BÀI TẬP ĐẠI SỐ CHƯƠNG 3_4

1.

a) $A = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix} \Rightarrow |A| = (3.5.8 + 1.6.1 + -4.2.4) - (-4.5.1 + 1.2.8 + 3.6.4) = 26$

b) $\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \Rightarrow |A| = (3.(-4).(-2) + 1.3.5 + 0.-2.4) - (0.-4.5 + 1.-2.-2 + 3.4.3) = -1$

c) $\begin{pmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{pmatrix} \Rightarrow |A| = (-2.1.4 + 7.-2.3 + 6.5.8) - (6.1.3 + 5.7.4 + -2.-2.8) = 0$

d) $\begin{pmatrix} -2 & 0 & 7 \\ 3 & 5 & 1 \\ -1 & 0 & 5 \end{pmatrix} \Rightarrow |A| = (-2.5.5 + 0.1.-1 + 7.3.0) - (7.5.(-1) + 0.3.5 + -2.1.0) = -15$

2. a) $(1,2,1), (1,0,2), (2,1,1)$

Và xét $|A| = (1.0.1 + 2.2.2 + 1.1.1) - (1.0.2 + 2.1.1 + 1.2.1) = 5 \neq 0$

$\Rightarrow$ Độc lập tuyến tính,

Có số chiều vectơ $= 3 =$ dim $\mathbb{R}^3 = 3 \Rightarrow$ Vậy hệ này là cơ sở của $\mathbb{R}^3$

VIBOOK

b) $(-1,3,2), (3,1,3), (2,10,2)$

Ta có $|A| = (-1.1.2 + 3.3.2 + 2.10.-3) - (2.1.2 + 3.(-3).2 + -1.2.10)$

$= 0 \Rightarrow$ Phụ thuộc tuyến tính $\Rightarrow$ không phải là cơ sở của $\mathbb{R}^3$

c/ $(1,2,1), (2,9,0), (3,3,4)$

Ta có: $|A| = (1.9.4 + 2.0.1 + 1.2.3) - (1.9.3 + 2.2.4 + 1.0.3)$

$= -1 \neq 0 \Rightarrow$ Độc lập tuyến tính

Có số vectơ $= 3 = \dim \mathbb{R}^3 \Rightarrow$ là cơ sở của $\mathbb{R}^3$

~~Bài~~ 3. $(1,1,0,0), (1,0,1,0), (0,0,1,1), (0,1,0,1)$

Đặt $u_1 = (1,1,0,0)$ $u_2 = (1,0,1,0)$, $u_3 = (0,0,1,1)$, $u_4 = (0,1,0,1)$

Ta có: $(u_1^T\ u_2^T\ u_3^T\ u_4^T \mid u^T)$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & x \\ 1 & 0 & 0 & 1 & y \\ 0 & 1 & 1 & 0 & z \\ 0 & 0 & 1 & 1 & t \end{array}\right) \xrightarrow{d_2 - d_1} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & x \\ 0 & -1 & 0 & 1 & y-x \\ 0 & 1 & 1 & 0 & z \\ 0 & 0 & 1 & 1 & t \end{array}\right)$$

$$\xrightarrow[\substack{d_3 - d_2 \\ d_1 - d_2}]{-d_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & y \\ 0 & 1 & 0 & -1 & -y+x \\ 0 & 0 & 1 & 1 & z+y-x \\ 0 & 0 & 1 & 1 & t \end{array}\right) \xrightarrow{d_4 - d_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & t \\ 0 & 1 & 0 & -1 & -y+x \\ 0 & 0 & 1 & 1 & z+y-x \\ 0 & 0 & 0 & 0 & t-z-y+x \end{array}\right)$$

với $u_0 = (1,1,1,1) \Rightarrow$ vô số nghiệm $\Rightarrow$ Mọi vectơ không phải là tập sinh của $\mathbb{R}^4$

$\Rightarrow$ không phải là cơ sở của $\mathbb{R}^4$

4. $B = \{u_1 = (2,1,1), u_2 = (2,-1,1), u_3 = (1,2,1)\}$

$C = \{v_1 = (3,1,-5), v_2 = (1,1,-3), v_3 = (-1,0,2)\}$

a) $(u_1^T\ u_2^T\ u_3^T \mid v_1^T\ v_2^T\ v_3^T) = \left(\begin{array}{ccc|ccc} 2 & 2 & 1 & 3 & 1 & -1 \\ 1 & -1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & -5 & -3 & 2 \end{array}\right)$

$$\begin{array}{l} d_1/2 \\ d_2-d_1 \\ d_3-d_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 1/2 & 3/2 & 1/2 & -1/2 \\ 0 & -2 & 3/2 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & -11/2 & -7/2 & 5/2 \end{array}\right)$$

$$3)\ \begin{array}{l} -d_2/2, \\ d_1-d_2 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 5/4 & 7/4 & 1/4 & -3/4 \\ 0 & 1 & -3/4 & -1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/2 & -13/2 & -7/2 & -5/2 \end{array}\right)$$

$$\begin{array}{l} d_3/(1/2) = 2d_3, \\ d_1-5/4\,d_3 \\ d_2+3/4\,d_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 18 & 9 & 11/2 \\ 0 & 1 & 0 & -10 & -5 & -7/2 \\ 0 & 0 & 1 & -13 & -7 & -5 \end{array}\right)$$

$$\Rightarrow (B \to C) = \begin{pmatrix} 18 & 9 & 11/2 \\ -10 & -5 & -7/2 \\ -13 & -7 & -5 \end{pmatrix}$$

5) $w = (-5, 8, -5)$ trong cơ sở B, $[w]_B$

Ta có $w = x_1u_1 + x_2u_2 + x_3u_3$

Ta có hệ pt

$$\begin{cases} 2x_1 + 2x_2 + x_3 = -5 \\ x_1 + -x_2 + 2x_3 = 8 \\ x_1 + x_2 + x_3 = -5 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & -5 \\ 2 & -1 & 2 & 8 \\ 1 & 1 & 1 & -5 \end{array}\right) \xrightarrow[d_3 - \frac{1}{2}d_1]{d_2 - \frac{1}{2}d_1} \left(\begin{array}{ccc|c} 2 & 2 & 1 & -5 \\ 0 & -2 & 3/2 & 21/5 \\ 0 & 0 & 1/2 & -5/2 \end{array}\right)$$

$$\begin{cases} 2x_1 + 2x_2 + x_3 = -5 \\ -2x_2 + \frac{3}{2}x_3 = 21/5 \\ \frac{1}{2}x_3 = -5/2 \end{cases} \Rightarrow \begin{cases} x_1 = 9 \\ x_2 = -9 \\ x_3 = -5 \end{cases}$$

Vậy toạ độ của w trong cơ sở B, $[w]_B$ là $(9, -9, -5)$

c) $w = (-5, 8, -5)$ trong cơ sở C, $[w]_C$ (

Ta có:

$$\left(\begin{array}{ccc|c} 3 & 1 & -1 & -5 \\ 1 & 1 & 0 & 8 \\ -5 & -3 & 2 & -5 \end{array}\right) \xrightarrow{d_1/3} \left(\begin{array}{ccc|c} 1 & 1/3 & -1/3 & -5/3 \\ 1 & 1 & 0 & 8 \\ -5 & 3 & 2 & -5 \end{array}\right)$$

$$\xrightarrow[d_3 + 5d_1]{d_2 - d_1} \left(\begin{array}{ccc|c} 1 & 1/3 & -1/3 & -5/3 \\ 0 & 2/3 & 1/3 & 29/3 \\ 0 & 14/3 & \frac{-1}{3} & \frac{-40}{3} \end{array}\right) \xrightarrow[\substack{d_3 - \frac{14}{3}d_2 \\ \cancel{d_3} \frac{-1}{2}d_3}]{\frac{3}{2}d_2} \left(\begin{array}{ccc|c} 1 & 1/3 & -1/3 & -5/3 \\ 0 & 1 & 1/2 & 29/2 \\ 0 & 0 & -2 & -81 \end{array}\right)$$

~~5) S là cơ sở chính tắc của $\mathbb{R}^3$,~~ B là cơ sở

$$\begin{cases} x_1 + \frac{1}{3}x_2 - \frac{1}{3}x_3 = -5/3 \\ x_2 + \frac{1}{2}x_3 = 29/2 \\ -2x_3 = -81 \end{cases} \Rightarrow \begin{cases} x_1 = 55/4 \\ x_2 = -23/4 \\ x_3 = \frac{-81}{2} \end{cases}$$

d) Toạ độ của vectơ $[w]_B = \left(\cancel{\frac{55}{4}, \frac{-23}{4}, \frac{-81}{2}}\right) \begin{pmatrix} 55/4 \\ -23/4 \\ -81/2 \end{pmatrix}$

S: S là cơ sở chính tắc của $\mathbb{R}^3$ $\Rightarrow S = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$

a) Tìm ma trận chuyển cơ sở từ B sang S

$(B\to S)$

Ta có: $(S\to B) = (u_1^T\ u_2^T\ u_3^T) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

$\Rightarrow (B\to S) = (S\to B)^{-1}$

$$\Leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow[d_3-d_1]{d_2-2d_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[-d_3]{d_3+2d_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \xrightarrow[d_1-3d_3]{d_2+3d_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\xrightarrow{d_1-2d_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

b) Từ S sang B

$(S\to B) = (u_1^T\ u_2^T\ u_3^T) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

c) $w = (5, -3, 1)$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 2 & 5 & 3 & -3 \\ 1 & 0 & 8 & 1 \end{array}\right) \xrightarrow[d_3-d_1]{d_2-2d_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & -3 & -13 \\ 0 & -2 & 5 & -4 \end{array}\right)$$

$$\xrightarrow{d_3+2d_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 3 & -13 \\ 0 & 0 & -1 & -30 \end{array}\right) \xrightarrow[d_2+3d_3]{-d_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 0 & 77 \\ 0 & 0 & 1 & 30 \end{array}\right)$$

$$\xrightarrow{d_1-3d_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & -85 \\ 0 & 1 & 0 & 77 \\ 0 & 0 & 1 & 30 \end{array}\right) \xrightarrow{d_1-2d_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -239 \\ 0 & 1 & 0 & 77 \\ 0 & 0 & 1 & 30 \end{array}\right)$$

$\Rightarrow [w]_B = \begin{pmatrix} -239 \\ 77 \\ 30 \end{pmatrix}$

$[w]_S = (S \to B)[w_B] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}\begin{pmatrix} -234 \\ 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$

~~$[w]_B = (B \to S)[w]_S = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} =$~~

d) $w = (3, -5, 0)$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow [w]_S = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix} \Rightarrow$ ~~$[w]_B =$~~

$\Rightarrow [w]_B = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} -200 \\ 64 \\ 25 \end{pmatrix}$

6 a) $u_1 = (1,1,1)$, $u_2 = (0,1,1)$, $u_3 = (0,0,1)$

Ta đặt $v_1 = u_1 = (1,1,1)$

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = (0,1,1) - \frac{0}{3}(1,1,1) = (0,1,1)$

$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2 =$ ~~$(0,0$~~

$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{2}(0,1,1) = (0,0,1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(0, \frac{1}{2}, \frac{1}{2}\right)$

$= \left(-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right) \Rightarrow$ Hệ vectơ trực giao

b) $u_1 = (1,1,1)$, $u_2 = (-1,1,0)$, $u_3 = (1,2,1)$

Ta đặt $v_1 = u_1 = (1,1,1)$

$v_2 = (-1,1,0) - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = (-1,1,0) - \frac{0}{3}(1,1,1) = (-1,1,0)$

$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2 = (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0)$

$= \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$

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=) Trực giao

Ta có: $\|v_1\| = \sqrt{3}$, $\|v_2\| = \sqrt{2}$, $\|v_3\| = \frac{\sqrt{6}}{6}$

=) $v_1' = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$v_2' = \frac{v_2}{\|v_2\|} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$v_3' = \frac{v_3}{\|v_3\|} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right)$

=) Hệ vectơ $\{v_1', v_2', v_3'\}$ là 1 hệ vectơ trực chuẩn

*) Bổ sung câu a:

Ta có: $\|v_1\| = \sqrt{3}$, $\|v_2\| = \sqrt{2}$, $\|v_3\| = \frac{\sqrt{6}}{6}$

=) $v_1' = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $v_2' = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$v_3' = \left(-\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$

=) Hệ vectơ $\{v_1', v_2', v_3'\}$ là 1 hệ vectơ trực chuẩn

c) $u_1 = (1, 0, 0)$, $u_2 = (3, 7, -2)$, $u_3 = (0, 4, 1)$

Ta có: $u_1 = v_1 = (1, 0, 0)$

$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1 = (3, 7, -2) - 3(1, 0, 0) = (0, 7, -2)$

$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2 = (0, 4, 1) - (0, 0, 0) - \frac{26}{53}(0, 7, -2)$

$= \left(-\frac{26}{53}, \frac{30}{53}, \frac{105}{53}\right)$ =) Hệ trực giao)

Ta có: $\|v_1\| = 1$, $\|v_2\| = \sqrt{53}$, $\|v_3\| = 2{,}12$

$v_1' = \frac{v_1}{\|v_1\|} = (1, 0, 0)$, $v_2' = \left(0, \frac{7}{\sqrt{53}}, \frac{-2}{\sqrt{53}}\right)$, $v_3' = (-0{,}23, 0{,}27, 0{,}93)$

=> Hệ trực chuẩn

$u_1 = (0,2,1,0)$, $u_2 = (1,-1,0,0)$, $u_3 = (1,2,0,-1)$, $u_4 = (1,0,0,1)$

$v_1 = u_1 = (0,2,1,0)$

$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1 = (1,-1,0,0) - \frac{-2}{5}(0,2,1,0) = (1, \frac{-1}{5}, \frac{+2}{5}, 0)$

$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2 = (1,2,0,-1) - \frac{4}{5}(0,2,1,0)$

$- \frac{\sqrt{30}}{10}(1,-1,0,0) = (0{,}45, 0{,}95, \frac{-4}{5}, \ldots)$

$v_4 = u_4 - \frac{\langle u_4, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_4, v_2\rangle}{\|v_2\|^2} v_2 - \frac{\langle u_4, v_3\rangle}{\|v_3\|^2} v_3$

$= (1,0,0,1) - \frac{5}{\sqrt{30}}(1, \frac{-1}{5}, \frac{2}{5}, 0) + 0{,}2 \cdot (0{,}41, 0{,}95, \frac{-4}{5}, -1)$

$= (0{,}18, 0{,}32, -0{,}53, 0{,}8)$ => Hệ trực giao

Ta có: $\|v_1\| = \sqrt{5}$, $\|v_2\| = \frac{\sqrt{30}}{5}$

$- \frac{1}{2}(1,-1,0,0) = (\frac{3}{2}, \frac{9}{10}, \frac{-4}{5}, -1)$

$v_4 = (1,0,0,1) - \frac{5}{6}(1, \frac{-1}{5}, \frac{2}{5}, 0) + \frac{8}{42}(\frac{3}{2}, \frac{9}{10}, \frac{-4}{5}, -1)$

$= (\frac{1}{141}, \frac{10}{141}, \frac{-35}{141}, -1)$

Ta có: $\|v_1\| = \sqrt{5}$, $\|v_2\| = \frac{\sqrt{30}}{5}$, $\|v_3\| = \frac{\sqrt{420}}{10}$, $\|v_4\| \approx 1{,}03$

$v_1' = \frac{v_1}{\|v_1\|} = (0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0)$

$v_2' = \frac{v_2}{\|v_2\|} = (\frac{\sqrt{30}}{5}, \frac{-\sqrt{30}}{25}, \frac{2}{5\sqrt{5}}, 0)$

$v_3' = \frac{v_3}{\|v_3\|} = (\cancel{0{,}4}, 0{,}69, 0{,}42, -0{,}36, \frac{-10}{\sqrt{420}})$

$v_4' = \frac{v_4}{\|v_4\|} = (0{,}007, 0{,}069, -0{,}24, \frac{-1}{1{,}03})$

Vậy hệ trực chuẩn

a) $\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2) = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases}$

• $\lambda_1 = 1 \Rightarrow A - \lambda_1 I = \begin{pmatrix} -2 & 3 \\ -2 & 3 \end{pmatrix}$

Xét $(A-\lambda I)x = 0$

$\Rightarrow \left(\begin{array}{cc|c} -2 & 3 & 0 \\ -2 & 3 & 0 \end{array}\right) \xrightarrow[d_2 - d_1]{-\frac{1}{2}d_1} \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 - 3/2\, x_2 = 0 \Rightarrow x_1 = \frac{3}{2}(x_2)$

$\Rightarrow$ Nghiệm tổng quát $x = \begin{pmatrix} 3/2\, x_2 \\ x_2 \end{pmatrix}$

Tập nghiệm $\left\{ x_2 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} \right\}$, đặt $x_2 = 1 \Rightarrow v_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$

• $\lambda_2 = 2$

$A - \lambda I = \begin{pmatrix} -3 & 3 \\ -2 & 2 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} -3 & 3 & 0 \\ -2 & 2 & 0 \end{array}\right) \xrightarrow[d_2 + 2d_1]{-\frac{d_1}{3}} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

$\Rightarrow$ nghiệm tổng quát $x = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix}$

Tập nghiệm $\left\{ x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, đặt $x_2 = 1 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Vậy trị riêng $\lambda_1 = 1 \Rightarrow$ vectơ riêng $v_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$

$\lambda_2 = 2 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} 5 & 2 \\ 9 & 2 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \lambda^2 - 7\lambda - 8 = 0 \Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = 8 \end{cases}$

• $\lambda_1 = -1 \Rightarrow A - \lambda_1 I = \begin{pmatrix} 6 & 2 \\ 9 & 3 \end{pmatrix}$

$(A - \lambda I)x = 0 \Rightarrow \left(\begin{array}{cc|c} 6 & 2 & 0 \\ 9 & 3 & 0 \end{array}\right) \xrightarrow{\frac{1}{6}d_1} \left(\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 9 & 3 & 0 \end{array}\right)$

$\xrightarrow{d_2 - 9d_1} \left(\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 + \frac{1}{3}x_2 = 0 \Rightarrow x_1 = \frac{-1}{3}x_2$

$\Rightarrow X = \begin{pmatrix} -1/3\, x_2 \\ x_2 \end{pmatrix}$, với $x_2 = 1 \Rightarrow v_1 = \begin{pmatrix} -1/3 \\ 1 \end{pmatrix}$

• $\lambda_2 = 8$

$(A - \lambda I)v = 0 \Rightarrow \left(\begin{array}{cc|c} -3 & 2 & 0 \\ 9 & -6 & 0 \end{array}\right) \xrightarrow{-1/3 d_1} \left(\begin{array}{cc|c} 1 & -2/3 & 0 \\ 9 & -6 & 0 \end{array}\right)$

$\xrightarrow{d_2 - 9d_1} \left(\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 - \frac{2}{3}x_2 = 0 \Rightarrow x_1 = \frac{2}{3}x_2$

$\Rightarrow X = \begin{pmatrix} 2/3\, x_2 \\ x_2 \end{pmatrix}$, với $x_2 = 1 \Rightarrow v_2 = \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \lambda^3 + 3\lambda^2 + 4\lambda - 12 \Rightarrow \begin{cases} \lambda_1 = -2 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{cases}$

• $\lambda_1 = -2$

$\Rightarrow \left(\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 1 & 5 & 1 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right) \xrightarrow[d_2 - d_1]{d_1/3} \left(\begin{array}{ccc|c} 1 & -1/3 & -1/3 & 0 \\ 0 & 16/3 & 4/3 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right) \xrightarrow[\frac{3}{16} d_2]{d_3 + 3d_1}$

$\left(\begin{array}{ccc|c} 1 & -1/3 & -1/3 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{d_1 + 1/3 d_2} \left(\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\begin{cases} x_1 - 1/4\, x_3 = 0 \\ x_2 + 1/4\, x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 1/4\, x_3 \\ x_2 = -1/4\, x_3 \end{cases}$

$\Rightarrow X = \begin{pmatrix} 1/4\, x_3 \\ -1/4\, x_3 \\ x_3 \end{pmatrix}$ với $x_3 = 1$, $v_1 = \begin{pmatrix} 1/4 \\ -1/4 \\ 1 \end{pmatrix}$

• $\lambda_2 = 2$

$\Rightarrow \left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ -3 & 1 & -3 & 0 \end{array}\right) \xrightarrow[d_2 - d_1]{-d_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 1 & -3 & 0 \end{array}\right)$

$\xrightarrow[\substack{d_3 \leftrightarrow d_2 \\ d_2/4}]{d_3 + 3d_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{d_1 - d_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = 0 \end{cases}$

$\Rightarrow X = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix}$ với $x_3 = 1 \Rightarrow v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

• $\lambda_3 = 3$

$\Rightarrow \left(\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 1 & -4 & 0 \end{array}\right) \xrightarrow[d_2 - d_1]{-\frac{1}{2}d_1} \left(\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ -3 & 1 & -4 & 0 \end{array}\right)$

$\xrightarrow{d_3 + 3d_1} \left(\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 5/2 & -5/2 & 0 \end{array}\right) \xrightarrow[d_3 - 5/2 d_2]{-2d_2} \left(\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\xrightarrow{d_1 - 1/2 d_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$

$\Rightarrow X = \begin{pmatrix} -x_3 \\ x_3 \\ x_3 \end{pmatrix} \Rightarrow$ với $x_3 = 1 \Rightarrow v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

d) $\begin{pmatrix} 5 & -1 & 1 \\ -1 & 2 & -2 \\ 1 & -2 & 2 \end{pmatrix} \Rightarrow \det(A - \lambda I) = -\lambda^3 + 9\lambda^2 - 18\lambda = 0 \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 3 \\ \lambda_3 = 6 \end{cases}$

• $\lambda_1 = 0$

$\Rightarrow \left(\begin{array}{ccc|c} 5 & -1 & 1 & 0 \\ -1 & 2 & -2 & 0 \\ 1 & -2 & 2 & 0 \end{array}\right) \xrightarrow{\frac{1}{5}d_1} \left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & \frac{1}{5} & 0 \\ -1 & 2 & -2 & 0 \\ 1 & -2 & 2 & 0 \end{array}\right)$

$\xrightarrow[d_3 - d_1]{d_2 + d_1} \left(\begin{array}{ccc|c} 1 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -\frac{9}{5} & \frac{9}{5} & 0 \end{array}\right) \xrightarrow[d_1 + \frac{1}{5}d_2]{d_3 + \frac{9}{5}d_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow X = \begin{pmatrix} 0 \\ x_3 \\ x_3 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} (x_3 = 1)$

• $\lambda_2 = 3$

$\Rightarrow \left(\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ -1 & -1 & -2 & 0 \\ 1 & -2 & -1 & 0 \end{array}\right) \xrightarrow[d_2 + d_1]{\frac{1}{2}d_1} \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & 0 \\ 1 & -2 & -1 & 0 \end{array}\right)$

$\xrightarrow[-\frac{2}{3}d_2]{d_3 - d_1} \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right) \xrightarrow[d_1 + \frac{1}{2}d_2]{d_3 + \frac{3}{2}d_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow X = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} (x_3 = 1)$

• $\lambda_3 = 6$

$\Rightarrow \left(\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ -1 & -4 & -2 & 0 \\ 1 & -2 & -4 & 0 \end{array}\right) \xrightarrow[d_2 + d_1]{-d_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & -3 & 0 \\ 1 & -2 & -4 & 0 \end{array}\right) \xrightarrow[-\frac{1}{3}d_2]{d_3 - d_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right)$

$\xrightarrow[\frac{-1}{3}d_2]{d_3-d_1}\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right) \xrightarrow[d_3-d_2]{d_1+2d_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{cases} x_1 - 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow X = \begin{pmatrix} 2x_3 \\ -x_3 \\ x_3 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} (x_3 = 1)$

Vậy $\lambda_1 = 0 \Rightarrow v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \lambda_2 = 3 \Rightarrow v_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \lambda_3 = 6 \Rightarrow v_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\lambda_3 = 3 \Rightarrow v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

j) $\begin{pmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \Rightarrow \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -2 \end{cases}$

• $\lambda_1 = 1$

$$\left(\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right) \xrightarrow{d_2 \leftrightarrow d_1} \left(\begin{array}{ccc|c} -3 & -6 & -3 & 0 \\ 0 & 3 & 3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right) \xrightarrow[d_3 - 3d_1,\ \frac{1}{3}d_2]{-\frac{1}{3}d_1}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right) \xrightarrow[d_1 - 2d_2]{d_3 + 3d_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

$\Rightarrow X = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} (x_3 = 1)$

• $\lambda_2 = -2$

$\left(\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right) \xrightarrow[\substack{d_2+3d_1 \\ d_3-3d_1}]{\frac{1}{3}d_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 + x_2 + x_3 = 0$

$\Rightarrow x_1 = -x_2 - x_3,\ x_2 = x_2,\ x_3 = x_3$

$\Rightarrow X = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow$ Với $x_2 = 1, x_3 = 0 \Rightarrow v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Với $x_2 = 0, x_3 = 1 \Rightarrow v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_3 = -2$, tương tự như $\lambda_2 = -2$, ta suy ra:

$v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},\ v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Vậy $\lambda_1 = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\lambda_2 = -2 \Rightarrow \begin{cases} v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{cases}$, $\lambda_3 = -2 \Rightarrow \begin{cases} v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{cases}$

g) $\begin{pmatrix} 3 & 4 & -4 \\ -2 & -1 & 2 \\ -2 & 0 & 1 \end{pmatrix} \Rightarrow \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + \cancel{\lambda} \cancel{\lambda} - 3 = -(\lambda - 1)(\lambda^2 - 2\lambda$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 3$

$\lambda_2 = -1$,

$\lambda_1 = 1$.

$\Rightarrow \left(\begin{array}{ccc} 2 & 4 & -4 \\ -2 & -2 & 2 \\ 2 & 0 & 0 \end{array}\right) \xrightarrow[d_2+2d_1]{\frac{1}{2}d_1} \left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & +2 & -2 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right)$

$$\xrightarrow[\frac{1}{2}d_2]{d_3+2d_1}\left(\begin{array}{ccc|c}1&2&-2&0\\0&1&-1&0\\0&4&-4&0\end{array}\right)\xrightarrow[d_1-2d_2]{d_3-4d_2}\left(\begin{array}{ccc|c}1&0&0&0\\0&1&-1&0\\0&0&0&0\end{array}\right)$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = -x_3 \end{cases} \Rightarrow X = \begin{pmatrix}0\\-x_3\\x_3\end{pmatrix} \Rightarrow v_1 = \begin{pmatrix}0\\-1\\1\end{pmatrix} (x_3 = 1)$$

• $\lambda_2 = -1$

$$\Rightarrow \left(\begin{array}{ccc|c}4&4&-4&0\\-2&0&2&0\\-2&0&2&0\end{array}\right)\xrightarrow[d_2+2d_1]{\frac{1}{4}d_1}\left(\begin{array}{ccc|c}1&1&-1&0\\0&2&0&0\\-2&0&2&0\end{array}\right)$$

$$\xrightarrow[\frac{1}{2}d_2]{d_3+2d_1}\left(\begin{array}{ccc|c}1&1&-1&0\\0&2&0&0\\0&2&0&0\end{array}\right)\xrightarrow[d_1-d_2]{d_3-2d_2}\left(\begin{array}{ccc|c}1&0&-1&0\\0&1&0&0\\0&0&0&0\end{array}\right)$$

$$\Rightarrow \begin{cases} x_1 - x_3 = 0 \\ x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = 0 \end{cases} \Rightarrow X = \begin{pmatrix}x_3\\0\\x_3\end{pmatrix} \Rightarrow v_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} (x_3 = 1)$$

$\lambda_3 = 3$

$$\Rightarrow \left(\begin{array}{ccc|c}0&4&-4&0\\-2&-4&2&0\\-2&0&2&0\end{array}\right)\xrightarrow{d_1 \leftrightarrow d_2}\left(\begin{array}{ccc|c}-2&-4&2&0\\0&4&-4&0\\-2&0&-2&0\end{array}\right)$$

$$\xrightarrow[d_3+2d_1]{-\frac{1}{2}d_1}\left(\begin{array}{ccc|c}1&2&-1&0\\0&4&-4&0\\0&4&-4&0\end{array}\right)\xrightarrow[d_3-4d_2]{\frac{1}{4}d_2}\left(\begin{array}{ccc|c}1&2&-1&0\\0&1&-1&0\\0&0&0&0\end{array}\right)$$

$$\xrightarrow{d_1-2d_2}\left(\begin{array}{ccc|c}1&0&1&0\\0&1&-1&0\\0&0&0&0\end{array}\right)\Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow X = \begin{pmatrix}-x_3\\x_3\\x_3\end{pmatrix} \Rightarrow v_3 = \begin{pmatrix}-1\\1\\1\end{pmatrix} (x_3 = 1)$$

Vậy $\lambda_1 = 1 \Rightarrow v_1 = \begin{pmatrix}0\\-1\\1\end{pmatrix}$

$\lambda_3 = 3 \Rightarrow v_3 = \begin{pmatrix}-1\\1\\1\end{pmatrix}$

$\lambda_2 = -1 \Rightarrow v_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}$

h) $\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \Rightarrow \det(A-\lambda I) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = (\lambda-1)(\lambda-2)^2 = 0$

$\Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = \lambda_3 = 2 \end{cases}$

• $\lambda_1 = 1$

$\Rightarrow \left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \xrightarrow[d_3 - d_1]{-d_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$x_1 + 2x_3 = 0$, $x_2 - x_3 = 0 \Rightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \end{cases} \Rightarrow X = \begin{pmatrix} -2x_3 \\ x_3 \\ x_3 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} (x_3 = 1)$

• $\lambda_2 = \lambda_3 = 2 \Rightarrow \left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[d_2 - d_1,\ d_3 - d_1]{-\frac{1}{2}d_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3 \Rightarrow X = \begin{pmatrix} -x_3 \\ x_2 \\ x_3 \end{pmatrix}$

Với $x_2 = 0$, $x_3 = 1 \Rightarrow v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = v_4$

Với $x_2 = 1$, $x_3 = 0 \Rightarrow v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = v_5$

Vậy với $\lambda_1 = 1 \Rightarrow v_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_2 = \lambda_3 = 2 \Rightarrow v_2 = v_4 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $v_3 = v_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

i) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix} \Rightarrow \det(A-\lambda I) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda-1)(\lambda^2 - 4\lambda + 4)$

$\Rightarrow -(\lambda-1)(\lambda-2)^2 = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = \lambda_3 = 2 \end{cases}$

$\Rightarrow \lambda_1$

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$\cdot\ \lambda_1 = 1 \Rightarrow \left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \xrightarrow[d_2 - d_1,\ d_3 - d_1]{-d_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{cases} x_1 + 2x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \end{cases} \Rightarrow X = \begin{pmatrix} -2x_3 \\ x_3 \\ x_3 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} (x_3 = 1)$

$\lambda_2 = \lambda_3 = 2 \Rightarrow \left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2} d_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$

$\xrightarrow[d_3 - d_1]{d_2 - d_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 + x_3 = 0 \Rightarrow X = \begin{pmatrix} -x_3 \\ x_2 \\ x_3 \end{pmatrix}$

với $x_2 = 0, x_3 = 1 \Rightarrow v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = v_2$

$x_2 = 1, x_3 = 0 \Rightarrow v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = v_3$

$\cdot$ ~~λ_3~~ vậy $\lambda_1 = 1 \Rightarrow v_1 = \Big($

i) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix} \Rightarrow \det(A - \lambda I) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda-1)(\lambda-2)^2 = 0$

$\Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = \lambda_3 = 2 \end{cases}$

$\cdot\ \lambda_1 = 1 \Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -3 & 5 & 1 & 0 \end{array}\right) \xrightarrow[d_3 + 3d_1]{d_1 \leftrightarrow d_2} \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 8 & 1 & 0 \end{array}\right) \xrightarrow[\frac{1}{8} d_2]{d_2 \leftrightarrow d_3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\xrightarrow{d_1 - d_2} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{8} & 0 \\ 0 & 1 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 - \frac{1}{8} x_3 = 0 \\ x_2 + \frac{1}{8} x_3 = 0 \end{cases} \Rightarrow X = \begin{pmatrix} +\frac{1}{8} x_3 \\ -\frac{1}{8} x_3 \\ x_3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1/8 \\ -1/8 \\ 1 \end{pmatrix} (x_3 = 1)$

$d_2: \lambda_2 = \lambda_3 = 2 \Rightarrow \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -3 & 5 & 0 & 0 \end{array}\right) \xrightarrow[d_2 - d_1]{-d_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 5 & 0 & 0 \end{array}\right)$

$\xrightarrow{d_2 \leftrightarrow d_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 5 & 0 & 0 \end{array}\right) \xrightarrow[\substack{d_3 \leftrightarrow d_2 \\ \frac{1}{5} d_2}]{d_3 + 3d_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow x_1 = x_2 = 0$

$\Rightarrow x = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (x_3 = 1) = v_3$

lấy $x_1 = 1 \Rightarrow v_1 = \begin{pmatrix} 1/8 \\ -1/8 \\ 1 \end{pmatrix}$

$\lambda_2 = \lambda_3 = 2 \Rightarrow v_2 = v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

j) $\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \Rightarrow \det(A - \lambda I) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$

$= -(\lambda - 1)(\lambda^2 - 5\lambda + 6) = -(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

$\Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases}, \lambda_3 = 3$

$\lambda_1 = 1 \Rightarrow \left(\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right) \xrightarrow[d_2 + 2d_1]{1/3 d_1} \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 0 & 2/3 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right)$

$\xrightarrow{d_3 + 2d_1} \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 2/3 & 0 \end{array}\right) \xrightarrow[d_3 - \frac{2}{3} d_2]{3/2 d_2} \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$$\xrightarrow{d_1 + \frac{1}{3} d_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases} \Rightarrow X = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (x_2 = 1)$$

$$\lambda_2 = 1 \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right) \xrightarrow[d_2 + 2d_1]{\frac{1}{2} d_1} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{d_3 + 2d_1} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-1 d_2} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x_1 + \frac{1}{2} x_3 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{1}{2} x_3 \\ x_2 = x_3 \end{cases} \Rightarrow X = \begin{pmatrix} -\frac{1}{3} x_3 \\ x_3 \\ x_3 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -\frac{1}{3} \\ 1 \\ 1 \end{pmatrix} (x_3 = 1)$$

$\lambda_3 = \ldots \Rightarrow (1 \quad 0 \quad 1 \quad 0$

$\lambda_3 = 3 \Rightarrow \left(\begin{array}{ccc|c} 1&0&1&0\\ -2&-2&0&0\\ -1&0&-2&0\end{array}\right) \xrightarrow[d_3+2d_1]{d_2+2d_1} \left(\begin{array}{ccc|c}1&0&1&0\\0&-2&2&0\\0&0&0&0\end{array}\right)$

$\xrightarrow{-1/2\, d_2} \left(\begin{array}{ccc|c}1&0&1&0\\0&1&-1&0\\0&0&0&0\end{array}\right) \Rightarrow \begin{cases} x_1+x_3=0\\ x_2-x_3=0\end{cases} \Rightarrow X=\begin{pmatrix}-x_3\\x_3\\x_3\end{pmatrix} \Rightarrow \cancel{v_3}$

$\Rightarrow v_3 = \begin{pmatrix}1\\1\\1\end{pmatrix} (x_3=1)$

Lấy $\lambda_1 = 1 \Rightarrow v_1 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \lambda_2 = 2 \Rightarrow v_2 = \begin{pmatrix}-1/3\\1\\1\end{pmatrix}$

$\lambda_3 = 3 \Rightarrow v_3 = \begin{pmatrix}-1\\1\\1\end{pmatrix}$

1)

e) $\begin{pmatrix}1&3&3\\-3&-5&-3\\3&3&1\end{pmatrix} \Rightarrow \det(A-\lambda I) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda-1)(\lambda^2+4\lambda+4)$

$= -(\lambda-1)(\lambda+2)^2 = 0 \Rightarrow \begin{cases}\lambda_1=1\\ \lambda_2=-2\end{cases}$

$\lambda_1 = 1 \Rightarrow \left(\begin{array}{ccc|c}0&3&3&0\\-3&-6&-3&0\\3&3&0&0\end{array}\right) \xrightarrow[-1/3\, d_1]{d_1 \leftrightarrow d_2} \left(\begin{array}{ccc|c}1&2&1&0\\0&3&3&0\\3&3&0&0\end{array}\right)$

$\xrightarrow{d_3-3d_1} \left(\begin{array}{ccc|c}1&2&1&0\\0&3&3&0\\0&-3&-3&0\end{array}\right) \xrightarrow[d_3+3d_2]{1/3\, d_2} \left(\begin{array}{ccc|c}1&2&1&0\\0&1&1&0\\0&0&0&0\end{array}\right)$

$\xrightarrow{d_1-2d_2} \left(\begin{array}{ccc|c}1&0&-1&0\\0&1&1&0\\0&0&0&0\end{array}\right) \Rightarrow \begin{cases}x_1=x_3\\x_2=-x_3\end{cases} \Rightarrow X=\begin{pmatrix}x_3\\-x_3\\x_3\end{pmatrix} \Rightarrow v_1=\begin{pmatrix}1\\-1\\1\end{pmatrix}(x_3=1)$

$\lambda_2=-2 \Rightarrow \left(\begin{array}{ccc|c}3&3&3&0\\-3&-3&-3&0\\3&3&3&0\end{array}\right) \xrightarrow[d_2+3d_1,\ d_3-3d_1]{1/3\, d_1} \left(\begin{array}{ccc|c}1&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right)$

$\Rightarrow x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3 \Rightarrow x = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix}$

$\Rightarrow$ Đặt $x_2 = 0, x_3 = 1 \Rightarrow v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$x_2 = 1, x_3 = 0 \Rightarrow v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Vậy $\lambda_1 = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -2 \Rightarrow v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$