10 detailed solutions *by Spencer Todd*

6.1 *#39*

Find $\left(f^{-1}\right)'(a)$ of $f(x) = 3x^3 + 4x^2 + 6x + 5$, $a = 5$.

Theorem 7 in Section 6.1 establishes that if f is one-to-one, its inverse is f^{-1}, and $f'\left(f^{-1}(a)\right) \neq 0$, then

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$

To begin, a sketch of the graph of f shows that it is one-to-one.

Next, we must find the derivative of $f(x)$, then apply Theorem 7. Additionally, $f(x) = 5$ when $x = 0$, so $f^{-1}(5) = 0$.

$$\begin{aligned} f'(x) &= 9x^2 + 8x + 6 \\ \left(f^{-1}\right)'(a) &= \frac{1}{f'\left(f^{-1}(5)\right)} \\ &= \frac{1}{f'(0)} \\ &= \frac{1}{6}. \end{aligned}$$

6.8 *#29*

Find the limit:

$$\lim_{x \to 0} \frac{\tanh x}{\tan x}$$

Since $\lim\limits_{x \to 0} \tanh x = 0$ and $\lim\limits_{x \to 0} \tan x = 0$, this limit is of the $\frac{0}{0}$ indeterminate form, so we can use L'Hôpital's

Rule.

$$\begin{aligned}
\lim_{x\to 0}\frac{\tanh x}{\tan x} &= \lim_{x\to 0}\frac{\frac{d}{dx}\tanh x}{\frac{d}{dx}\tan x} \\
&= \lim_{x\to 0}\frac{\operatorname{sech}^2 x}{\sec^2 x} \\
&= \frac{1^2}{1^2} \\
&= 1.
\end{aligned}$$

7.1 *#9*

Evaluate the integral:

$$\int \cos^{-1} x \, dx$$

Since $\cos^{-1} x$ has an easy derivative, we can use integration by parts by differentiating the $\cos^{-1} x$ and integrating the dx.

$$\int u \, dv = u\, v - \int v \, du$$

$$\begin{aligned}
u &= \cos^{-1} x & dv &= dx \\
du &= \frac{-dx}{\sqrt{1-x^2}} & v &= x
\end{aligned}$$

$$\int \cos^{-1} x \, dx = x\cos^{-1} x - \int \frac{-x}{\sqrt{1-x^2}} \, dx$$

We can set everything inside the radical to a new variable, a.

$$\begin{aligned}
a &= 1 - x^2 \\
da &= -2x \, dx \\
-x \, dx &= \frac{1}{2} \, da
\end{aligned}$$

$$\begin{aligned}
\int \cos^{-1} x \, dx &= x\cos^{-1} x - \int \frac{da}{2\sqrt{a}} \\
&= x\cos^{-1} x - \sqrt{a} + C \\
&= x\cos^{-1} x - \sqrt{1-x^2} + C.
\end{aligned}$$

7.2 *#1*

Evaluate the integral:

$$\int \sin^2 x \cos^3 x \, dx$$

Since the power of cos is odd, $z = \sin x$ will work.

$$\begin{aligned} z &= \sin x \\ \frac{dz}{dx} &= \cos x \\ dx &= \frac{dz}{\cos x} \end{aligned}$$

We can use a trigonometric identity to handle the cos.

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \cos^2 x &= 1 - \sin^2 x \\ &= 1 - z^2 \end{aligned}$$

Arranging the expression as follows will highlight our choices of substitutions:

$$\begin{aligned} &\int (\sin x)^2 \left(\cos^2 x\right) (\cos x \, dx) \\ &\int z^2 \left(1 - z^2\right) \left(\cos x \, \frac{dz}{\cos x}\right) \\ &\int z^2 - z^4 \, dz \\ &\frac{z^3}{3} - \frac{z^5}{5} + C \\ &\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C. \end{aligned}$$

9.5 #7

Solve the differential equation:

$$y' + y = x \tag{1}$$

We are given an inhomogenous equation, so we must first solve the corresponding homogenous equation:

$$\begin{aligned} \frac{dy_0}{dx} + y_0 &= 0 \\ \frac{dy_0}{dx} &= -y_0 \\ \frac{dy_0}{y_0} &= -dx \\ \int \frac{dy_0}{y_0} &= -\int dx \\ \ln |y_0| &= -x + A \\ |y_0| &= e^{-x+A} \\ y_0 &= Be^{-x}, \text{ whereas } B = \pm e^A \end{aligned}$$

Returning to the original equation, we can replace B with a differable function, u.

$$y = u\,e^{-x} \tag{2}$$

$$y' = u'e^{-x} - u\,e^{-x} \tag{3}$$

Now, we can plug (2) and (3) in to (1) and solve for u. Notice that a nice cancellation occurs.

$$\begin{aligned} u'e^{-x} - u\,e^{-x} + u\,e^{-x} &= x \\ u'e^{-x} &= x \\ u' &= x\,e^{x} \\ \int u &= \int x\,e^{x}\,dx \end{aligned}$$

We may proceed using integration by parts $\left(\int g\,dh = g\,h - \int h\,dg\right)$.

$$\begin{aligned} g &= x \qquad dh = e^{x}\,dx \\ dg &= dx \qquad h = e^{x} \end{aligned}$$

$$\begin{aligned} u &= x\,e^{x} - \int e^{x}\,dx \\ &= x\,e^{x} - e^{x} + C \end{aligned} \tag{4}$$

Finally, we can plug (4) into (2) for our final answer.

$$\begin{aligned} y &= e^{-x}\left(x\,e^{x} - e^{x} + C\right) \\ &= x - 1 + Ce^{-x} \end{aligned}$$

10.2 *#15*

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$x = t - \ln t$$

$$y = t + \ln t$$

By the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

$$\begin{aligned}\frac{dx}{dt} &= 1 - \frac{1}{t}\\ \frac{dy}{dt} &= 1 + \frac{1}{t}\\ \frac{\frac{dy}{dt}}{\frac{dx}{dt}} &= \frac{1-\frac{1}{t}}{1+\frac{1}{t}}\\ \frac{dy}{dx} &= \frac{t-1}{t+1}\end{aligned}$$

By the chain rule, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\dfrac{dy}{dx} = \dfrac{d}{dx}\dfrac{dy}{dt}\dfrac{dt}{dx} = \dfrac{d}{dt}\dfrac{dy}{dx}\dfrac{dt}{dx} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$.

$$\begin{aligned}\frac{d}{dt}\left(\frac{dy}{dx}\right) &= 1\\ \frac{dx}{dt} &= 1 - \frac{1}{t}\\ \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} &= \frac{1}{1-\frac{1}{t}}\\ \frac{d^2y}{dx^2} &= \frac{t}{t-1}\end{aligned}$$

10.4 *#2*

Find the area of the region that is bounded by the given curve and lies in the specified sector.

$$r = \cos\theta,\ 0 \le \theta \le \frac{\pi}{6}$$

The area of a sector of a circle, $\left(A = \frac{1}{2}\theta r^2\right)$, leads us to the equation for the integral of polar equations:

$$\frac{1}{2}\int_a^b r\left(\theta\right)^2 d\theta$$

$$\frac{1}{2}\int_0^{\frac{\pi}{6}} \cos^2\theta \, d\theta$$

Since the power of sin (0) and cos (2) are both even, we may proceed using a double angle formula for cosine.

$$\begin{aligned}&\frac{1}{4}\int_0^{\frac{\pi}{6}} 1 + \cos 2\theta \, d\theta\\ &\frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right|_0^{\frac{\pi}{6}}\\ &\frac{1}{4}\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right)\end{aligned}$$

Notice that the expression evaluated at 0 is 0.

$$\frac{\pi}{24} + \frac{\sqrt{3}}{16}$$