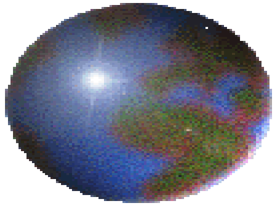
Matrix Algebra



- *Systems of Linear Equations*
- *Vectors*
- *Matrix*
- *Matrix Decomposition*
- *Applications*

SYSTEM OF LINEAR EQUATIONs

SYSTEM OF LINEAR EQUATIONS

- A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers, usually known in advance.

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables — say, $x_1, \ldots, x_n$.

SYSTEM OF LINEAR EQUATIONS

- A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

- The set of all possible solutions is called the **solution set** of the linear system.

- Two linear systems are called **equivalent** if they have the same solution set.

Solutions for system of linear equations

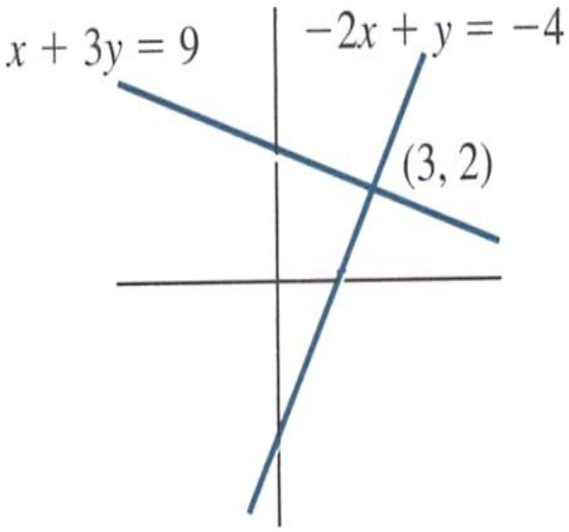


Figure 1.1

Unique solution

$x + 3y = 9$
$-2x + y = -4$

Lines intersect at (3, 2)

Unique solution:

$x = 3, y = 2.$

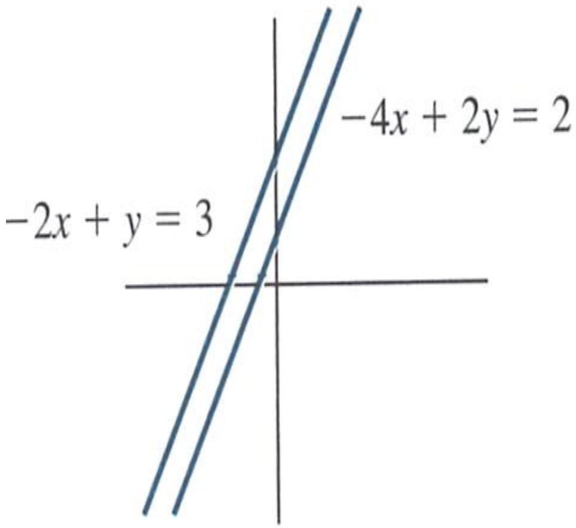


Figure 1.2

No solution

$-2x + y = 3$
$-4x + 2y = 2$

Lines are parallel.

No point of intersection.

No solutions.

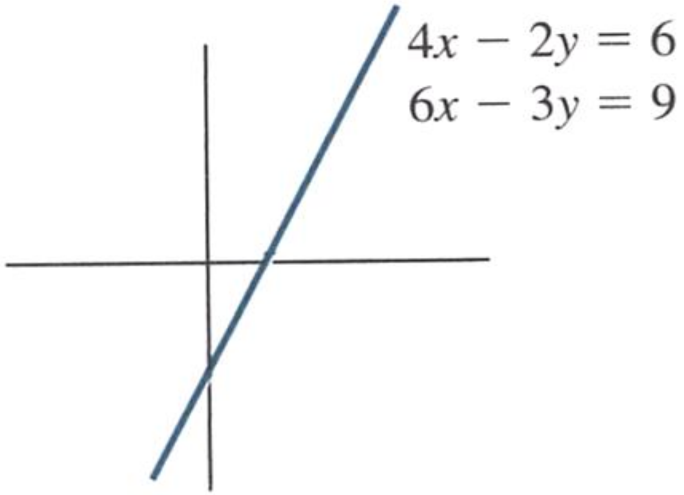


Figure 1.3

Many solution

$4x - 2y = 6$
$6x - 3y = 9$

Both equations have the same graph. Any point on the graph is a solution.

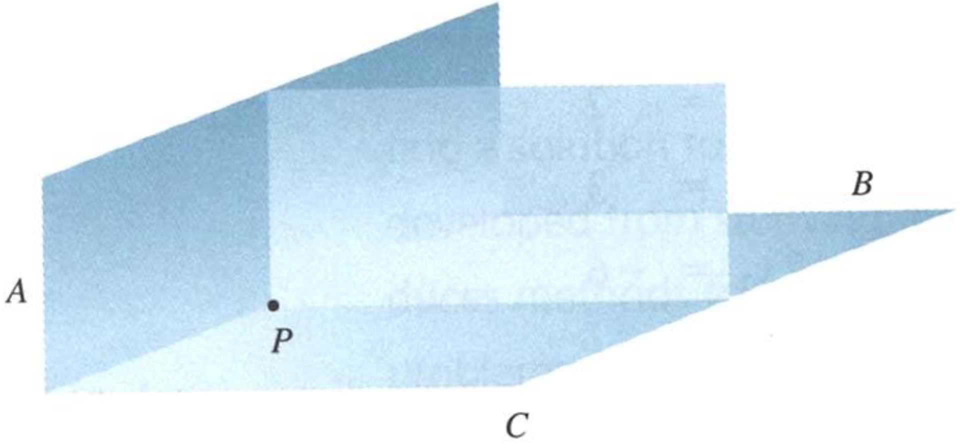
Many solutions.

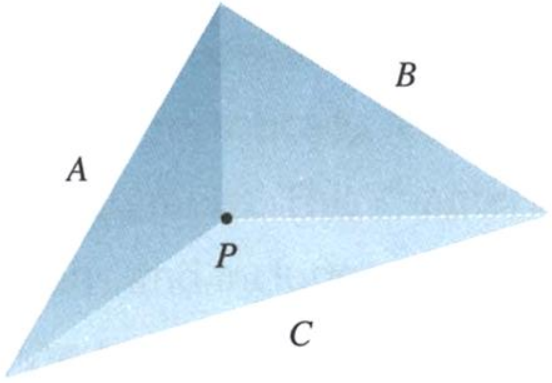
Solutions for system of linear equations

A linear equation in three variables corresponds to a plane in three-dimensional space.

※ Systems of three linear equations in three variables:

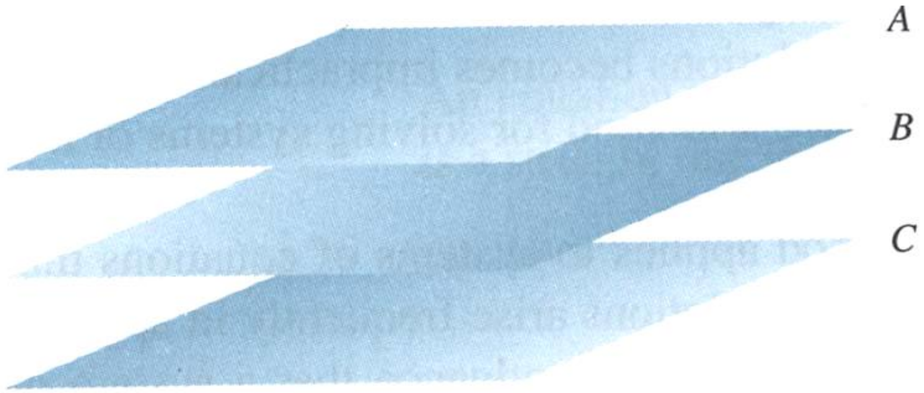
- *Unique solution*

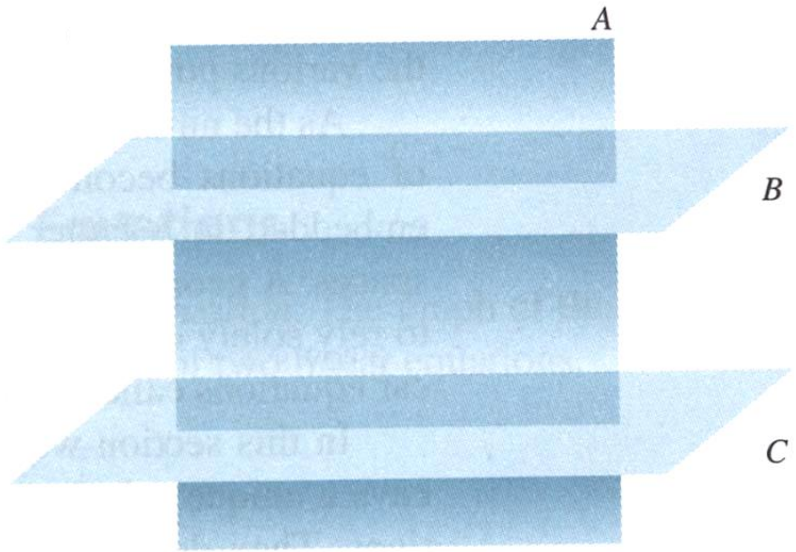




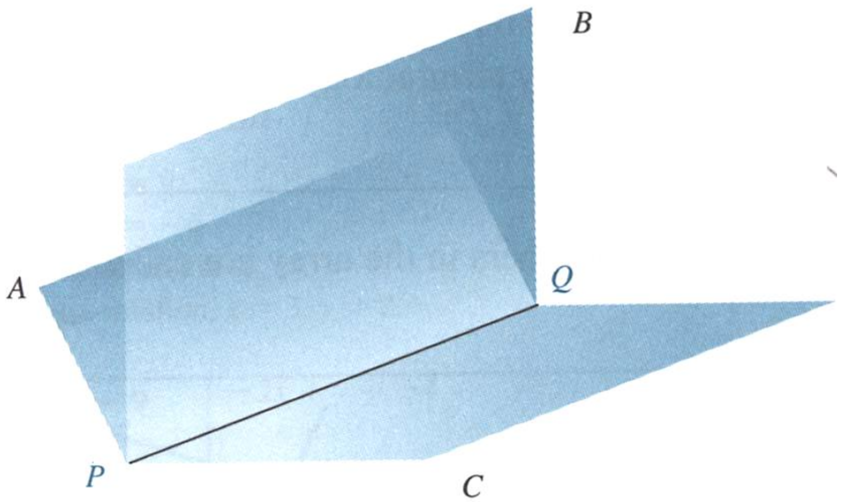
Solutions for system of linear equations

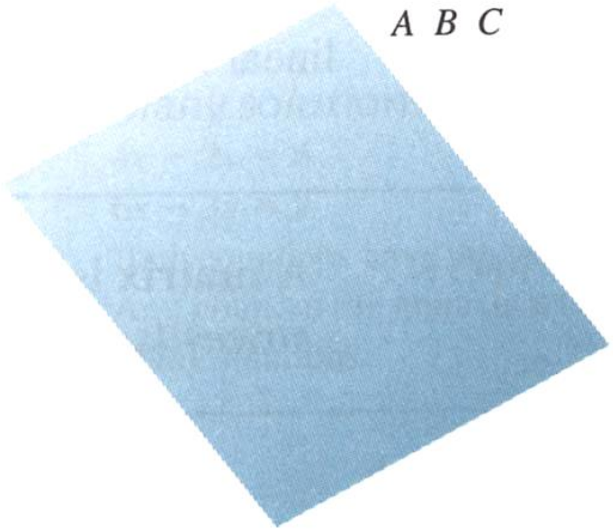
- *No solutions*





- *Many solutions*





Relations between system of linear equations and matrices

- **matrix of coefficient and augmented matrix**

Relations between system of linear equations and matrices

- The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**.
- Given the system,

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9, \end{aligned}$$

with the coefficients of each variable aligned in columns, the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

is called the **coefficient matrix** (or matrix of **coefficients**) of the system

Relations between system of linear equations and matrices

- An **augmented matrix** of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.
- For the given system of equations,

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

is called the **augmented matrix** of the system.

Relations between system of linear equations and matrices

- **matrix of coefficient and augmented matrix**

$$\begin{aligned} x_1 + \quad x_2 + \quad x_3 &= 2 \\ 2x_1 + 3x_2 + \quad x_3 &= 3 \\ x_1 - \quad x_2 - 2x_3 &= -6 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

matrix of coefficient

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

augmented matrix

$$\begin{cases} 2x_1 - 3x_2 + 5x_3 - \quad x_4 = 2 \\ -x_1 - 2x_2 + 3x_3 + 4x_4 = 0 \\ 3x_1 + 8x_2 - 5x_3 + 3x_4 = -2 \\ \quad -4x_2 + 2x_3 - 7x_4 = 9 \end{cases}$$

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Definition:** If A is an $m \times n$ matrix, with columns $\mathbf{a}_1$, …, $\mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of** A **and** $\mathbf{x}$, denoted by $A\mathbf{x}$, **is the linear combination of the columns of** A **using the corresponding entries in** $\mathbf{x}$ **as weights**; that is,

$$A\mathrm{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n.$$

- Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$.

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Example:** For $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\mathrm{v}_1 - 5\mathrm{v}_2 + 7\mathrm{v}_3$ as a matrix times a vector.

- **Solution:** Place $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ into the columns of a matrix A and place the weights 3, -5, and 7 into a vector $\mathbf{x}$.
- That is,

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]\begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathbf{x}$$

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- Now, write the system of linear equations as a vector equation involving a linear combination of vectors.
- For example, the following system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned} \qquad (1)$$

is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad (2)$$

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- As in the example, the linear combination on the left side is a matrix times a vector, so that (2) becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad (3)$$

- Equation (3) has the form $A\mathbf{x} = \mathbf{b}$. Such an equation is called a **matrix equation**, to distinguish it from a vector equation such as shown in (2).

MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{pmatrix}$$

SOLUTIONS OF LINEAR SYSTEMS

The basic strategy for solving a linear system is to *replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve*.

Gauss-Jordan Elimination

- System of linear equations
  $\Rightarrow$ augmented matrix
  $\Rightarrow$ reduced echelon form
  $\Rightarrow$ solution

Elementary Row Operations of Matrices

- Elementary Transformation

1. Interchange two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a multiple of one equation to another equation.

- Elementary Row Operation

1. Interchange two rows of a matrix.
2. Multiply the elements of a row by a nonzero constant.
3. Add a multiple of the elements of one row to the corresponding elements of another row.

- In the definitions that follow,

a *nonzero* row or column in a matrix means a row or column that contains at least one nonzero entry;

a **leading entry** of a row refers to the leftmost nonzero entry (in a nonzero row).

$$\begin{bmatrix} 1 & -2 & 1 & -2 & 0 \\ 0 & 2 & -8 & 0 & 0 \\ -4 & 5 & 9 & -9 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

ECHELON FORM

- A rectangular matrix is in **echelon form** (or **row echelon form, REF**) if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.

Examples

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1/2 & -2 & 2 \\ 0 & 0 & 2 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

ECHELON FORM

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form, RREF**):

    4. The leading entry in each nonzero row is 1.

    5. Each leading 1 is the only nonzero entry in its column.

- An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form.)

Gauss-Jordan Elimination

Definition

A matrix is in ***reduced echelon form*** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each other row is **1**. This element is called a ***leading* 1**.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.
4. All other elements in a column that contains a leading 1 are zero.

Examples

Echelon Forms

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Reduced Echelon Forms

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

- Examples for reduced echelon form

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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- elementary row operations, reduced echelon form
- The reduced echelon form of a matrix is unique.

Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Solution

pivot (leading 1)

$$\underset{R1 \leftrightarrow R2}{\approx} \begin{bmatrix} 3 & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix} \underset{\left(\frac{1}{3}\right)R1}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

$$\underset{R3+(-4)R1}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \underset{\left(\frac{1}{2}\right)R2}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

pivot

$$\underset{\substack{R1+R2 \\ R3+(-2)R2}}{\approx} \begin{bmatrix} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix} \underset{\substack{R1+(-2)R3 \\ R2+R3}}{\approx} \begin{bmatrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

The matrix is the reduced echelon form of the given matrix.

Example 2

Solve, if possible, the system of equations

$$\begin{aligned} 3x_1 - 3x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= 7 \end{aligned}$$

Solution

$$\begin{bmatrix} 3 & -3 & 3 & 9 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix} \underset{\left(\frac{1}{3}\right)R1}{\approx} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix} \underset{\substack{R2+(-2)R1 \\ R3+(-3)R1}}{\approx} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & -2 \end{bmatrix}$$

$$\underset{\substack{R1+R2 \\ R3+2R2}}{\approx} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 3x_3 &= 4 \\ x_2 + 2x_3 &= 1 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -3x_3 + 4 \\ x_2 &= -2x_3 + 1 \end{aligned}$$

The general solution to the system is

$$x_1 = -3r + 4$$

$$x_2 = -2r + 1$$

$x_3 = r$, where r is real number (called a parameter).

HOMOGENEOUS LINEAR SYSTEMS

- A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{b}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.
- Such a system $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely, $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$).
  - This zero solution is usually called the **trivial solution**.
- The homogenous equation $A\mathbf{x} = \mathbf{0}$, the important question is whether there exists a **nontrivial solution**, that is, a nonzero vector x that satisfies $A\mathbf{x} = \mathbf{0}$.
- N ote: $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution iff. the equation has at least one free variable.

HOMOGENEOUS LINEAR SYSTEMS

- **Example 1:** Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

- **Solution:** Let A be the matrix of coefficients of the system and row reduce the augmented matrix $[A \quad \mathbf{0}]$ to echelon form:

HOMOGENEOUS LINEAR SYSTEMS

- $\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- Since x_3 is a free variable, $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions (one for each choice of x_3.)
- Continue the row reduction of $[A \quad \mathbf{0}]$ to *reduced* echelon form:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 - \tfrac{4}{3}x_3 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

HOMOGENEOUS LINEAR SYSTEMS

- Solve for the basic variables x_1 and x_2 to obtain

$$x_1 = \frac{4}{3}x_3,\ x_2 = 0, \text{ with } x_3 \text{ free.}$$

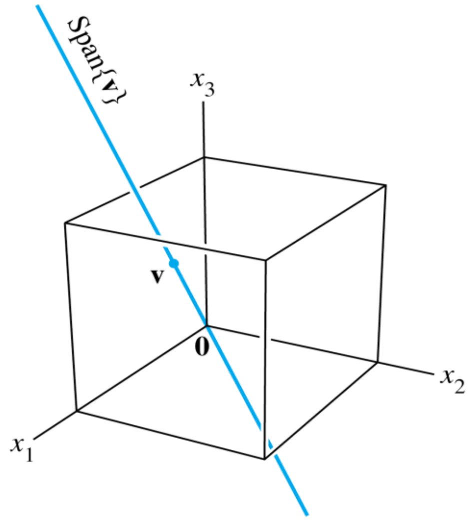
- As a vector, the general solution of $A\mathbf{x} = \mathbf{0}$ has the form given below.

- $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\mathbf{v}$, where $\mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$

HOMOGENEOUS LINEAR SYSTEMS

- Here x_3 is factored out of the expression for the general solution vector.
- This shows that every solution of $A\mathbf{x} = 0$ in this case is a scalar multiple of $\mathbf{v}$.
- The trivial solution is obtained by choosing $x_3 = 0$.
- Geometrically, the solution set is a line through 0 in $\mathbb{R}^3$.

See Fig.1 below.



HOMOGENEOUS LINEAR SYSTEMS

- **Example 2:** A single linear equation can be treated as a very simple system of equations. Describe all solutions of the homogeneous "system"

$$10x_1 - 3x_2 - 2x_3 = 0 \qquad (1)$$

- **Solution:** There is no need for matrix notation. Solve for the basic variable x_1 in terms of the free variables. The general solution is

$$x_1 = .3x_2 + .2x_3$$

with x_2 and x_3 free.

HOMOGENEOUS LINEAR SYSTEMS

- As a vector, the general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$x_2 \begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix} = x_2\mathbf{u} + x_3\mathbf{v}, \text{ with } x_2, x_3 \text{ free.} \quad (2)$$

Since neither u nor v is a scalar multiple of the other, the solution set is a plane through the origin.

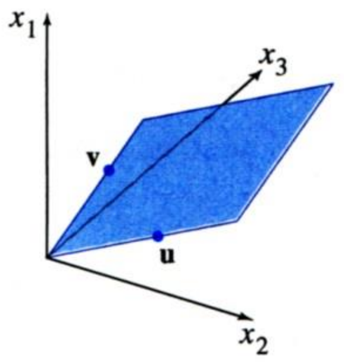


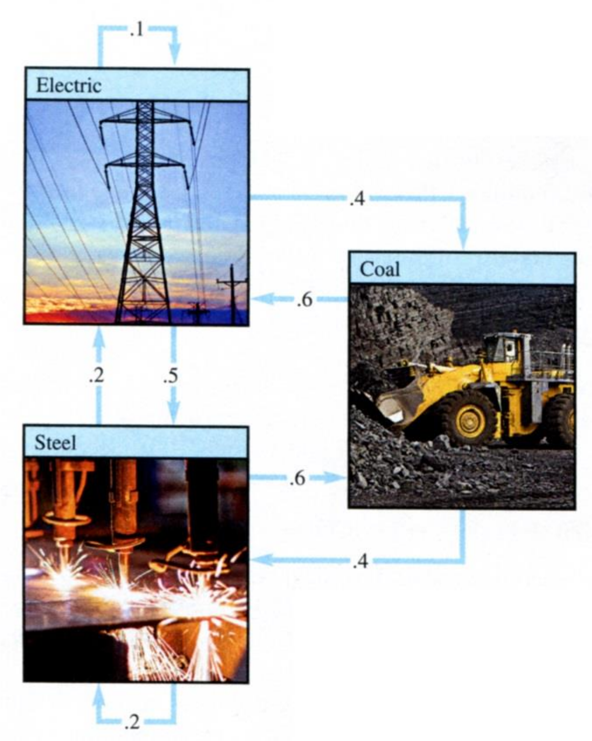
FIGURE 2

Applications

- Economics
- Chemistry
- Network flow

A Homogeneous System in Economics

- Leontief "input-output" model



Suppose an economy consists of the Coal, Electric (power), and Steel sectors, and the output of each sector is distributed among the various sectors as shown in Table 1, where the entries in a column represent the fractional parts of a sector's total output.

TABLE 1 A Simple Economy

Distribution of Output from:			
Coal	**Electric**	**Steel**	**Purchased by:**
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

- Denote the prices (i.e., dollar values) of the total annual outputs of the Coal, Electric, and Steel sectors by p_C, p_E, and p_S, respectively.
- p_C=.4pE+.6pS
- p_E=.6pC +.1pE+.2pS
- p_S=.4pC +.5pE+.2pS

TABLE 1 A Simple Economy

Distribution of Output from:			
Coal	**Electric**	**Steel**	**Purchased by:**
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

$$\begin{aligned} p_C - .4p_E - .6p_S &= 0 \\ -.6p_C + .9p_E - .2p_S &= 0 \\ -.4p_C - .5p_E + .8p_S &= 0 \end{aligned}$$

$$\mathbf{p} = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = \begin{bmatrix} .94p_S \\ .85p_S \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}$$

Balancing Chemical Equations

- $(x1)C_3H_8 + (x2)O_2 \rightarrow (x3)CO_2 + (x4)H_2O$

$$C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$$

Network Flow

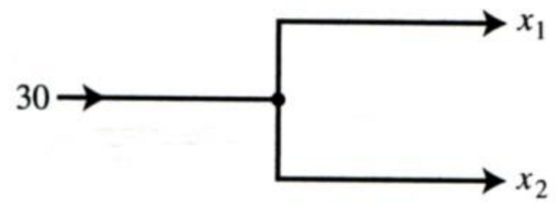


FIGURE 1
A junction, or node.

Intersection	Flow in		Flow out
A	$300 + 500$	$=$	$x_1 + x_2$
B	$x_2 + x_4$	$=$	$300 + x_3$
C	$100 + 400$	$=$	$x_4 + x_5$
D	$x_1 + x_5$	$=$	600

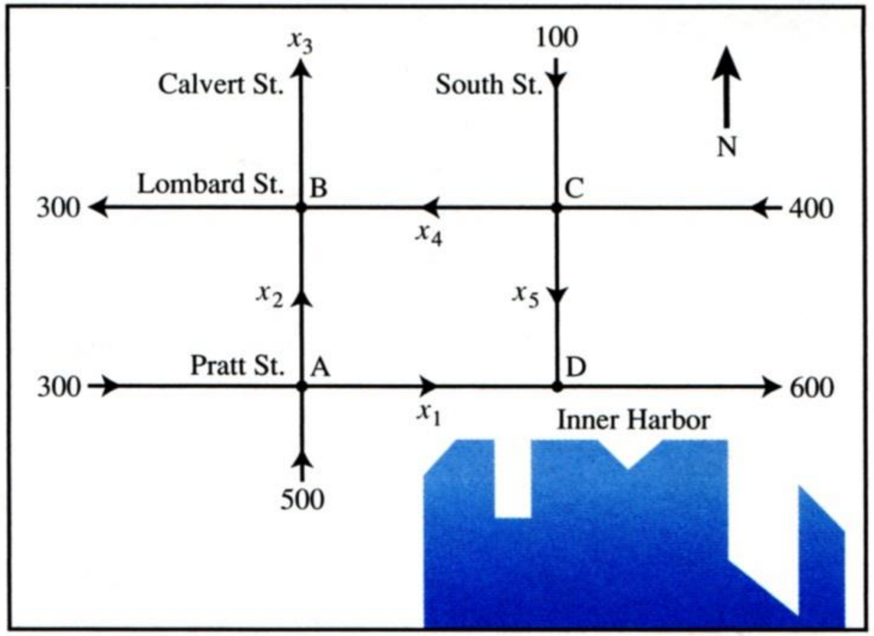


FIGURE 2 Baltimore streets.

$$\begin{cases} x_1 = 600 - x_5 \\ x_2 = 200 + x_5 \\ x_3 = 400 \\ x_4 = 500 - x_5 \\ x_5 \text{ is free} \end{cases}$$