When Perron-Frobenius meet Shannon: Nonconvex Power Control in Multiuser Communication Systems

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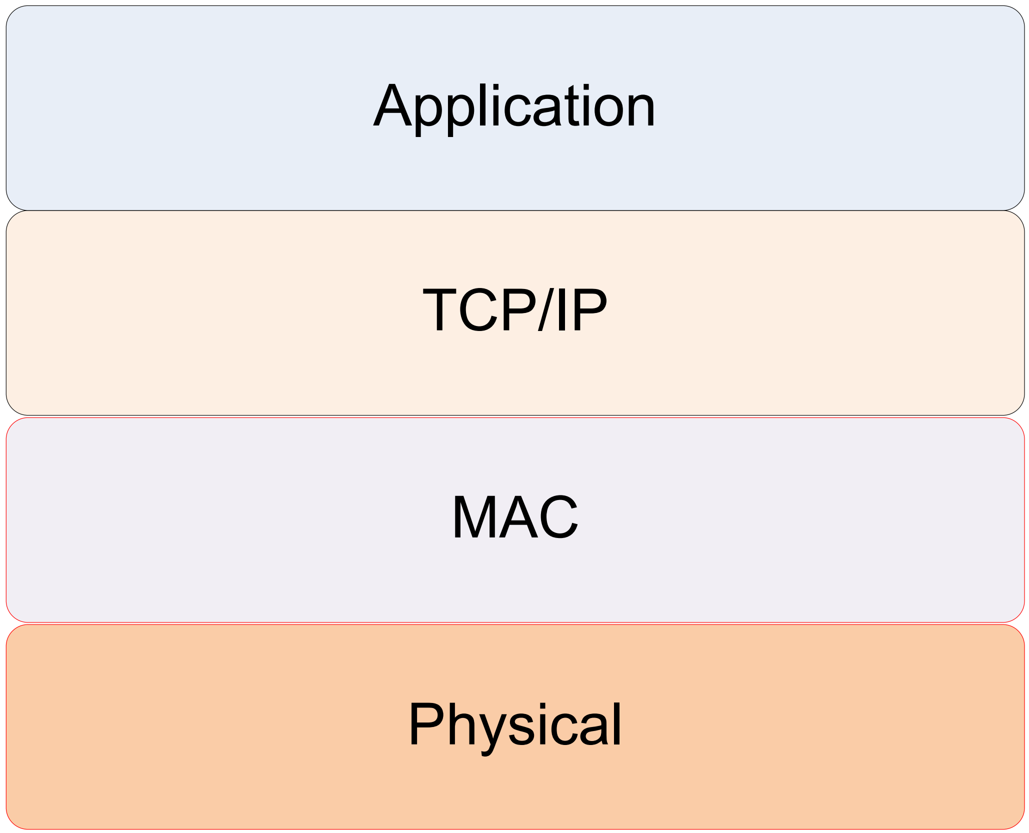
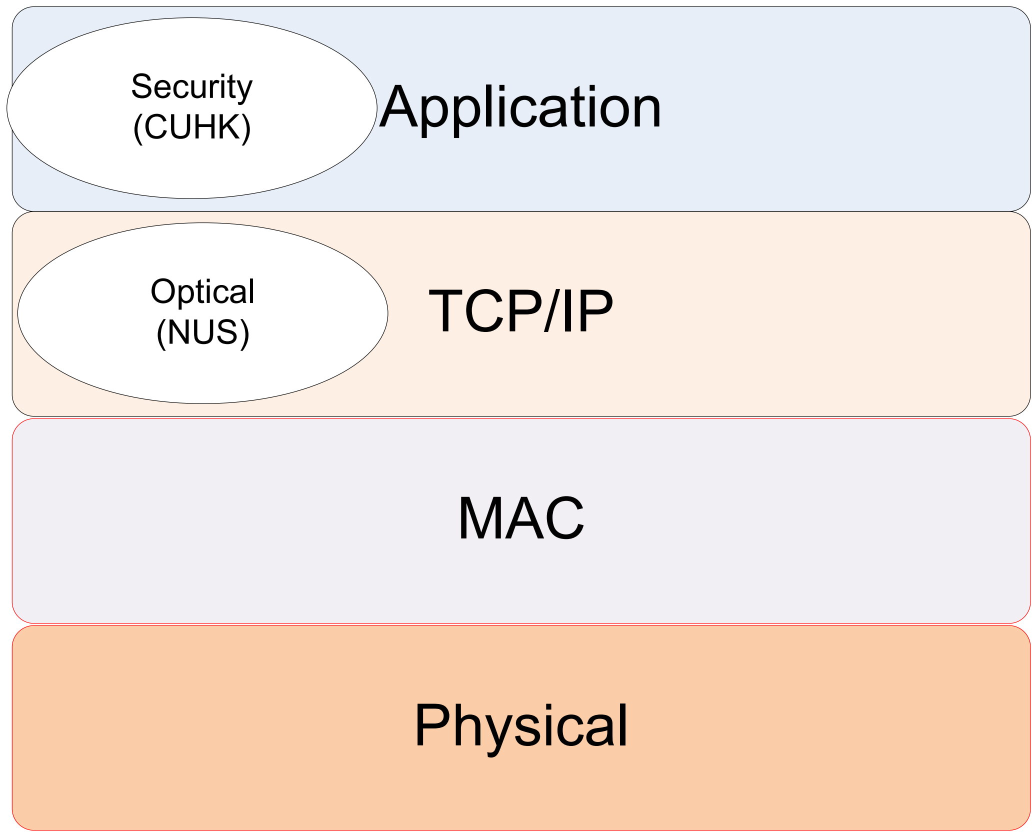
Research



Research



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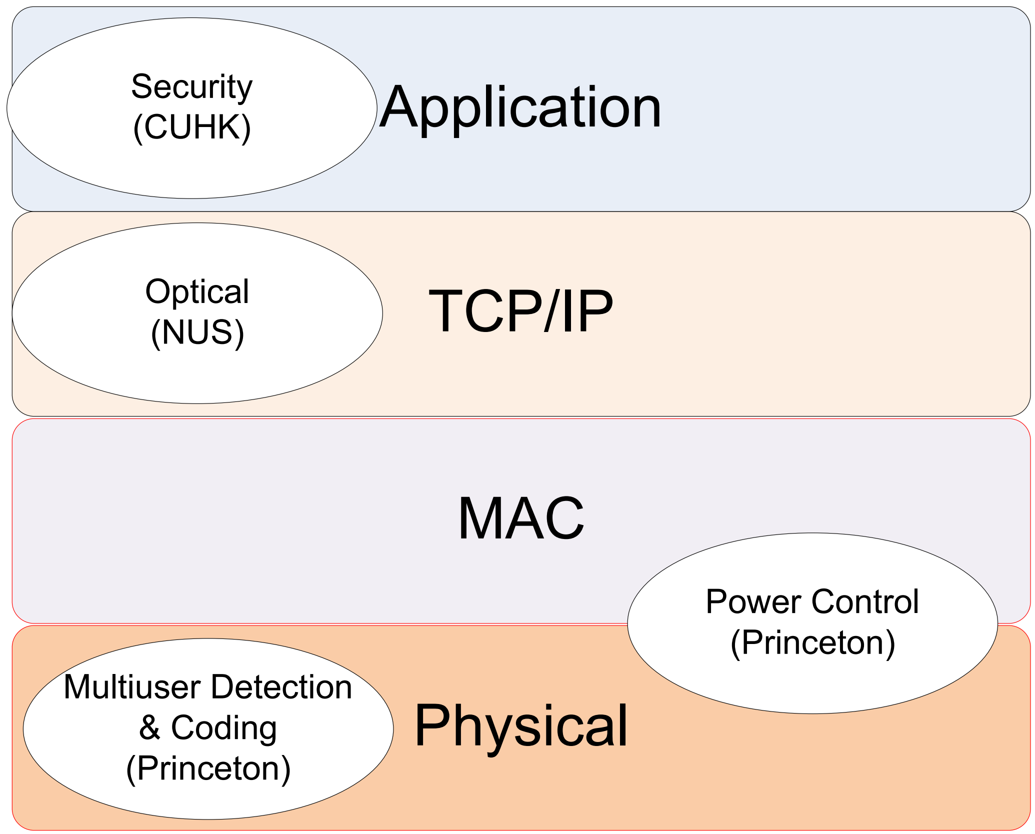
Security
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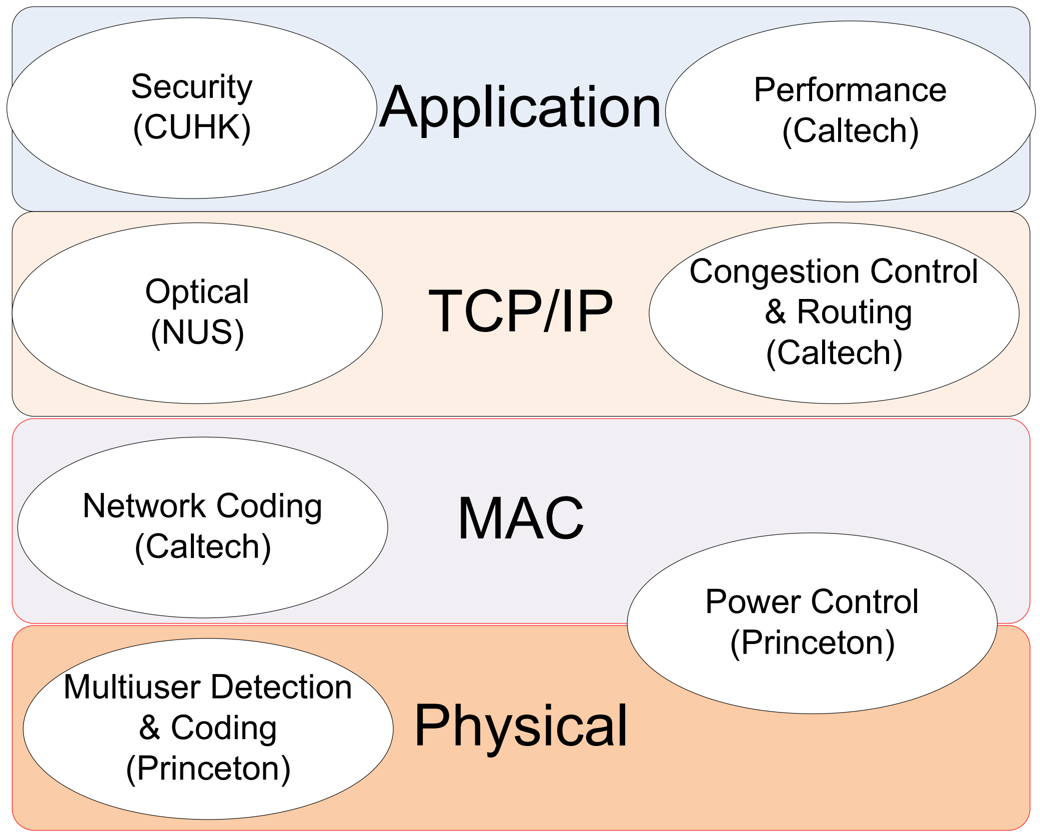
Application
Optical
(NUS)
TCP/IP
MAC
Power Control
(Princeton)
Multiuser Detection
& Coding
(Princeton)
Physical

Research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Outline

- Motivations
- System Model and Basic Power Control Problems
- Nonconvex Power Control Problems
- Power Control Algorithms with Performance Guarantees
- Global Optimization of Sum Shannon Rates
- Balancing Energy Efficiency and Robustness

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What makes a problem easy or hard

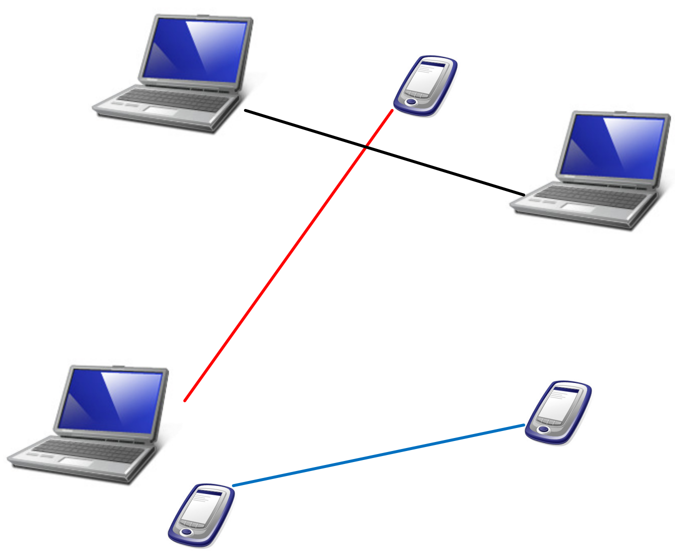
`... the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.`

– *SIAM Review 1993*, R. Rockafellar

- Linear inequality theory & nonconvex integer programming (1947)
- Semidefinite matrix theory & nonconvex quadratic programming (1995)
- Nonnegative matrix theory & nonconvex cone programming (this talk)

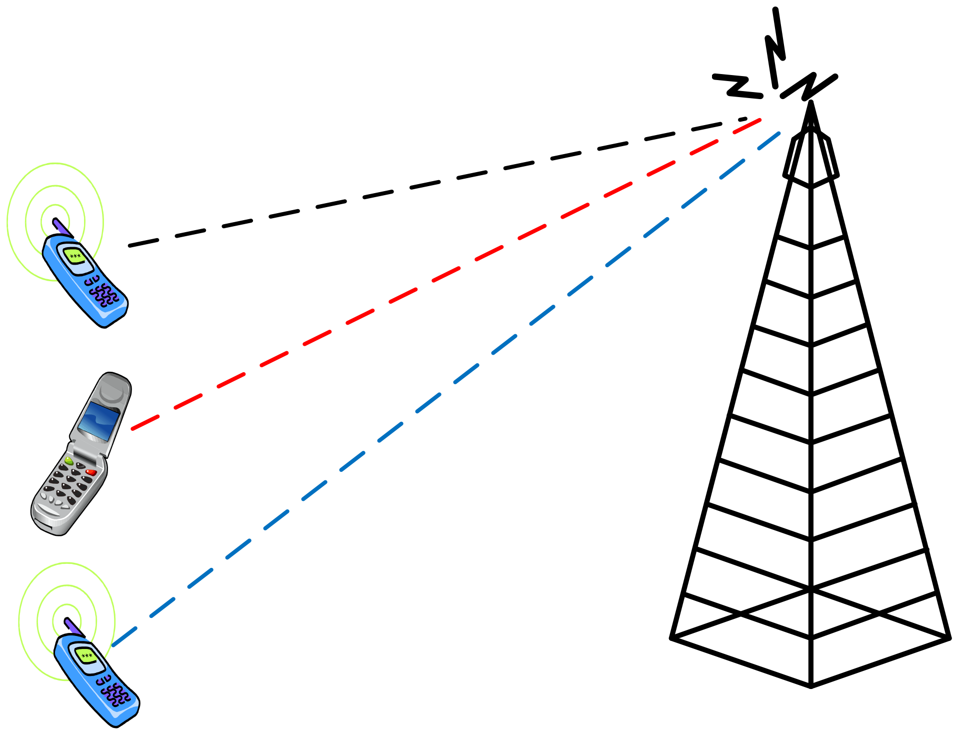
Motivation 1

- IEEE 802.11b ad hoc network cross-layer (TCP/IP/MAC)
- TCP/IP and application layers demand data rate
- MAC/Physical layers build variable capacity 'pipes' as supply



Motivation 2

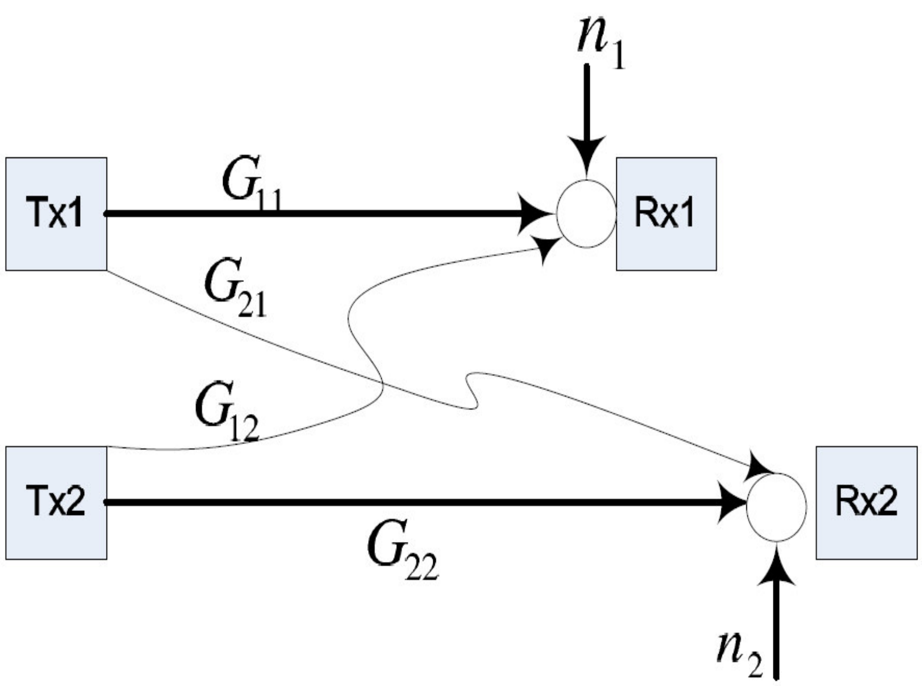
- 3G CDMA2000 EV-DO cellular network link adaptation maximizes uplink/downlink rate using power control



System Model

- Interference channel with single-user decoding: Treat interference as additive Gaussian noise

- Control interference and meet objective using power control



Performance Metrics

- Signal-to-Interference Ratio:

$$\mathsf{SIR}_l(\mathbf{p}) = \frac{G_{ll}p_l}{\sum_{j \neq l} G_{lj}p_j + n_l}$$

  with G_{lj} the channel gains from transmitter j to receiver l and n_l the additive white Gaussian noise (AWGN) power at receiver l

- Attainable data rate (nats per channel use) is a function of SIR, e.g., Shannon capacity formula $r_l = \log(1 + \mathsf{SIR}_l)$

- Mean Squared Error (MSE) of received signal, e.g., $(1 + \mathsf{SIR}_l)^{-1}$

- Power constraints $\mathbf{p} \leq \bar{\mathbf{p}}$

Interference Parameters

- Let $\mathbf{F}$ be a nonnegative matrix with entries:

$$F_{ij} = \begin{cases} 0, & \text{if } i = j \\ \frac{G_{ij}}{G_{ii}}, & \text{if } i \neq j \end{cases}$$

and

$$\mathbf{v} = \left(\frac{n_1}{G_{11}}, \frac{n_2}{G_{22}}, \ldots, \frac{n_L}{G_{LL}} \right)^\top .$$

- $\mathbf{F}$ is irreducible (each user has at least one interferer)

Zander's Max-min SIR Problem 1992

$$\max_{p_l \geq 0\ \forall\, l} \quad \min_l \frac{G_{ll} p_l}{\sum_{j \neq l} G_{lj} p_j}.$$

J. Zander, *Distributed Cochannel Interference Control in Cellular Radio Systems*,

IEEE Trans. Vehicular Technology, 1992

- Wielandt's characterization of spectral radius:

$$\rho(\mathbf{F}) = \max_{\mathbf{p} \geq \mathbf{0}} \ \min_l \frac{(\mathbf{F}\mathbf{p})_l}{p_l} = \min_{\mathbf{p} \geq \mathbf{0}} \ \max_l \frac{(\mathbf{F}\mathbf{p})_l}{p_l}$$

- optimal SIR: $1/\rho(\mathbf{F})$, optimal power: $\mathbf{x}(\mathbf{F})$

Foschini's Power Minimization 1993

$$\begin{array}{ll} \text{minimize} & \sum_l p_l \\ \text{subject to} & \frac{G_{ll} p_l}{\sum_{j \neq l} G_{lj} p_j + n_l} \geq \gamma_l \ \ \forall \, l. \end{array}$$

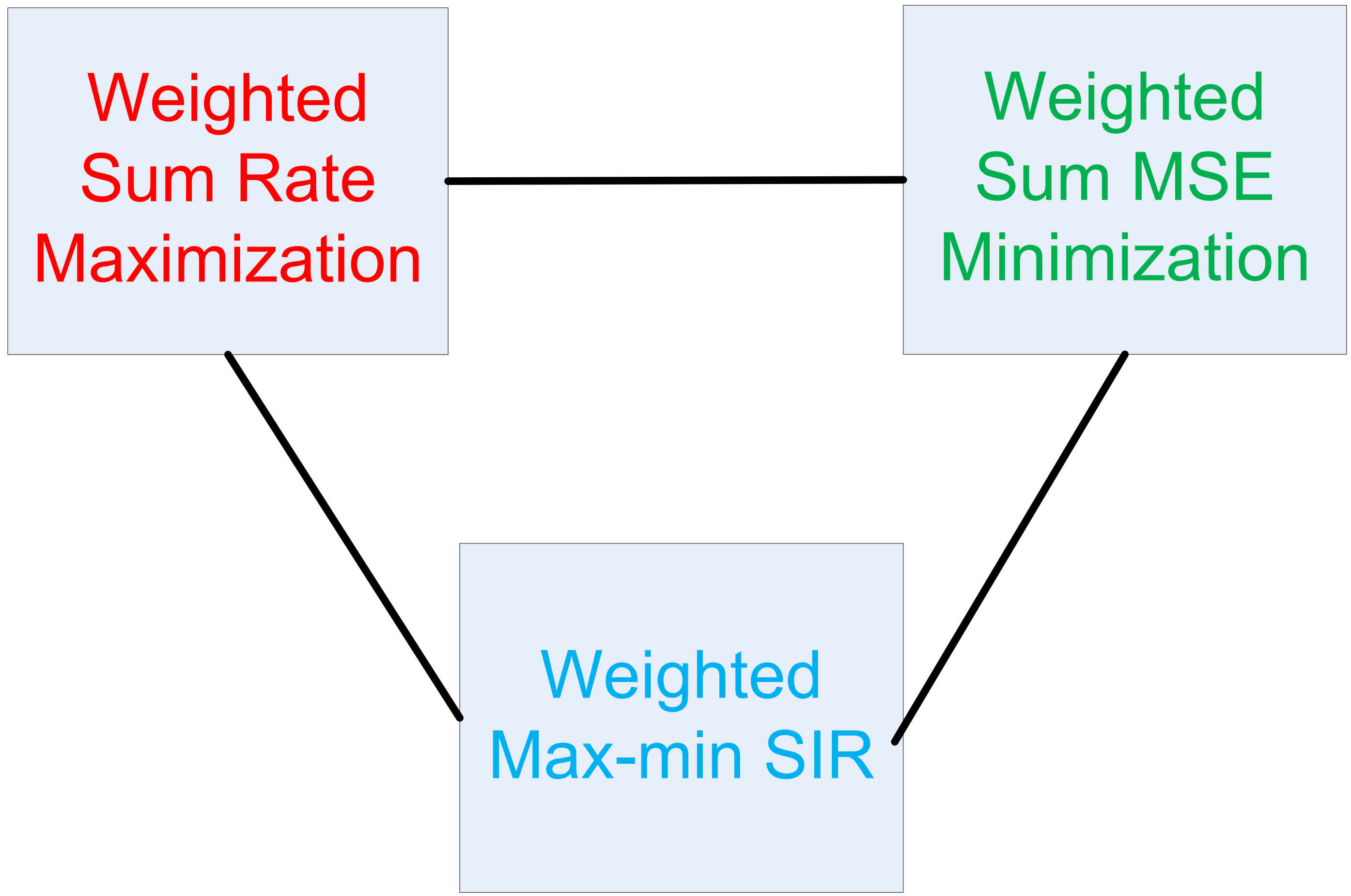
G. J. Foschini and Z. Miljanic, *A Simple Distributed Autonomous Power Control Algorithm and its Convergence*, IEEE Trans. Vehicular Technology, 1993

- Distributed Power Control (DPC) algorithm:

$$p_l(k+1) = \frac{\gamma_l}{\mathsf{SIR}_l(\mathbf{p}(k))} p_l(k) \ \ \forall \, l.$$

- IS-95 CDMA Systems, Qualcomm 3G Systems

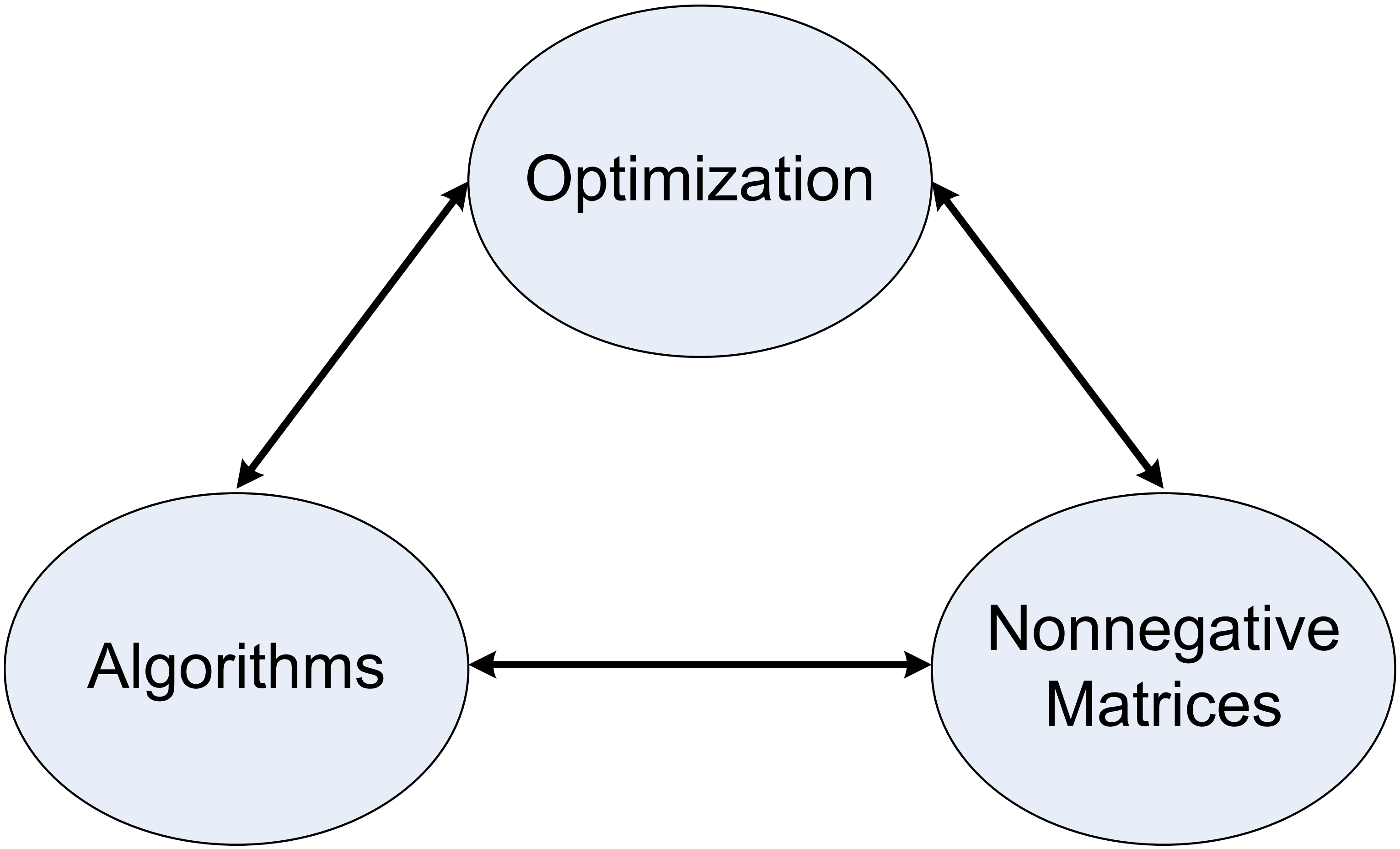
Power Control Problems



System Considerations

- How to solve optimally nonconvex power control problems?
- How many ways to characterize optimality?
- How to design distributed power control algorithms with fast convergence and good performance guarantees?
- How fast is fast?
- Can we leverage existing technology?
- What is the industry impact?

Interplay of Mathematical Tools



Problem: Maximize Sum Shannon Rates

- Find $\mathbf{p}^\star = \arg\max_{\mathbf{0}\le\mathbf{p}\le\bar{\mathbf{p}}} \sum_l w_l \log(1 + \mathsf{SIR}_l(\mathbf{p}))$ where $\mathbf{1}^\top \mathbf{w} = 1$

- Characterize the achievable rate region: $r_l = \log(1 + \mathsf{SIR}_l(\mathbf{p}))\ \forall\, l$

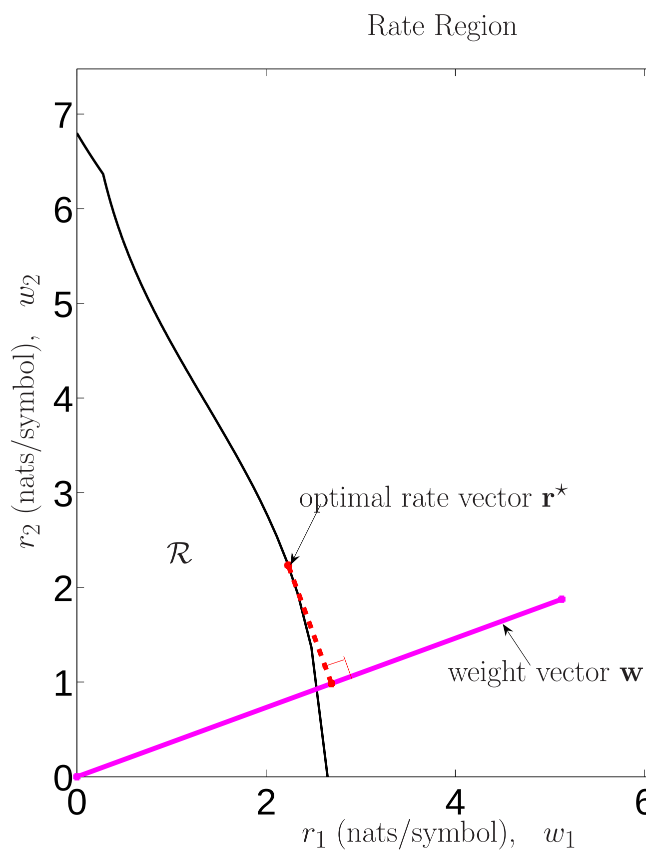
- Two-User case:

$$\max \quad w_1 \log\left(1 + \frac{G_{11}p_1}{G_{12}p_2 + n_1}\right) + w_2 \log\left(1 + \frac{G_{22}p_2}{G_{21}p_1 + n_2}\right)$$

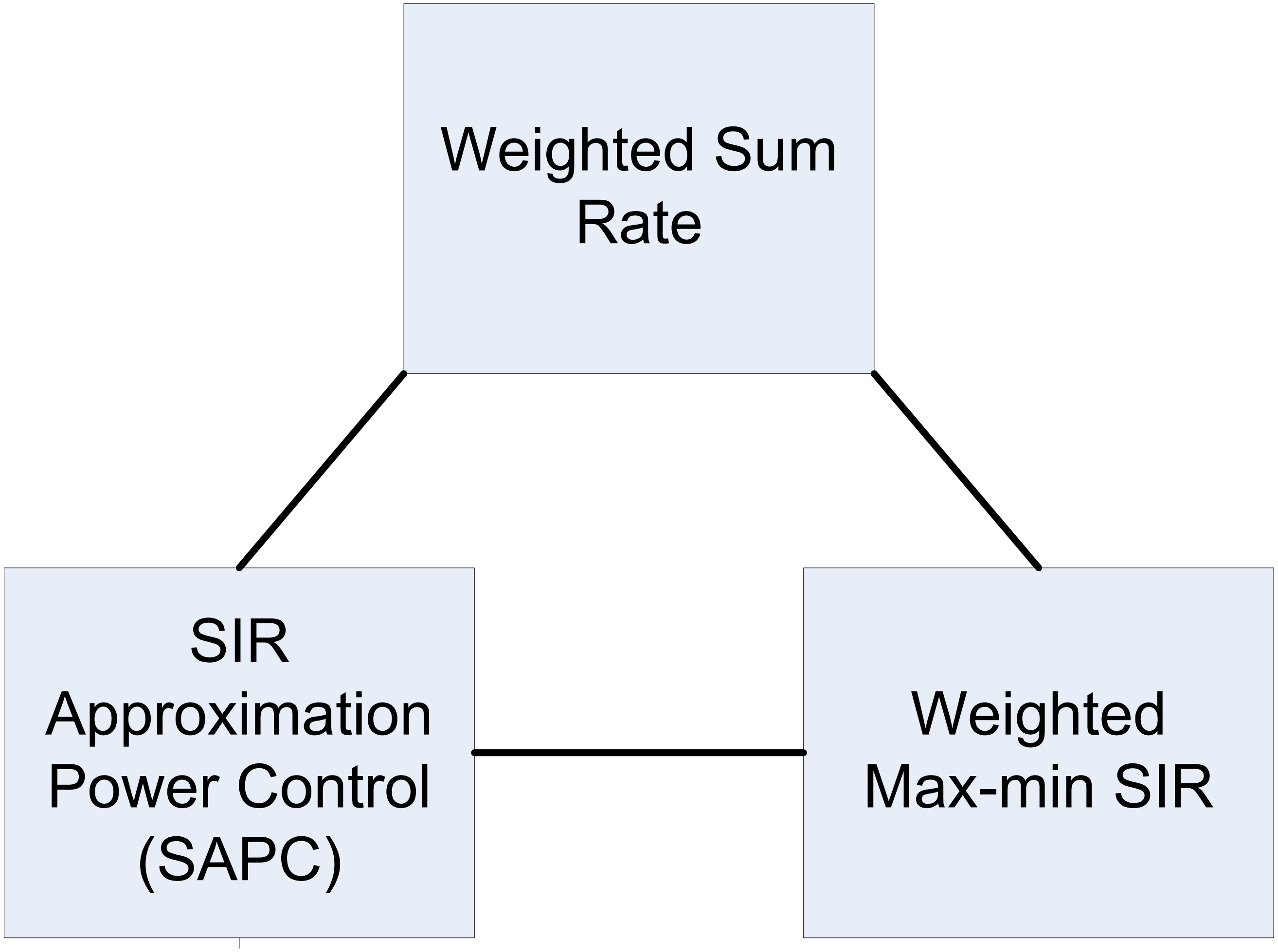
subject to: $\quad 0 \le p_1 \le \bar{p}_1,\ \ 0 \le p_2 \le \bar{p}_2$

Sum Rate Geometry Illustration

$$\begin{array}{rl} \text{maximize} & \sum_l w_l \log(1+\mathsf{SIR}_l(\mathbf{p})) \;=\; \sum_l w_l\, r_l \\ \text{subject to} & 0 \le p_l \le \bar{p}_l \quad \forall\, l, \\ \text{variables:} & p_l \quad \forall\, l. \end{array}$$



Fast Algorithms with Performance Guarantees



Tan, Chiang and Srikant, *Fast Algorithms and Performance Bounds for Sum Rate Maximization in Wireless Networks*, IEEE INFOCOM, 2009

SAPC: New Perspective

- Drop the '1' approach:

$$\begin{array}{rl} \text{maximize} & \sum_l w_l \log \mathsf{SIR}_l(\mathbf{p}) \\ \text{subject to} & 0 \le p_l \le \bar{p}_l \quad \forall\, l, \\ \text{variables:} & p_l \quad \forall\, l. \end{array}$$

- Geometric programming ($\tilde{p}_l = \log p_l$)

  Chiang, Tan, Palomar, O'Neill, Julian, *Power Control by Geometric Programming*, IEEE Trans Wireless Comms, 2007

- 1) Connection with Weighted max-min SIR
  2) New algorithm with faster convergence

SAPC: Algorithm

- **Algorithm 1. [SAPC Algorithm]**

  1. Update $\mathbf{p}(k+1)$*:*

$$p_l(k+1) = \min\left\{ w_l / \left(\sum_{j \neq l} \frac{w_j F_{jl} \mathsf{SIR}_j(\mathbf{p}(k))}{p_j} \right), \bar{p}_l \right\}$$

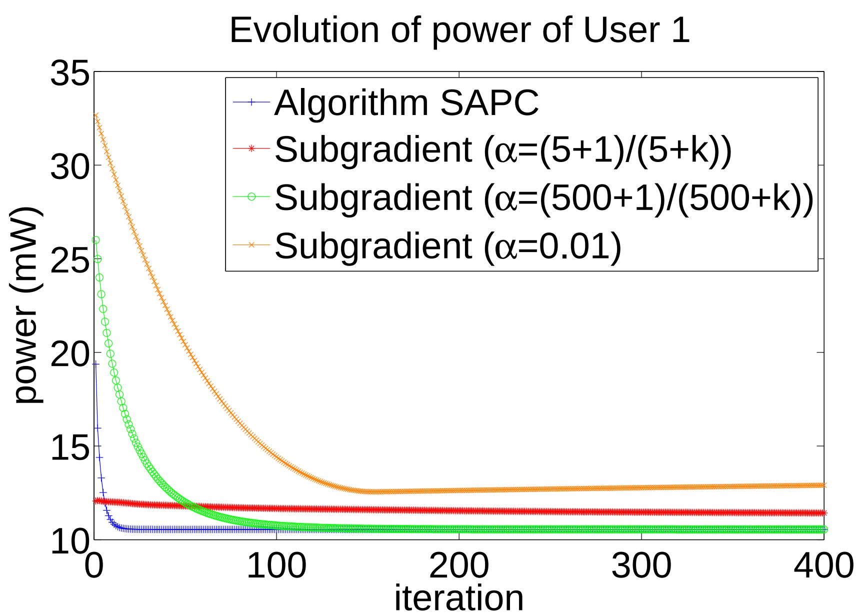
  for all l*, where* k *indexes discrete time slots.*

  Theorem 1. *Starting from any initial point* $\mathbf{p}(0)$*,* $\mathbf{p}(k)$ *in Algorithm 1 converges to* $\mathbf{p}'$ *asymptotically, the optimal solution to SAPC under synchronous and asynchronous updates.*

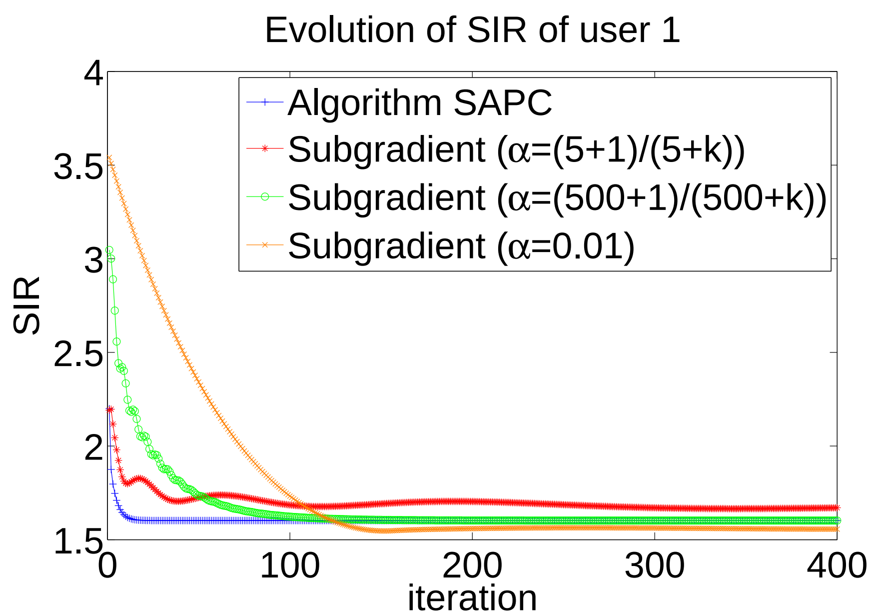
- Geometrically fast when the initial point is $\overline{\mathbf{p}}$.

SAPC: Examples

- Algorithm SAPC is faster than the gradient algorithm (stepsize α)



(a)



(b)

Weighted Max-Min SIR

- Consider $\max_{\mathbf{p}\geq\mathbf{0}} \min_l \frac{\mathsf{SIR}_l(\mathbf{p})}{\beta_l}$ subject to $p_l \leq \bar{p}_l \quad \forall\, l$

- **Theorem 2.** *The optimal solution is such that the value* SIR_l/β_l *for all users are equal. The optimal weighted max-min* SIR *is given by*

$$\gamma^* = \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))},$$

where

$$i = \arg\min_l \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))}.$$

Further, all links i *transmit at peak power* $\bar{p}_i$ *and the rest do not. Further, the optimal* $\mathbf{p}$*, denoted by* $\mathbf{p}^*$*, is*
$t\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))$ *for some constant* $t > 0$.

Max-min SIR: Primal-Dual Algorithm

- **Algorithm 2. [Weighted Max-min** SIR **Algorithm]**

  1. *Initialize an arbitrarily positive* $\mathbf{w}(t)$ *and small* $\epsilon, \alpha(1)$.
  2. *Set* $\mathbf{p}(0) = \bar{\mathbf{p}}$. *Repeat*

$$p_l(k+1) = \min\left\{ w_l(t) \Big/ \left(\sum_{j \neq l} \frac{w_j(t) F_{jl} \mathsf{SIR}_j(\mathbf{p}(k))}{p_j(k))} \right), \bar{p}_l \right\}$$

     until $\|\mathbf{p}(k+1) - \mathbf{p}(k)\| \leq \epsilon$.

  3. *Compute*

$$\begin{aligned} w_l(t+1) = \max\{w_l(t) \quad & +\alpha(t)(\textstyle\sum_j w_j(t) \log(\mathsf{SIR}_j(\mathbf{p}(k+1))/\beta_j) - \\ & \log(\mathsf{SIR}_l(\mathbf{p}(k+1))/\beta_l)),\, 0\} \end{aligned}$$

     for all l, *where* t *indexes discrete time slots much larger than* k.

  4. *Normalize* $\mathbf{w}(t+1)$ *so that* $\mathbf{1}^\top \mathbf{w}(t+1) = 1$. *Go to Step 2.*

Connecting SAPC & Max-min SIR

- Let $\mathbf{x}$ and $\mathbf{y}$ be the Perron and left eigenvectors of $\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top$ respectively, where $i = \arg\min_l \dfrac{1}{\rho(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)}$

- Set $\mathbf{w} = \mathbf{x} \circ \mathbf{y}$ in SAPC:

$$
\begin{array}{rl}
\text{maximize} & \sum_l x_l y_l \log \mathsf{SIR}_l(\mathbf{p}) \\
\text{subject to} & 0 \leq p_l \leq \bar{p}_l \quad \forall\, l, \\
\text{variables:} & p_l \quad \forall\, l.
\end{array}
$$

$\mathbf{p}^* = \mathbf{x}$ (unique up to a scaling constant)

Nonlinear Perron-Frobenius Theory

- Find $(\check{\lambda}, \check{\mathbf{s}})$ in

$$\lambda \mathbf{s} = \mathbf{A}\mathbf{s} + \mathbf{b}, \quad \lambda \in \mathbb{R}, \quad \mathbf{s} \geq \mathbf{0}, \quad \|\mathbf{s}\| = 1,$$

  where $\mathbf{A}$ and $\mathbf{b}$ is a square irreducible nonnegative matrix and nonnegative vector, respectively and $\|\cdot\|$ a monotone vector norm.

- $(\check{\lambda}, \check{\mathbf{s}})$ is the Perron-Frobenius eigenvalue and vector pair of $\mathbf{A} + \mathbf{b}\mathbf{c}_*^\top$, where

$$\mathbf{c}_* = \arg \max_{\|\mathbf{c}\|_*=1} \rho(\mathbf{A} + \mathbf{b}\mathbf{c}^\top),$$

  where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$, and $\check{\mathbf{s}} = (\mathbf{A}\check{\mathbf{s}} + \mathbf{b})/\|\mathbf{A}\check{\mathbf{s}} + \mathbf{b}\|$.

V. D. Blondel, L. Ninove and P. Van Dooren, *An affine eigenvalue problem on the nonnegative orthant*, Linear Algebra & its Applications, 2005

Nonlinear Perron-Frobenius Theory: Max-min SIR

- Individual power constraints ($\bar{p}_1 = \bar{p}_2 = \cdots = \bar{p}_L$):

$$\mathsf{SIR}_l(\mathbf{p}^*) = \tau^* \beta_l \;\Rightarrow\; \frac{(p_l^*/\bar{p}_i)}{\sum_{j \neq l} F_{lj}(p_l^*/\bar{p}_i) + (v_l/\bar{p}_i)} = \tau^* \beta_l$$

Let $\mathbf{s}^* = (1/\bar{p}_i)\mathbf{p}^*$:

$$(1/\tau^*)\mathbf{s}^* = \mathsf{diag}(\boldsymbol{\beta})\mathbf{F}\mathbf{s}^* + (1/\bar{p}_i)\mathsf{diag}(\boldsymbol{\beta})\mathbf{v}, \quad \|\mathbf{s}\|_\infty = 1$$

- ▪ $s_l = p_l/\bar{p}_l$, $\mathbf{A} = \mathrm{diag}(\boldsymbol{\beta})\mathbf{F}$, $\mathbf{b} = (1/\bar{p}_i)\,\mathrm{diag}(\boldsymbol{\beta})\mathbf{v}$ and $\lambda = 1/\tau^*$
  - ▪ $\|\cdot\| = \|\cdot\|_\infty \longleftrightarrow \|\cdot\|_* = \|\cdot\|_1$ & $\mathbf{c}_* = \mathbf{e}_i$
  - ▪ $(\check{\lambda}, \check{\mathbf{s}})$ is the Perron-Frobenius eigenvalue and vector pair of $\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i{}^\top)$

A Faster Max-min SIR Algorithm

- **Algorithm 3. [Equal power constrained Max-min** SIR**]**

  1. *Update power* $\mathbf{p}(k+1)$*:*

$$p_l(k+1) = \frac{\beta_l}{\mathsf{SIR}_l(\mathbf{p}(k))} p_l(k) \ \ \forall\, l.$$

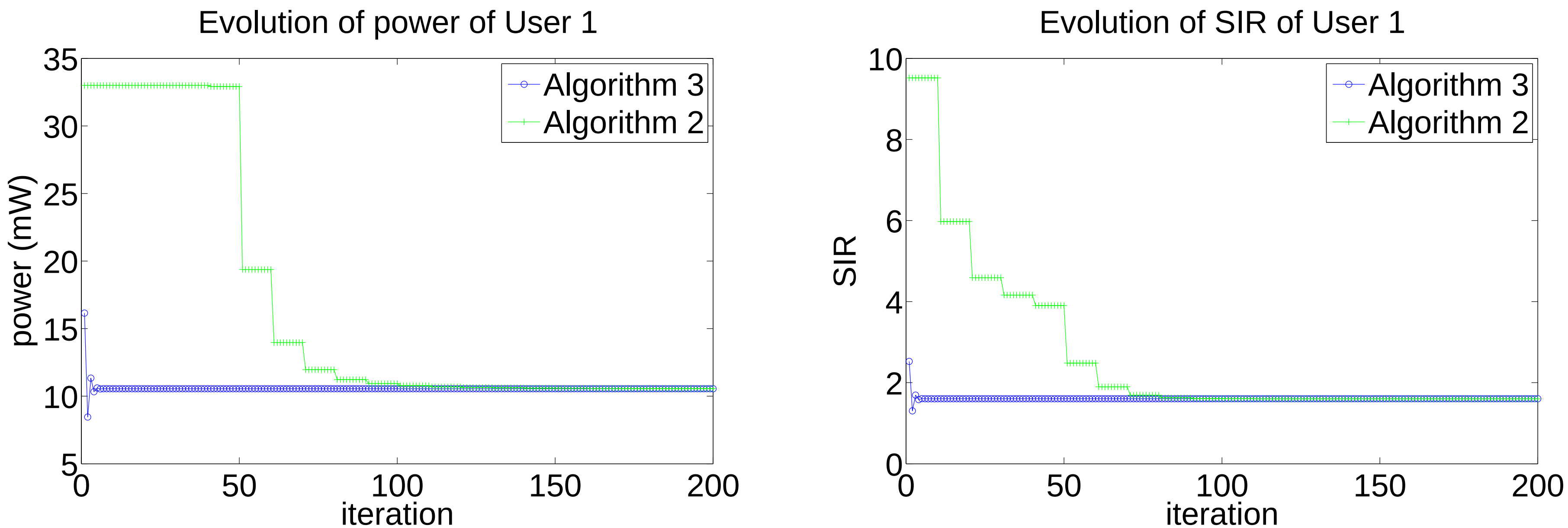
  2. *Normalize* $\mathbf{p}(k+1)$*:*

$$p_l(k+1) = p_l(k+1) / \max_j p_j(k+1) \cdot \bar{p}_i \ \ \forall\, l.$$

- **Theorem 3.** *Starting from any initial point* $\mathbf{p}(0)$, $\mathbf{p}(k)$ *in Algorithm 3 converges geometrically fast to* $\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$ *(unique up to a scaling constant).*

Max-min SIR: Examples

- The nonlinear Perron-Frobenius theory based algorithm is much faster than the subgradient algorithm



(a) (b)

Goodness of Suboptimality

- Max-min SIR, SAPC, All-$\bar{\mathbf{p}}$, One-On-Others-Off, . . .

- Good suboptimal solutions may have attractive implementation quality
  - Simplicity, distributed protocol, fairness, backward compatibility

- Positive Duality Gap: Standard optimization theory is limited

- Strongly NP-hard and Inapproximability of nonconvex problem **[LuoZhang07]**

  Z.-Q. Luo and S. Zhang, *Dynamic Spectrum Management: Complexity and Duality*, IEEE J. of Selected Topics in Signal Processing, 2007

A Negative Result: Hardness

- Inapproximability of nonconvex problem **[LuoZhang07]**

  When $L \geq 3$ and $w_l = w_j \; \forall \, l \neq j$, there is a positive constant c such that no polynomial time L^{-c}-approximation algorithm exists, unless $P = NP$

- η-approximation algorithm:

$$\text{Objective}(\mathbf{p}_{\text{approx}}) \leq \text{Objective}(\mathbf{p}^{\star}) \leq \eta \cdot \text{Objective}(\mathbf{p}_{\text{approx}})$$

  where $\eta \geq 1$

Quasi-Inverse of Nonnegative Matrices

- Definition **[Wong54]**: $\mathbf{B}$ is a quasi-inverse of $\tilde{\mathbf{B}} \geq \mathbf{0}$ if $\mathbf{B} - \tilde{\mathbf{B}} = \mathbf{B}\tilde{\mathbf{B}} = \tilde{\mathbf{B}}\mathbf{B} \geq \mathbf{0}$
- Instrumental property of Minkowski-Leontief (ML) matrices in mathematical economy **[Wong54]**
- $(\mathbf{I} + \mathbf{B})^{-1}\mathbf{B} = \mathbf{B}(\mathbf{I} + \mathbf{B})^{-1} \geq \mathbf{0}$
- $\rho(\tilde{\mathbf{B}}) = \frac{\rho(\mathbf{B})}{1+\rho(\mathbf{B})}$
- $\mathbf{x}(\tilde{\mathbf{B}}) = \mathbf{x}(\mathbf{B})$ & $\mathbf{y}(\tilde{\mathbf{B}}) = \mathbf{y}(\mathbf{B})$

Interference & SNR Regime

- Consider the matrix

$$\mathbf{B} = \mathbf{F} + \sum_l \frac{1}{\mathbf{1}^\top \overline{\mathbf{p}}} \mathbf{v}\mathbf{e}_l^\top$$

- (High SNR regime) $\tilde{\mathbf{B}}$ does not exist

  or any nonnegative matrix with a zero trace & positive off-diagonals

- (Low SNR regime) $\tilde{\mathbf{B}}$ always exists

  or any nonnegative matrix that is a dyad

- (Low interference/moderate SNR regime) $\tilde{\mathbf{B}}$ almost always exists

Tight Upper Bound: Key Theorem

- If $\tilde{\mathbf{B}} \geq \mathbf{0}$, then

$$\sum_l w_l \log(1 + \mathsf{SIR}_l(\mathbf{p}^\star)) \leq \|\mathbf{w}\|_\infty^{\mathbf{x}\circ\mathbf{y}} \log\left(1 + 1/\rho(\mathbf{B})\right),$$

  where $\mathbf{x}, \mathbf{y}$ are the Perron and left eigenvectors of $\mathbf{B}$ respectively.

- Main ideas of proof:
  - Relaxation of nonconvexity
  - Quasi-invertibility of nonnegative matrix **[Wong54]**
  - Friedland-Karlin Inequalities **[FriedlandKarlin75]**

- Physical and operational meaning of upper bound

Physical Interpretation of Upper Bound (I)

- $\sum_l w_l \log(1 + \mathsf{SIR}_l(\mathbf{p}^\star)) \leq \|\mathbf{w}\|_\infty^{\mathbf{x}\circ\mathbf{y}} \log\left(1 + 1/\rho(\mathbf{B})\right).$

- ▪ $1 \leq \|\mathbf{w}\|_\infty^{\mathbf{x}\circ\mathbf{y}} \leq \frac{1}{\min_l\, (\mathbf{x}\circ\mathbf{y})_l}$ as an approximation ratio using

$$
\begin{array}{rl}
\text{maximize} & \min\limits_l \mathsf{SIR}_l(\mathbf{p}) \\
\text{subject to} & \mathbf{1}^\top \mathbf{p} \leq \mathbf{1}^\top \overline{\mathbf{p}} \\
\text{variables:} & \mathbf{p}.
\end{array}
$$

  - Closed-form solution (via Nonlinear Perron-Frobenius Theory):

$$
\begin{array}{rc}
\text{Optimal solution :} & 1/\rho(\mathbf{B}), \quad \mathbf{B} = \mathbf{F} + (1/\mathbf{1}^\top \bar{\mathbf{p}})\mathbf{v}\mathbf{1}^\top; \\
\text{Optimizer :} & \mathbf{x}(\mathbf{B})
\end{array}
$$

General Bounds

- A subset of users $\mathcal{C} = \{l \,|\, l = 1, \ldots, L\}$ with $|\mathcal{C}| \leq L$. Users in $\mathcal{C}$ transmit with positive power. Users that belong to $\bar{\mathcal{C}}$ are removed (delete rows/columns of $\mathbf{B}$)
- L users $\Rightarrow \sum_{l=1}^{L-2} \binom{L}{l} + 2$ possible configurations
- General upper bound (subject to $\tilde{\mathbf{B}}_{\mathcal{C}} \geq \mathbf{0}$):

$$\sum_{l=1}^{L} w_l \log(1 + \mathsf{SIR}_l(\mathbf{p}^\star))$$
$$\leq \max_{l \in \mathcal{C}} \frac{w_l}{(\mathbf{x}(\mathbf{B}_{\mathcal{C}}) \circ \mathbf{y}(\mathbf{B}_{\mathcal{C}}))_l} \log\left(1 + \frac{1}{\rho(\mathbf{B}_{\mathcal{C}})}\right) + \sum_{l \in \bar{\mathcal{C}}} w_l \log(1 + \frac{G_{ll}\bar{p}_l}{n_l})$$

Performance Guarantee: Weighted Max-min SIR

- **Theorem 4.** *Suppose* $\tilde{\mathrm{B}} \geq \mathbf{0}$*. Let*

$$\eta = \frac{\sum_l w_l \log(1 + w_l/\rho(\mathit{diag}(\mathbf{w})(\mathbf{F} + (1/\overline{p}_i)\mathbf{v}\mathbf{e}_i^\top)))}{\|\mathbf{w}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})} \log(1 + 1/\rho(\mathbf{B}))},$$

where

$$i = \arg\min_l \frac{1}{\rho(\mathit{diag}(\mathbf{w})(\mathbf{F} + (1/\overline{p}_l)\mathbf{v}\mathbf{e}_l^\top))}.$$

Then, η *is an approximation ratio by solving the constrained max-min weighted SIR problem:*

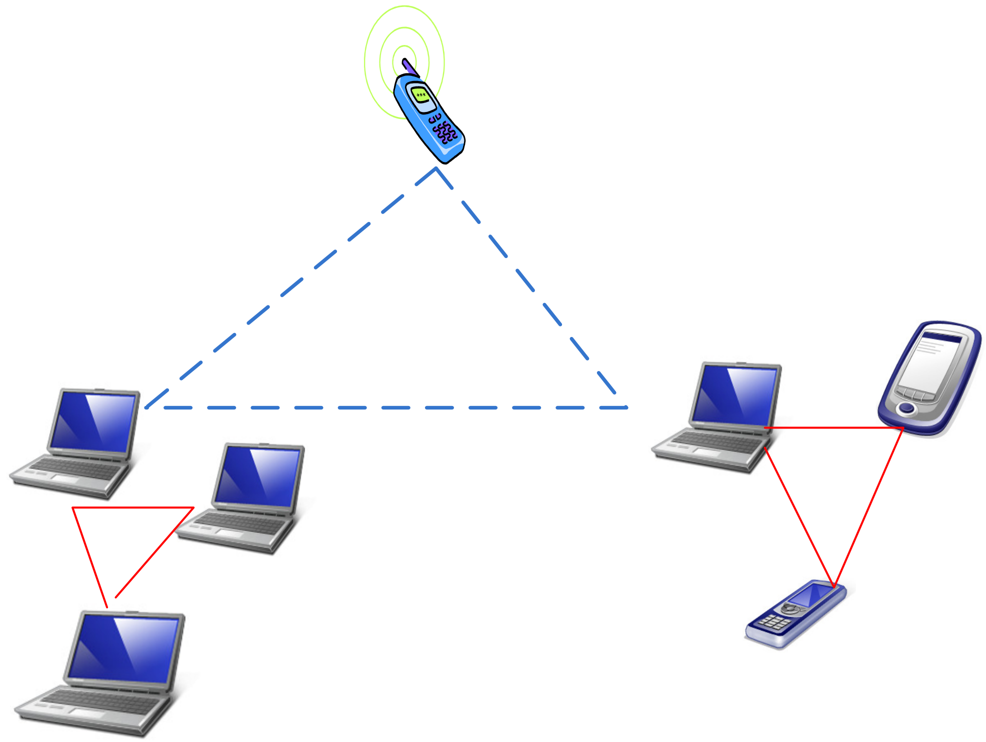
$$\begin{array}{rl} \textit{maximize} & \min_l \dfrac{\mathsf{SIR}_l(\mathbf{p})}{w_l} \\ \textit{subject to} & \mathbf{p} \leq \overline{\mathbf{p}} \\ \textit{variables:} & \mathbf{p}. \end{array}$$

Quasi-invertibility in Wireless Network: Examples

Parameter	Avg. % of $\tilde{\mathbf{B}} \geq \mathbf{0}$	SAPC (η)	Max-min SIR (η)	On-off sched. (η)
$\bar{p}_l = 33$mW $\forall l$ SNR $= 7$dB	99	0.97 (0.93)	0.99 (0.96)	0.89 (0.84)
$\bar{p}_l = 1$W $\forall l$ SNR $= 40$dB	65	0.87 (0.82)	0.91 (0.83)	0.87 (0.82)

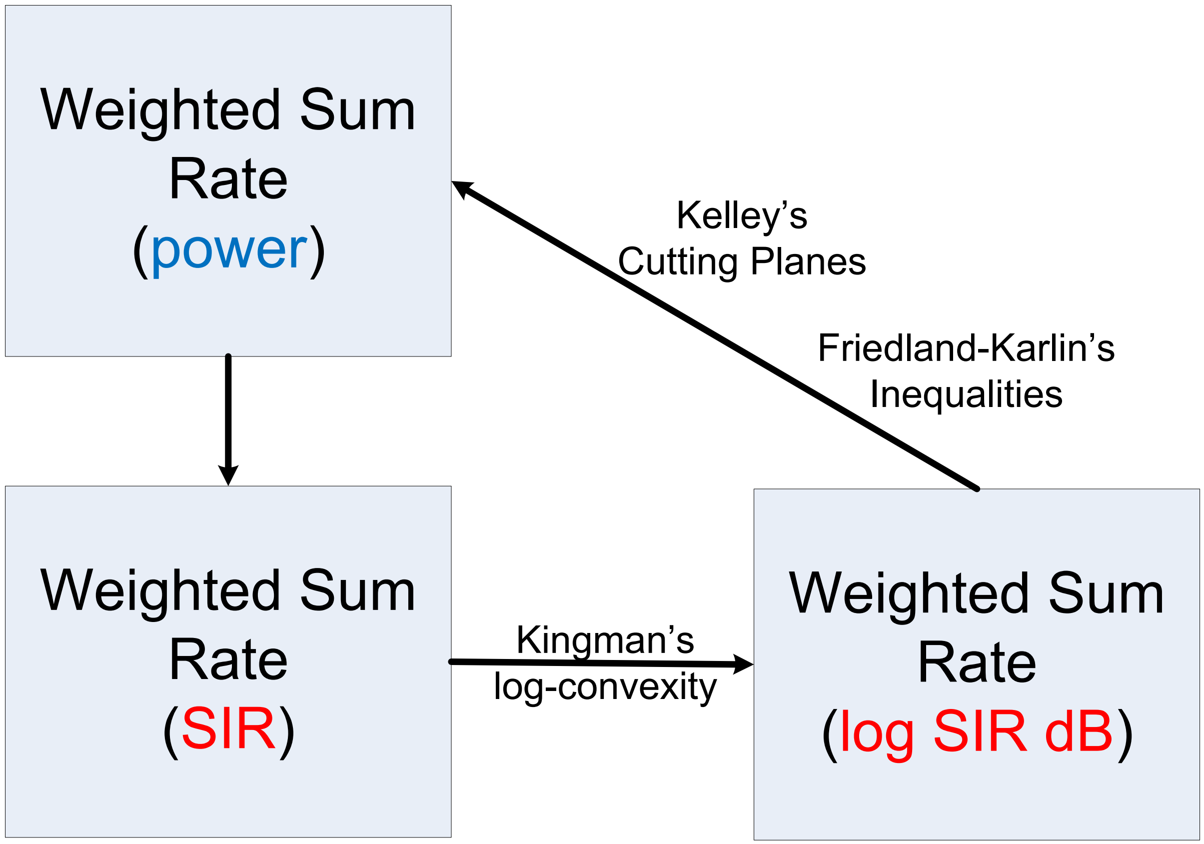
Table 1: A typical numerical example in a ten-user network with two different maximum power constraint settings.

Industry Impact/Adoption



- Qualcomm Flarion Technologies Flash-OFDM
- Telcordia Research Lab's Defense Advanced Research Projects Agency (DARPA) project

Solution Map



Friedland and Tan, *Maximizing Sum Rates in Gaussian Interference-limited Channels*, submitted to IEEE Transactions on Information Theory, 2008

As Eigenvalue Problem: SIR Domain

Theorem 5. *Consider the following maximization problem:*

$$\begin{array}{rl} \textit{maximize} & \sum_l w_l \log(1+\gamma_l) \\ \textit{subject to} & \rho(\textit{diag}(\boldsymbol{\gamma})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \leq 1 \quad \forall\, l, \\ \textit{variables:} & \gamma_l, \quad \forall\, l. \end{array}$$

The optimal SIR *vector* $\boldsymbol{\gamma}^\star$ *is related to the optimal power vector* $\mathbf{p}^\star$ *as follows:*

$$\mathbf{p}^\star = (\mathbf{I} - \textit{diag}(\boldsymbol{\gamma}^\star)\mathbf{F})^{-1}\, \textit{diag}(\boldsymbol{\gamma}^\star)\mathbf{v}.$$

Further, there exists a link i *such that*

$$\rho(\textit{diag}(\boldsymbol{\gamma}^\star)(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \leq \rho(\textit{diag}(\boldsymbol{\gamma}^\star)(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)) = 1$$

for all l. *Further,* $\mathbf{p}^\star$ *is the Perron eigenvector of* $\textit{diag}(\boldsymbol{\gamma}^\star)(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$ *for some* i *corresponding to Perron eigenvalue of 1.*

Nonnegative Matrix Theory: Minimax Theorem

- **Theorem 6.** *Friedland-Karlin inequality [**FriedlandKarlin'75**]: For any irreducible nonnegative matrix* $\mathbf{A}$*,*

$$\prod_l ((\mathbf{A}\mathbf{z})_l/z_l)^{x_l y_l} \geq \rho(\mathbf{A})$$

for all strictly positive $\mathbf{z}$*, where* $\mathbf{x}$ *and* $\mathbf{y}$ *are the Perron and left eigenvectors of* $\mathbf{A}$ *respectively. Equality holds in (1) if and only if* $\mathbf{z} = a\mathbf{x}$ *for some positive* a*.*

- Donsker-Varadhan's variational principle (1975):

$$\max_{\boldsymbol{\lambda}\geq \mathbf{0}, \mathbf{1}^\top \boldsymbol{\lambda}=1} \min_{\mathbf{p}\geq\mathbf{0}} \sum_l \lambda_l \frac{(\mathbf{A}\mathbf{p})_l}{p_l} = \min_{\mathbf{p}\geq\mathbf{0}} \max_{\boldsymbol{\lambda}\geq \mathbf{0}, \mathbf{1}^\top \boldsymbol{\lambda}=1} \sum_l \lambda_l \frac{(\mathbf{A}\mathbf{p})_l}{p_l}$$

- Extensions (see **[FriedlandTan'08]**)

Sum Shannon Rate Global Optimization

- Convert into concave minimization (dB domain)

$$
\begin{array}{rl}
\text{maximize} & \sum_l w_l \log(1+\exp(\tilde{\gamma}_l)) \\
\text{subject to} & \log \rho(\mathsf{diag}(\exp(\tilde{\boldsymbol{\gamma}}))(\mathbf{F}+(1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \leq 0 \quad \forall\, l, \\
\text{variables:} & \tilde{\gamma}_l, \quad \forall\, l.
\end{array}
$$

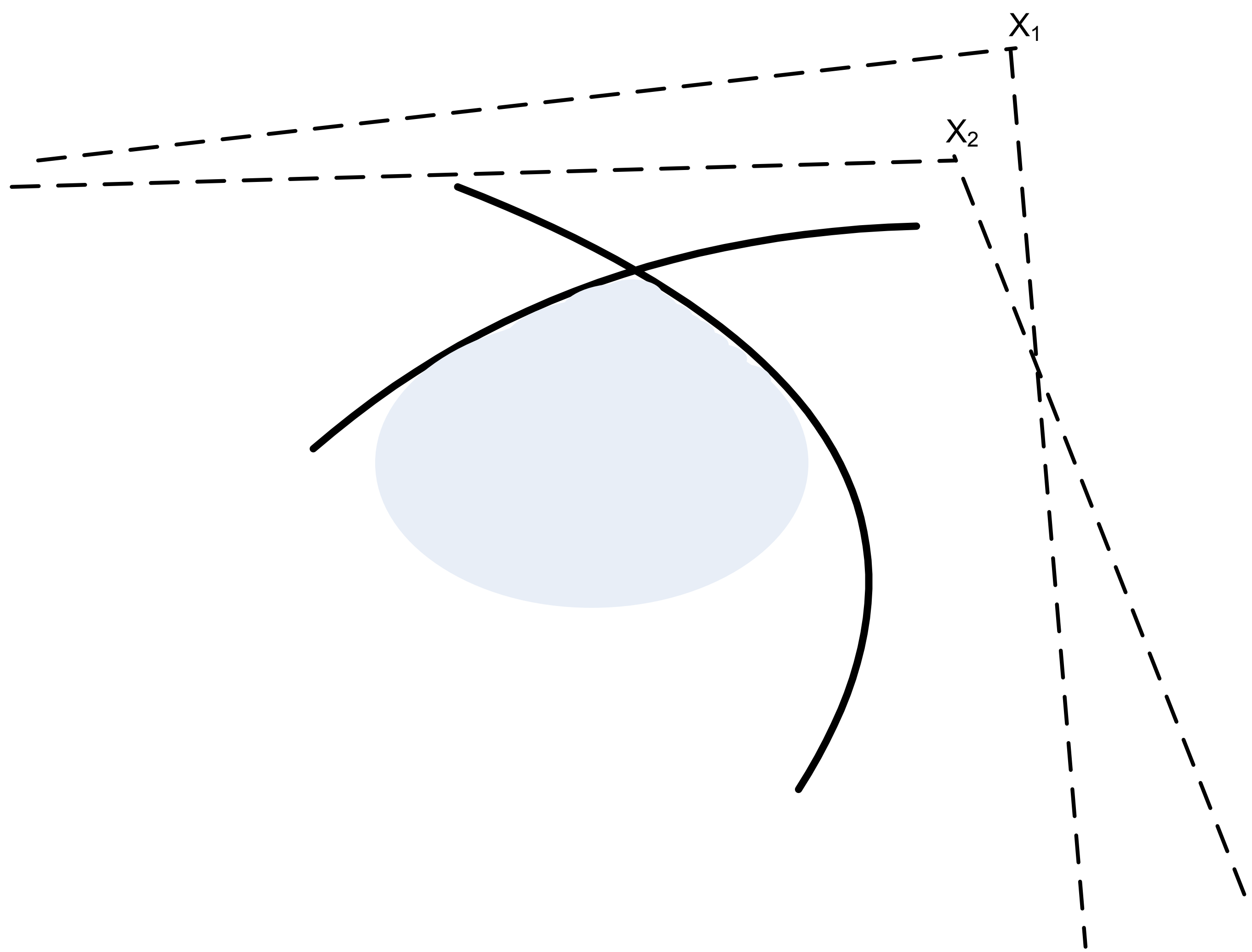
- Relaxation of the constraint set by the Friedland-Karlin Inequalities:

$$\prod_l \gamma_l^{x_l(\mathbf{A})y_l(\mathbf{A})}\rho(\mathbf{A}) \leq \rho(\mathsf{diag}(\boldsymbol{\gamma})\mathbf{A})$$

$$\sum_l x_l(\mathbf{A})y_l(\mathbf{A})\tilde{\gamma}_l + \log\rho(\mathbf{A}) \leq \log\rho(\mathsf{diag}(\exp(\tilde{\boldsymbol{\gamma}}))\mathbf{A}) \quad \text{(dB domain)}.$$

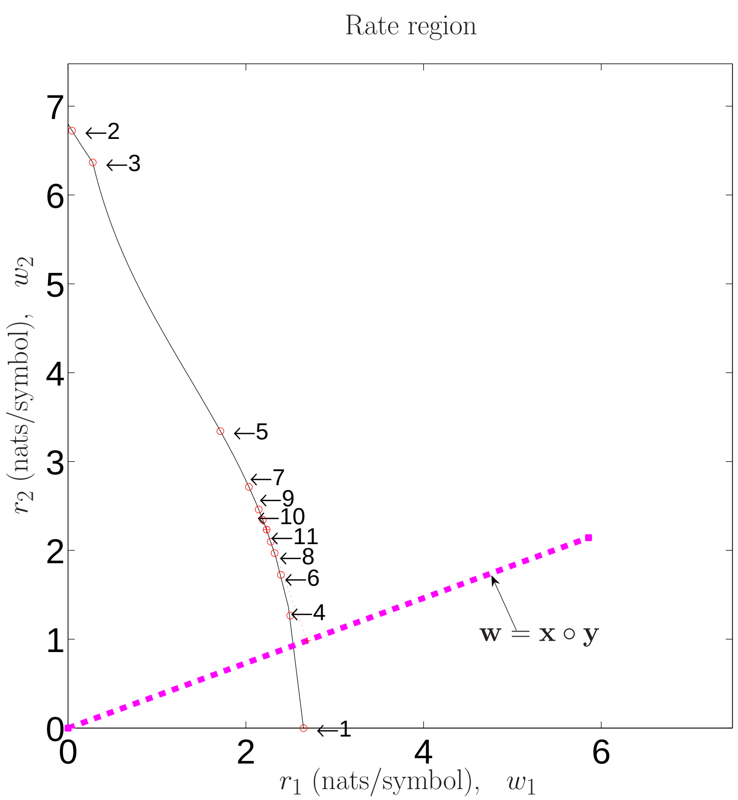
- Outer approximation algorithm (Kelley's cutting planes)

Outer Approximation Algorithm Illustration



Global Optimizing Sum Rate: Examples

- $\lim_{k \to \infty} \min \left\{ \left(\mathbf{I} - \mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))\mathbf{F} \right)^{-1} \mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))\mathbf{v}, \bar{p}_l \right\} = \mathbf{p}^\star$
- Efficient and fast for small to medium-sized networks



Extensions 1

- How to realize $\tilde{\mathbf{B}} \geq \mathbf{0}$:
  - Joint scheduling & power control (INFOCOM 2009)
  - Joint beamforming & power control (IEEE ISIT 2009 submission)

- Cone nonnegativity and nonconvexity (American Institute of Mathematics (AIM) Workshop 2008)

- Extension to multiple tone model, e.g., OFDM,DSL:

$$\max_{\sum_{k=1}^{K} p_{l,k} \leq \bar{p}_l \ \forall\, l} \sum_{l=1}^{L} w_l \sum_{k=1}^{K} \log(1 + \gamma_{l,k}(\mathbf{p})),$$

where

$$\gamma_{l,k} = g_{ll,k} p_{l,k} / (\sum_{j \neq l} g_{lj,k} p_{j,k} + n_{l,k})$$

is the SIR of lth user at tone k

Extensions 2

- $$\begin{array}{rl} \text{minimize} & \sum_{l=1}^{L} w_l \frac{1}{1+\mathsf{SIR}_l(\mathbf{p})} \\ \text{subject to} & \sum_{l=1}^{L} p_l \leq \bar{P}, \quad p_l \geq 0 \quad \forall\, l. \\ \text{variables:} & p_l \quad \forall\, l \end{array}$$

- C. W. Tan, M. Chiang and R. Srikant, *Sum Rate Maximization and MSE Minimization on Multiuser Downlink: Optimality, Fast Algorithms and Equivalence via Max-min SIR*, submitted to IEEE Intern. Symp. of Information Theory, 2009.

- Max-min SIR problem: First distributed fast algorithm with geometric convergence rate to optimize transmit beamforming and power

Robust, Fragile or Optimal?

- 'Optimal yet Fragile' power control algorithm:
  - Distributed Power Control (DPC):

$$p_l(k+1) = \frac{\gamma_l}{\mathsf{SIR}_l(\mathbf{p}(k))} p_l(k) \quad \forall\, l.$$

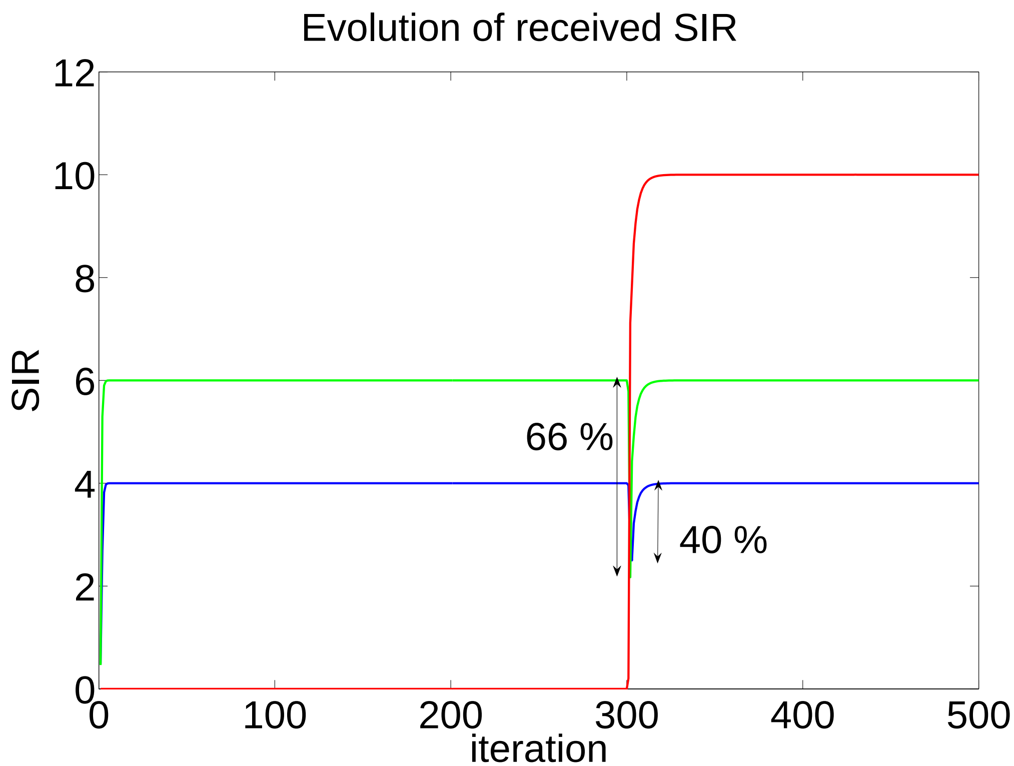
- 'Robust yet Fragile' power control algorithm

- 'Robust yet Optimal' power control algorithm

  Tan, Palomar and Chiang, *Energy-Robustness Tradeoff in Cellular Network Power Control*, IEEE/ACM Transactions on Networking, 2009

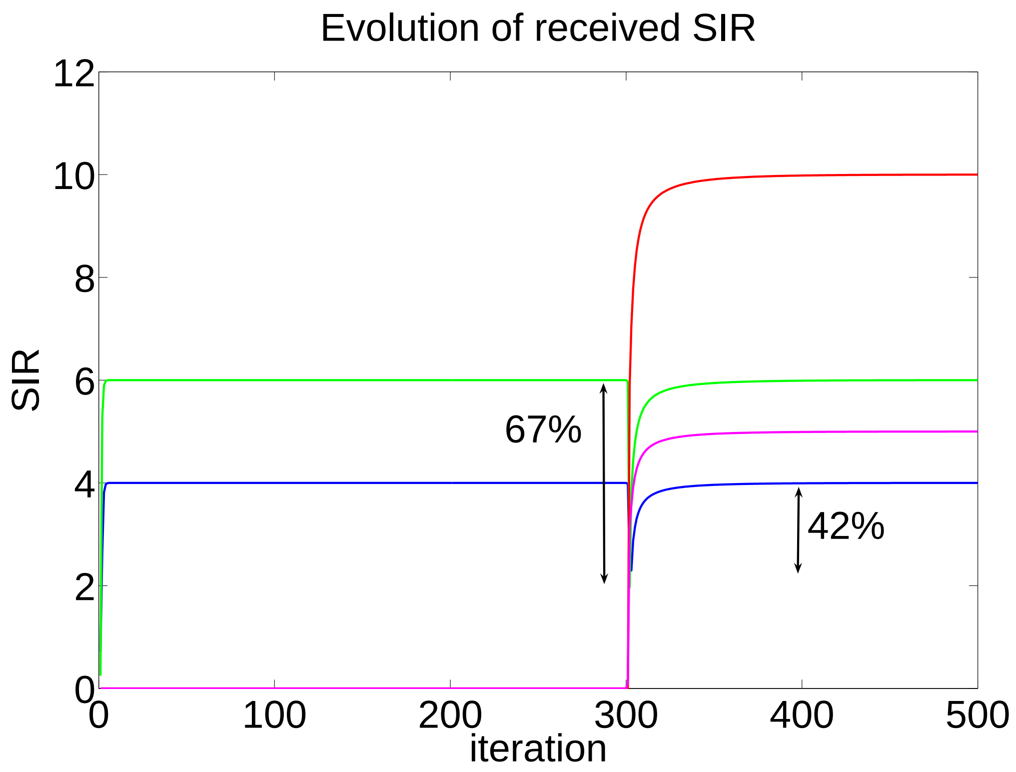
SIR Outage When A New User Enters

- SIR's drop below $40 - 66\%$!



SIR Outage When Two New Users Enter

- SIR's drop below $42 - 67\%$! Dip worsens with congestion!



Algorithm DPC/ALP

- Distributed Power Control with Active Link Protection (DPC-ALP) by Bambos et al, IEEE/ACM Transaction on Networking, 2000

- Each user updates the transmitter powers $p_l(k+1)$ at the $(k+1)$th step according to the following rule:

$$p_l(k+1) = \begin{cases} \frac{(1+\epsilon)\gamma_l}{\mathsf{SIR}_l(k)} p_l(k), & \text{if } \mathsf{SIR}_l(k) \geq \gamma_l \\ (1+\epsilon) p_l(k), & \text{if } \mathsf{SIR}_l(k) < \gamma_l \end{cases}$$

- Open question: how to tune ϵ?

Energy-Robustness Tradeoff

- $\text{SIR}_l(\mathbf{p}^\star) = \gamma_l$ for all l. Tightening or loosening constraint affects network objective $\sum_l p_l^\star$

- Introduce protection margin to SIR thresholds:
  - $\text{SIR}_l \geq \gamma_l$ for reliable transmission
  - $\text{SIR}_l \geq (1 + \epsilon)\gamma_l$ for robust protection against disturbances in network

- Tradeoff between robustness & power saving

Sensitivity Analysis

- Energy-Robustness Tradeoff Theorem:
  - Tightening the lth SIR threshold constraint by β_l percent increases the total power by approximately $\beta_l \nu_l^\star / \mathbf{1}^T \mathbf{p}^\star$ percent, for β_l small
  - Total power increment $\approx \sum_l \beta_l \, \nu_l^\star / \mathbf{1}^T \mathbf{p}^\star$ percent

Uplink-Downlink Algorithm and Duality

- Power update:

$$\mathbf{p}(k+1) = \mathrm{diag}(\boldsymbol{\gamma})\mathbf{F}\mathbf{p}(k) + \mathrm{diag}(\boldsymbol{\gamma})\mathbf{v}$$

- Auxiliary variable update:

$$\mathbf{x}(k+1) = (\mathrm{diag}(\boldsymbol{\gamma})\mathbf{F})^T\mathbf{x}(k) + \mathbf{1}$$

- Dual variable update (Schur product):

$$\boldsymbol{\nu}(k+1) = \mathbf{x}(k+1) \circ \mathbf{p}(k+1)$$

Robust Power Control Problem

- Robust Power Control Problem:

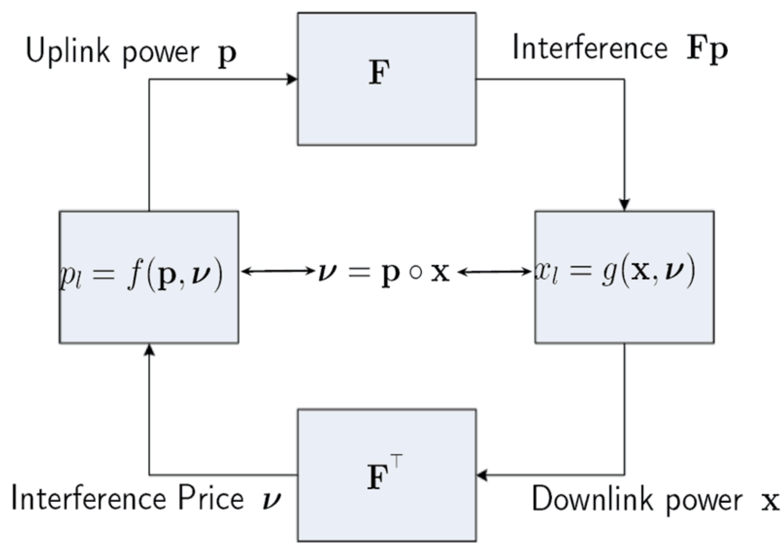
$$\begin{array}{rl} \text{minimize} & \sum_l p_l + \phi(\epsilon) \\ \text{subject to} & \text{SIR}_l(\mathbf{p}) \geq \gamma_l(1+\epsilon) \quad \forall l, \\ & \epsilon \geq 0, p_l \geq 0 \quad \forall l \\ \text{variables:} & p_l \, \forall l, \ \epsilon \end{array}$$

- Appropriate choice of $\phi(\epsilon)$ to balance tradeoff in power cost

- Problem is nonconvex, but convex after log change of variables ($\mathbf{p}$ & ϵ) and if $\frac{\partial^2 \phi(z)/\partial z^2}{\partial \phi(z)/\partial z} \geq -1/z$

Algorithm RDPC

- Algorithm Robust Distributed Power Control (RDPC)
- Optimize $\mathbf{p}, \epsilon$ by inferring congestion, aided by Interference Price $\boldsymbol{\nu}$ and uplink-downlink duality



Algorithm RDPC (Uplink)

- Updates the transmitter powers $p_l(k+1)$ at the $(k+1)$th step according to the following rule:

$$p_l(k+1) = \begin{cases} \frac{(1+\epsilon(k))\gamma_l}{\mathsf{SIR}_l(k)} p_l(k), & \text{if } \mathsf{SIR}_l(k) \geq \gamma_l \\ (1+\epsilon(k)) p_l(k), & \text{if } \mathsf{SIR}_l(k) < \gamma_l \end{cases}$$

Algorithm RDPC (Downlink)

- Computes $x_l(k+1)$, the lth component of $\mathbf{x}(k+1)$, using

$$\mathbf{x}(k+1) = (1+\epsilon(k))\mathbf{F}^T \text{diag}(\boldsymbol{\gamma})\mathbf{x}(k) + \mathbf{1},$$

- Computes

$$\nu_l(k+1) = x_l(k+1)p_l(k+1) \quad \forall l,$$

- Updates $\epsilon(k+1)$ by solving

$$-\frac{\partial \phi(\epsilon)}{\partial \epsilon}\bigg|_{\epsilon=\epsilon(k+1)} (1+\epsilon(k+1)) = \mathbf{1}^T \boldsymbol{\nu}(k+1).$$

Optimal Energy-Robustness Tradeoff

- Choose $\phi(\epsilon) = \delta \log(1 + 1/\epsilon)$ (energy efficiency)

- Engineering interpretation:
  network can tolerate at most an increase of $\delta/(\mathbf{1}^T\mathbf{p}^\star)$ percent in total power

- From Energy-Robustness Tradeoff Theorem:

$$\epsilon^\star(\mathbf{1}^T\boldsymbol{\nu}^\star) = \delta/(\mathbf{1}^T\mathbf{p}^\star)$$

- Applications
  - Interference Rise-Over-Thermal in cellular network
  - Interference Temperature in cognitive radio network

Non-equilibrium Engineering Implication

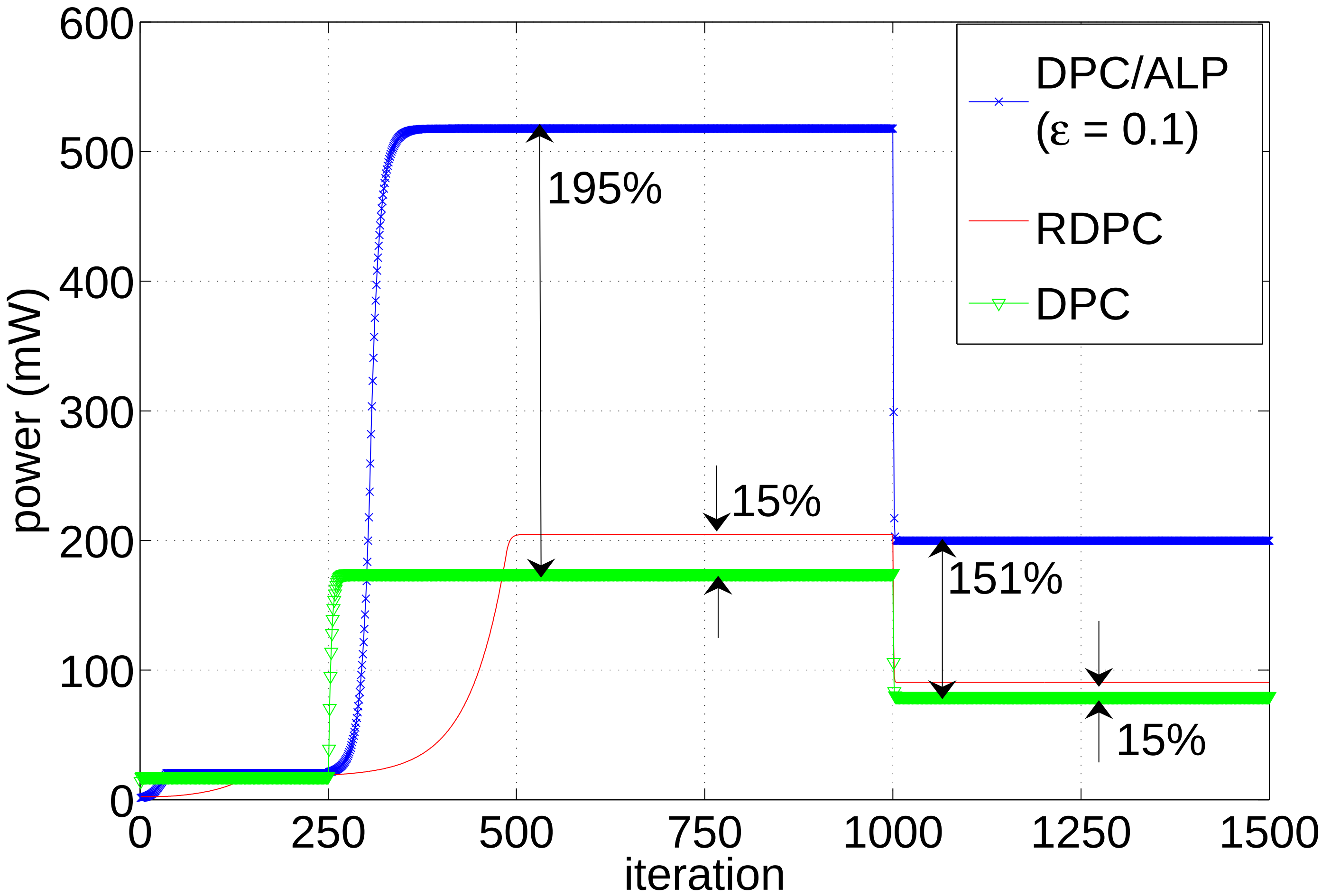
$$\epsilon(k) \;\propto\; \frac{1}{\mathbf{1}^T \boldsymbol{\nu}(k)} \quad \text{for all } k$$

- Quantifies the remark on choosing the parameter ϵ in **[Bambos00]**:

  "ϵ should be chosen such that $(1+\epsilon)$ should be larger when the network is uncongested, so that users power up fast, and grow smaller as congestion builds up to have more users power up more gently."

- Different $\phi(\epsilon)$'s to relate ϵ to aggregate 'congestion price' $\mathbf{1}^T\boldsymbol{\nu}$

Numerical Example



Thank You

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Sum Rate Global Optimization: Algorithm

Algorithm 4. [Sum Rate Outer Approximation Algorithm]

1. *Compute the vertices of the enclosing linear polyhedron* $D^{(0)}$*, described by the set of constraints:*

$$\sum_j (\mathbf{x}(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top) \circ \mathbf{y}(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))_j \tilde{\gamma}_j + \log \rho(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top) \leq 0,$$

   and $\tilde{\gamma}_l \geq -K$ *for all* l*. Let* $V^{(0)}$ *be the set of vertices of* $D^{(0)}$*. Set* $k = 1$ *and go to Step 2.*

2. *Iteration* k*: Solve the problem:*

$$\begin{array}{rl} \textit{maximize} & \sum_l w_l \log(1 + e^{\tilde{\gamma}_l}) \\ \textit{subject to} & \tilde{\gamma}_l \in D^{(k-1)} \end{array}$$

   by selecting $\max\left\{\sum_l w_l \log(1 + e^{\tilde{\gamma}_l}) : v \in V^{(k-1)}\right\}$*. Let* $\tilde{\boldsymbol{\gamma}}^k$ *be the optimizer to (1).*

3. *Compute*

$$\mathbf{p}^k = \Big(\mathbf{I} - \mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))\mathbf{F}\Big)^{-1} \mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))\mathbf{v}.$$

4. *If* $\mathbf{p}^k \leq \bar{\mathbf{p}}$*, stop:* $\tilde{\boldsymbol{\gamma}}^k$ *is the solution to (1) and* $\mathbf{p}^k$ *is the solution to (1). Otherwise, let*

$$\begin{aligned} J^k \quad &= \{l : \log \rho(\mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \\ &= \max\nolimits_{1\leq j \leq L} \log \rho(\mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))(\mathbf{F} + (1/\bar{p}_j)\mathbf{v}\mathbf{e}_j^\top))\} \end{aligned} \tag{1}$$

and choose any $j^k \in J^k$*.*

5. *Compute the left eigenvector* $\mathbf{y}_{j^k}$ *and right (Perron) eigenvector* $\mathbf{x}_{j^k}$ *of* $\mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))(\mathbf{F} + (1/\bar{p}_{j^k})\mathbf{v}\mathbf{e}_{j^k}^\top)$*. Set*

$$\begin{aligned} G_{j^k}^k(\tilde{\boldsymbol{\gamma}}) \quad &= \log \rho(\mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))(\mathbf{F} + (1/\bar{p}_{j^k})\mathbf{v}\mathbf{e}_{j^k}^\top)) + \\ &\frac{[\exp(\tilde{\boldsymbol{\gamma}}^k) \circ \mathbf{x}_{j^k} \circ \mathbf{y}_{j^k}]^\top (\tilde{\boldsymbol{\gamma}} - \tilde{\boldsymbol{\gamma}}^k)}{\rho(\mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))(\mathbf{F} + (1/\bar{p}_{j^k})\mathbf{v}\mathbf{e}_{j^k}^\top))}. \end{aligned} \tag{2}$$

6. *Set* $D^{(k)} = D^{(k-1)} \cap \{\tilde{\boldsymbol{\gamma}} : G^k_{j^k}(\tilde{\boldsymbol{\gamma}}) \leq 0\}$, $V^{(k)} = \{$*extreme points of* $D^{(k)}\}$.

7. *Set* $k \leftarrow k + 1$. *Go to Step 2.*

- Step 3 yields a feasible power vector $\hat{\mathbf{p}}^k$: $\hat{p}^k_l = \min\{p^k_l, \bar{p}_l\}$ for all l.

Convergence of Algorithm RDPC

- **Stability Theorem**:
  The RDPC algorithm with $\phi = \delta \log(1 + 1/\epsilon)$ (energy efficiency) is locally asymptotically stable if and only if $(1 + \Delta/(\mathbf{p}^{\star T}\mathbf{x}^\star))\rho(\mathbf{F}) < 1$.

- Nonlinear iterative map ($\mathbf{z} = [\mathbf{p}^T \ \mathbf{x}^T]^T$, $\Delta = \delta/\mathbf{1}^T\mathbf{p}^\star$):

$$f(\mathbf{z}) = \begin{bmatrix} \left(1 + \frac{\Delta}{\mathbf{p}^T\mathbf{x}}\right)(\mathbf{F}\mathbf{p} & + & \mathbf{v}) \\ \left(1 + \frac{\Delta}{\mathbf{p}^T\mathbf{x}}\right)\mathbf{F}^T\mathbf{x} & + & \mathbf{1} \end{bmatrix}$$