Coordinated Max-Min SIR Optimization in Multicell Downlink - Duality and Algorithm

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***Abstract*—Typical formulations of max-min weighted SIR problems involve either a total power constraint or individual power constraints. These formulations are unable to handle the complexities in multicell networks where each base station can be subject to its own sum power constraint. This paper considers the max-min weighted SIR problem subject to multiple weighted-sum power constraints, where the weights can represent relative power costs of serving different users. First, we derive the uplink-downlink duality principle by applying Lagrange duality to the single-constraint problem. Next, we apply nonlinear Perron-Frobenius theory to derive a closed-form solution for the multiple-constraint problem. Then, by exploiting the structure of the closed-form solution, we relate the multiple-constraint problem with its single-constraint subproblems and establish the dual uplink problem. Finally, we further apply nonlinear Perron-Frobenius theory to derive an algorithm which converges geometrically fast to the optimal solution.**

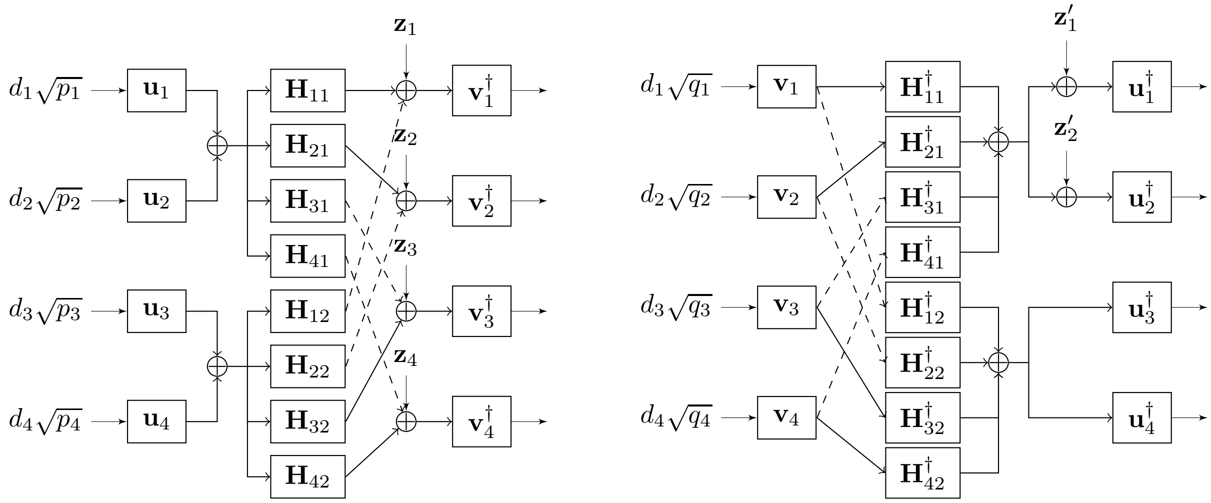
I. Introduction

This paper considers a wireless network with arbitrary topology, in which each node is equipped with multiple antennas. The goal is to maximize the minimum weighted signal-to-interference-plus-noise ratio (SIR) subject to multiple weighted-sum power constraints. The weights can represent relative power costs of serving different users. Previous work on the max-min weighted SIR problem have considered only the cases of a total power constraint or individual power constraints [1]. However, multiple power constraints can arise in a variety of scenarios. For instance, in multicell networks, each base station can be subject to its own transmit power constraint [2]. One strategy for dealing with multiple power constraints was proposed in the context of Multiple-Input-Single-Output (MISO) sum-rate maximization [3]. It involves forming a relaxed single-constraint problem by taking a parameterized sum of the original sum power constraints. It turns out that the solution to the relaxed problem is a tight upper bound to the solution of the original multiple-constraint problem. Hence, the multiple-constraint problem can be solved using a computationally-intensive inner-outer (subgradient) iterative algorithm [3].

Multiple-Input-Multiple-Output (MIMO) systems provide spatial diversity which can be exploited by using beamforming to mitigate the interference between users and increase the overall performance while having users share a common bandwidth. It is well-known that while receive beamforming is a straightforward problem, transmit beamforming is a significantly more difficult problem due to the mutual coupling between the transmit beamformers. One popular method for handling transmit beamforming is to use uplink-downlink duality to convert the downlink transmit beamforming problem into an equivalent uplink receive beamforming problem which is easier to solve. This has motivated numerous studies on uplink-downlink duality [4]–[6]. In particular, an interesting connection was discovered between uplink-downlink duality and linear program (LP) duality in the context of the total power minimization problem [5].

This paper studies the max-min weighted SIR problem subject to multiple weighted-sum power constraints. First, in Section III, we study uplink-downlink duality by applying Lagrange duality to the max-min weighted SIR problem subject to a single weighted-sum power constraint. Next, in Section IV, we study the multiple-constraint problem. By reformulating the multiple constraints as a single norm constraint, we show that nonlinear Perron-Frobenius theory can be used to derive a closed-form solution to the multiple-constraint problem. Then, we exploit the structure of the closed-form solution to relate the multiple-constraint problem with its single-constraint subproblems and establish the dual uplink problem. Finally, we further apply nonlinear Perron-Frobenius theory to propose an algorithm for the multiple-constraint problem which converges geometrically fast to the optimal solution. Numerical results are provided to illustrate the performance of the proposed algorithm in a multicell scenario.

The following notations are used. Boldface upper-case letters denote matrices, boldface lowercase letters denote column vectors, italics denote scalars, $\mathbf{u} \geq \mathbf{v}$ denotes componentwise inequality between vectors $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{u} \circ \mathbf{v}$ denotes the Schur product between $\mathbf{u}$ and $\mathbf{v}$. The Perron-Frobenius eigenvalue of a nonnegative matrix $\mathbf{F}$ is denoted as $\rho(\mathbf{F})$, and the Perron (right) and left eigenvectors of $\mathbf{F}$ associated with $\rho(\mathbf{F})$ are denoted by $\mathbf{x}(\mathbf{F})$ and $\mathbf{y}(\mathbf{F})$ respectively. The diagonal matrix formed by the components of a vector $\mathbf{v}$ is denoted as $\mathrm{diag}(\mathbf{v})$. The superscripts $(\cdot)^\mathsf{T}$ and $(\cdot)^\dagger$ denote transpose and complex conjugate transpose respectively. The set of positive real numbers is denoted by $\mathbb{R}_+$ and the set of nonnegative real numbers is denoted by $\mathbb{R}_{++}$. The set of complex numbers is denoted by $\mathbb{C}$.



(a) Downlink, $\mathbf{z}_l \sim \mathcal{CN}(\mathbf{0}, n_l\mathbf{I})$.

(b) Dual uplink (assuming $\mathbf{w}_1^\mathsf{T}\mathbf{p} = \bar{P}_1$ at optimality), $\mathbf{z}_l' \sim \mathcal{CN}(\mathbf{0}, w_{1l}\mathbf{I})$.

Fig. 1. Block diagrams of a two-cell MIMO downlink system and its dual uplink system (assuming $\mathbf{w}_1^\mathsf{T}\mathbf{p} = \bar{P}_1$ at optimality). Each cell serves two users. The per-cell power constraints in the downlink are given by $\mathbf{w}_1^\mathsf{T}\mathbf{p} \le \bar{P}_1$ and $\mathbf{w}_2^\mathsf{T}\mathbf{p} \le \bar{P}_2$, where $\mathbf{w}_1 = [w_{11}\ w_{12}\ 0\ 0]^\mathsf{T}$ and $\mathbf{w}_2 = [0\ 0\ w_{23}\ w_{24}]^\mathsf{T}$. The dual uplink system has a total power constraint given by $\mathbf{n}^\mathsf{T}\mathbf{q} \le \bar{P}_1$. Dotted lines denote the inter-cell interfering paths.

II. System Model

We consider a MIMO network where L independent data streams are transmitted over a common frequency band. The transmitter and receiver of the lth stream, denoted by t_l and r_l respectively, are equipped with N_{t_l} antennas and N_{r_l} antennas respectively. The downlink channel can be modeled as a Gaussian broadcast channel given by

$$\mathbf{y}_l = \sum_{i=1}^{L} \mathbf{H}_{li}\mathbf{x}_i + \mathbf{z}_l, \quad l = 1, \ldots, L, \tag{1}$$

where $\mathbf{y}_l \in \mathbb{C}^{N_{r_l}}$ is the received signal at r_l, $\mathbf{H}_{li} \in \mathbb{C}^{N_{r_l} \times N_{t_i}}$ is the channel vector between t_i and r_l, $\mathbf{x}_i \in \mathbb{C}^{N_{t_i}}$ is the transmitted signal vector of the ith stream, and $\mathbf{z}_l \sim \mathcal{CN}(\mathbf{0}, n_l\mathbf{I})$ is the circularly symmetric Gaussian noise vector with covariance $n_l\mathbf{I}$ at r_l. The transmitted signal vector of the ith stream can be written as $\mathbf{x}_i = d_i\sqrt{p_i}\mathbf{u}_i$, where $\mathbf{u}_i \in \mathbb{C}^{N_{t_i}}$ is the normalized transmit beamformer, and d_i and $p_i \in \mathbb{R}_+$ are the information signal and transmit power respectively for that stream. The lth stream is decoded using a normalized receive beamformer $\mathbf{v}_l \in \mathbb{C}^{N_{r_l}}$. Define the power vector $\mathbf{p} = [p_1, \ldots, p_L]^\mathsf{T}$, the sets of beamforming matrices $\mathbb{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_L\}$ and $\mathbb{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_L\}$, and the noise vector $\mathbf{n} = [n_1, \ldots, n_L]^\mathsf{T}$. Define the matrix $\mathbf{G} \in \mathbb{R}_{++}^{L \times L}$ where the entry $G_{li} = \left|\mathbf{v}_l^\dagger \mathbf{H}_{li}\mathbf{u}_i\right|^2$ is the effective link gain between t_i and r_l. The SIR of the lth stream can be expressed as

$$\mathsf{SIR}_l^{\mathsf{DL}}(\mathbf{p}) = \frac{p_l G_{ll}}{\sum_{i \neq l} p_i G_{li} + n_l}. \tag{2}$$

Define the priority vector $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_L]^\mathsf{T}$ where $\beta_l \in \mathbb{R}_+$ is the priority assigned by the network to the lth stream. We are interested in the J-constraint max-min weighted SIR problem in the downlink given by

$$\mathcal{S}^{\mathsf{DL}} = \begin{cases} \underset{\mathbf{p}}{\max} & \underset{l}{\min} \frac{\mathsf{SIR}_l^{\mathsf{DL}}(\mathbf{p})}{\beta_l} \\ \text{subject to} & \mathbf{w}_j^\mathsf{T}\mathbf{p} \le \bar{P}_j, \quad j = 1, \ldots, J, \\ & \mathbf{p} > \mathbf{0}, \end{cases} \tag{3}$$

where $\bar{P}_j \in \mathbb{R}_+$ is the given jth power constraint and $\mathbf{w}_j = [w_{j1}, \ldots, w_{jL}]^\mathsf{T}$ is the weight vector such that $w_{jl} \in \mathbb{R}_{++}$ is the weight associated with p_l in the jth power constraint. It is easy to see that at the optimal solution to $\mathcal{S}^{\mathsf{DL}}$, all the weighted SIR's are equal, i.e., $\mathsf{SIR}_l^{\mathsf{DL}}(\mathbf{p})/\beta_l$ have the same value for all l. Moreover, at least one power constraint must be tight.

We define the (cross channel interference) matrix $\mathbf{F} \in \mathbb{R}_+^{L \times L}$ and the normalized priority vector $\hat{\boldsymbol{\beta}} \in \mathbb{R}_+^L$ with the following entries:

$$F_{li} = \begin{cases} 0, & \text{if} \quad l = i, \\ G_{li}, & \text{if} \quad l \neq i, \end{cases} \tag{4}$$

$$\hat{\boldsymbol{\beta}} = \left(\frac{\beta_1}{G_{11}}, \ldots, \frac{\beta_L}{G_{LL}}\right)^\mathsf{T}. \tag{5}$$

Hence, $\mathcal{S}^{\mathsf{DL}}$ can be rewritten as

$$\mathcal{S}^{\mathsf{DL}} = \begin{cases} \underset{\mathbf{p}}{\max} & \underset{l}{\min} \frac{p_l}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}+\mathbf{n}))_l} \\ \text{subject to} & \mathbf{w}_j^\mathsf{T}\mathbf{p} \le \bar{P}_j, \quad j = 1, \ldots, J, \\ & \mathbf{p} > \mathbf{0}. \end{cases} \tag{6}$$

For reasons that will become clear later, we also define the single-constraint downlink subproblem of $\mathcal{S}^{\mathsf{DL}}$ given by

$$\mathcal{S}_j^{\mathsf{DL}} = \begin{cases} \underset{\mathbf{p}}{\max} & \underset{l}{\min} \frac{p_l}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}+\mathbf{n}))_l} \\ \text{subject to} & \mathbf{w}_j^\mathsf{T}\mathbf{p} \le \bar{P}_j, \quad \mathbf{p} > \mathbf{0}. \end{cases} \tag{7}$$

Similarly, at the optimal solution to $\mathcal{S}_j^{\mathsf{DL}}$, all the weighted SIR's are equal. Moreover, the sole power constraint in $\mathcal{S}_j^{\mathsf{DL}}$ must be tight.

III. Duality with Single Power Constraint

In this section, we establish the dual uplink problem for $\mathcal{S}_j^{\mathsf{DL}}$ by mapping the necessary and sufficient conditions for optimality of $\mathcal{S}_j^{\mathsf{DL}}$ into those for a dual uplink problem. Introducing an auxiliary variable τ, we can rewrite (7) in epigraph form as

$$\mathcal{S}_j^{\mathsf{DL}} = \begin{cases} \max\limits_{\tau,\mathbf{p}} & \tau \\ \text{subject to} & \tau \leq \frac{p_l}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}+\mathbf{n}))_l}, \\ & \mathbf{w}_j^{\mathsf{T}}\mathbf{p} \leq \bar{P}_j, \quad \mathbf{p} > \mathbf{0}. \end{cases} \tag{8}$$

By making a change of variables $\tilde{\tau} = \log \tau$ and $\tilde{p}_l = \log p_l$ for all l, we arrive at the following equivalent convex problem:

$$\begin{array}{ll} \min\limits_{\tilde{\tau},\tilde{\mathbf{p}}} & -\tilde{\tau} \\ \text{subject to} & \log\left(\frac{e^{\tilde{\tau}}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}e^{\tilde{\mathbf{p}}}+\mathbf{n}))_l}{(e^{\tilde{\mathbf{p}}})_l}\right) \leq 0, \quad l = 1,\ldots,L, \\ & \log\left(\frac{1}{\bar{P}_j}\mathbf{w}_j^{\mathsf{T}}e^{\tilde{\mathbf{p}}}\right) \leq 0. \end{array} \tag{9}$$

The Lagrangian associated with (9) is

$$\mathcal{L} = -\tilde{\tau} + \sum_{l=1}^{L} \lambda_l \log\left(\frac{e^{\tilde{\tau}}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}e^{\tilde{\mathbf{p}}}+\mathbf{n}))_l}{(e^{\tilde{\mathbf{p}}})_l}\right) + \lambda \log\left(\frac{1}{\bar{P}_j}\mathbf{w}_j^{\mathsf{T}}e^{\tilde{\mathbf{p}}}\right), \tag{10}$$

where λ_l and λ are the Lagrange dual variables.

Next, we use the Karush-Kuhn Tucker (KKT) conditions for (9) to obtain the necessary and sufficient conditions for optimality of $\mathcal{S}_j^{\mathsf{DL}}$. Observe that λ_l and λ must be strictly positive in order for $\tilde{p}_l$ to be greater than $-\infty$ and for $\mathcal{L}$ to be bounded from below. Moreover, at the optimal solution to (9), all the inequality constraints must be active. Hence, we can replace the inequality constraints (in the feasbility conditions) with equality and drop the complementary slackness conditions in the KKT conditions. Making a change of variables back into τ and $\mathbf{p}$, we arrive at the following necessary and sufficient conditions for optimality of $\mathcal{S}_j^{\mathsf{DL}}$:

$$\tau = \frac{p_l}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}+\mathbf{n}))_l}, \quad l = 1,\ldots,L, \tag{11}$$

$$\mathbf{w}_j^{\mathsf{T}}\mathbf{p} = \bar{P}_j, \tag{12}$$

$$\tau = \frac{q_l}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}^{\mathsf{T}}\mathbf{q}+\mathbf{w}_j))_l}, \quad l = 1,\ldots,L, \tag{13}$$

$$\mathbf{1}^{\mathsf{T}}\boldsymbol{\lambda} = 1, \tag{14}$$

$$\mathbf{q} = \frac{\tau\bar{P}_j}{\lambda} \cdot \left(\frac{\lambda_1\hat{\beta}_1}{p_1},\ldots,\frac{\lambda_L\hat{\beta}_L}{p_L}\right)^{\mathsf{T}}, \tag{15}$$

where (13) was obtained from $\frac{\partial\mathcal{L}}{\partial p_l}$ and we introduced the auxiliary variable $\mathbf{q}$. Consider rewriting (11)-(15) as follows. First, rewrite (13) in vector form as

$$\mathbf{F}^{\mathsf{T}}\mathbf{q} + \mathbf{w}_j = \frac{1}{\tau}\mathrm{diag}(\hat{\boldsymbol{\beta}})^{-1}\mathbf{q}. \tag{16}$$

Taking the inner product with $\mathbf{p}$, we get

$$\mathbf{q}^{\mathsf{T}}\mathbf{F}\mathbf{p} + \mathbf{w}_j^{\mathsf{T}}\mathbf{p} = \frac{1}{\tau}\mathbf{q}^{\mathsf{T}}\mathrm{diag}(\hat{\boldsymbol{\beta}})^{-1}\mathbf{p}. \tag{17}$$

Substituting for $\mathbf{F}\mathbf{p}$ and $\mathbf{w}_j^{\mathsf{T}}\mathbf{p}$ from (11) and (12) gives

$$\mathbf{n}^{\mathsf{T}}\mathbf{q} = \bar{P}_j. \tag{18}$$

Hence, we can rewrite (11)-(15) as

$$\tau = \frac{q_l}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}^{\mathsf{T}}\mathbf{q}+\mathbf{w}_j))_l}, \quad l = 1,\ldots,L, \tag{19}$$

$$\mathbf{n}^{\mathsf{T}}\mathbf{q} = \bar{P}_j, \tag{20}$$

$$\tau = \frac{p_l}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}+\mathbf{n}))_l}, \quad l = 1,\ldots,L, \tag{21}$$

$$\mathbf{1}^{\mathsf{T}}\boldsymbol{\lambda} = 1, \tag{22}$$

$$\mathbf{p} = \frac{\tau\bar{P}_j}{\lambda} \cdot \left(\frac{\lambda_1\hat{\beta}_1}{q_1},\ldots,\frac{\lambda_L\hat{\beta}_L}{q_L}\right)^{\mathsf{T}}. \tag{23}$$

Comparing (19)-(23) with (11)-(15), it can be seen that (19)-(23) can be obtained from (11)-(15) by making the following substitutions:

$$\begin{array}{c}\text{Uplink-Downlink} \\ \text{Duality Mapping}\end{array} : \begin{cases} \mathbf{p} & \longleftrightarrow & \mathbf{q}, \\ \mathbf{w}_j & \longleftrightarrow & \mathbf{n}, \\ \mathbf{n} & \longleftrightarrow & \mathbf{w}_j, \\ \mathbf{F} & \longleftrightarrow & \mathbf{F}^{\mathsf{T}}, \\ \bar{P}_j & \longleftrightarrow & \bar{P}_j. \end{cases} \tag{24}$$

Hence, (19)-(23) are the necessary and sufficient conditions for the optimal solution to the max-min weighted SIR problem of a hypothetical network where the network variables are defined by the duality mappings in (24).[1] We state this result in the following theorem.

Theorem 1: The optimal value of the single-constraint *downlink* subproblem $\mathcal{S}_j^{\mathsf{DL}}$ is also the optimal value of an *uplink* problem given by

$$\mathcal{S}_j^{\mathsf{UL}} = \begin{cases} \max\limits_{\mathbf{q}} & \min\limits_{l} \frac{\mathsf{SIR}_l^{\mathsf{UL},j}(\mathbf{q})}{\beta_l} \\ \text{subject to} & \mathbf{n}^{\mathsf{T}}\mathbf{q} \leq \bar{P}_j, \quad \mathbf{q} > \mathbf{0}, \end{cases} \tag{25}$$

where the dual uplink SIR is given by

$$\mathsf{SIR}_l^{\mathsf{UL},j}(\mathbf{q}) = \frac{q_l G_{ll}}{\sum_{i\neq l} q_i G_{il} + w_{jl}}. \tag{26}$$

Furthermore, the optimal primal and dual variables are related by

$$\boldsymbol{\lambda}_* = \frac{\lambda_*}{\tau_*\bar{P}_j}\mathbf{p}_* \circ \mathrm{diag}(\hat{\boldsymbol{\beta}})^{-1}\mathbf{q}_*. \tag{27}$$

It is straightforward to see that the hypothetical network in $\mathcal{S}_j^{\mathsf{UL}}$ can be interpreted as the dual uplink network of $\mathcal{S}_j^{\mathsf{DL}}$. Denote the transmitter and receiver of the lth stream in the hypothetical network by $\tilde{t}_l$ and $\tilde{r}_l$ respectively. From (25), it can be seen that the effective link gain between $\tilde{t}_i$ and $\tilde{r}_l$ is given by $G_{il} = |\mathbf{v}_i^{\dagger}\mathbf{H}_{il}\mathbf{u}_l|^2 = |\mathbf{u}_l^{\dagger}\mathbf{H}_{il}^{\dagger}\mathbf{v}_i|^2$. Hence, it can be inferred that $\mathbf{v}_i$ and $\mathbf{u}_l$ now have the role of transmit and receive beamformers of the ith and lth stream respectively. Moreover, since the channel between $\tilde{t}_i$ and $\tilde{r}_l$ is given by $\mathbf{H}_{il}^{\dagger}$, which is the conjugate transposed channel matrix between t_l and r_i, the transmitters and receivers in the original downlink network now have the role of receivers and transmitters respectively in the hypothetical network. Hence, we conclude that $\mathcal{S}_j^{\mathsf{UL}}$ is the dual uplink problem of the original downlink problem $\mathcal{S}_j^{\mathsf{DL}}$. The dual uplink power is denoted by $\mathbf{q}$. Moreover, the uplink receiver of the lth stream suffers from

[1] A more illuminating result is given in Section IV-A using spectral radius relations.

a noise power of w_{jl}, and q_l is weighted by n_l in the power constraint. In fact, since Theorem 1 holds for any given $\mathbb{U}$, $\mathbb{V}$, $\boldsymbol{\beta}$, $\mathbf{n}$, $\mathbf{w}_j$, and $\bar{P}_j$, it can be generalized to the network duality result in [5] which was derived based on studying the total power minimization problem.

IV. Multiple Power Constraints

In this section, we study the multiple-constraint max-min weighted SIR problem $\mathcal{S}^{\mathsf{DL}}$. We begin in Section IV-A by deriving the closed-form solution to $\mathcal{S}^{\mathsf{DL}}$ based on nonlinear Perron-Frobenius theory. Next, we exploit the structure of the closed-form solution to obtain the relationship between $\mathcal{S}^{\mathsf{DL}}$ and its single-constraint subproblems $\mathcal{S}_j^{\mathsf{DL}}$, and we eventually establish the dual uplink problem for $\mathcal{S}^{\mathsf{DL}}$. Then, in Section IV-B, we further exploit nonlinear Perron-Frobenius theory to derive an algorithm for $\mathcal{S}^{\mathsf{DL}}$ which converges geometrically fast to the optimal solution.

A. Optimal Solution and Duality

In this section, we derive the optimal solution and the dual uplink problem of $\mathcal{S}^{\mathsf{DL}}$. Unfortunately, unlike in the single-constraint case, $\mathcal{S}^{\mathsf{DL}}$ does not straightforwardly lead to a dual uplink problem. Nevertheless, we can extend Theorem 1 to a dual uplink result for $\mathcal{S}^{\mathsf{DL}}$.

Define $\mathbf{B}_j = \mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}+(1/\bar{P}_j)\mathbf{n}\mathbf{w}_j^{\mathsf{T}})$. By exploiting nonlinear Perron-Frobenius theory [7] and the algebraic structure of $\mathcal{S}^{\mathsf{DL}}$, we derive a closed-form expression for the optimal solution of $\mathcal{S}^{\mathsf{DL}}$, which we give in the following theorem.

Theorem 2: Let $k = \arg\min_j (1/\rho(\mathbf{B}_j))$. The optimal objective and solution of $\mathcal{S}^{\mathsf{DL}}$ is given by

$$\mathcal{S}^{\mathsf{DL}} = 1/\rho(\mathbf{B}_k), \tag{28}$$

$$\mathbf{p} = (\bar{P}_k/\mathbf{w}_k^{\mathsf{T}}\mathbf{x}(\mathbf{B}_k))\mathbf{x}(\mathbf{B}_k). \tag{29}$$

Proof: Refer to Appendix A.

A key step in the proof of Theorem 2 is the introduction of a weighted maximum norm in order to reformulate the multiple power constraints as a single norm constraint. We demonstrate later that this reformulation is also useful in studying the convex reformulation of $\mathcal{S}^{\mathsf{DL}}$.

Using Theorem 2, it is easily seen that the optimal value and solution of the single-constraint subproblem $\mathcal{S}_j^{\mathsf{DL}}$ are given by $1/\rho(\mathbf{B}_j)$ and $(\bar{P}_j/\mathbf{w}_j^{\mathsf{T}}\mathbf{x}(\mathbf{B}_j))\mathbf{x}(\mathbf{B}_j)$ respectively. Hence, it can be inferred that $\mathcal{S}^{\mathsf{DL}}$ can be decoupled into J single-constraint subproblems $\mathcal{S}_j^{\mathsf{DL}}$, where the optimal solution to $\mathcal{S}^{\mathsf{DL}}$ is given by the optimal solution of the subproblem with the lowest optimal value. This observation, together with Theorem 1, leads us to the following result.

Theorem 3: The optimal value of the multiple-constraint *downlink* problem $\mathcal{S}^{\mathsf{DL}}$ can be obtained from at least one of J *downlink* problems or one of J *uplink* problems. Specifically,

$$\mathcal{S}^{\mathsf{DL}} = \min_j \mathcal{S}_j^{\mathsf{DL}} = \min_j \mathcal{S}_j^{\mathsf{UL}}. \tag{30}$$

Remark 1: In [8], the authors studied the max-min weighted SIR problem subject to a total power constraint and provided a simple derivation of uplink-downlink duality by applying spectral radius relations. We demonstrate that a similar line of analysis applies to $\mathcal{S}^{\mathsf{DL}}$. Using the fact that $\rho(\mathbf{A}) = \rho(\mathbf{A}^{\mathsf{T}})$ and $\rho(\mathbf{AC}) = \rho(\mathbf{CA})$ for any irreducible nonnegative matrices $\mathbf{A}$ and $\mathbf{C}$, we have

$$\rho(\mathbf{B}_k) = \rho(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F} + (1/\bar{P}_k)\mathbf{n}\mathbf{w}_k^{\mathsf{T}})) \tag{31}$$

$$= \rho(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}^{\mathsf{T}} + (1/\bar{P}_k)\mathbf{w}_k\mathbf{n}^{\mathsf{T}})), \tag{32}$$

where k was defined in Theorem 2. (31) and (32) implies that an uplink channel with noise vector $\mathbf{w}_j$ and power constraint $\mathbf{n}^{\mathsf{T}}\mathbf{q} \le \bar{P}_k$ achieves the same max-min weighted SIR as $\mathcal{S}^{\mathsf{DL}}$. This duality relationship is illustrated in Fig. 1.

Next, we study the convex reformulation of $\mathcal{S}^{\mathsf{DL}}$ and exploit the weighted maximum norm introduced in the proof of Theorem 2 to relate Theorem 2 and 3 with Section III. By taking similar steps as in Section III, we can reformulate $\mathcal{S}^{\mathsf{DL}}$ as a convex problem. From the partial Lagrangian of the convex problem, we obtain the dual problem given by

$$\max_{\boldsymbol{\mu}\ge\mathbf{0},\mathbf{1}^{\mathsf{T}}\boldsymbol{\mu}=1} \ \min_{\|\mathbf{p}\|_\xi=1} \sum_{l=1}^{L} \mu_l \log\left(\frac{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}+\mathbf{n}))_l}{p_l}\right), \tag{33}$$

where $\|\cdot\|_\xi$ is defined in Appendix A. Applying nonlinear Perron-Frobenius minimax characterization [**?**], the optimal value of (33) is given by $\log\rho(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}+\mathbf{n}\mathbf{c}_*^{\mathsf{T}})$ where

$$\mathbf{c}_* = \arg\max_{\|\mathbf{c}\|_\xi^{\mathsf{D}}=1} \log\rho(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}+\mathbf{n}\mathbf{c}^{\mathsf{T}})) \tag{34}$$

$$= \arg\max_{\mathbf{w}_j\forall j} \rho(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}+(1/\bar{P}_j)\mathbf{n}\mathbf{w}_j^{\mathsf{T}})). \tag{35}$$

Furthermore, the optimal $\mathbf{p}$ is given by $\mathbf{p}_* = \mathbf{x}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}+\mathbf{n}\mathbf{c}_*^{\mathsf{T}}))$ normalized such that $\|\mathbf{p}_*\|_\xi = 1$, and the optimal $\boldsymbol{\mu}$ is given by $\boldsymbol{\mu}_* = \mathbf{x}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}+\mathbf{n}\mathbf{c}_*^{\mathsf{T}})) \circ \mathbf{y}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}+\mathbf{n}\mathbf{c}_*^{\mathsf{T}}))$ normalized such that $\mathbf{1}^{\mathsf{T}}\boldsymbol{\mu}_* = 1$. Finally, the optimal dual uplink power $\mathbf{q}$ is equal to $\mathrm{diag}(\boldsymbol{\beta})\mathbf{y}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}+\mathbf{n}\mathbf{c}_*^{\mathsf{T}}))$ up to a scaling constant. Hence, $\boldsymbol{\mu}_*$ can also be written as

$$\boldsymbol{\mu}_* = \mathbf{p}_* \circ \mathrm{diag}(\hat{\boldsymbol{\beta}})^{-1}\mathbf{q}_*, \tag{36}$$

where the equality denotes equality up to a scaling constant. Comparing (36) with (27), we see that $\boldsymbol{\mu}_* = \boldsymbol{\lambda}_*$ where $\boldsymbol{\lambda}_*$ is the dual variable of the single-constraint subproblem $\mathcal{S}_k^{\mathsf{DL}}$ and k was defined in Theorem 2.

B. Coordinated Power Control Algorithm

In this section, we further exploit the nonlinear Perron-Frobenius theory [7], [9] to give an iterative algorithm for $\mathcal{S}^{\mathsf{DL}}$.

Algorithm 1: Coordinated max-min weighted SIR

1) Update auxiliary variables $\boldsymbol{\lambda}^{(n+1)}$:
$$p_l^{(n+1)} = \left(\frac{\beta_l}{\mathsf{SIR}_l^{\mathsf{DL}}(\mathbf{p}^{(n)})}\right) p_l^{(n)}, \quad l = 1, \ldots, L.$$
2) Update power $\mathbf{p}^{(n+1)}$:
$$\mathbf{p}^{(n+1)} \leftarrow \mathbf{p}^{(n+1)} \times \min_j \frac{\bar{P}_j}{\mathbf{w}_j^{\mathsf{T}}\mathbf{p}^{(n+1)}}.$$

Theorem 4: Starting from any initial point $\mathbf{p}^{(0)}$, $\mathbf{p}^{(n)}$ in Algorithm 1 converges geometrically fast to the optimal solution (29) given in Theorem 2 with a monotonically increasing minimum weighted SIR, $\min_l(\mathsf{SIR}_l^{\mathsf{DL}}(\mathbf{p}^{(n)})/\beta_l)$.

Proof: Refer to Appendix B.

Remark 2: Observe that step 1 of the algorithm resembles the distributed power control (DPC) algorithm and step 2 simply normalizes the auxiliary variables such that all the power constraints are satisfied.

V. Application in a Multicell System

In this section, we apply the results on $\mathcal{S}^{\mathrm{DL}}$ established in the previous section to a cellular network with J cells in which the jth base station is subject to a sum power constraint $\bar{P}_j$. Hence, the entries of the weight vectors are either 0 or 1, and the indexes of the nonzero entries of the jth weight vector indicate the links associated with the jth base station. Let $\mathcal{I}_j$ contain the indexes of the nonzero entries of the jth weight vector. Step 2 of Algorithm 1 can be rewritten as

$$\mathbf{p}^{(n+1)} \leftarrow \mathbf{p}^{(n+1)} \times \min_j \frac{\bar{P}_j}{\sum_{l\in\mathcal{I}_j} p_l^{(n+1)}}. \quad (37)$$

It is easily seen that $\bar{P}_j / \sum_{l\in\mathcal{I}_j} p_l^{(n+1)}$ can be completely computed at the jth base station and it represents that base station's effective normalization factor (ENF). Hence, Algorithm 1 requires the base stations to share their ENFs at each time step and normalize their transmit powers according to the minimum ENF. This coordination could be carried out using high-speed backhaul links and gossip algorithms.

In practice, the large size of the network imply that having the base stations *fully coordinate* their power updates could lead to high latencies. Hence, it is natural to consider the simpler approach with *partial coordination* in which each base station coordinates with only nearby base stations, e.g., each base station could compute the minimum ENF only among itself and adjacent base stations. This idea can be further extended to the approach of *no coordination* in which each base station normalizes by its own ENF, i.e., it seeks to satisfy its own power constraint only. The importance of devising an efficient and practical algorithm makes it imperative that we study the tradeoffs in performance of the various approaches.

We illustrate the performance of all three approaches in a 19-cell network with 2 randomly located users per cell. The cell radius is 1km. Each base station is assumed to have a total power constraint of 10W and each user is served one independent data stream from its base station. It will be assumed for simplicity that every data stream has the same priority, i.e., $\boldsymbol{\beta} = \mathbf{1}$. The base stations are equipped with $N_{t_l} = 4$ antennas and each user is equipped with $N_{r_l} = 2$ antennas. The base stations and the users are assumed to carry out maximum ratio transmission (MRT) and maximum ratio combining (MRC) beamforming respectively. The noise power spectral density is set to -162dBm/Hz. Each user communicates with the base station over independent MIMO Rayleigh fading channels, with a path loss of $L = 128.1 + 37.6 \log_{10}(d)$dB, where d is the distance in kilometers, and with 8dB log-normal shadowing.

Fig. 2 shows the total power consumption of the network as a function of the iteration index n and Table I shows the final minimum SIR.[2] It can be seen that with no coordination,

[2]It has not been proven that the *partial coordination* and *no coordination* schemes converge.

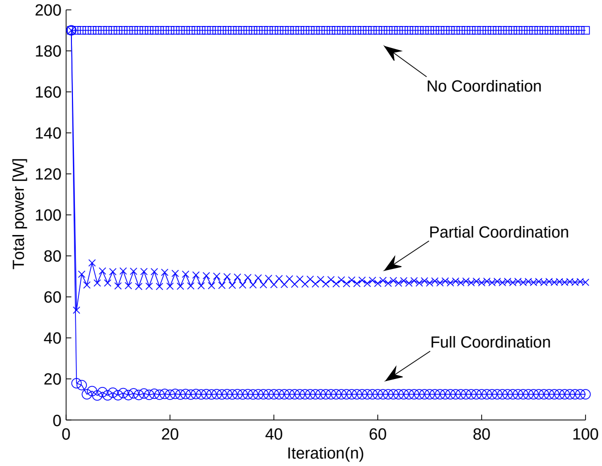


Fig. 2. Total power consumption of a 19-cell network. Each cell is assigned a total power constraint of 10W.

TABLE I
Improvement Over No Coordination

Scheme	Improvement in minimum SIR
Partial Coordination	11.2864dB
Full Coordination	13.1885dB

each base station maintains a constant total transmit power of 10W throughout the iterations, while with full coordination, only one base station transmits with a total power of 10W and the other base stations transmit with less power to reduce the interference on users in neighboring cells. This back-off behavior enables the network to achieve a minimum SIR 13dB higher while using 170W less power than having no coordination. With partial coordination, in which each base station shares its ENF with adjacent base stations only, the minimum SIR is 10dB higher than having no coordination while using 120W less power.

VI. Conclusion

We conducted a study of the max-min weighted SIR problem in the downlink subject to multiple weighted-sum power constraints. First, we provided a derivation of uplink-downlink duality in the context of the max-min weighted SIR problem by applying Lagrange duality. Then, by representing the multiple weighted-sum power constraints as a weighted-maximum-norm constraint, we applied nonlinear Perron-Frobenius theory to derive a closed-form expression for the optimal solution and a coordinated algorithm with geometric convergence rate. We demonstrated the performance of the proposed algorithm in a multicell scenario in which each base station is subject to its own sum power constraint. In contrast to having each base station optimize its powers independently, the coordinated optimization approach significantly reduced power consumption while achieving a higher minimum SIR for all users.

APPENDIX

A. Proof of Theorem 2

Proof: First, we define the weighted maximum norm $\|\cdot\|_\xi$ on $\mathbb{R}^L$ and its dual norm $\|\cdot\|_\xi^{\mathsf{D}}$ as follows:

$$\|\mathbf{s}\|_\xi = \max_j \left((1/\bar{P}_j) \sum_{l=1}^{L} w_{jl}|s_l| \right), \tag{38}$$

$$\|\mathbf{s}\|_\xi^{\mathsf{D}} = \max_{\|\mathbf{x}\|_\xi=1} |\mathbf{s}^\mathsf{T}\mathbf{x}|. \tag{39}$$

It is clear that $\|\cdot\|_\xi$ satisfies the properties of a norm [10]. Moreover, it is clear that $\|\cdot\|_\xi$ is a monotone norm. The dual of the vector $\mathbf{s} \gneqq \mathbf{0}$, $\|\mathbf{s}\|_\xi = 1$ with respect to $\|\cdot\|_\xi$ is defined as $\{\mathbf{c} \in \mathbb{R}^L : \|\cdot\|_\xi^{\mathsf{D}} = \mathbf{c}^\mathsf{T}\mathbf{s} = 1\}$, which can be rewritten as

$$\left\{ \mathbf{c} \in \mathbb{R}^L : \max_{\|\mathbf{x}\|_\xi=1} |\mathbf{c}^\mathsf{T}\mathbf{x}| = \mathbf{c}^\mathsf{T}\mathbf{s} = 1 \right\}.$$

For any vector $\mathbf{s} \gneqq \mathbf{0}$ such that $\|\mathbf{s}\|_\xi = 1$, there exists at least one vector in the dual of $\mathbf{s}$ which has the form $\mathbf{c}_j = (1/\bar{P}_j)\mathbf{w}_j$ for some $j \in \{1,\dots,J\}$. We can particularize Theorem 6 of [7] in the following lemma:

Lemma 1: Let $\mathbf{A}$ be a nonnegative matrix and $\mathbf{b}$ be a positive vector. Let $\|\cdot\|_\xi$ be the weighted maximum norm and $\|\cdot\|_\xi^{\mathsf{D}}$ be the dual norm as defined in (38) and (39). Let $\mathbf{c}_j = (1/\bar{P}_j)\mathbf{w}_j$ and $k = \arg\max_j \rho(\mathbf{A} + \mathbf{b}\mathbf{c}_j^\mathsf{T})$. If $\rho(\mathbf{A}+\mathbf{b}\mathbf{c}_k^\mathsf{T}) > \rho(\mathbf{A})$, then the conditional eigenvalue problem

$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}, \quad \lambda \in \mathbb{R}, \quad \mathbf{x} \geq \mathbf{0}, \quad \|\mathbf{x}\|_\xi = 1, \tag{40}$$

has a unique solution $(\lambda_*, \mathbf{x}_*)$, where $\lambda_* = \rho(\mathbf{A}+\mathbf{b}\mathbf{c}_k^\mathsf{T})$ and $\mathbf{x}_*$ is the unique normalized Perron vector of $\mathbf{A}+\mathbf{b}\mathbf{c}_k^\mathsf{T}$. Moreover, $(\mathbf{x}_*, \mathbf{c}_k)$ is a dual pair with respect to $\|\cdot\|_\xi$. Also, $\mathbf{x}_*$ is the unique nonnegative solution of the equation

$$\mathbf{x} = \frac{\mathbf{A}\mathbf{x}+\mathbf{b}}{\|\mathbf{A}\mathbf{x}+\mathbf{b}\|_\xi}. \tag{41}$$

Next we will show how Lemma 1 can be used to prove Theorem 2. Let the optimal objective of $\mathcal{S}^{\mathsf{DL}}$ be τ_*, and the optimal variable be $\mathbf{p}_*$. Since the SIR constraints must be tight at optimality, we have

$$(1/\tau_*)\mathbf{p}_* = \mathrm{diag}(\hat{\boldsymbol{\beta}})\mathbf{F}\mathbf{p}_* + \mathrm{diag}(\hat{\boldsymbol{\beta}})\mathbf{n}, \tag{42}$$

where $(1/\tau_*) \in \mathbb{R}$ and $\mathbf{p}_* \geq \mathbf{0}$. Moreover, since at least one power constraint in $\mathcal{S}^{\mathsf{DL}}$ must be tight, we have that $\|\mathbf{p}_*\|_\xi = 1$. Applying Lemma 1 to (42) and using the fact that $\mathbf{c}_k$ and the normalized Perron vector of $\mathbf{B}_k$ form a dual pair, we complete the proof of Theorem 2. □

B. Proof of Theorem 4

Proof: From the proof of Theorem 2, it is easily seen that $\mathcal{S}^{\mathsf{DL}}$ is equivalent to solving the conditional eigenvalue problem given by (42), where $(1/\tau_*) \in \mathbb{R}$, $\mathbf{p}_* \geq \mathbf{0}$, and $\|\mathbf{p}_*\|_\xi = 1$. Applying Theorem 1 of [9] to (42), we conclude that Algorithm 1 converges geometrically fast to $\mathbf{p}_*$.

Moreover, it can be shown that Algorithm 1 converges with a monotonically increasing minimum weighted SIR, $\min_l(\mathsf{SIR}_l^{\mathsf{DL}}(\mathbf{p}^{(n)})/\beta_l)$. First we prove the following result.

Theorem 5: Consider the fixed-point iteration $\mathbf{p}^{(n+1)} = (1/\|f(\mathbf{p}^{(n)})\|)f(\mathbf{p}^{(n)})$, where $\|\cdot\|$ denotes a monotone norm on $\mathbb{R}^L$. Define $\tau^{(n)} = \min_i(p_i^{(n)}/f_i(\mathbf{p}^{(n)}))$. If $f_i(\mathbf{p})$ is concave, i.e., $f(\alpha\mathbf{x}+(1-\alpha\mathbf{y})) \geq \alpha f(\mathbf{x})+(1-\alpha)f(\mathbf{y})$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}_+^L$ and all $\alpha \in [0,1]$, and monotone, i.e., $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y}$ implies $\mathbf{0} \leq f(\mathbf{x}) \leq f(\mathbf{y})$, then

$$\tau^{(n+1)} \geq \tau^{(n)}, \tag{43}$$

for all n, with equality if and only if $\tau^{(n)} = p_j^{(n)}/f_j(\mathbf{p}^{(n)})$ for all j.

Proof: First, we consider the case when $\tau^{(n)} < p_j^{(n)}/f_j(\mathbf{p}^{(n)})$ for some j. Denote as $\mathbf{x} \geq_* \mathbf{y}$ the componentwise inequality between any two vectors $\mathbf{x} \in \mathbb{R}^L$ and $\mathbf{y} \in \mathbb{R}^L$ such that at least one of the inequalities is satisfied with equality. From the definition of $\tau^{(n)}$, we have

$$\mathbf{p}^{(n)} \geq_* \tau^{(n)} f(\mathbf{p}^{(n)}). \tag{44}$$

Note that $\geq_*$ possesses all the properties of the standard inequality symbol $\geq$. (44) can be rewritten as

$$\mathbf{p}^{(n)} \geq_* \tau^{(n)} \|f(\mathbf{p}^{(n)})\| \mathbf{p}^{(n+1)}. \tag{45}$$

Taking the norm on both sides, we have that $\tau^{(n)}\|f(\mathbf{p}^{(n)})\| < 1$. Hence, due to the concavity and monotonicity of $f(\mathbf{p})$, (45) can be rewritten as

$$\mathbf{p}^{(n+1)} > \tau^{(n)} \|f(\mathbf{p}^{(n+1)})\| \mathbf{p}^{(n+2)}. \tag{46}$$

However, note that (45) written at the $(n+1)$th time step gives

$$\mathbf{p}^{(n+1)} \geq_* \tau^{(n+1)} \|f(\mathbf{p}^{(n+1)})\| \mathbf{p}^{(n+2)}. \tag{47}$$

From (46) and (47), it is necessary that $\tau^{(n+1)} > \tau^{(n)}$. Equality in (43) is proved by applying the same arguments to $\mathbf{p}^{(n)} = \tau^{(n)} f(\mathbf{p}^{(n)})$. □

The rest of Theorem 4 is proved by recognizing that Algorithm 1 can be written as the fixed-point iteration in Theorem 5 using $f(\mathbf{p}^{(n)}) = \mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}^{(n)}+\mathbf{n})$ and $\|\cdot\|_\xi$. The minimum weighted SIR at the nth time step is given by $\tau^{(n)}$. Moreover, $\tau^{(n)}$ is strictly increasing unless $\mathbf{p}^{(n)} = \tau^{(n)} f(\mathbf{p}^{(n)})$, in which case all the weighted SIR's are equal so the algorithm has converged. □

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