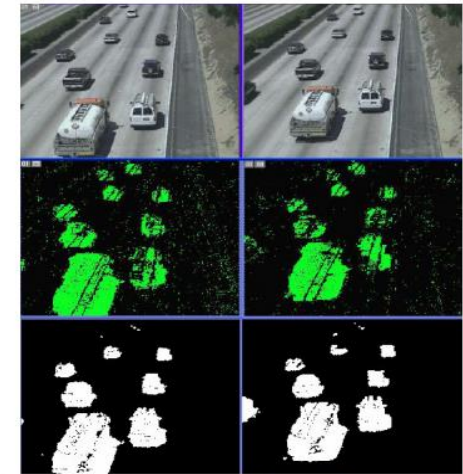


Figure 5. From top to bottom: original frames, foreground object detection using GMM (taken from [11]) and the proposed approach.

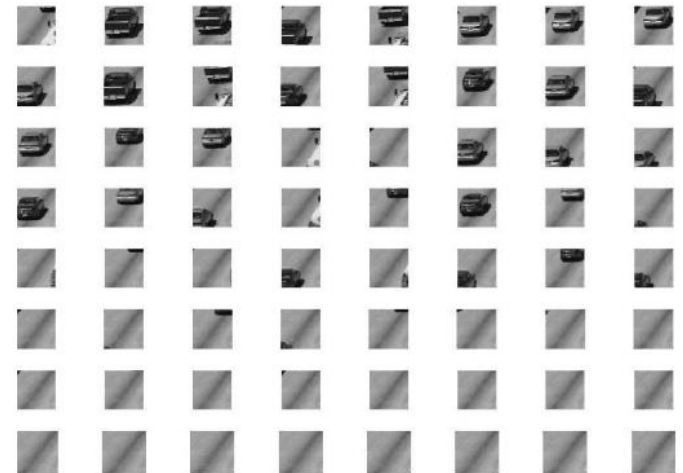


Figure 3. Rearrangement of blocks in Figure 2 based on their R-values' order. A can be seen, background frames has been shifted to the end.