Optimal Max-min Fairness Rate Control in Wireless Networks: Perron-Frobenius Characterization and Algorithms

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Abstract—Rate adaptation and power control are two key resource allocation mechanisms in multiuser wireless networks. In the presence of interference, how do we jointly optimize end-to-end source rates and link powers to achieve weighted max-min rate fairness for all sources in the network? This optimization problem is hard to solve as physical layer link rate functions are nonlinear, nonconvex, and coupled in the transmit powers. We show that the weighted max-min rate fairness problem can, in fact, be decoupled into separate fairness problems for flow rate and power control. For a large class of physical layer link rate functions, we characterize the optimal solution analytically by a nonlinear Perron-Frobenius theory (through solving a conditional eigenvalue problem) that captures the interaction of multiuser interference. We give an iterative algorithm to compute the optimal flow rate that converges geometrically fast without any parameter configuration. Numerical results show that our iterative algorithm is computationally fast for both the Shannon capacity, CDMA, and piecewise linear link rate functions.

Index Terms— Max-min fairness, power control, wireless network, convex optimization, nonnegative matrix theory, nonlinear Perron-Frobenius theory

I. INTRODUCTION

Wireless spectrum has become a highly valued and scarce resource due to the exponential growth of mobile data usage over the last decade. As more wireless communication networks deploy spectrum reuse due to limited link budgets, multiuser interference is increasingly becoming a dominant feature. Wireless networks such as 4G heterogeneous CDMA cellular networks and IEEE 802.11 ad-hoc and cognitive radio networks have to be provisioned for maximum efficiency and fairness. However, resource provisioning in a wireless network is a difficult problem due to the unreliable physical channel and predominant multiuser interference that can degrade the transmission quality. Link adaptation schemes are often used in wireless transmission to compensate for the negative effects of multiuser interference. Thus, the fairness of the resource provisioning scheme depends strongly on the wireless transmission factors such as the chosen modulation, power allocation, interference level and maximum power constraints.

Since the seminal paper by Kelly *et al.* in [1], rate control at the flow level has been viewed as solving distributively a global optimization problem that maximizes a system utility, where the fairness of the rate allocation depends on the utility functions [2], [3], [4]. When a mobile user adapts its flow sending rate at the network layer, lower layer mechanisms interact with the network flow mechanism to maximize the fairness of all traffic flows subject to transmission rate constraints at the links, which can be viewed as implicitly solving a system-wide optimization problem in a distributed fashion [2], [3]. In wireless networks, link adaptation depends strongly on the physical layer implementation and interference coupling across multiple links. Widely-studied physical layer transmission rate functions are often nonconvex and coupled in the transmission powers and interference. Achieving end-to-end flow rate fairness in a wireless network is thus a challenging nonlinear problem even for a fixed number of flows and links.

In this paper, we study the weighted max-min end-to-end flow rate fairness problem where link adaptation by power control determines the link capacity. All the links are subject to affine power constraints. This is useful for modeling a large number of practical systems. For instance, a single-hop network with a single total power constraint models a downlink cellular system, and interference temperature constraints (that cap the interference level at each link) can be modeled suitably by multiple affine power constraints. Max-min rate fairness is an egalitarian approach by which the rate of a flow can be increased only if it will not decrease the rate of an already smaller flow [5], [3]. Various methods have been proposed to optimally maximize the fairness related to wireless data rate in [6], [7], [8], [9], [10], but these methods are either not general enough to allow a diverse set of physical layer transmission rate functions and constraints or require centralized computation with a fair amount of parameter configuration and tuning.

Overall, the contributions of the paper are as follows:

1. We cast the solution of the classical max-min rate fairness optimization as computing a Perron-Frobenius eigenvector obtained through an iterative power method algorithm.
2. We solve the weighted max-min flow rate fairness problem subject to affine power constraints for a large class of physical layer link rate functions. We show that this problem can be decoupled into two subproblems

on fairness - one at the network layer and one at the link layer - whose optimal value and solutions can be characterized using nonlinear Perron-Frobenius theory and nonnegative matrix theory. Geometrically fast convergent algorithms with no parameter configuration are proposed to compute the optimal solution.

This paper is organized as follows. We introduce the system model in Section II. In Section III-A, we give a linear Perron-Frobenius theoretic interpretation of the classical max-min rate waterfilling algorithm. In Sections III-B and III-C, we solve the weighted max-min rate and power control problem for a large class of physical layer rate functions using nonlinear Perron-Frobenius theory. In Section IV, numerical examples are given to illustrate the performance of the algorithms. We conclude the paper in Section V.

The following notations are used.

- Boldface uppercase letters denote matrices, boldface lowercase letters denote column vectors, italics denote scalars.
- $(\mathbf{M}\mathbf{x})_l$ denotes the lth element of the vector $\mathbf{M}\mathbf{x}$.
- For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, let $\mathbf{x} \le \mathbf{y}$ denote $x_i \le y_i$ for $1 \le i \le N$, let $\mathbf{x} < \mathbf{y}$ denote $x_i < y_i$ for $1 \le i \le N$, let $\mathbf{x} \lneq \mathbf{y}$ denote $\mathbf{x} \le \mathbf{y}$ and $\mathbf{x} \ne \mathbf{y}$, and let $\mathbf{x} \not< \mathbf{y}$ if there exists some $i \in \{1, \ldots, N\}$ such that $x_i \ge y_i$.
- The super-script $(\cdot)^\top$ denotes transpose.
- The vector $\mathbf{e}_l$ denotes the lth unit coordinate vector.
- Suppose $\mathcal{I} \subset \{1, \ldots, N\}$ and $\mathcal{J} \subset \{1, \ldots, M\}$. We denote by $\mathbf{x}_\mathcal{I} \in \mathbb{R}^{|\mathcal{I}|}$ the subvector formed by the components indexed by $\mathcal{I}$ of a vector $\mathbf{x} \in \mathbb{R}^N$, and by $\mathbf{M}_{\mathcal{I}, \mathcal{J}} \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{J}|}$ the submatrix formed by the components indexed row-wise by $\mathcal{I}$ and column-wise by $\mathcal{J}$ of a matrix $\mathbf{M} \in \mathbb{R}^{N \times M}$.
- Suppose $\mathbf{d} \in \mathbb{R}^M_{>0}$ and $\mathbf{B} \in \mathbb{R}^{M \times N}_{\ge 0}$ such that there is at least one nonzero element in each column of $\mathbf{B}$. We let $\|\cdot\|^{\mathbf{B}}_{\infty, \mathbf{d}}$ denote the norm on $\mathbb{R}^N$ defined by:

$$\|\mathbf{z}\|^{\mathbf{B}}_{\infty, \mathbf{d}} = \max_{m=1, \ldots, M} \frac{(\mathbf{B}|\mathbf{z}|)_m}{d_m}.$$

- The Perron-Frobenius eigenvalue of a nonnegative matrix $\mathbf{F}$ is denoted by $\rho(\mathbf{F})$, and the associated right eigenvector of $\mathbf{F}$ is denoted by $\mathbf{x}(\mathbf{F})$.

II. System Model

We present the model for the network and physical layers of the multiuser interference network. We consider a wireless network with L uni-directional links and S source-destination pairs. Links are indexed by l and sources by s. The network topology is represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{L}$ is the set of uni-directional links. This fluid model of data applies to traffic with adaptive rates such as non real-time file sharing and email applications. A path on $\mathcal{G}$ is selected for each source using source-based routing, and this is captured by a routing matrix $\mathbf{R}$ with entries:

$$R_{ls} = \begin{cases} 1, & \text{if link } l \text{ is in a path of source } s \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the path from source s to its destination node is given by $\mathbf{R}\mathbf{e}_s = [R_{1s}, \ldots, R_{Ls}]^\top$.

In the physical layer, a common spectrum bandwidth is shared by all the nodes in the network. The transmit power of the transmitter of link l is denoted by p_l. The signal-to-interference-plus-noise ratio (SINR) at the receiver of link l is given by

$$\mathsf{SINR}_l(\mathbf{p}) = \frac{G_{ll}p_l}{\sum_{j \ne l} G_{lj}p_j + n_l}$$

where G_{lj} are the channel gains from the transmitter of link j to the receiver of link l and n_l is the additive white Gaussian noise (AWGN) power at the receiver of link l. The channel gain matrix $\mathbf{G}$ takes into account propagation loss, spreading loss and other transmission modulation factors. We also make the following assumption on $\mathbf{G}$.

Assumption 1: There is nonzero interference between all channels, i.e., $G_{lj} > 0$ for all l and j.

We assume that all receivers use singe-user decoding (i.e., treating interference as additive Gaussian noise) and have perfect channel state information. We also assume that the coherence time of the channel is less than the duration of the whole transmission by any user. This assumption is valid for example when fading occurs sufficiently slowly in the channel, i.e., flat-fading, so that the channel can be considered essentially fixed during transmission. We model the achievable data rate of link l, denoted by $c_l(\mathbf{p})$, as a function of the SINR:

$$c_l(\mathbf{p}) = C(\mathsf{SINR}_l(\mathbf{p})).$$

In this paper, we consider the case where C satisfies the following conditions:

C1. $\mathbf{dom}\, C = \mathbb{R}_{\ge 0}$.
C2. C is a continuously differentiable and strictly increasing function starting from 0 and $C'(0) < \infty$.
C3. $x/C(x)$ is a non-decreasing function.

Condition C1 guarantees that $C(\mathsf{SINR}_l(\mathbf{p}))$ is well-defined for all possible values of the SINR. It is reasonable to expect the achievable data rate of a link to depend on the SINR according to condition C2. Condition C3 can be interpreted as a bound on the rate of increase of the link rate as a function of the SINR since it can be rewritten as $C'(x) \le C(x)/x$. This is characteristic of the diminishing marginal effect of the SINR on the achievable link rate (Chap. 5, [11]).

To demonstrate the relevance of conditions C1-C3, we consider one commonly-used model for C [12]:

$$C(\mathsf{SINR}_l(\mathbf{p})) = \log(1 + \mathsf{SINR}_l(\mathbf{p})) \tag{1}$$

which is the Shannon capacity function. It is easily seen that (1) satisfies conditions C1-C2 and one can verify that it also satisfies condition C3. Another commonly-used model for the rate function is the linear model in CDMA networks (which is also the linear approximation of (1) for small values of the SINR) [13], [14]:

$$C(\mathsf{SINR}_l(\mathbf{p})) = \mathsf{SINR}_l(\mathbf{p}). \tag{2}$$

It is straightforward to see that (2) satisfies conditions C1-C3. Note that (1) and (2) are by no means exhaustive of the possible functions for link rates introduced here.

For notational brevity, we define a $L \times L$ nonnegative matrix $\mathbf{F}$ with entries:

$$F_{lj} = \begin{cases} 0, & \text{if } l = j \\ \frac{G_{lj}}{G_{ll}}, & \text{if } l \neq j \end{cases}$$

and a $L \times 1$ vector

$$\mathbf{v} = \left(\frac{n_1}{G_{11}}, \frac{n_2}{G_{22}}, \ldots, \frac{n_L}{G_{LL}} \right)^\top.$$

Hence, $\mathsf{SINR}_l(\mathbf{p})$ can be written as $p_l / ((\mathbf{Fp})_l + v_l)$.

In practice, there is typically a maximum transmit power allowed at each transmitter. We assume that the feasible set of powers can be represented by a linear constraint:

$$\mathbf{Dp} \leq \bar{\mathbf{p}} \tag{3}$$

where $\bar{\mathbf{p}}$ is a $M \times 1$ positive vector of constraint values and $\mathbf{D}$ is a $M \times L$ nonnegative weight matrix. We make the following assumption on the links.

Assumption 2: Every link is involved in at least one power constraint, i.e., each column of $\mathbf{D}$ has at least one nonzero entry.

This is a reasonable assumption for typical physical setups. Moreover, even if the transmit power of a link were physically unconstrained, we could simply augment $\mathbf{D}$ and $\bar{\mathbf{p}}$ with an arbitrarily large individual power constraint on that link. Two possible scenarios included under (3) are that of individual power constraints on each link in an uplink network (with $M = L$ and $\mathbf{D} = \mathbf{I}$) or that of a single total power constraint on all links in a downlink network (with $M = 1$ and $\mathbf{D} = \mathbf{1}^\top$). Linear power constraints also appear in other kinds of ad-hoc settings, e.g., interference temperature constraints in wireless cognitive radio settings.

III. JOINT RATE AND POWER CONTROL FOR WEIGHTED MAX-MIN FAIRNESS

Since the different sources could have different service requirements, the network could assign different weights to the sources for rate allocation. Let r_s denote the sending rate of source s and $\nu_s > 0$ denote the weight assigned to source s. The flow rate allocation and power control problem is to jointly optimize the sending rates of all sources and the transmit power at all transmitters to achieve weighted max-min rate fairness for all sources in the network:

$$\begin{aligned} \underset{\mathbf{r}, \mathbf{p}}{\text{maximize}} \quad & \min_s \frac{r_s}{\nu_s} \\ \text{subject to} \quad & \mathbf{Rr} \leq \mathbf{c}(\mathbf{p}) \quad \text{(link constraint)} \\ & \mathbf{Dp} \leq \bar{\mathbf{p}}, \ \mathbf{p} \geq \mathbf{0}. \quad \text{(power constraint)} \end{aligned} \tag{4}$$

While (4) can be reformulated as a convex problem for certain choices of C (e.g. where C is given by (1) or (2)), the case where C is a general function satisfying conditions C1-C3 is, to our knowledge, an open problem. We will derive an iterative algorithm that converges to the optimal solution of (4) and characterize its optimal solution using nonlinear Perron-Frobenius theory.

A. Flow Rate Allocation

We first consider the subproblem with fixed link transmit powers:

$$\begin{aligned} \underset{\mathbf{r}}{\text{maximize}} \quad & \min_s \frac{r_s}{\nu_s} \\ \text{subject to} \quad & \mathbf{Rr} \leq \mathbf{c}. \end{aligned} \tag{5}$$

Note that (5) could have multiple optimal solutions. Here, our objective is to characterize one particular solution in relation to Perron-Frobenius theory. The purpose of analyzing this simple problem is to motivate as well as provide the intuition for the subsequent application of nonlinear Perron-Frobenius theory to the joint optimization of flow rates and power.

Observe that one solution to (5) can be obtained by fixing $\mathbf{r}_* = \tau_* \boldsymbol{\nu}$ and finding the maximal τ_* such that $\mathbf{Rr}_* \leq \mathbf{c}$ is satisfied. Clearly, this solution is given by

$$\tau_* = \min_{l=1,\ldots,L} \frac{c_l}{(\mathbf{R}\boldsymbol{\nu})_l}.$$

The l' which achieves this minimum corresponds to the link with a tight link rate constraint, i.e., $(\mathbf{Rr}_*)_{l'} = c_{l'}$. By substituting this observation into $\mathbf{r}_* = \tau_* \boldsymbol{\nu}$, we can write:

$$\frac{1}{\tau_*} \mathbf{r}_* = \left(\frac{1}{c_{l'}} \boldsymbol{\nu} \mathbf{e}_{l'}^\top \mathbf{R} \right) \mathbf{r}_*.$$

Note that $(1/c_{l'}) \boldsymbol{\nu} \mathbf{e}_{l'}^\top \mathbf{R}$ is a rank-one nonnegative matrix. It is straightforward to compute its spectral radius which is given by its single non-zero eigenvalue $(1/c_{l'})(\mathbf{R}\boldsymbol{\nu})_{l'}$. The associated right eigenvector is given by $\boldsymbol{\nu}$. Hence, τ_* is given by the inverse of the spectral radius and $\mathbf{r}_*$ is given by the right eigenvector normalized such that $(\mathbf{Rr}_*)_{l'} = c_{l'}$. It follows that the computation of τ_* can be rewritten as a spectral radius optimization problem over the set of L rank-one matrices:

$$\tau_* = \min_{l=1,\ldots,L} \frac{1}{\rho \left((1/c_l) \boldsymbol{\nu} \mathbf{e}_l^\top \mathbf{R} \right)}.$$

Now, since $\mathbf{r}_*$ is given by the right eigenvector of a matrix, one might ask whether the iterative power method for computing the dominant eigenvector could be used to compute $\mathbf{r}_*$ (although this method is unlikely to be useful since we can expect to know $\boldsymbol{\nu}$). But since $(1/c_{l'}) \boldsymbol{\nu} \mathbf{e}_{l'}^\top \mathbf{R}$ is a rank-one matrix, for any vector $\mathbf{x}$, we have that $(1/c_{l'}) \boldsymbol{\nu} \mathbf{e}_{l'}^\top \mathbf{Rx} = (\mathbf{e}_{l'}^\top \mathbf{Rx}/c_{l'}) \boldsymbol{\nu}$. Hence, in this case, starting from any $\mathbf{x} > \mathbf{0}$, the power method converges in one step to the right eigenvector $\boldsymbol{\nu}$. Now all of the above conclusions can be drawn by first casting (5) as a conditional eigenvalue problem, that of finding $(\tau_*, \mathbf{r}_*)$ such that

$$\frac{1}{\tau_*} \mathbf{r}_* = \boldsymbol{\nu}, \ \ \tau_* \in \mathbb{R}, \ \ \mathbf{r}_* > \mathbf{0}, \ \ \mathbf{Rr}_* \leq \mathbf{c}, \ \ \mathbf{Rr}_* \not< \mathbf{c}$$

and then applying the results in Appendix A. Because our analysis of (5) is not the main result of this paper, we will not describe in further detail the solving of the conditional eigenvalue problem, except mention that we will later use a similar technique to solve the joint flow rate and power control problem.

For completeness, we will extend the Perron-Frobenius characterization of (5) to the classical weighted max-min

fairness problem. Recall that there could be multiple solutions to (5). The discussions above pertain to only one of the (possibly) multiple solutions. It is easy to see that there exists a solution to (5) which satisfies the following definition.

Definition 1 (Weighted Max-Min Fairness [15], [3]): A flow rate allocation $\mathbf{r}$ that satisfies the link rate constraints $\mathbf{Rr} \leq \mathbf{c}$ is said to be weighted max-min fair when for each source s, any increase in r_s would cause a decrease in the rate $r_{j'}$, for some source j' satisfying $r_{j'}/\nu_{j'} \leq r_s/\nu_s$.

Now the weighted max-min fairness problem has been widely studied in the networking literature [5], [3], and there are well-established algorithms to compute the solution. Based on the discussions in this section, it is easy to understand the following Perron-Frobenius characterization of the waterfilling algorithm [5], [15].

Algorithm 1 (Waterfilling):
Initialization: $\mathcal{L}(0) = \mathcal{L}$, $\mathbf{c}(0) = \mathbf{c}$, $\boldsymbol{\nu}(0) = \boldsymbol{\nu}$, $\mathbf{R}(0) = \mathbf{R}$, and $k = 0$.

1) Find the set of most constrained links:
$$\mathcal{L}'(k) = \arg \max_{l \in \mathcal{L}(k)} \rho((1/c_l(k))\boldsymbol{\nu}(k)\mathbf{e}_l^\top \mathbf{R}(k)).$$

2) Choose any $l \in \mathcal{L}'(k)$. Set bandwidth $\mathbf{r}(k)$ to $\boldsymbol{\nu}(k)$, normalized such that $(\mathbf{R}(k)\mathbf{r}(k))_l = c_l(k)$:
$$\mathbf{r}(k) = \left(\frac{c_l(k)}{(\mathbf{R}(k)\boldsymbol{\nu}(k))_l} \right) \boldsymbol{\nu}(k).$$

3) For every source s which uses a link indexed in $\mathcal{L}'(k)$, set its sending rate to $r_s(k)$. Then form $\boldsymbol{\nu}(k+1)$ by removing from $\boldsymbol{\nu}(k)$ all these flows. Renumber the indexes of the remaining flows in consecutive order.

4) Form $\mathbf{c}(k+1)$ by removing from $\mathbf{c}(k)$ all the links indexed in $\mathcal{L}'(k)$.

5) Form $\mathbf{R}(k+1)$ by removing from $\mathbf{R}(k)$ all the links and flows that have been removed.

6) Set $\mathcal{L}(k+1) = \mathcal{L}(k) \backslash \mathcal{L}'(k)$. If the dimension of $\boldsymbol{\nu}(k+1)$ is greater than 0, update $k \leftarrow k+1$ and repeat from step 1.

B. Joint Flow Rate and Power Control

Next, we proceed to solve (4). Observe that one complication arises from the coupling between the flow rates and the link powers through the rate constraints on the links. It turns out that, due to the structure of the optimal rates, this coupling between the two groups of variables can be broken. By doing so, (4) can be converted into an optimization problem over the link powers only, and the optimal flow rates can be easily computed from the optimal link powers.

Begin by introducing an auxiliary variable τ and rewriting (4) as:
$$\begin{aligned} \underset{\mathbf{r},\, \mathbf{p},\, \tau}{\text{maximize}} \quad & \tau \\ \text{subject to} \quad & \tau\boldsymbol{\nu} \leq \mathbf{r} \\ & \mathbf{Rr} \leq \mathbf{c}(\mathbf{p}) \\ & \mathbf{Dp} \leq \bar{\mathbf{p}},\ \mathbf{p} \geq \mathbf{0}. \end{aligned} \quad (6)$$

From the constraints $\tau\boldsymbol{\nu} \leq \mathbf{r}$ and $\mathbf{Rr} \leq \mathbf{c}(\mathbf{p})$, we infer that $\tau\mathbf{R}\boldsymbol{\nu} \leq \mathbf{c}(\mathbf{p})$. Introduce the latter as a redundant constraint and divide (6) into an outer optimization over $\mathbf{p}$ and τ and an inner optimization over $\mathbf{r}$ as follows:
$$\max_{\substack{\mathbf{p},\, \tau: \\ \tau\mathbf{R}\boldsymbol{\nu} \leq \mathbf{c}(\mathbf{p}) \\ \mathbf{Dp} \leq \bar{\mathbf{p}} \\ \mathbf{p} \geq \mathbf{0}}} \quad \max_{\substack{\mathbf{r}: \\ \tau\boldsymbol{\nu} \leq \mathbf{r} \\ \mathbf{Rr} \leq \mathbf{c}(\mathbf{p})}} \tau. \quad (7)$$

Now, given any feasible $\mathbf{p}$ and τ for the outer problem, the optimal value of the inner problem is τ as long as it is feasible. But the flow rate $\mathbf{r} = \tau\boldsymbol{\nu}$ is always feasible for the inner problem. Hence, (7) can be rewritten as:
$$\begin{aligned} \underset{\mathbf{p},\, \tau}{\text{maximize}} \quad & \tau \\ \text{subject to} \quad & \tau(\mathbf{R}\boldsymbol{\nu})_l \leq C(\mathsf{SINR}_l(\mathbf{p}))\ \forall\, l \\ & \mathbf{Dp} \leq \bar{\mathbf{p}},\ \mathbf{p} \geq \mathbf{0}. \end{aligned} \quad (8)$$

Let $\mathbf{p}_*$ and $\mathbf{r}_*$ be a pair of optimal powers and flow rates of (4) and let τ_* be the optimal value. Notice that the flow rates do not appear in (8). Hence, we can solve (8) for $\mathbf{p}_*$ and τ_* independent of the flow rates. Thereafter, we can compute a vector of optimal flow rates by choosing $\mathbf{r}_* = \tau_*\boldsymbol{\nu}$. Note that this decoupling is independent of the functional form of the link rate function C.

Next, we present a sequence of lemmas to simplify (8). Define
$$\mathcal{K} = \{\, l \mid (\mathbf{R}\boldsymbol{\nu})_l > 0 \,\}.$$
It is easy to obtain an optimal value of p_l for all $l \notin \mathcal{K}$.

Lemma 1: Suppose C satisfies conditions C1 and C2. Then there exists a $\mathbf{p}_*$ such that $p_{*l} = 0$ for all $l \notin \mathcal{K}$.

Proof of Lemma 1: Let $(\tau_*, \mathbf{p}_*)$ be an optimal solution to (8) and define
$$p'_{*l} = \begin{cases} p_{*l}, & \text{if } l \in \mathcal{K} \\ 0, & \text{if } l \notin \mathcal{K}. \end{cases}$$

Clearly, $\mathbf{Dp}'_* \leq \mathbf{Dp}_* \leq \bar{\mathbf{p}}$ so $\mathbf{p}'_*$ is a feasible power allocation. Moreover, we have that $\mathsf{SINR}_l(\mathbf{p}'_*) \geq \mathsf{SINR}_l(\mathbf{p}_*)$ for all $l \in \mathcal{K}$, and since C is a strictly increasing function of the SINR (by condition C2), it follows that $C(\mathsf{SINR}_l(\mathbf{p}'_*)) \geq C(\mathsf{SINR}_l(\mathbf{p}_*))$ for all $l \in \mathcal{K}$. Now define $\mathcal{T}(\mathbf{p}) = \{\, \tau \mid \tau(\mathbf{R}\boldsymbol{\nu})_l \leq C(\mathsf{SINR}_l(\mathbf{p}))\ \forall\, l \in \mathcal{L} \,\}$. It is straightforward to see that $\mathcal{T}(\mathbf{p}'_*) \supset \mathcal{T}(\mathbf{p}_*)$, so we have that $\tau'_* = \max\{\, \tau \mid \tau \in \mathcal{T}(\mathbf{p}'_*) \,\} \geq \max\{\, \tau \mid \tau \in \mathcal{T}(\mathbf{p}_*) \,\} = \tau_*$. But $(\tau_*, \mathbf{p}_*)$ is an optimal solution so $(\tau'_*, \mathbf{p}'_*)$ is also an optimal solution. ∎

From now on, we assume that $p_l = 0$ for all $l \notin \mathcal{K}$ and consider only the subvector $\mathbf{p}_\mathcal{K}$. Whenever we refer to $\mathbf{p}$, it is understood that we have substituted $p_l = 0$ for all $l \notin \mathcal{K}$. Define
$$\mathcal{M} = \{\, m \mid \exists\, l \in \mathcal{K} \text{ such that } D_{ml} > 0 \}.$$

Hence, $\mathcal{M}$ denotes the set of indexes of power constraints associated with the links in $\mathcal{K}$. We can solve for $\mathbf{p}_{*\mathcal{K}}$ by considering the following optimization problem over link rates:
$$\begin{aligned} \underset{\mathbf{p}_\mathcal{K}}{\text{maximize}} \quad & \min_{l \in \mathcal{K}} \frac{C(\mathsf{SINR}_l(\mathbf{p}_\mathcal{K}))}{(\mathbf{R}\boldsymbol{\nu})_l} \\ \text{subject to} \quad & \mathbf{D}_{\mathcal{M},\mathcal{K}}\mathbf{p}_\mathcal{K} \leq \bar{\mathbf{p}}_\mathcal{M},\ \mathbf{p}_\mathcal{K} \geq \mathbf{0}. \end{aligned} \quad (9)$$

For notational brevity, we have omitted the dependence of the SINR on p_l for all $l \notin \mathcal{K}$. The following two lemmas characterize the structure of an optimal solution to (9).

Lemma 2: Suppose C satisfies conditions C1 and C2. Then $\mathbf{D}_{\mathcal{M},\mathcal{K}}\mathbf{p}_{*\mathcal{K}} \not< \bar{\mathbf{p}}_{\mathcal{M}}$.

Lemma 3: Suppose C satisfies conditions C1 and C2. Then $C(\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}}))/(\mathbf{R}\boldsymbol{\nu})_l$ for all $l \in \mathcal{K}$ are equal.

Proof of Lemma 2: Let $\mathbf{p}_{*\mathcal{K}}$ be an optimal solution to (9). Suppose $\mathbf{D}_{\mathcal{M},\mathcal{K}}\mathbf{p}_{*\mathcal{K}} < \bar{\mathbf{p}}_{\mathcal{M}}$. Now let $\mathbf{p}'_{*\mathcal{K}} = (\min\{ \bar{p}_m/(\mathbf{Dp}_*)_m \mid m \in \mathcal{M} \})\mathbf{p}_{*\mathcal{K}}$. Clearly, $\mathbf{p}'_{*\mathcal{K}}$ is a feasible power allocation and $\mathbf{p}'_{*\mathcal{K}} > \mathbf{p}_{*\mathcal{K}}$. Moreover, since $\mathsf{SINR}_l(\lambda\mathbf{p}_{*\mathcal{K}})$ is strictly increasing in λ, we have that $\mathsf{SINR}_l(\mathbf{p}'_{*\mathcal{K}}) > \mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}})$ for all l. Since C is a strictly increasing function of the SINR (by condition C2), it follows that $C(\mathsf{SINR}_l(\mathbf{p}'_{*\mathcal{K}})) > C(\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}}))$ for all l. This contradicts the optimality of $\mathbf{p}_*$. ∎

Proof of Lemma 3: Suppose not and let $l' \in \mathcal{K}$ be the link with the largest weighted link rate. Recall that $G_{lj} > 0$ for all l and j. Hence, $\mathsf{SINR}_{l'}(\mathbf{p}_{*\mathcal{K}})$ is strictly increasing in $p_{*l'}$, and for all $l \neq l'$, we have that $\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}})$ is strictly decreasing in $p_{*l'}$. Since C is a strictly increasing and continuous function of the SINR, it is possible to decrease $p_{*l'}$ such that $\min_{l \in \mathcal{K}} C(\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}}))/(\mathbf{R}\boldsymbol{\nu})_l$ increases.[1] ∎

We now present the following algorithm for solving (4).

Algorithm 2 (Weighted Max-Min Fair Rate Control):
Initialization:

- Identify the busy links: $\mathcal{K} = \{ l \mid (\mathbf{R}\boldsymbol{\nu})_l > 0 \}$.
- Identify the power constraints associated with the busy links: $\mathcal{M} = \{ m \mid \exists l \in \mathcal{K} \text{ such that } D_{ml} > 0 \}$.
- Turn off the idle links: $p_l(0) = 0 \ \forall l \notin \mathcal{K}$.
- Initialize the powers of the busy links: $p_l(0) > 0 \ \forall l \in \mathcal{K}$.
- Set $k = 0$.

1) Update power of busy links:
$$p_l(k + 1) = \left(\frac{(\mathbf{R}\boldsymbol{\nu})_l}{C(\mathsf{SINR}_l(\mathbf{p}(k)))} \right) p_l(k) \ \forall l \in \mathcal{K}.$$

2) Normalize power of busy links:
$$p_l(k+1) \leftarrow \left(\min_{m \in \mathcal{M}} \frac{\bar{p}_m}{(\mathbf{Dp}(k+1))_m} \right) p_l(k+1) \ \forall l \in \mathcal{K}.$$

3) Allocate flow rates:
$$\mathbf{r}(k + 1) = \left(\min_{l \in \mathcal{K}} \frac{C(\mathsf{SINR}_l(\mathbf{p}(k + 1)))}{(\mathbf{R}\boldsymbol{\nu})_l} \right) \boldsymbol{\nu}.$$

4) Update $k \leftarrow k + 1$ and repeat from step 1.

Theorem 1: Suppose C satisfies conditions C1-C3. Furthermore, suppose that $x/C(x)$ is concave.[2] Then $\mathbf{p}(k)$ and $\mathbf{r}(k)$ in Algorithm 2 converge to an optimal solution of (4) from any $\mathbf{p}_{\mathcal{K}}(0) > \mathbf{0}$. Moreover, $\mathbf{p}_{\mathcal{K}}(k)$ converges geometrically fast.

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[1]Even if we do not have $G_{lj} > 0$ for all l and j, it is still possible to prove Lemma 3 if $\mathbf{F}_{\mathcal{K},\mathcal{K}}$ is irreducible.

[2]It is easy to check that the link rate functions given in (1) and (2) satisfy the condition that $x/C(x)$ is concave.

Proof of Theorem 1: From Lemma 3, we have that $(\tau_*, \mathbf{p}_*)$ must necessarily satisfy the following condition:
$$\tau_* = \frac{C(\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}}))}{(\mathbf{R}\boldsymbol{\nu})_l} \ \forall l \in \mathcal{K}.$$

By combining this with Lemma 2, we can cast (9) into the following eigenvalue problem: find $(\tau_*, \mathbf{p}_{*\mathcal{K}})$ such that
$$\frac{1}{\tau_*}\mathbf{p}_{*\mathcal{K}} = \mathbf{f}_{\mathcal{K}}(\mathbf{p}_{*\mathcal{K}}), \qquad \tau_* \in \mathbb{R}, \qquad \mathbf{p}_{*\mathcal{K}} > \mathbf{0},$$
$$\mathbf{D}_{\mathcal{M},\mathcal{K}}\mathbf{p}_{*\mathcal{K}} \leq \bar{\mathbf{p}}_{\mathcal{M}}, \qquad \mathbf{D}_{\mathcal{M},\mathcal{K}}\mathbf{p}_{*\mathcal{K}} \not< \bar{\mathbf{p}}_{\mathcal{M}} \qquad (10)$$

and τ_* is maximal. Here $\mathbf{f} : \mathbb{R}_{\geqslant 0}^{|\mathcal{K}|} \to \mathbb{R}_{\geqslant 0}^L$ is defined by:
$$f_l(\mathbf{p}_{\mathcal{K}}) = \begin{cases} \left(\frac{(\mathbf{R}\boldsymbol{\nu})_l}{C(\mathsf{SINR}_l(\mathbf{p}_{\mathcal{K}}))} \right) p_l, & \text{if } l \in \mathcal{K} \\ 0, & \text{if } l \notin \mathcal{K}. \end{cases}$$

Note that (10) is necessary but not sufficient for optimality unless the resulting τ_* is maximal. However, we will show that there is a unique solution to (10) so this necessary condition is also sufficient.

We refer the reader to Lemma 5 in the appendix. We will show that for all $l \in \mathcal{K}$, we have that $f_l(\mathbf{p}_{\mathcal{K}})$ is concave and that $f_l(\mathbf{p}_{\mathcal{K}}) > 0$ for all $\mathbf{p}_{\mathcal{K}} \gneqq \mathbf{0}$, which implies by Lemma 5 that (10) has a unique solution which can be found using a fixed-point iteration. First rewrite $f_l(\mathbf{p}_{\mathcal{K}})$ as:
$$f_l(\mathbf{p}_{\mathcal{K}}) = (\mathbf{e}_l)_{\mathcal{K}}^{\top}(\mathbf{F}_{\mathcal{K},\mathcal{K}}\mathbf{p}_{\mathcal{K}} + \mathbf{v}_{\mathcal{K}})\frac{(\mathbf{R}\boldsymbol{\nu})_l\mathsf{SINR}_l(\mathbf{p}_{\mathcal{K}})}{C(\mathsf{SINR}_l(\mathbf{p}_{\mathcal{K}}))}.$$

To show that f_l is concave, we need the following result from [16]:

Lemma 4: Suppose $h : \mathbb{R}^m \to \mathbb{R}$ is concave, and $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$, and $d \in \mathbb{R}$. Define
$$g(\mathbf{x}) = (\mathbf{c}^{\top}\mathbf{x} + d)h\left((1/(\mathbf{c}^{\top}\mathbf{x} + d))(\mathbf{A}\mathbf{x} + \mathbf{b}) \right)$$

with $\mathbf{dom}\, g = \{ \mathbf{x} \mid \mathbf{c}^{\top}\mathbf{x} + d > 0, \ (1/(\mathbf{c}^{\top}\mathbf{x} + d))(\mathbf{A}\mathbf{x} + \mathbf{b}) \in \mathbf{dom}\, h \}$. Then $g(\mathbf{x})$ is concave.

Lemma 4 follows from the fact that the perspective and the affine composition of a function preserves the concavity of the function. Let $h(x) = x/C(x)$, and $\mathbf{A} = (\mathbf{e}_l)_{\mathcal{K}}^{\top}$, $\mathbf{b} = \mathbf{0}$, $\mathbf{c} = (\mathbf{e}_l)_{\mathcal{K}}^{\top}\mathbf{F}_{\mathcal{K},\mathcal{K}}$, and $d = v_l$ in Lemma 4. Then $f_l(\mathbf{p}_{\mathcal{K}}) = (\mathbf{R}\boldsymbol{\nu})_l g(\mathbf{p}_{\mathcal{K}})$. Since $x/C(x)$ is concave and $(\mathbf{R}\boldsymbol{\nu})_l > 0$, it follows that f_l is concave.

Next, we show that $f_l(\mathbf{p}_{\mathcal{K}}) > 0$ for all $\mathbf{p}_{\mathcal{K}} \gneqq \mathbf{0}$. First note that $x/C(x) > 0$ for all $x > 0$ since $C(x) > 0$ (by condition C2), which implies that $\mathsf{SINR}_l(\mathbf{p}_{\mathcal{K}})/C(\mathsf{SINR}_l(\mathbf{p}_{\mathcal{K}})) > 0$ for all $\mathbf{p}_{\mathcal{K}} \gneqq \mathbf{0}$ such that $\mathsf{SINR}_l(\mathbf{p}_{\mathcal{K}}) > 0$. Now suppose, $\mathbf{p}_{\mathcal{K}} \gneqq \mathbf{0}$ is such that $\mathsf{SINR}_l(\mathbf{p}_{\mathcal{K}}) = 0$. By L'Hopital's rule, we have that:
$$f_l(\mathbf{p}_{\mathcal{K}}) = (\mathbf{e}_l)_{\mathcal{K}}^{\top}(\mathbf{F}_{\mathcal{K},\mathcal{K}}\mathbf{p}_{\mathcal{K}} + \mathbf{v}_{\mathcal{K}}) \lim_{\mathbf{p}'_{\mathcal{K}} \to \mathbf{p}_{\mathcal{K}}} \frac{(\mathbf{R}\boldsymbol{\nu})_l}{C'(\mathsf{SINR}_l(\mathbf{p}'_{\mathcal{K}}))}$$

which is strictly positive since C' is a continuous function and $C'(\mathsf{SINR}_l(\mathbf{p}_{\mathcal{K}})) < \infty$ (by condition C2).

Hence, we can use the normalized fixed-point iteration in Lemma 5 to compute $\mathbf{p}_{*\mathcal{K}}$:

$$\mathbf{p}_{\mathcal{K}}(k+1) = \frac{\mathbf{f}_{\mathcal{K}}(\mathbf{p}_{\mathcal{K}}(k))}{\|\mathbf{f}_{\mathcal{K}}(\mathbf{p}_{\mathcal{K}}(k))\|_{\infty,\bar{\mathbf{p}}_{\mathcal{M}}}^{\mathbf{D}_{\mathcal{M},\mathcal{K}}}}$$

which converges geometrically fast to $\mathbf{p}_{*\mathcal{K}}$ for any $\mathbf{p}_{\mathcal{K}}(0) > \mathbf{0}$. This iteration can be rewritten as steps 1 and 2 of Algorithm 2. Note that step 2 always gives a feasible power allocation. From the discussions following (8), it is easy to see that the flow rate allocation scheme in step 3 is also always feasible for the power allocation computed from step 2. To see that $\mathbf{r}(k)$ converges to an optimal rate vector, note that

$$\lim_{k\to\infty} \mathbf{r}(k) = \left(\min_{l\in\mathcal{K}} \frac{C\left(\lim_{k\to\infty} \mathsf{SINR}_l(\mathbf{p}(k))\right)}{(\mathbf{R}\boldsymbol{\nu})_l} \right) \boldsymbol{\nu}$$

$$= \left(\min_{l\in\mathcal{K}} \frac{C\left(\mathsf{SINR}_l\left(\lim_{k\to\infty} \mathbf{p}(k)\right)\right)}{(\mathbf{R}\boldsymbol{\nu})_l} \right) \boldsymbol{\nu} = \mathbf{r}_*$$

since C is a continuous function and the SINR's are continuous functions of the powers. This proves Theorem 1. ∎

Remark 1: Although we do not show this here, it is possible to prove, under the conditions in Theorem 1, that $\mathbf{r}(k)$ in Algorithm 2 increases monotonically.

Next, we present the following insight on the structure of the optimal solution.

Theorem 2: If C satisfies conditions C1 and C2, then the optimal flow rate allocation $\mathbf{r}_*$ is unique and is a weighted max-min fair flow rate allocation.

Proof of Theorem 2: First, we show that the optimal flow rate allocation is a weighted max-min fair allocation. Lemma 3 implies that

$$\mathbf{r}_* = \frac{C(\mathsf{SINR}_l(\mathbf{p}_*))}{(\mathbf{R}\boldsymbol{\nu})_l}\boldsymbol{\nu} \quad \forall\, l \in \mathcal{K}. \tag{11}$$

It follows that

$$(\mathbf{R}\mathbf{r}_*)_l = \begin{cases} C(\mathsf{SINR}_l(\mathbf{p}_*)), & \text{if } l \in \mathcal{K} \\ 0, & \text{if } l \notin \mathcal{K} \end{cases}$$

and hence $(\mathbf{R}\mathbf{r}_*)_l = c_l(\mathbf{p}_*)$ for all $l \in \mathcal{L}$. Hence, the link rate constraints are all tight so any increase in the sending rate of some source can only be achieved by a decrease in the sending rate of some other source.

Next, we show that $\mathbf{r}_*$ is the unique optimal flow rate allocation. Substituting τ_* and $\mathbf{p}_*$ into (7), the flow rate optimization problem consists of finding a flow rate allocation $\mathbf{r}$ that satisfies $\tau_*\boldsymbol{\nu} \leq \mathbf{r}$ and $\mathbf{R}\mathbf{r} \leq \mathbf{c}(\mathbf{p}_*)$. From the above discussions, it is clear that $\mathbf{r}_*$ is the only such feasible flow rate allocation. ∎

We now interpret (9) to provide the intuition behind the decoupling of the powers and the flow sending rates. Consider the case where all sources have equal priority, that is, $\boldsymbol{\nu} = \mathbf{1}$. Then, the term $(\mathbf{R}\boldsymbol{\nu})_l$ denotes the number of sources using

link l and the links for which $(\mathbf{R}\boldsymbol{\nu})_l = 0$ can be interpreted as idle. Hence, the power control problem (9) can be interpreted as maximizing the per-source rate at each busy link. But in the weighted max-min flow rate problem (5), there will always be one optimal solution which is a scalar multiple of $\boldsymbol{\nu}$, which in this case, corresponds to all the sources having equal sending rates. Hence, to solve the joint flow rate and power control problem, it suffices to solve two separate problems on fairness - one at the network layer and one at the link layer - where the latter problem assumes the case where all sources in the network layer are sending at a common rate.

Next, we further characterize the optimal solution of (9). The following theorem follows from Lemma 6.

Theorem 3: Let $\mathbf{S}_m = (1/\bar{p}_m)\mathbf{f}_{\mathcal{K}}(\mathbf{p}_{*\mathcal{K}})(\mathbf{e}_m)_{\mathcal{M}}^{\top}\mathbf{D}_{\mathcal{M},\mathcal{K}}$ where $\mathbf{f}: \mathbb{R}_{\geq 0}^{|\mathcal{K}|} \to \mathbb{R}_{\geq 0}^{L}$ is defined by

$$f_l(\mathbf{p}_{\mathcal{K}}) = \begin{cases} \left(\frac{(\mathbf{R}\boldsymbol{\nu})_l}{C(\mathsf{SINR}_l(\mathbf{p}_{\mathcal{K}}))}\right) p_l, & \text{if } l \in \mathcal{K} \\ 0, & \text{if } l \notin \mathcal{K}. \end{cases}$$

The optimal value and unique solution of (9) are given, respectively, by $1/\rho(\mathbf{S}_i)$ and $\mathbf{x}(\mathbf{S}_i)$, where $i = \arg\max_{m\in\mathcal{M}} \rho(\mathbf{S}_m)$.

It turns out that when the achievable data rate of a link is modeled as a linear function of the SINR, the optimal value and solution given in Theorem 3 simplify into closed-form solutions.

Theorem 4: Suppose $C(\mathsf{SINR}_l(\mathbf{p})) = \mathsf{SINR}_l(\mathbf{p})$. Let $\mathbf{S}_m = \mathrm{diag}((\mathbf{R}\boldsymbol{\nu})_{\mathcal{K}})(\mathbf{F}_{\mathcal{K},\mathcal{K}} + (1/\bar{p}_m)\mathbf{v}_{\mathcal{K}}(\mathbf{e}_m)_{\mathcal{K}}^{\top}\mathbf{D}_{\mathcal{M},\mathcal{K}})$. Then the optimal value and unique solution of (9) are given, respectively, by $1/\rho(\mathbf{S}_i)$ and $\mathbf{x}(\mathbf{S}_i)$, where $i = \arg\max_{m\in\mathcal{M}} \rho(\mathbf{S}_m)$.

Remark 2: The index i in Theorems 3 and 4 correspond to the power constraint in (9) which is tight at optimality.

We summarize in Table I the Perron-Frobenius characterizations of the optimal solution to the subproblems of the weighted max-min fairness optimization.

C. Extension to Piecewise Linear Link Rate Functions

Next, we consider a larger class of link rate functions and prove the optimality of Algorithm 2 for these functions. The following theorem is the main result of this section.

Theorem 5: Suppose C satisfies conditions C1-C3. Furthermore suppose that there are at least three active links, i.e., $|\mathcal{K}| \geq 3$, and suppose that C is continuously differentiable on an open neighborhood of $\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}})$ for every $l \in \mathcal{K}$. Then $\mathbf{p}(k)$ and $\mathbf{r}(k)$ in Algorithm 2 converge to an optimal solution of (4) from any $\mathbf{p}_{\mathcal{K}}(0) > \mathbf{0}$. Moreover, $\mathbf{p}_{\mathcal{K}}(k)$ converges geometrically fast.

Remark 3: Note that the link rate functions in (1) and (2) satisfy the conditions in Theorem 5. However, Theorem 5 does not subsume Theorem 1. For instance, if C is a piecewise function such that $x/C(x)$ is concave[3] but C is not continuously differentiable on an open neighborhood of $\mathsf{SINR}_l(\mathbf{p}_*)$

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[3]Consider the following function:

$$C(x) = \begin{cases} \log_2(1+x), & \text{if } 0 \leq x \leq 1 \\ \frac{5x}{x+4}, & \text{if } 1 \leq x. \end{cases}$$

This function is not continuously differentiable everywhere but $x/C(x)$ is concave.

	$\mathbf{z}$	$\mathbf{A}$	$\mathbf{b}(\mathbf{z})$	$\mathbf{B}$	$\mathbf{d}$
Max-Min Weighted Flow Rate	$\mathbf{r}$	$\mathbf{0}$	$\boldsymbol{\nu}$	$\mathbf{R}$	$\mathbf{c}$
Max-Min Weighted Link Rate	$\mathbf{p}_\mathcal{K}$	$\mathbf{0}$	$\left(\left(\frac{(\mathbf{R}\boldsymbol{\nu})_l}{C(\mathsf{SINR}_l(\mathbf{p}_\mathcal{K}))}\right) p_l : l \in \mathcal{K}\right)$	$\mathbf{D}_{\mathcal{M},\mathcal{K}}$	$\bar{\mathbf{p}}_\mathcal{M}$
Max-Min Weighted SINR	$\mathbf{p}_\mathcal{K}$	$\mathrm{diag}((\mathbf{R}\boldsymbol{\nu})_\mathcal{K})\mathbf{F}_{\mathcal{K},\mathcal{K}}$	$\mathrm{diag}((\mathbf{R}\boldsymbol{\nu})_\mathcal{K})\mathbf{v}_\mathcal{K}$	$\mathbf{D}_{\mathcal{M},\mathcal{K}}$	$\bar{\mathbf{p}}_\mathcal{M}$

TABLE I: Nonlinear Perron-Frobenius characterization of subproblems in the weighted max-min fairness optimization. Here, $\mathbf{A}$, $\mathbf{b}(\mathbf{z})$, $\mathbf{B}$, and $\mathbf{d}$ are the parameters of the conditional eigenvalue problem given in (13). Hence, each subproblem can be mapped into problem (13).

for every l, then C would still satisfy Theorem 1 but it would not satisfy Theorem 5. On the other hand, it is also possible to construct a link rate function that could satisfy the conditions in Theorem 5 but violates the conditions in Theorem 1. An example is the following piecewise linear function:

$$C(x) = \begin{cases} x, & \text{if } 0 \le x \le 1 \\ \frac{1}{2}x + \frac{1}{2}, & \text{if } 1 \le x. \end{cases}$$

Clearly, this function satisfies conditions C1 and C2. Moreover, we have that $x/C(x)$ is non-decreasing and hence satisfies condition C3 but it is not concave.

Proof of Theorem 5: We continue to consider only the case where $p_l = 0$ for all $l \notin \mathcal{K}$ and solve for $\mathbf{p}_\mathcal{K}$. First, we argue that (9) must achieve its optimal value. Since $\mathsf{SINR}_l(\mathbf{p}_\mathcal{K})$ is continuous in $\mathbf{p}_\mathcal{K}$ and C is a continuous function (by condition C2), we have that $C(\mathsf{SINR}_l(\mathbf{p}_\mathcal{K}))$ is continuous in $\mathbf{p}_\mathcal{K}$. Now for any two continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, the function $\min\{f(x), g(x)\}$ is continuous in x since it can be rewritten as $\min\{f(x), g(x)\} = (f(x) + g(x))/2 - |f(x) - g(x)|/2$. Hence, the objective function in (9) is continuous in $\mathbf{p}_\mathcal{K}$. Clearly, the feasible set of (9) is compact. Any continuous function must achieve its maximum on a compact set, so we have that (9) must achieve its optimal value.

We repeat the following eigenvalue problem from the proof of Theorem 1: find $(\tau_*, \mathbf{p}_{*\mathcal{K}})$ such that

$$\frac{1}{\tau_*}\mathbf{p}_{*\mathcal{K}} = \mathbf{f}_\mathcal{K}(\mathbf{p}_{*\mathcal{K}}), \qquad \tau_* \in \mathbb{R}, \qquad \mathbf{p}_{*\mathcal{K}} > \mathbf{0},$$
$$\mathbf{D}_{\mathcal{M},\mathcal{K}}\mathbf{p}_{*\mathcal{K}} \le \bar{\mathbf{p}}_\mathcal{M}, \qquad \mathbf{D}_{\mathcal{M},\mathcal{K}}\mathbf{p}_{*\mathcal{K}} \not< \bar{\mathbf{p}}_\mathcal{M} \qquad (12)$$

where $\mathbf{f} : \mathbb{R}_{>0}^{|\mathcal{K}|} \to \mathbb{R}_{>0}^L$ is defined by:

$$f_l(\mathbf{p}_\mathcal{K}) = \begin{cases} \left(\frac{(\mathbf{R}\boldsymbol{\nu})_l}{C(\mathsf{SINR}_l(\mathbf{p}_\mathcal{K}))}\right) p_l, & \text{if } l \in \mathcal{K} \\ 0, & \text{if } l \notin \mathcal{K}. \end{cases}$$

Since (9) must achieve its optimal value, it follows that there exists a solution to (12). Clearly, $\mathbf{f}_\mathcal{K}$ is continuous for all $\mathbf{p}_\mathcal{K} > \mathbf{0}$. We will show that $\mathbf{f}_\mathcal{K}$ satisfies the rest of condition 2 in Lemma 5.

Next, we show that $\mathbf{f}_\mathcal{K}$ is subhomogeneous on $\mathbb{R}_{>0}^{|\mathcal{K}|}$. For every $\mathbf{p}_\mathcal{K} > \mathbf{0}$ and every $\lambda \in [0,1]$, we have that $\mathsf{SINR}_l(\mathbf{p}_\mathcal{K}) \ge \mathsf{SINR}_l(\lambda\mathbf{p}_\mathcal{K})$. Since C is a strictly increasing function (by condition C2), it follows that:

$$\lambda f_l(\mathbf{p}_\mathcal{K}) = \frac{\lambda p_l(\mathbf{R}\boldsymbol{\nu})_l}{C(\mathsf{SINR}_l(\mathbf{p}_\mathcal{K}))} \le \frac{\lambda p_l(\mathbf{R}\boldsymbol{\nu})_l}{C(\mathsf{SINR}_l(\lambda\mathbf{p}_\mathcal{K}))} = f_l(\lambda\mathbf{p}_\mathcal{K}).$$

for all $\mathbf{p}_\mathcal{K} > \mathbf{0}$ and every $l \in \mathcal{K}$.

Now, we show that $\mathbf{f}_\mathcal{K}$ is order-preserving on $\mathbb{R}_{>0}^{|\mathcal{K}|}$. Suppose $\mathbf{f}_\mathcal{K}$ is differentiable at some $\mathbf{p}'_\mathcal{K} \in \mathbb{R}_{>0}^{|\mathcal{K}|}$. The jth entry of the derivative of f_l with respect to $\mathbf{p}_\mathcal{K}$ at $\mathbf{p}'_\mathcal{K}$ is given by

$(\nabla f_l(\mathbf{p}'_\mathcal{K}))_j$
$$= \frac{(\mathbf{R}\boldsymbol{\nu})_l}{C(\mathsf{SINR}_l(\mathbf{p}'_\mathcal{K}))^2} \left(C'(\mathsf{SINR}_l(\mathbf{p}'_\mathcal{K}))\mathsf{SINR}_l(\mathbf{p}'_\mathcal{K})^2 (\mathbf{e}_l)_\mathcal{K}^\top \mathbf{F}_{\mathcal{K},\mathcal{K}}\mathbf{e}_j\right.$$
$$\left. + (C(\mathsf{SINR}_l(\mathbf{p}'_\mathcal{K})) - C'(\mathsf{SINR}_l(\mathbf{p}'_\mathcal{K}))\mathsf{SINR}_l(\mathbf{p}'_\mathcal{K}))(\mathbf{e}_l)_\mathcal{K}^\top\mathbf{e}_j\right).$$

Since C is a strictly increasing function (by condition C2), the first term is positive. Moreover, since $x/C(x)$ is a non-decreasing function (by condition C3), we have that $C(\mathsf{SINR}_l(\mathbf{p}'_\mathcal{K})) - C'(\mathsf{SINR}_l(\mathbf{p}'_\mathcal{K}))\mathsf{SINR}_l(\mathbf{p}'_\mathcal{K}) \ge 0$. Hence, $\nabla f_l \ge \mathbf{0}$ wherever f_l is differentiable. Since f_l is continuous, it follows that f_l increases monotonically on $\mathbb{R}_{\ge 0}^{|\mathcal{K}|}$.

Finally, we show that $\nabla\mathbf{f}(\mathbf{p}_{*\mathcal{K}})$ is a primitive matrix. Let $\mathrm{diag}_\mathcal{K}(x_l)$ denote the diagonal matrix formed from the vector $(x_l, l \in \mathcal{K})$. We have that:

$\nabla\mathbf{f}(\mathbf{p}_{*\mathcal{K}})$
$$= \mathrm{diag}_\mathcal{K}\left(\frac{(\mathbf{R}\boldsymbol{\nu})_l}{C(\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}}))^2}\right)$$
$$\left(\mathrm{diag}_\mathcal{K}\left(C'(\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}}))\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}})^2\right)\mathbf{F}_{\mathcal{K},\mathcal{K}}\right.$$
$$\left. + \mathrm{diag}_\mathcal{K}\left(C(\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}})) - C'(\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}}))\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}}))\right)\right).$$

Since C is a strictly increasing function (by condition C2), we have that $C'(\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}})) > 0$. Recall that, by condition C3, we have that $C(\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}})) - C'(\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}}))\mathsf{SINR}_l(\mathbf{p}_{*\mathcal{K}}) \ge 0$. Hence, if $\mathbf{F}_{\mathcal{K},\mathcal{K}}$ is a primitive matrix, then it follows that $\nabla\mathbf{f}(\mathbf{p}_{*\mathcal{K}})$ is a primitive matrix. Note that since $G_{lj} > 0$ for all l and j, we have that $F_{lj} > 0$ for all $l \ne j$. Consider $\mathbf{F}_{\mathcal{K},\mathcal{K}}\mathbf{F}_{\mathcal{K},\mathcal{K}}$. We have that

$$(\mathbf{e}_i)_\mathcal{K}^\top(\mathbf{F}_{\mathcal{K},\mathcal{K}}\mathbf{F}_{\mathcal{K},\mathcal{K}})(\mathbf{e}_k)_\mathcal{K} = \sum_{j\in\mathcal{K}} F_{ij}F_{jk} = \sum_{\substack{j\in\mathcal{K}\\j\ne i, j\ne k}} F_{ij}F_{jk}.$$

Now this last term is strictly positive whenever $|\mathcal{K}| \ge 3$. $\blacksquare$

IV. NUMERICAL EXAMPLES

We simulate the joint flow rate and power control algorithm for a small example consisting of four multihop connections and four flows:
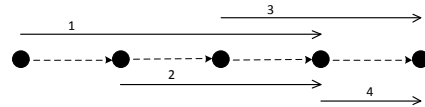
The dotted lines denote the logical wireless links, which are indexed from left to right. The path losses $G_{ij} = G_{ji}$ are
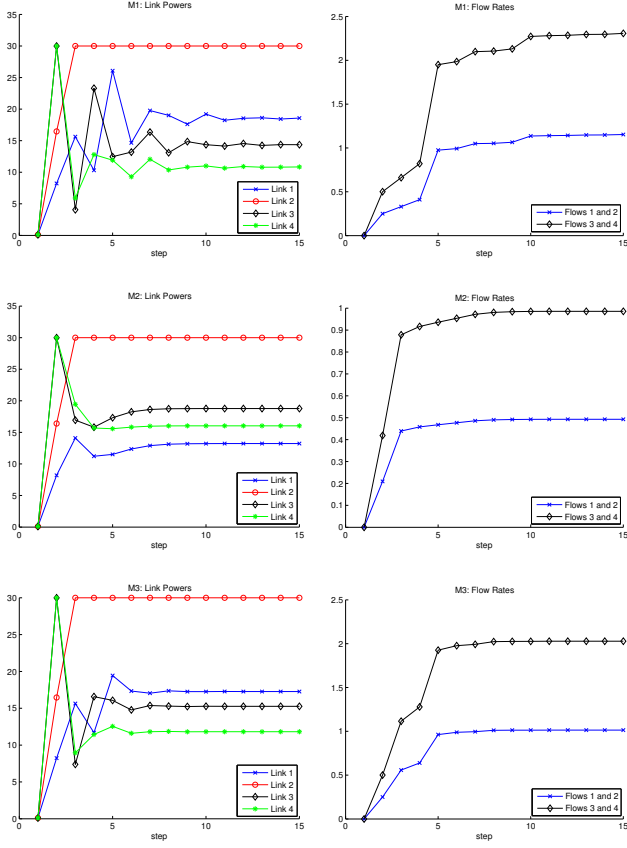
Fig. 1: An example of the weighted max-min joint flow rate and link power control. Each row corresponds to one of the rate functions M1, M2, and M3 (from left to right). The graphs on the left column show the link powers over time while the graphs on the right column show the flow rates over time.

chosen to vary with the physical distance d_{ij} as $G_{ij} = d_{ij}^{-4}$. We assume that each transmitter is subject to a peak power constraint. Flows from links 3 and 4 are assigned twice the priority of the flows from links 1 and 2. We consider three link rate functions:

M1. CDMA rate function: $C(x) = x$.
M2. Shannon capacity function: $C(x) = \log(1 + x)$.
M3. Piecewise linear function: $C(x) = \sum_{i=0}^{\lfloor x \rfloor - 1} (9/10)^i + (9/10)^{\lfloor x \rfloor}(x - \lfloor x \rfloor)$.

The results are shown in Figure 1. Observe that there is rapid convergence for all the rate functions, which implies that the static channel assumption is a fair approximation. Observe also that for rate functions M1 and M2, the flow rates increase monotonically (cf. remark 1). For the settings used in this example, it is not surprising that there is little difference in optimal power and rate allocation between the CDMA and piecewise linear rate functions. However, it is interesting to note that this is not true for the Shannon capacity function. The optimal transmit power of link 1 is significantly lower under the Shannon capacity function than under the other two rate functions.

V. CONCLUSIONS

We considered the weighted max-min fairness flow rate allocation problem in a multiuser wireless network for a large class of link rate functions which includes the well-known Shannon capacity function and the CDMA rate function. We showed that the joint flow rate and power control problem can be decoupled into two separate problems on fairness - one at the network layer and one at the link layer - where the latter problem assumes the case where all sources in the network layer are sending at a common rate. Then, we characterized the optimal value and solution using nonlinear Perron-Frobenius theory, and proposed an iterative algorithm for the joint flow rate and power control problem which converges geometrically fast to the optimal solution.

APPENDIX

A. Conditional Eigenvalue Problem with Affine Constraints

Let $\mathbf{A} \in \mathbb{R}_{\geqslant 0}^{L \times L}$ be either a nonnegative irreducible square matrix or the all-zero matrix, $\mathbf{B} \in \mathbb{R}_{\geqslant 0}^{M \times L}$ be a nonnegative rectangular matrix with at least one nonzero element in each column, $\mathbf{b} : \mathbb{R}_{\geqslant 0}^{L} \to \mathbb{R}_{\geqslant 0}^{L}$ be such that $\mathbf{b}(\mathbf{z}) > \mathbf{0}$ for all $\mathbf{z} \gneq \mathbf{0}$, and $\mathbf{d} \in \mathbb{R}_{>0}^{M}$ be a positive vector. Consider the problem of finding $(\lambda, \mathbf{z})$ such that

$$\lambda\mathbf{z} = \mathbf{A}\mathbf{z} + \mathbf{b}(\mathbf{z}), \quad \lambda \in \mathbb{R}, \quad \mathbf{z} > \mathbf{0}, \quad \mathbf{B}\mathbf{z} \leq \mathbf{d}, \quad \mathbf{B}\mathbf{z} \not< \mathbf{d}. \quad (13)$$

In this appendix, we present two lemmas pertaining to this problem. These lemmas will be used to derive most of the results in this paper.

Before presenting the lemmas, we first state some definitions. A norm $\|\cdot\|$ is monotone if $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y}$ implies $\|\mathbf{x}\| \leq \|\mathbf{y}\|$. A map $\mathbf{f} : \mathbb{R}_{\geqslant 0}^{L} \to \mathbb{R}_{\geqslant 0}^{L}$ is concave if $\mathbf{f}(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda\mathbf{f}(\mathbf{x}) + (1-\lambda)\mathbf{f}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}_{\geqslant 0}^{L}$ and $\lambda \in [0, 1]$. The map is order-preserving on a set U if $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in U$ such that $\mathbf{x} \leq \mathbf{y}$. The map is subhomogeneous on U if $\lambda\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\lambda\mathbf{x})$ for all $\mathbf{x} \in U$ and every $\lambda \in [0, 1]$. A matrix $\mathbf{A} \in \mathbb{R}^{L \times L}$ is primitive if there exists $k \in \mathbb{Z}_{\geq 1}$ such that $\mathbf{A}^k > \mathbf{0}$.

Lemma 5: Suppose either of the following two conditions hold:

1. $\mathbf{b}$ is concave.
2. $\mathbf{b}$ is a continuous, subhomogeneous, and order-preserving map on the set $\{ \mathbf{z} \in \mathbb{R}_{>0}^{L} \mid \|\mathbf{z}\| = 1 \}$. There exists a solution $(\lambda_*, \mathbf{z}_*)$ to (13). And $\mathbf{b}$ is continuously differentiable on an open neighborhood of $\mathbf{z}_*$ and the matrix $\mathbf{A} + \nabla\mathbf{b}(\mathbf{z}_*)$ is primitive.

Then there exists a unique solution $(\lambda_*, \mathbf{z}_*)$ to (13) and the following iteration:

$$\mathbf{z}(k+1) = \frac{\mathbf{A}\mathbf{z}(k) + \mathbf{b}(\mathbf{z}(k))}{\|\mathbf{A}\mathbf{z}(k) + \mathbf{b}(\mathbf{z}(k))\|_{\infty, \mathbf{d}}^{\mathbf{B}}}$$

converges to $\mathbf{z}_*$ geometrically fast from any initial point $\mathbf{z}_0 \gneq \mathbf{0}$.

Lemma 6: Suppose either conditions 1 or 2 in Lemma 5 hold. Then the unique solution $(\lambda_\star, \mathbf{z}_\star)$ to (13) is given by

$$\lambda_\star = \rho\left(\mathbf{A} + (1/d_i)\mathbf{b}(\mathbf{z}_\star)\mathbf{e}_i^\top\mathbf{B}\right)$$

and
$$\mathbf{z}_\star = \mathbf{x}\left(\mathbf{A} + (1/d_i)\mathbf{b}(\mathbf{z}_\star)\mathbf{e}_i^\top\mathbf{B}\right)$$

where $\mathbf{z}_\star$ is unique up to a scaling constant and $i = \arg\max_{m=1,\ldots,M}\rho(\mathbf{A} + (1/d_m)\mathbf{b}(\mathbf{z}_\star)\mathbf{e}_m^\top\mathbf{B})$.

Proof of Lemma 5: Recall that $\mathbf{B}$ is a nonnegative matrix with at least one nonzero element in each column and $\mathbf{d}$ is a positive vector. Hence, $\|\cdot\|_{\infty,\mathbf{d}}^{\mathbf{B}}$ is a monotone norm. Observe that the constraints $\mathbf{Bz} \le \mathbf{d}$ and $\mathbf{Bz} \not< \mathbf{z}$ can be rewritten as $\|\mathbf{z}\|_{\infty,\mathbf{d}}^{\mathbf{B}} = 1$. Now our lemma follows from the following two nonlinear Perron-Frobenius theorems.

Theorem 6 ([17]): Let $\|\cdot\|$ be a monotone vector norm on $\mathbb{R}^L$ and $S = \{\mathbf{x} \in \mathbb{R}_{\ge 0}^L : \|\mathbf{x}\| = 1\}$. Let $\mathbf{f} : \mathbb{R}_{\ge 0}^L \to \mathbb{R}_{\ge 0}^L$ be a concave map with $\mathbf{f}(\mathbf{x}) > \mathbf{0}$ for $\mathbf{x} \gneq \mathbf{0}$. Then $\mathbf{f}$ has a unique eigenvector $\mathbf{x}_* \in S$. Furthermore, $\mathbf{x}_*$ is the unique fixed point of the normalized map $\mathbf{u} : S \to S$ defined by $\mathbf{u}(\mathbf{x}) = (1/\|\mathbf{f}(\mathbf{x})\|)\mathbf{f}(\mathbf{x})$ and all the orbits of $\mathbf{u}$ converge geometrically fast to $\mathbf{x}_*$ from any initial point $\mathbf{x}_0 \gneq \mathbf{0}$.

Theorem 7 ([18]): Let $\|\cdot\|$ be a monotone vector norm on $\mathbb{R}_L$ and $S = \{\mathbf{x} \in \mathbb{R}_{>0}^L : \|\mathbf{x}\| = 1\}$. Let $\mathbf{f} : \mathbb{R}_{>0}^L \to \mathbb{R}_{>0}^L$ be a continuous, order-preserving map which is subhomogeneous on S. Moreover, suppose that if $\mathbf{x} \in S$ is an eigenvector of $\mathbf{f}$, then $\mathbf{f}$ is continuously differentiable on an open neighborhood of $\mathbf{x}$ and the matrix $\nabla\mathbf{f}(\mathbf{x})$ is nonnegative and irreducible. Then $\mathbf{f}$ has at most one eigenvector $\mathbf{x}_* \in S$. Furthermore, the normalized map $\mathbf{u} : S \to S$ defined by $\mathbf{u}(\mathbf{x}) = (1/\|\mathbf{f}(\mathbf{x})\|)\mathbf{f}(\mathbf{x})$ has at most one fixed point. Moreover, if $\mathbf{f}$ has an eigenvector $\mathbf{x}_* \in S$, and if $\nabla\mathbf{f}(\mathbf{x}_*)$ is a primitive matrix, then all the orbits of $\mathbf{u}$ converge geometrically fast to $\mathbf{x}_*$ from any initial point $\mathbf{x}_0 > \mathbf{0}$. ∎

Proof of Lemma 6: First, we show that $\rho(\mathbf{A}) < \lambda$. Note that since $\mathbf{Az} + \mathbf{b}(\mathbf{z}) > \mathbf{0}$ for all $\mathbf{z} > \mathbf{0}$, we have that $\lambda > 0$. Then, using the fact that $\mathbf{b}(\mathbf{z}) > \mathbf{0}$ for all $\mathbf{z} > \mathbf{0}$, we have that $\lambda\mathbf{z} > \mathbf{Az}$. Now the fact that $\rho(\mathbf{A}) < \lambda$ follows from the following theorem which we state from [19].

Theorem 8 ([19]): Let $\mathbf{A} \in \mathbb{R}_{\ge 0}^{L \times L}$ be a nonnegative matrix and $\mathbf{z} \in \mathbb{R}_{\ge 0}^L$ be a positive vector. If $\alpha, \beta \ge 0$ are such that $\alpha\mathbf{z} \le \mathbf{Az} \le \beta\mathbf{z}$, then $\alpha \le \rho(\mathbf{A}) \le \beta$. If $\alpha\mathbf{z} < \mathbf{Az}$, then $\alpha < \rho(\mathbf{A})$; if $\mathbf{Az} < \beta\mathbf{z}$, then $\rho(\mathbf{A}) < \beta$.

Next, write $\lambda\mathbf{z} = \mathbf{Az} + \mathbf{b}(\mathbf{z})$ as $\mathbf{z} = (\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}(\mathbf{z})$ and substitute into $\mathbf{Bz} \le \mathbf{d}$ to get

$$\mathbf{B}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}(\mathbf{z}) \le \mathbf{d}. \qquad (14)$$

We now determine a necessary and sufficient condition under which (14) holds when $\rho(\mathbf{A}) < \lambda$. Under this condition, $(\lambda\mathbf{I} - \mathbf{A})^{-1}$ is a nonnegative matrix. Consider the following:

$$\begin{aligned}
&\det(\lambda\mathbf{I} - (\mathbf{A} + (1/d_m)\mathbf{b}(\mathbf{z})\mathbf{e}_m^\top\mathbf{B})) \\
=~ &\det(\lambda\mathbf{I} - \mathbf{A})\det(\mathbf{I} - (1/d_m)(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}(\mathbf{z})\mathbf{e}_m^\top\mathbf{B}) \\
=~ &\det(\lambda\mathbf{I} - \mathbf{A})(1 - (1/d_m)\mathbf{e}_m^\top\mathbf{B}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}(\mathbf{z})).
\end{aligned}$$

Since $\rho(\mathbf{A}) < \lambda$, we have that $\det(\lambda\mathbf{I} - (\mathbf{A} + (1/d_m)\mathbf{b}(\mathbf{z})\mathbf{e}_m^\top\mathbf{B})) \ge 0$ if and only if $\mathbf{e}_m^\top\mathbf{B}(\lambda\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}(\mathbf{z}) \le d_m$. But $\det(\lambda\mathbf{I} - (\mathbf{A} + (1/d_m)\mathbf{b}(\mathbf{z})\mathbf{e}_m^\top\mathbf{B})) \ge 0$ if and only if $\rho(\mathbf{A} + (1/d_m)\mathbf{b}(\mathbf{z})\mathbf{e}_m^\top\mathbf{B}) \le \lambda$. Hence, (14) is satisfied if and only if

$$\rho\left(\mathbf{A} + (1/d_m)\mathbf{b}(\mathbf{z})\mathbf{e}_m^\top\mathbf{B}\right) \le \lambda \quad \forall~m. \qquad (15)$$

Moreover, since we also have $\mathbf{Bz} \not< \mathbf{d}$, it follows that (15) must hold with equality for some m. Hence, λ must be given by the smallest possible value which satisfies (15), that is, $\rho(\mathbf{A} + (1/d_i)\mathbf{b}(\mathbf{z})\mathbf{e}_i^\top\mathbf{B})$ where

$$i = \arg\max_{m=1,\ldots,M}\rho\left(\mathbf{A} + (1/d_m)\mathbf{b}(\mathbf{z})\mathbf{e}_m^\top\mathbf{B}\right). \qquad (16)$$

In particular, for this index i, we have that $\mathbf{e}_i^\top\mathbf{Bz} = d_i$. Substituting this into $\lambda\mathbf{z} = \mathbf{Az} + \mathbf{b}(\mathbf{z})$, we have that:

$$\rho(\mathbf{A} + (1/d_i)\mathbf{b}(\mathbf{z})\mathbf{e}_i^\top\mathbf{B}))\mathbf{z} = (\mathbf{A} + (1/d_i)\mathbf{b}(\mathbf{z})\mathbf{e}_i^\top\mathbf{B}))\mathbf{z}.$$

Consider the case where $\mathbf{A}$ is irreducible.[4] Then, $(\mathbf{A} + (1/d_i)\mathbf{b}(\mathbf{z})\mathbf{e}_i^\top\mathbf{B})$ is irreducible so by the Perron-Frobenius Theorem (see [20]), we have that $\mathbf{z}$ is uniquely given by $\mathbf{x}\left(\mathbf{A} + (1/d_i)\mathbf{b}(\mathbf{z})\mathbf{e}_i^\top\mathbf{B}\right)$ which is strictly positive. Next, consider the case where $\mathbf{A}$ is the all-zero matrix. Then, $((1/d_i)\mathbf{b}(\mathbf{z})\mathbf{e}_i^\top\mathbf{B})$ is a rank-one matrix so it has a unique right eigenvector given by $\mathbf{b}(\mathbf{z})$ (up to a scaling constant) which is strictly positive for all $\mathbf{z} > \mathbf{0}$. ∎

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