

10 detailed solutions *by Spencer Todd*

6.1 #39

Find $(f^{-1})'(a)$ if $f(x) = 3x^3 + 4x^2 + 6x + 5$ and $a = 5$.

Theorem 7 in Section 6.1 establishes that if f is one-to-one, its inverse is f^{-1} , and $f'(f^{-1}(a)) \neq 0$, then

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

(A glance at f' indicates that f is monotonic increasing.) First, we must find the derivative of $f(x)$, then apply Theorem 7. Additionally, $f(x) = 5$ when $x = 0$, so $f^{-1}(5) = 0$.

$$\begin{aligned} f'(x) &= 9x^2 + 8x + 6 \\ (f^{-1})'(a) &= \frac{1}{f'(f^{-1}(5))} \\ &= \frac{1}{f'(0)} \\ &= \frac{1}{6} \end{aligned}$$

6.8 #29

Find the limit:

$$\lim_{x \rightarrow 0} \frac{\tanh x}{\tan x}.$$

Since $\lim_{x \rightarrow 0} \tanh x = 0$ and $\lim_{x \rightarrow 0} \tan x = 0$, this limit is of the $\frac{0}{0}$ indeterminate form, so we can use L'Hôpital's Rule.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \tanh x}{\frac{d}{dx} \tan x} \\ \lim_{x \rightarrow 0} \frac{\operatorname{sech}^2 x}{\sec^2 x} \\ \frac{1^2}{1^2} \\ 1 \end{aligned}$$

7.1 #9

Evaluate the integral:

$$\int \cos^{-1} x \, dx.$$

Since $\cos^{-1} x$ has an easy derivative, we can use integration by parts by differentiating $\cos^{-1} x$ and integrating dx .

$$\int u \, dv = u \, v - \int v \, du$$

$$\begin{aligned} u &= \cos^{-1} x & dv &= dx \\ du &= \frac{-dx}{\sqrt{1-x^2}} & v &= x \end{aligned}$$

$$\int \cos^{-1} x \, dx = x \cos^{-1} x - \int \frac{-x}{\sqrt{1-x^2}} \, dx$$

We can set everything inside the radical to a new variable, a .

$$a = 1 - x^2$$

$$da = -2x \, dx$$

$$\frac{da}{2} = -x \, dx$$

$$\begin{aligned} \int \cos^{-1} x \, dx &= x \cos^{-1} x - \int \frac{da}{2\sqrt{a}} \\ &= x \cos^{-1} x - \sqrt{a} + C \\ &= x \cos^{-1} x - \sqrt{1-x^2} + C \end{aligned}$$

7.2 #1

Evaluate the integral:

$$\int \sin^2 x \cos^3 x \, dx.$$

Since the power of \cos is odd, $z = \sin x$ will work.

$$\begin{aligned} z &= \sin x \\ \frac{dz}{dx} &= \cos x \\ dx &= \frac{dz}{\cos x} \end{aligned}$$

We can use a trigonometric identity to handle the \cos .

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \cos^2 x &= 1 - \sin^2 x \\ &= 1 - z^2 \end{aligned}$$

Arranging the expression as follows will highlight our choices of substitutions:

$$\begin{aligned} \int \sin^2 x \cos^3 x \, dx &= \int (\sin x)^2 (\cos^2 x) (\cos x \, dx) . \\ &= \int z^2 (1 - z^2) \left(\cos x \frac{dz}{\cos x} \right) \\ &= \int z^2 - z^4 \, dz \\ &= \frac{z^3}{3} - \frac{z^5}{5} + C \\ &= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \end{aligned}$$

9.5 #7

Solve the differential equation:

$$y' + y = x. \tag{1}$$

We are given an inhomogenous equation, so we must first solve the corresponding homogenous

equation:

$$\frac{dy_0}{dx} + y_0 = 0.$$

$$\frac{dy_0}{y_0} = -dx$$

$$\int \frac{dy_0}{y_0} = - \int dx$$

$$\ln |y_0| = -x + A$$

$$|y_0| = e^{-x+A}$$

$$y_0 = B e^{-x}, \text{ whereas } B = \pm e^A$$

Returning to the original equation, we can replace B with a differentiable function, u .

$$y = u e^{-x} \tag{2}$$

$$y' = u' e^{-x} - u e^{-x} \tag{3}$$

Now, we can plug (2) and (3) in to (1) and solve for u . Notice that a nice cancellation occurs.

$$u' e^{-x} - u e^{-x} + u e^{-x} = x$$

$$u' e^{-x} = x$$

$$u' = x e^x$$

$$\int du = \int x e^x dx$$

We may proceed using integration by parts ($\int g dh = g h - \int h dg$).

$$g = x \quad dh = e^x dx$$

$$dg = dx \quad h = e^x$$

$$\begin{aligned} u &= x e^x - \int e^x dx \\ &= x e^x - e^x + C \end{aligned} \tag{4}$$

Finally, we can plug (4) into (2) for our final answer.

$$\begin{aligned}y &= e^{-x} (x e^x - e^x + C) \\&= x - 1 + C e^{-x}\end{aligned}$$

10.2 #15

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$x = t - \ln t$$

$$y = t + \ln t$$

By the chain rule, $\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

$$\begin{aligned}\frac{dy}{dt} &= 1 + \frac{1}{t} \\ \frac{dx}{dt} &= 1 - \frac{1}{t} \\ \frac{\frac{dy}{dt}}{\frac{dx}{dt}} &= \frac{1 + \frac{1}{t}}{1 - \frac{1}{t}} \\ \frac{dy}{dx} &= \frac{t + 1}{t - 1}\end{aligned}$$

By the chain rule, $\frac{d^2y}{dx^2} = \frac{d}{dx} \frac{dy}{dx} = \frac{d}{dt} \frac{dt}{dx} \frac{dy}{dx} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}$.

$$\begin{aligned}\frac{d}{dt} \left(\frac{dy}{dx} \right) &= \frac{(t-1)(1) - (t+1)(1)}{(t-1)^2} \\&= \frac{-2}{(t-1)^2} \\ \frac{dx}{dt} &= 1 - \frac{1}{t} \\ \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} &= \frac{-2}{(t-1)^2} \cdot \frac{1}{1 - \frac{1}{t}} \\ \frac{d^2y}{dx^2} &= \frac{-2t}{(t-1)^3}\end{aligned}$$

10.4 #2

Find the area of the region that is bounded by the given curve and lies in the specified sector.

$$r(\theta) = \cos \theta, \quad 0 \leq \theta \leq \frac{\pi}{6}$$

The area of a sector of a circle ($A = \frac{1}{2}r^2\theta$) leads us to the equation for the integral of polar equations:

$$\frac{1}{2} \int_a^b r(\theta)^2 d\theta.$$

$$\frac{1}{2} \int_0^{\frac{\pi}{6}} \cos^2 \theta d\theta$$

Since the power of $\sin(0)$ and $\cos(2)$ are both even, we may proceed using the double angle formula for cosine.

$$\begin{aligned} & \frac{1}{4} \int_0^{\frac{\pi}{6}} 1 + \cos 2\theta d\theta \\ & \frac{1}{4} \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_0^{\frac{\pi}{6}} \\ & \frac{1}{4} \left(\frac{\pi}{6} + \frac{1}{2} \sin \frac{\pi}{3} \right) \end{aligned}$$

Notice that the expression evaluated at 0 is 0.

$$\frac{\pi}{24} + \frac{\sqrt{3}}{16}$$