10 detailed solutions *by Spencer Todd*

6.1 *#39*

Find $\left(f^{-1}\right)'(a)$ if $f(x) = 3x^3 + 4x^2 + 6x + 5$ and $a = 5$.

Theorem 7 in Section 6.1 establishes that if f is one-to-one, its inverse is f^{-1}, and $f'\left(f^{-1}(a)\right) \neq 0$, then

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$

(A glance at f' indicates that f is monotonic increasing.) First, we must find the derivative of $f(x)$, then apply Theorem 7. Additionally, $f(x) = 5$ when $x = 0$, so $f^{-1}(5) = 0$.

$$\begin{aligned} f'(x) &= 9x^2 + 8x + 6 \\ \left(f^{-1}\right)'(a) &= \frac{1}{f'\left(f^{-1}(5)\right)} \\ &= \frac{1}{f'(0)} \\ &= \frac{1}{6} \end{aligned}$$

6.8 *#29*

Find the limit:

$$\lim_{x\to 0} \frac{\tanh x}{\tan x}.$$

Since $\lim_{x\to 0} \tanh x = 0$ and $\lim_{x\to 0} \tan x = 0$, this limit is of the $\frac{0}{0}$ indeterminate form, so we can use L'Hôpital's Rule.

$$\lim_{x\to 0} \frac{\frac{d}{dx}\tanh x}{\frac{d}{dx}\tan x}$$

$$\lim_{x\to 0} \frac{\operatorname{sech}^2 x}{\sec^2 x}$$

$$\frac{1^2}{1^2}$$

$$1$$

7.1 *#9*

Evaluate the integral:

$$\int \cos^{-1} x \, dx.$$

Since $\cos^{-1} x$ has an easy derivative, we can use integration by parts by differentiating $\cos^{-1} x$ and integrating dx.

$$\int u \, dv = u \, v - \int v \, du$$

$$u = \cos^{-1} x \qquad dv = dx$$

$$du = \frac{-dx}{\sqrt{1 - x^2}} \qquad v = x$$

$$\int \cos^{-1} x \, dx = x \cos^{-1} x - \int \frac{-x}{\sqrt{1 - x^2}} \, dx$$

We can set a new variable equal to everything inside the radical.

$$a = 1 - x^2$$

$$da = -2x \, dx$$

$$\frac{da}{2} = -x \, dx$$

$$\begin{aligned}\int \cos^{-1} x \, dx &= x \cos^{-1} x - \int \frac{da}{2\sqrt{a}} \\ &= x \cos^{-1} x - \sqrt{a} + C \\ &= x \cos^{-1} x - \sqrt{1 - x^2} + C\end{aligned}$$

7.2 *#1*

Evaluate the integral:

$$\int \sin^2 x \cos^3 x \, dx.$$

Since the power of cos is odd, the substitution $z = \sin x$ will work.

$$\begin{aligned} z &= \sin x \\ \frac{dz}{dx} &= \cos x \\ \frac{dz}{\cos x} &= dx \end{aligned}$$

We can use a trigonometric identity to handle the cos.

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \cos^2 x &= 1 - \sin^2 x \\ &= 1 - z^2 \end{aligned}$$

Arranging the integrand as follows will highlight our choices of substitutions:

$$\begin{aligned} \int \sin^2 x \cos^3 x \, dx &= \int (\sin x)^2 \left(\cos^2 x\right) (\cos x \, dx) \,. \\ &= \int z^2 \left(1 - z^2\right) \left(\cos x \frac{dz}{\cos x}\right) \\ &= \int z^2 - z^4 \, dz \end{aligned}$$

Now we may finish the integral using the power rule.

$$\begin{aligned} &= \frac{z^3}{3} - \frac{z^5}{5} + C \\ &= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \end{aligned}$$

9.5 #7

Solve the differential equation:

$$y' + y = x. \tag{1}$$

We are given an inhomogenous equation, so we must first solve the corresponding homogenous equation:

$$\frac{dy_0}{dx} + y_0 = 0.$$

The homogenous equation is seperable, and we can solve it using the integral $\int \frac{du}{u} = \ln|u|$.

$$\begin{aligned}
\frac{dy_0}{y_0} &= -dx \\
\int \frac{dy_0}{y_0} &= -\int dx \\
\ln|y_0| &= -x + A \\
|y_0| &= e^{-x+A} \\
y_0 &= Be^{-x}, \text{ whereas } B = \pm e^A
\end{aligned}$$

Returning to the original equation, we can replace the constant B with a differentiable function, u.

$$y = u\,e^{-x} \tag{2}$$

$$y' = u'e^{-x} - u\,e^{-x} \tag{3}$$

Now, we can plug equations (2) and (3) in to equation (1) and solve for u. Notice that a nice cancellation occurs.

$$\begin{aligned}
u'e^{-x} - u\,e^{-x} + u\,e^{-x} &= x \\
u'e^{-x} &= x \\
u' &= x\,e^{x} \\
\int du &= \int x\,e^{x}\,dx
\end{aligned}$$

We may proceed using integration by parts ($\int g\,dh = g\,h - \int h\,dg$).

$$\begin{aligned}
g &= x \qquad dh = e^x\,dx \\
dg &= dx \qquad h = e^x
\end{aligned}$$

$$
\begin{aligned}
u &= x\,e^x - \int e^x\,dx \\
&= x\,e^x - e^x + C \qquad (4)
\end{aligned}
$$

Finally, we can plug equation (4) in to equation (2) for our final answer.

$$
\begin{aligned}
y &= e^{-x}\left(x\,e^x - e^x + C\right) \\
&= x - 1 + Ce^{-x}
\end{aligned}
$$

10.2 *#15*

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

$$
\begin{aligned}
x &= t - \ln t \\
y &= t + \ln t
\end{aligned}
$$

By the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

$$
\begin{aligned}
\frac{dy}{dt} &= 1 + \frac{1}{t} \\
\frac{dx}{dt} &= 1 - \frac{1}{t} \\
\frac{\frac{dy}{dt}}{\frac{dx}{dt}} &= \frac{1 + \frac{1}{t}}{1 - \frac{1}{t}} \\
\frac{dy}{dx} &= \frac{t + 1}{t - 1}
\end{aligned}
$$

The second derivative $\left(\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx}\right)$, by the chain rule, equals $\frac{d}{dt}\frac{dt}{dx}\frac{dy}{dx}$ or $\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$.

$$\begin{aligned}
\frac{d}{dt}\left(\frac{dy}{dx}\right) &= \frac{(t-1)(1)-(t+1)(1)}{(t-1)^2} \\
&= \frac{-2}{(t-1)^2} \\
\frac{dx}{dt} &= 1 - \frac{1}{t} \\
\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} &= \frac{-2}{(t-1)^2} \cdot \frac{1}{1-\frac{1}{t}} \\
\frac{d^2y}{dx^2} &= \frac{-2t}{(t-1)^3}
\end{aligned}$$

10.4 *#2*

Find the area of the region that is bounded by the given curve and lies in the specified sector.

$$r(\theta) = \cos\theta,\ 0 \le \theta \le \frac{\pi}{6}$$

The area of a sector of a circle $\left(A = \frac{1}{2}r^2\theta\right)$ leads us to the equation for the integral of polar equations:

$$\frac{1}{2}\int_a^b r(\theta)^2\, d\theta.$$

$$\frac{1}{2}\int_0^{\frac{\pi}{6}} \cos^2\theta\, d\theta$$

Since the power of sin (0) and cos (2) are both even, we may proceed using the double angle formula for cosine, $\cos^2 x = \frac{1}{2}(1+\cos 2x)$.

$$\begin{aligned}
&\frac{1}{4}\int_0^{\frac{\pi}{6}} 1+\cos 2\theta\, d\theta \\
&\frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right|_0^{\frac{\pi}{6}} \\
&\frac{1}{4}\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right)
\end{aligned}$$

Notice that the expression evaluated at 0 is 0.

$$\frac{\pi}{24} + \frac{\sqrt{3}}{16}$$

10.6 *#10*

Find the eccentricity, identify the conic, and give an equation of the directrix.

$$r = \frac{1}{2 + \sin\theta}$$

Theorem 6 in Section 10.6 tells us that a conic section with eccentricity e can be represented by a polar equation

$$r = \frac{ed}{1 \pm e\cos\theta} \text{ or } r = \frac{ed}{1 \pm e\sin\theta}.$$

It also says that the conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$. We must manipulate the given equation so that it matches the form $1 \pm e\sin\theta$ in the denominator.

$$\begin{aligned} r &= \frac{1}{2\left(1 + \frac{1}{2}\sin\theta\right)} \\ &= \frac{\frac{1}{2}}{1 + \frac{1}{2}\sin\theta} \end{aligned}$$

Our equation gives $e = \frac{1}{2}$ and $ed = \frac{1}{2}$, so $d = 1$, the directrix is $y = 1$, and the equation represents an ellipse, because $e < 1$.

11.4 *#3*

Determine whether $\sum_{n=1}^{\infty} a_n$ converges or diverges.

$$a_n = \frac{1}{n^3 + 8}$$

We can compare a_n to another series, $b_n = \frac{1}{n^3}$, to determine convergence.

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 8} < \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Since $a_n < b_n$ for any positive integer n, a_n will converge if b_n converges. The series b_n is of the p-series form $\sum \frac{1}{n^p}$, which converges if $p > 1$. In the case of b_n, $p = 3 > 1$, so b_n and a_n converge.

11.10 *#78*

Find the sum of the series.

$$1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \cdots$$

Writing the equation for the series will clearly show us a necessary substitution.

$$\sum_{n=0}^{\infty} (-1)^n \frac{(\ln 2)^n}{n!}$$

Since the factors -1 and $\ln 2$ both are raised to power n, the -1 can multiply in to the numerator to make $(-\ln 2)^n$. Then we can make the substitution $x = -\ln 2$, resulting in the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We know that this is also the power series for e^x evaluated at any complex number x. So, the solution is e^x evaluated at $x = -\ln 2$, or $\frac{1}{2}$.