Optimal Algorithms in Wireless Utility Maximization: Proportional Fairness Decomposition and Nonlinear Perron-Frobenius Theory Framework

Liang Zheng, *Student Member, IEEE*, and Chee Wei Tan, *Senior Member, IEEE*

***Abstract*—We study the network utility maximization problems in wireless networks for service differentiation that optimize the Signal-to-Interference-plus-Noise Radio (SINR) and reliability under Rayleigh fading. Though seemingly nonconvex, we show that these problems can be decomposed into an optimization framework where each user calculates a payment for a given resource allocation, and the network uses the payment to optimize the performance of the user. We study three important examples of this utility maximization, namely the weighted sum logarithmic SINR maximization, the weighted sum inverse SINR minimization and the weighted sum logarithmic reliability maximization. These problems have hitherto been solved suboptimally in the literature. By exploiting the positivity, quasi-concavity and homogeneity properties in these problems and using the nonlinear Perron-Frobenius theory, we propose fixed-point algorithms that converge geometrically fast to the globally optimal solution. Numerical evaluations show that our algorithms are stable (free of parameter configuration) and computationally fast.**

***Index Terms*—Optimization, network utility maximization, resource allocation, nonlinear Perron-Frobenius theory.**

I. INTRODUCTION

UTILITY maximization in wireless networks is more complicated than its wireline counterpart, e.g., the rate control in the Internet, because of factors such as time-varying channel fading, interference, reliability requirements and the ease of low complexity implementation. Solving the wireless utility maximization thus requires the design of resource allocation and interference management algorithms with good convergence properties to the global optimal solution. However, the performance metrics and constraint set in these problems are often nonconvex and nonlinear, thus making it generally hard to design algorithms to solve them optimally.

Network utility maximization has its roots in the seminal work [1], [2] by Frank Kelly who established an optimization framework that decomposes the overall system problem into separable smaller subproblems that are solved by each user and a network controller. A novelty in this optimization framework is the use of proportional fairness as an intermediate mechanism for resource allocation, particularly to design rate control algorithms in [1]. This proportional fairness framework has been generalized using Lagrange duality for TCP flow control in the Internet wireline network [3]. This framework was later extended in [4], [5] to cross-layer optimization for wireless networks by using convex approximation to circumvent the nonconvexity in wireless utility maximization.

On the other hand, the authors in [6] showed that the utility maximization problem in [4], [5] can be solved optimally, albeit with algorithms that are more coupled and require appropriate step-size tuning. Subsequently, there have been efforts to jointly address the decoupling of constraints and algorithm design in wireless utility maximization, e.g., using optimization decomposition in [7]–[9] and decomposition by Gibbs sampling in [10]. The authors in [11] studied a wireless utility that is the sum of inverse SINR or nonlinear interference functions. The authors in [12] proposed a game-theoretic algorithm to maximize a utility that is a weighted sum of logarithmic reliability. The authors in [13] proposed a sum inverse SINR heuristic similar to [11] to solve the utility maximization problem in [4].

A recent effort to overcome the nonconvexity in special cases of wireless utility maximization is the use of the nonlinear Perron-Frobenius theory in [14]. The authors in [15]–[19] solved the wireless utility maximization problem whose objective is to guarantee a max-min fairness for commonly-used wireless performance metric using fixed-point algorithms. The applications are broad ranging from solving the max-min SINR fairess problem in wireless cellular networks to cross-layer rate fairness optimization. The authors in [20] showed that the wireless utility maximization in wireless cognitive network can be solved systematically using the max-min fairness optimization as a proxy. Different from these works, this paper leverages the nonlinear Perron-Frobenius theory to design new service differentiation algorithms that solve nonconvex utility maximization problems (other than the max-min fairness) that have previously been solved suboptimally, e.g., in [12], [13] or partially in special cases, e.g., in [11]. Using convex decomposition and convex reformulation technique, which is in part inspired by [1], the utility maximization problem is decomposed into multiple user subproblems and a network subproblem that iteratively adapts the link performance metric for convergence to the global optimality.

Manuscript received May 31, 2013; revised September 5, 2013; accepted November 13, 2013. The associate editor coordinating the review of this paper and approving it for publication was G. Abreu.

The work in this paper was partially supported by grants from the Research Grants Council of Hong Kong Project No. RGC CityU 125212, SRFDP & RGC ERG Joint Research Scheme M-CityU107/13, Qualcomm Inc., the Science, Technology and Innovation Commission of Shenzhen Municipality, Project No. JCYJ20120829161727318 on Green Communications in Small-cell Mobile Networks, and Project No. JCYJ20130401145617277 on Adaptive Spectrum Access Resource Allocation in Cognitive Radio Networks.

The material in this paper was presented in part at the 11th International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks, Tsukuba Science City, Japan, May 13-17, 2013.

The authors are with the Department of Computer Science, City University of Hong Kong, Tat Chee Ave., Hong Kong (e-mail: liangzheng.hkcityu@gmail.com, cheewtan@cityu.edu.hk).

Digital Object Identifier 10.1109/TWC.2013.020714.130980

The main contributions of this paper are summarized as follows.

- We propose a novel technique based on the nonlinear Perron-Frobenius theory to decompose and solve the nonconvex utility maximization problems that hitherto have been solved suboptimally or partially in [11]–[13]. New characterizations of the optimality properties overcome the limitations associated with nonconvexity.
- Our analysis demonstrates that four properties, namely positivity, homogeneity, monotonicity and quasi-concavity, that are inherent in these utility maximization problems can be exploited to design fixed-point algorithms that converge geometrically fast.

The rest of this paper is organized as follows. We present the system model in Section II. In Section III, we formulate the utility maximization problems. In Section IV, we decompose the utility maximization problem into simpler subproblems using Kelly's proportional fairness decomposition technique. Next, in Section V, by exploiting the optimality conditions and leveraging the nonlinear Perron-Frobenius theory, we propose fixed-point algorithms to solve the decomposed subproblems with applications to the weighted sum logarithmic SINR maximization, the weighted sum inverse SINR minimization and the weighted sum logarithmic reliability maximization. We evaluate the performance of our algorithms numerically in Section VI. We conclude the paper in Section VII.

The following notation is used in our paper. Column vectors and matrices are denoted by boldfaced lowercase and uppercase respectively. Let $\rho(\mathbf{A})$ denote the Perron-Frobenius eigenvalue of a nonnegative matrix $\mathbf{A}$, and $\mathbf{x}(\mathbf{A})$ and $\mathbf{y}(\mathbf{A})$ denote the Perron right and left eigenvectors of $\mathbf{A}$ associated with $\rho(\mathbf{A})$. Furthermore, we denote $\mathbf{x}\circ\mathbf{y}$ as the Schur product of $\mathbf{x}$ and $\mathbf{y}$. We let $\mathbf{a}_l$ denote the lth column vector of matrix $\mathbf{A}$ and let $\mathbf{e}_l$ and $\mathbf{I}$ denote the lth unit coordinate vector and the identity matrix respectively. Let $\mathbf{1} = (1,\ldots,1)^\top \in \mathbb{R}^L$ be an all-one vector. The super-script $(\cdot)^\top$ denotes transpose. For a given vector $\mathbf{x} = (x_1,\ldots,x_L)^\top$, $\mathrm{diag}(\mathbf{x})$ is a diagonal matrix $\mathrm{diag}(x_1,\ldots,x_L)$, $e^{\mathbf{x}}$ denotes $e^{\mathbf{x}} = (e^{x_1},\ldots,e^{x_L})^\top$, and $\log\mathbf{x}$ denotes $\log\mathbf{x} = (\log x_1,\ldots,\log x_L)^\top$.

II. System Model

We consider a multiuser communication wireless network with L users (transmitter/receiver pairs) transmitting simultaneously on a shared spectrum. Let $\mathbf{G} = [G_{lj}]_{l,j=1}^L > 0_{L\times L}$ represent the channel gain, where G_{lj} is the channel gain from the jth transmitter to the lth receiver, and $\mathbf{n} = (n_1,\ldots,n_L)^\top > \mathbf{0}$, where n_l is the noise power at the lth user. The vector $\mathbf{p} = (p_1,\ldots,p_L)^\top$ is the transmit power vector. An important link metric is the received SINR of the lth user under frequency-flat fading given by:

$$\mathsf{SINR}_l(\mathbf{p}) = \frac{G_{ll}p_l}{\sum_{j\neq l} G_{lj}p_j + n_l}. \tag{1}$$

Now, we define a nonnegative matrix $\mathbf{F}$ with entries:

$$F_{lj} = \begin{cases} 0, & \text{if } l = j \\ G_{lj}/G_{ll}, & \text{if } l \neq j \end{cases} \tag{2}$$

and the vector $\mathbf{v} = (n_1/G_{11}, n_2/G_{22},\ldots,n_L/G_{LL})^\top$. Moreover, we assume that $\mathbf{F}$ is irreducible, i.e., each link has at least one interferer. Then, the SINR can be represented compactly as: $\gamma_l(\mathbf{p}) = p_l/(\mathbf{F}\mathbf{p}+\mathbf{v})_l$, where we denote the vector $\boldsymbol{\gamma}$ as the SINR for all the users.

In the case of Rayleigh fading, the received power from the jth transmitter at the lth receiver is given by $G_{lj}R_{lj}p_j$ where R_{lj} is a random variable due to Rayleigh fading. In particular, R_{lj} are independent and exponentially distributed with unit mean, i.e., $E[G_{lj}R_{lj}p_j] = G_{lj}p_j$ [21]. The received SINR of the lth user under Rayleigh fading is a random variable in terms of $\mathbf{p}$:

$$\mathsf{SINR}_l(\mathbf{p}) = \frac{R_{ll}p_l}{\sum_{j\neq l} F_{lj}R_{lj}p_j + n_l}. \tag{3}$$

An outage occurs when the received SINR of the lth user falls below β_l, a minimum SINR threshold for reliable communication. This means that when $\mathsf{SINR}_l(\mathbf{p}) \geq \beta_l$, the transmission at the lth receiver is successful; otherwise, the transmission fails. Clearly, link reliability is an important factor in wireless networks. Assuming independent Rayleigh fading at all the signals, this reliability function of the lth user is given as the complement of the outage probability [21]:

$$\Psi_l(\mathbf{p}) \triangleq \mathrm{Prob}(\mathsf{SINR}_l(\mathbf{p}) \geq \beta_l) = e^{\frac{-v_l\beta_l}{p_l}} \prod_{j=1}^{L}\left(1+\frac{\beta_l F_{lj}p_j}{p_l}\right)^{-1}. \tag{4}$$

Note that whenever we use (4), the SINR function in (4) comes from (3) and not from (1). We study the network utility maximization problem subject to a set of maximum power constraints in the following.

III. Utility Maximization Problem Formulation

The overall network utility, denoted by $U(f(\mathbf{p}))$, is a network-wide objective function whose argument $f(\mathbf{p})$ is a positive vector function with entries $f_l(\mathbf{p})$ that is a bijective mapping from the transmit power $\mathbf{p}$ to a link metric of the lth user. Commonly-used link metrics $f_l(\mathbf{p})$ are the SINR in (1) or the reliability function in (4). We assume that the overall network utility function is separable and continuous, i.e.,

$$U(\mathbf{p}) = \sum_{l=1}^{L} U_l(f_l(\mathbf{p})). \tag{5}$$

For example, $U(\cdot)$ can be given by the α-fairness utility [22]:

$$U(\boldsymbol{\gamma}) = \begin{cases} \sum_{l=1}^L \log\gamma_l, & \text{if } \alpha = 1, \\ \sum_{l=1}^L (1-\alpha)^{-1}\gamma_l^{1-\alpha}, & \text{if } \alpha > 1. \end{cases} \tag{6}$$

To guarantee the power efficiency and also avoid overwhelming interference, we assume that all the users are constrained by a weighted power constraint set $\mathcal{P}$ given by:

$$\mathcal{P} = \{\mathbf{p}\,|\,\mathbf{a}_l^\top\mathbf{p} \leq \bar{p}_l,\ \ l = 1,\ldots,L\}, \tag{7}$$

where $\bar{\mathbf{p}}$ is the upper bound for the weighted power constraints, and $\mathbf{a}_l$ can be any positive vectors. Letting $\mathbf{a}_l$ be the lth column vector of a nonnegative matrix $\mathbf{A}$, (7) can also be expressed as $\mathcal{P} = \{\mathbf{p}\,|\,\mathbf{A}^\top\mathbf{p} \leq \bar{\mathbf{p}}\}$. As special cases, when $\mathbf{a}_l = \mathbf{e}_l$ (i.e., $\mathbf{A} = \mathbf{I}$), we have the individual power constraints $\mathcal{P} = \{\mathbf{p}\,|\,\mathbf{p} \leq \bar{\mathbf{p}}\}$, and when $\mathbf{a}_l = \mathbf{1}$ (i.e., $\mathbf{A}$ is an all-one matrix), we have a total power constraint $\mathcal{P} = \{\mathbf{p}\,|\,\mathbf{1}^\top\mathbf{p} \leq \min_{l=1,\ldots,L} \bar{p}_l\}$, i.e., the total energy consumed by all the users in the network should be constrained under a given threshold. Thus, $\mathcal{P}$ in (7) is general enough to model most practical power constraints encountered in wireless networks.

We study the general problem of maximizing the overall network utility that can be formulated as:

$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} U_l(f_l(\mathbf{p})) \\ \text{subject to} & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p}. \end{array} \tag{8}$$

In general, depending on the utility in (5), (8) can be nonconvex, and thus it may be hard to solve (8) optimally. Using a logarithmic mapping of variable, for $\mathbf{p} = (p_1, \ldots, p_L)^\top > 0$, let $\tilde{p}_l = \log p_l$ for all l, i.e., $\mathbf{p} = e^{\tilde{\mathbf{p}}}$, (8) can be transformed to the following equivalent optimization problem:

$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} U_l(f_l(e^{\tilde{\mathbf{p}}})) \\ \text{subject to} & e^{\tilde{\mathbf{p}}} \in \mathcal{P}, \\ \text{variables:} & \tilde{\mathbf{p}}. \end{array} \tag{9}$$

Note that, in (9), the constraint set is no longer a polyhedron. However, $\mathcal{P} = \{\tilde{\mathbf{p}} \,|\, \mathbf{A}^\top e^{\tilde{\mathbf{p}}} \leq \bar{\mathbf{p}}\}$ is a set of sum of exponential functions. Now, (9) may still be nonconvex, but we look at important special cases of (9) that can be solved optimally (cf. Assumption 1 in Section IV). These special cases hitherto have been tackled suboptimally in the literature when viewed as a nonconvex problem in (8). In the following, we overcome the nonconvexity hurdle by decomposing (9) (exploiting the separability of (9)) in Section IV and then applying the nonlinear Perron-Frobenius theory to solve the subproblems in Section V. As a by-product, fixed-point algorithms with geometric convergence rate can be systematically obtained.

IV. Service Differentiation by Proportional Fairness

As a first step to overcome the hurdle of nonconvexity in solving (8), we decompose (9) and then exploit the optimality conditions of (9). This approach is motivated by the Kelly's decomposition in [1], where we iteratively solve two different sets of subproblems over discrete time slots:

1) The user subproblem: At each time slot, each user computes a positive payment that is associated with the current resource allocated by a centralized network controller. This payment can be interpreted as the willingness to pay for a particular allocated resource.
2) The network subproblem: At the next time slot, based on the payments computed by all the users, the network controller maximizes the weighted sum of link metrics, where the payment received from each user plays the role of weight. Then, the controller returns the optimal resource allocation solution to all the users.

The above two subproblems are then solved iteratively until the network reaches an equilibrium. At this equilibrium, we say that the resource allocation is proportionally fair to the equilibrium payments.

By introducing an auxiliary variable $\boldsymbol{\sigma}$, let us rewrite (9) as:

$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} U_l(\sigma_l) \\ \text{subject to} & \sigma_l \leq f_l(e^{\tilde{\mathbf{p}}}), \quad l = 1, \ldots, L, \\ & e^{\tilde{\mathbf{p}}} \in \mathcal{P}, \\ \text{variables:} & \tilde{\mathbf{p}}, \boldsymbol{\sigma}. \end{array} \tag{10}$$

Note that (10) is nonconvex in general. By using a logarithmic transformation of variable, i.e., $\boldsymbol{\sigma} = e^{\tilde{\boldsymbol{\sigma}}}$, (10) is equivalent to:

$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} U_l(e^{\tilde{\sigma}_l}) \\ \text{subject to} & \tilde{\sigma}_l \leq \log f_l(e^{\tilde{\mathbf{p}}}), \quad l = 1, \ldots, L, \\ & e^{\tilde{\mathbf{p}}} \in \mathcal{P}, \\ \text{variables:} & \tilde{\mathbf{p}}, \tilde{\boldsymbol{\sigma}}. \end{array} \tag{11}$$

In this paper, we study the utility functions that satisfy the following assumption.

Assumption 1: The following are sufficient for (11) to be convex: 1) $\sum_{l=1}^{L} U_l(e^{\tilde{\sigma}_l})$ is concave in $\tilde{\sigma}_l$, 2) $\log f_l(e^{\tilde{\mathbf{p}}})$ is concave in $\tilde{\mathbf{p}}$.

Lemma 1: Both $f(e^{\tilde{\mathbf{p}}})$, i.e., $\mathsf{SINR}(e^{\tilde{\mathbf{p}}})$ in (1) and $\Psi_l(e^{\tilde{\mathbf{p}}})$ in (4), satisfy the second condition in Assumption 1.

Next, we form the partial Lagrangian of (11) by introducing the dual variable $\boldsymbol{\mu} \in \mathbb{R}^L_+$:

$$\mathcal{L}(\tilde{\mathbf{p}}, \tilde{\boldsymbol{\sigma}}, \boldsymbol{\mu}) = \sum_{l=1}^{L} U_l(e^{\tilde{\sigma}_l}) + \sum_{l=1}^{L} \mu_l \big(\log f_l(e^{\tilde{\mathbf{p}}}) - \tilde{\sigma}_l\big). \tag{12}$$

By taking the partial derivative of (12) with respect to $\tilde{\sigma}_l$ and using Lagrange duality, we can obtain the following result.

Lemma 2: The optimal solution $\tilde{\boldsymbol{\sigma}}^\star$ and $\tilde{\mathbf{p}}^\star$ of (11) satisfy $e^{\tilde{\sigma}_l^\star} = f_l(e^{\tilde{\mathbf{p}}^\star})$ for all l. In other words, the optimal solution $\boldsymbol{\sigma}^\star$ and $\mathbf{p}^\star$ of (10) satisfy $\sigma_l^\star = f_l(\mathbf{p}^\star)$ for all l. We also have $\mu_l^\star = \sigma_l^\star \nabla_{\sigma_l} U_l(\sigma_l^\star)$.

According to the results stated in Lemma 2, we can decompose (9) as follows. For any feasible power vector $\mathbf{p}$, each user calculates a payment as follows:

$$w_l = f_l(\mathbf{p}) \nabla_{f_l} U_l(f_l(\mathbf{p})), \tag{13}$$

where w_l denotes the payment made by the lth user to the network controller, and $\nabla_{f_l} U_l(f_l(\mathbf{p}))$ is the first order derivative with respect to $f_l(\mathbf{p})$. Suppose the network controller receives the payment $\mathbf{w}$ from all the users, it solves a problem that maximizes the weighted sum of link metrics $f_l(\mathbf{p})$ for all l, where the payment $\mathbf{w}$ acts as a weight. In particular, the network controller solves the following problem:

$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} w_l \log f_l(\mathbf{p}) \\ \text{subject to} & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p}. \end{array} \tag{14}$$

In the above, $\mathbf{w}$ is a payment vector calculated by each user for a given power vector. We say that $\mathbf{w}$ is proportionally fair if $w_l = f_l(\mathbf{p}^\star) \nabla_{f_l} U_l(f_l(\mathbf{p}^\star))$, where $\mathbf{p}^\star$ is the optimal solution of (8). Interestingly, at optimality, both (8) and (14) have the same optimal solution.

Figure 1 gives a geometric interpretation of this proportional fairness equilibrium: a hyperplane intersecting the optimal point is perpendicular to $\mathbf{w}$ when $\mathbf{w}$ is proportionally fair. By leveraging this relationship, we propose the following algorithm that computes the optimal solution of (9).

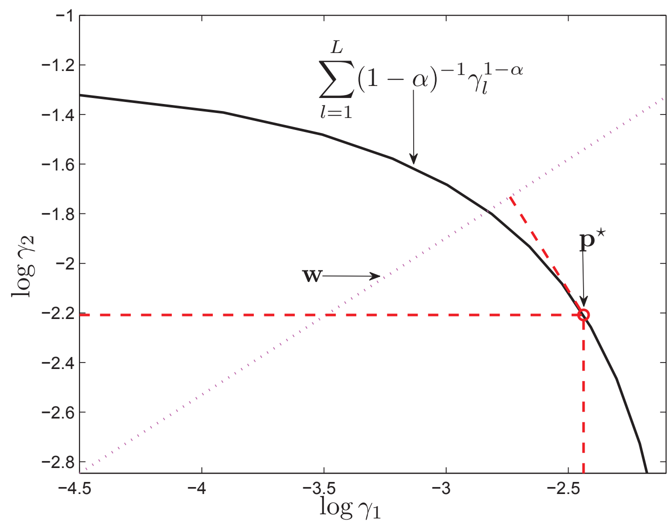


Fig. 1. Illustration of the proportional fairness in the $\log\mathsf{SINR}$ domain. We use a two-user example, and the objective utility function is the α-fairness utility function with $\alpha = 3$. The channel gains are given by $G_{11} = 0.75$, $G_{12} = 0.12$, $G_{21} = 0.13$, $G_{22} = 0.70$ and the weights for the power constraints are $\mathbf{a}_1 = [4.50,\ 5.20]^\top$ and $\mathbf{a}_2 = [2.80,\ 2.40]^\top$ respectively. The upper bound for the weighted power constraints is $\bar{\mathbf{p}} = [1.20,\ 1.00]^\top$ W, and the noise power for both users are 1 W. We solve the utility maximization problem to obtain the optimal power $\mathbf{p}^\star$, and use this $\mathbf{p}^\star$ to calculate the proportional fairness which is $w_l = \gamma_l^{\star\,-2}$ in this case. The perpendicular of the proportional fairness $\mathbf{w}$ intersects optimal solution of the utility maximization problem at the boundary of the feasible region.

Algorithm 1 (Utility Maximization):

1) Each lth user updates the payment:
$$w_l(k+1) = f_l(\mathbf{p}(k))\nabla_{f_l} U_l(f_l(\mathbf{p}(k))).$$
2) The controller solves the following problem:
$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} w_l(k+1)\log f_l(\mathbf{p}) \\ \text{subject to} & \mathbf{p}\in\mathcal{P}, \\ \text{variables:} & \mathbf{p}, \end{array}$$
whose optimal solution is denoted as $\mathbf{p}(k+1)$.

Theorem 1: Algorithm 1 converges to the optimal solution of (8) from any initial point $\mathbf{p}(0)$.

Remark 1: The proof of Theorem 1 is based on successive convex approximation (cf. Appendix Sec. E). At Step 1, the computation of payment can be made distributed by each user. The optimization problem in Step 2 is solved optimally by the network controller in a centralized manner using optimal algorithms proposed in the following section.

V. Optimal Fixed-Point Algorithms

In this section, we particularize solving (14) for the weighted sum logarithmic SINR maximization, the weighted sum inverse SINR minimization and the weighted sum reliability maximization. In particular, the stationarity of the Lagrangians for these problems share a common interesting property: They can be expressed as fixed-point problems involving concave self-mapping functions for the optimal solution. This enables us to propose tuning-free fixed-point algorithms that converge geometrically fast by leveraging the nonlinear Perron-Frobenius theory in [14] (cf. Appendix Sec. B).

A. Weighted Sum Logarithmic SINR Maximization

Now, we let $f_l(\mathbf{p})$ be $\mathsf{SINR}_l(\mathbf{p})$, and thus (14) is specified to be the weighted sum logarithmic SINR maximization problem given by:
$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} w_l \log\mathsf{SINR}_l(\mathbf{p}) \\ \text{subject to} & \mathbf{p}\in\mathcal{P}, \\ \text{variables:} & \mathbf{p}. \end{array} \tag{15}$$
We denote the optimal solution to (15) by $\mathbf{p}^\star$. Note that (15) is nonconvex due to the nonconvex objective function.

Although (15) is nonconvex, it is equivalent to a convex problem by a change of variable $\tilde{\mathbf{p}} = \log\mathbf{p}$. Hence, (15) is equivalent to the following convex optimization problem:
$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} w_l \log\mathsf{SINR}_l(e^{\tilde{\mathbf{p}}}) \\ \text{subject to} & \log\left((1/\bar{p}_l)\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right) \le 0, \quad l = 1,\dots,L, \\ \text{variables:} & \tilde{\mathbf{p}}. \end{array} \tag{16}$$
The Lagrangian associated with (16) is given by:
$$\mathcal{L}(\tilde{\mathbf{p}},\boldsymbol{\lambda}) = \log\prod_{l=1}^{L}\left((\mathbf{F}e^{\tilde{\mathbf{p}}}+\mathbf{v})_l e^{-\tilde{p}_l}\right)^{w_l} + \sum_{l=1}^{L}\lambda_l\log\left(\frac{1}{\bar{p}_l}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right), \tag{17}$$
where $\boldsymbol{\lambda}\in\mathbb{R}_+^L$ is the dual variable vector for $\log\left(\frac{1}{\bar{p}_l}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right) \le 0,\ \ l = 1,\dots,L$. By taking first order derivative of (17) with respect to p_l, setting it to zero and substituting $\mathbf{p} = e^{\tilde{\mathbf{p}}}$ back, we have the following equations satisfied by the optimal power $\mathbf{p}^\star$ to (15) and the optimal dual solution $\boldsymbol{\lambda}^\star$ to (16):
$$p_l^\star = \frac{w_l}{\sum_{j=1}^{L}\frac{w_j F_{jl}}{(\mathbf{F}\mathbf{p}^\star+\mathbf{v})_j} + \sum_{j=1}^{L}\frac{\lambda_j^\star A_{lj}}{\mathbf{a}_j^\top\mathbf{p}^\star}}, \quad l = 1,\dots,L, \tag{18}$$
and
$$\sum_{j=1}^{L}\frac{A_{lj}p_l^\star}{\mathbf{a}_j^\top\mathbf{p}^\star}\lambda_j^\star = w_l - p_l^\star\sum_{j\neq l}\frac{w_j F_{jl}}{(\mathbf{F}\mathbf{p}^\star+\mathbf{v})_j}, \quad l = 1,\dots,L. \tag{19}$$

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

Note that if $\boldsymbol{\lambda}^\star$ is the only unknown in (19), $\boldsymbol{\lambda}^\star$ can be easily computed by solving a system of linear equations (by keeping $\mathbf{p}^\star$ fixed). Moreover, as the right-hand side of (19) is positive, (19) is in fact a positive system of linear equations in $\mathbf{p}^\star$ and $\boldsymbol{\lambda}^\star$.

Next, let us define the following nonnegative matrix:

$$\mathbf{B}(\mathbf{p}) = \mathbf{F} + \sum_{l=1}^{L} \frac{\hat{\lambda}_l}{\bar{p}_l}\mathbf{v}\mathbf{a}_l^\top, \tag{20}$$

where $\hat{\boldsymbol{\lambda}} \in \mathbb{R}_+^L$ is the normalized dual variable vector (i.e., $\hat{\boldsymbol{\lambda}} = \boldsymbol{\lambda}/\mathbf{1}^\top\boldsymbol{\lambda}$). Note that $\mathbf{B}$ defined in (20) is a function of the transmit power $\mathbf{p}$, since $\boldsymbol{\lambda}$ in (19) depends on $\mathbf{p}$.

By using the complementary slackness at optimality, since $\hat{\lambda}_l^\star > 0 \Rightarrow \frac{1}{\bar{p}_l}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}^\star} = 1$ or $\hat{\lambda}_l^\star = 0 \Rightarrow \frac{1}{\bar{p}_l}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}^\star} < 1$, only those constraints that are tight at optimality participate in forming $\mathbf{B}$. This leads to $\sum_{l=1}^{L}(\hat{\lambda}_l^\star/\bar{p}_l)\mathbf{v}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}^\star} = 1$, which implies that $\mathbf{F}e^{\tilde{\mathbf{p}}^\star} + \mathbf{v} = \mathbf{B}(e^{\tilde{\mathbf{p}}^\star})e^{\tilde{\mathbf{p}}^\star}$.

Furthermore, using $\mathbf{B}$ in (20) and the complementary slackness condition, (16) can then be rewritten as:

$$\begin{array}{ll} \text{minimize} & \prod_{l=1}^{L}\left(\frac{(\mathbf{B}(e^{\tilde{\mathbf{p}}})e^{\tilde{\mathbf{p}}})_l}{e^{\tilde{p}_l}}\right)^{w_l} \\ \text{subject to} & \log\left((1/\bar{p}_l)\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right) \le 0, \quad l = 1,\ldots,L, \\ \text{variables:} & \tilde{\mathbf{p}}. \end{array} \tag{21}$$

Theorem 2: The optimal power $\mathbf{p}^\star$ of (15) satisfies

$$p_l^\star = \frac{w_l}{\sum_{j=1}^{L} w_j B_{jl}(\mathbf{p}^\star)/(\mathbf{B}(\mathbf{p}^\star)\mathbf{p}^\star)_j}, \quad l = 1,\ldots,L. \tag{22}$$

Moreover, $\tilde{p}_l^\star = \log p_l^\star$ solves (21) for all l.

Observe that the right-hand side of (22) is positive, homogeneous of degree one and quasi-concave. Thus, it is a concave function (cf. Appendix Sec. A). We can exploit this fact together with the nonlinear Perron-Frobenius theory in [14] (cf. Appendix Sec. B) to propose the following fixed-point algorithm that solves (15) optimally.

Algorithm 2 (Weighted Sum Log SINR Maximization):

---

1) Compute $\boldsymbol{\lambda}(k+1)$ by solving the following equations for a given $\mathbf{p}(k)$:
$$\sum_{j=1}^{L}\frac{A_{lj}p_l(k)}{\mathbf{a}_j^\top\mathbf{p}(k)}\lambda_j(k+1) = w_l - p_l(k)\sum_{j=1}^{L}\frac{w_jF_{jl}}{(\mathbf{F}\mathbf{p}(k)+\mathbf{v})_j}, \quad l = 1,\ldots,L.$$
2) Normalize $\boldsymbol{\lambda}(k+1)$ to $\hat{\boldsymbol{\lambda}}(k+1)$ and update the nonnegative matrix $\mathbf{B}(k+1)$:
$$\hat{\boldsymbol{\lambda}}(k+1) = \boldsymbol{\lambda}(k+1)/\mathbf{1}^\top\boldsymbol{\lambda}(k+1),$$
$$\mathbf{B}(k+1) = \mathbf{F} + \sum_{l=1}^{L}\frac{\hat{\lambda}_l(k+1)}{\bar{p}_l}\mathbf{v}\mathbf{a}_l^\top.$$
3) Update the power $p_l(k+1)$ for the lth user by:
$$p_l(k+1) = \frac{w_l}{\sum_{j=1}^{L} w_j B_{jl}(k+1)/(\mathbf{B}(k+1)\mathbf{p}(k))_j}.$$
4) Normalize $\mathbf{p}(k+1)$:
$$\mathbf{p}(k+1) \leftarrow \frac{\mathbf{p}(k+1)}{\max\limits_{l=1,\ldots,L}\{\mathbf{a}_l^\top\mathbf{p}(k+1)/\bar{p}_l\}}.$$

Corollary 1: Algorithm 2 converges geometrically fast to the fixed point $\mathbf{p}^\star$ in Theorem 2 from any initial point $\mathbf{p}(0)$.

B. Weighted Sum Inverse SINR Minimization

The weighted sum inverse SINR minimization problem is given by:

$$\begin{array}{ll} \text{minimize} & \sum_{l=1}^{L} w_l \frac{1}{\mathsf{SINR}_l(\mathbf{p})} \\ \text{subject to} & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p}. \end{array} \tag{23}$$

In this subsection, we denote the optimal solution to (23) by $\mathbf{p}^\star$. Although (23) has a different objective from (15), it is closely related to (15) due to the following reasons: 1) Solving (23) can be viewed as an alternative for solving (15) [23], [24]; 2) The optimal solution obtained by solving (23) provides a useful upper bound to the optimal value of (15); 3) The optimal solution of (15) and (23) can be the same under special cases, and this connection between (15) and (23) will be discussed in details later in this section.

Similar to the convexification technique used on (15) to get (16), (23) is equivalent to:

$$\begin{array}{ll} \text{minimize} & \sum_{l=1}^{L} w_l \frac{1}{\mathsf{SINR}_l(e^{\tilde{\mathbf{p}}})} \\ \text{subject to} & \log\left((1/\bar{p}_l)\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right) \le 0, \quad l = 1,\ldots,L, \\ \text{variables:} & \tilde{\mathbf{p}}. \end{array} \tag{24}$$

The optimization problem in (24) is convex (cf. Appendix Sec. H). By introducing the dual variable $\boldsymbol{\lambda} \in \mathbb{R}_+^L$, we form the Lagrangian for (24), given by:

$$\mathcal{L}(\tilde{\mathbf{p}}, \boldsymbol{\lambda}) = \sum_{l=1}^{L} w_l(\mathbf{F}e^{\tilde{\mathbf{p}}} + \mathbf{v})_l e^{-\tilde{p}_l} + \sum_{l=1}^{L}\lambda_l \log\frac{1}{\bar{p}_l}\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}. \tag{25}$$

Taking the partial derivative of (25) with respect to $\tilde{p}_l$, setting it to zero and substituting $\mathbf{p} = e^{\tilde{\mathbf{p}}}$ back, we have the following equations satisfied by the optimal solution $\mathbf{p}^\star$ to (23) and the optimal dual solution $\boldsymbol{\lambda}^\star$ to (24):

$$p_l^\star = \sqrt{\frac{w_l(\mathbf{F}\mathbf{p}^\star + \mathbf{v})_l}{\sum_{j\neq l} w_j F_{jl}/p_j^\star + \sum_{j=1}^{L}\lambda_j^\star A_{lj}/\mathbf{a}_j^\top\mathbf{p}^\star}}, \quad l = 1,\ldots,L, \tag{26}$$

and

$$\sum_{j=1}^{L}\frac{A_{lj}p_l^\star}{\mathbf{a}_j^\top\mathbf{p}^\star}\lambda_j^\star = \frac{w_l(\mathbf{F}\mathbf{p}^\star + \mathbf{v})_l}{p_l^\star} - p_l^\star\sum_{j\neq l}\frac{w_jF_{jl}}{p_j^\star}, \quad l = 1,\ldots,L. \tag{27}$$

Note that (27) is a system of linear equations in $\mathbf{p}^\star$ and $\boldsymbol{\lambda}^\star$ with a positive right-hand side, and hence a positive linear system.

Recall the same definition of $\mathbf{B}$ in (20). In this case, $\hat{\boldsymbol{\lambda}}^\star \in \mathbb{R}_+^L$ is the normalized dual variable for (24). Notice that (24) can be further rewritten as:

$$\begin{array}{ll} \text{minimize} & \sum_{l=1}^{L} w_l \frac{(\mathbf{B}(e^{\tilde{\mathbf{p}}})e^{\tilde{\mathbf{p}}})_l}{e^{\tilde{p}_l}} \\ \text{subject to} & \log\left((1/\bar{p}_l)\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right) \le 0, \quad l = 1,\ldots,L, \\ \text{variables:} & \tilde{\mathbf{p}}. \end{array} \tag{28}$$

Theorem 3: The optimal power $\mathbf{p}^\star$ of (23) satisfies

$$p_l^\star = \sqrt{\frac{w_l\sum_{j\neq l}B_{lj}(\mathbf{p}^\star)p_j^\star}{\sum_{j\neq l}w_jB_{jl}(\mathbf{p}^\star)/p_j^\star}}, \quad l = 1,\ldots,L. \tag{29}$$

As in the previous, the following fixed-point algorithm computes the optimal solution of (23) in Theorem 3 by using the nonlinear Perron-Frobenius theory in [14] (cf. Appendix Sec. B).

Algorithm 3 (Weighted Sum Inverse SINR Minimization):

1) Compute $\boldsymbol{\lambda}(k+1)$ by solving the following equations for a given $\mathbf{p}(k)$:
$$\sum_{j=1}^{L} \frac{A_{lj} p_l(k)}{\mathbf{a}_j^\top \mathbf{p}(k)} \lambda_j(k+1) = \frac{w_l (\mathbf{F}\mathbf{p}(k)+\mathbf{v})_l}{p_l(k)} - p_l(k) \sum_{j \neq l} \frac{w_j F_{jl}}{p_j(k)}, \quad l = 1, \ldots, L.$$
2) Normalize $\boldsymbol{\lambda}(k+1)$ to $\hat{\boldsymbol{\lambda}}(k+1)$ and update the nonnegative matrix $\mathbf{B}(k+1)$:
$$\hat{\boldsymbol{\lambda}}(k+1) = \boldsymbol{\lambda}(k+1)/\mathbf{1}^\top \boldsymbol{\lambda}(k+1),$$
$$\mathbf{B}(k+1) = \mathbf{F} + \sum_{l=1}^{L} \frac{\hat{\lambda}_l(k+1)}{\bar{p}_l} \mathbf{v}\mathbf{a}_l^\top.$$
3) Update the power $p_l(k+1)$ for the lth user by:
$$p_l(k+1) = \sqrt{\frac{w_l \sum_{j \neq l} B_{lj}(k+1) p_j(k)}{\sum_{j \neq l} w_j B_{jl}(k+1)/p_j(k)}}.$$
4) Normalize $\mathbf{p}(k+1)$:
$$\mathbf{p}(k+1) \leftarrow \frac{\mathbf{p}(k+1)}{\max_{l=1,\ldots,L} \{\mathbf{a}_l^\top \mathbf{p}(k+1)/\bar{p}_l\}}.$$

Corollary 2: Algorithm 3 converges geometrically fast to the fixed point $\mathbf{p}^\star$ in Theorem 3 from any initial point $\mathbf{p}(0)$.

Remark 2: We connect the weighted sum logarithmic SINR maximization in Section V-A with (23). In particular, by applying the arithmetic-geometric mean inequality and the Friedland-Karlin inequality in [25], we have
$$\sum_{l=1}^{L} w_l \frac{(\mathbf{B}(\mathbf{p})\mathbf{p})_l}{p_l} \geq \prod_{l=1}^{L} \left(\frac{(\mathbf{B}(\mathbf{p})\mathbf{p})_l}{p_l} \right)^{w_l} \geq \rho(\mathbf{B}(\mathbf{p})). \quad (30)$$
The equality holds for both the first and the second inequalities in (30) if and only if $(\mathbf{B}(\mathbf{p})\mathbf{p})_1/p_1 = \cdots = (\mathbf{B}(\mathbf{p})\mathbf{p})_L/p_L$, i.e., $\mathsf{SINR}_1(\mathbf{p}) = \cdots = \mathsf{SINR}_L(\mathbf{p})$.

Interestingly, the weighted sum inverse SINR minimization in (23) has the same optimal solution $\mathbf{p}^\star$ as the weighted sum logarithmic SINR maximization in (15) when
$$\mathbf{w} = \mathbf{x}(\mathbf{B}(\mathbf{p}^\star)) \circ \mathbf{y}(\mathbf{B}(\mathbf{p}^\star)), \quad (31)$$
where $\mathbf{x}(\mathbf{B}(\mathbf{p}^\star))$ and $\mathbf{y}(\mathbf{B}(\mathbf{p}^\star))$ are respectively the Perron right and left eigenvectors of $\mathbf{B}(\mathbf{p}^\star)$, and $\mathbf{x}(\mathbf{B}(\mathbf{p}^\star)) \circ \mathbf{y}(\mathbf{B}(\mathbf{p}^\star))$ is a probability vector. The physical interpretation of (31) is that both the Perron right and left eigenvectors of $\mathbf{B}(\mathbf{p}^\star)$ provide the proportional fairness *direction* as shown in Figure 1 for its perpendicular to intersect with the optimal solution of the utility maximization problem.

Finally, the same technique as in Section V-A and V-B can be used to solve the weighted sum logarithmic reliability maximization in Section V-C, i.e., finding a concave self-mapping, and then leveraging the nonlinear Perron-Frobenius theory to propose a fixed-point algorithm.

C. Weighted Sum Logarithmic Reliability Maximization

Let $f_l(\mathbf{p})$ be the reliability $\Psi_l(\mathbf{p})$. Then the weighted sum logarithmic reliability maximization problem is given by:
$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} w_l \log \Psi_l(\mathbf{p}) \\ \text{subject to} & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p}. \end{array} \quad (32)$$
We denote the optimal solution to (32) by $\mathbf{p}^\star$.

Through a logarithmic transformation of variable on (32), we obtain:
$$\begin{array}{ll} \text{maximize} & \sum_{l=1}^{L} w_l \log \Psi_l(e^{\tilde{\mathbf{p}}}) \\ \text{subject to} & \log\left((1/\bar{p}_l)\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right) \leq 0, \quad l = 1, \ldots, L, \\ \text{variables:} & \tilde{\mathbf{p}}. \end{array} \quad (33)$$
Next, let us define the nonnegative matrix $\mathbf{C}$ with the entries (that are function of $\mathbf{p}$):
$$C_{lj}(\mathbf{p}) = \begin{cases} 0, & \text{if } l = j, \\ \dfrac{p_l}{\beta_l p_j} \log\left(1 + \dfrac{\beta_l F_{lj} p_j}{p_l}\right), & \text{if } l \neq j. \end{cases} \quad (34)$$
We then form the Lagrangian for (33) by introducing the dual variable $\boldsymbol{\lambda} \in \mathbb{R}_+^L$ for the L inequality constraints in (33) to obtain:
$$\mathcal{L}(\tilde{\mathbf{p}}, \boldsymbol{\lambda}) = \sum_{l=1}^{L} w_l \left(v_l \beta_l e^{-\tilde{p}_l} + \sum_{j=1}^{L} \log\left(1 + \beta_l F_{lj} e^{\tilde{p}_j - \tilde{p}_l}\right) \right) + \sum_{l=1}^{L} \lambda_l \log \frac{1}{\bar{p}_l} \mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}. \quad (35)$$

Taking the partial derivative of (35) with respect to $\tilde{p}_l$, setting it to zero and then substituting $\mathbf{p} = e^{\tilde{\mathbf{p}}}$ back, we have the following equations for the optimal power $\mathbf{p}^\star$:
$$p_l^{\star 2} \sum_{j=1}^{L} \frac{A_{lj}}{\mathbf{a}_j^\top \mathbf{p}^\star} \lambda_j = w_l v_l \beta_l + \sum_{j \neq l} \frac{w_l \beta_l F_{lj} p_j^\star p_l^\star}{p_l^\star + \beta_l F_{lj} p_j^\star} - p_l^{\star 2} \sum_{j \neq l} \frac{w_j \beta_j F_{jl}}{p_j^\star + \beta_j F_{jl} p_l^\star}, \quad l = 1, \ldots, L. \quad (36)$$
As in the previous, (36) is a positive system of linear equations in the optimal transmit power $\mathbf{p}^\star$ to (32) and the optimal dual solution $\boldsymbol{\lambda}^\star$ to (33) that characterizes the optimality conditions of (32).

In Section V-A, dual variables are used to construct a nonnegative matrix $\mathbf{B}$ in (20). Likewise, we define a nonnegative matrix $\mathbf{D}$ given by:
$$\mathbf{D}(\mathbf{p}) = \mathbf{C}(\mathbf{p}) + \sum_{l=1}^{L} \frac{\hat{\lambda}_l}{\bar{p}_l} \mathbf{v}\mathbf{a}_l^\top, \quad (37)$$
where $\hat{\boldsymbol{\lambda}} \in \mathbb{R}_+^L$ is the normalized dual variable vector of (33) such that $\hat{\boldsymbol{\lambda}} = \boldsymbol{\lambda}/\mathbf{1}^\top \boldsymbol{\lambda}$. We can then rewrite (33) as
$$\begin{array}{ll} \text{minimize} & \sum_{l=1}^{L} w_l \dfrac{(\mathrm{diag}(\boldsymbol{\beta})\mathbf{D}(e^{\tilde{\mathbf{p}}})e^{\tilde{\mathbf{p}}})_l}{e^{\tilde{p}_l}} \\ \text{subject to} & \log\left((1/\bar{p}_l)\mathbf{a}_l^\top e^{\tilde{\mathbf{p}}}\right) \leq 0, \quad l = 1, \ldots, L, \\ \text{variables:} & \tilde{\mathbf{p}}. \end{array} \quad (38)$$
Theorem 4: The optimal power $\mathbf{p}^\star$ of (32) satisfies
$$p_l^\star = \sqrt{\frac{w_l \sum_{j \neq l} \left(v_l \beta_l \hat{a}_j(\mathbf{p}^\star) p_j^\star + \dfrac{\beta_l F_{lj} p_j^\star p_l^\star}{p_l^\star + \beta_l F_{lj} p_j^\star} \right)}{\sum_{j \neq l} w_j \left(v_j \beta_j \hat{a}_l(\mathbf{p}^\star)/p_j^\star + \dfrac{\beta_j F_{jl}}{p_j^\star + \beta_j F_{jl} p_l^\star} \right)}} \quad (39)$$
for all l, where $\hat{\mathbf{a}}$ is given by
$$\hat{\mathbf{a}} = \sum_{l=1}^{L} \frac{\hat{\lambda}_l^\star}{\bar{p}_l} \mathbf{a}_l.$$

Similar to (20), the function of $\hat{\mathbf{a}}$ involving $\hat{\lambda}_l^\star$ is also a function of $\mathbf{p}^\star$. Moreover, $\tilde{p}_l^\star = \log p_l^\star$ solves (38) for all l.

It can be verified that the right-hand side of (39) is positive,

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

quasi-concave and homogeneous of degree one, and hence it is concave (cf. Appendix Sec. A). We thus leverage the nonlinear Perron-Frobenius theory in [14] (cf. Appendix Sec. B) to propose the following algorithm.

Algorithm 4 (Weighted Sum Log Reliability Maximization):

---

1) Compute $\boldsymbol{\lambda}(k+1)$ by solving the following equations for a given $\mathbf{p}(k)$:
$$\sum_{j=1}^{L} \frac{A_{lj} p_l(k)}{\mathbf{a}_j^\top \mathbf{p}(k)} \lambda_j(k+1) = w_l v_l \beta_l / p_l(k)$$
$$+ \sum_{j \neq l} \frac{w_l \beta_l F_{lj} p_j(k)}{p_l(k) + \beta_l F_{lj} p_j(k)} - \sum_{j \neq l} \frac{w_j \beta_j F_{jl} p_l(k)}{p_j(k) + \beta_j F_{jl} p_l(k)},$$
$$l = 1, \ldots, L.$$

2) Normalize $\boldsymbol{\lambda}(k+1)$ to $\hat{\boldsymbol{\lambda}}(k+1)$ and update the vector $\hat{\mathbf{a}}(k+1)$:
$$\hat{\boldsymbol{\lambda}}(k+1) = \boldsymbol{\lambda}(k+1) / 1^\top \boldsymbol{\lambda}(k+1),$$
$$\hat{\mathbf{a}}(k+1) = \sum_{l=1}^{L} \frac{\hat{\lambda}_l(k+1)}{\bar{p}_l} \mathbf{a}_l.$$

3) Update the power $p_l(k+1)$ for the lth user by:
$$p_l(k+1) = \sqrt{\frac{w_l \sum_{j \neq l} \left(v_l \beta_l \hat{a}_j(k+1) p_j(k) + \frac{\beta_l F_{lj} p_j(k) p_l(k)}{p_l(k) + \beta_l F_{lj} p_j(k)} \right)}{\sum_{j \neq l} w_j \left(v_j \beta_j \hat{a}_l(k+1) / p_j(k) + \frac{\beta_j F_{jl}}{p_j(k) + \beta_j F_{jl} p_l(k)} \right)}}.$$

4) Normalize $\mathbf{p}(k+1)$:
$$\mathbf{p}(k+1) \leftarrow \frac{\mathbf{p}(k+1)}{\max\limits_{l=1,\ldots,L} \{\mathbf{a}_l^\top \mathbf{p}(k+1) / \bar{p}_l\}}.$$

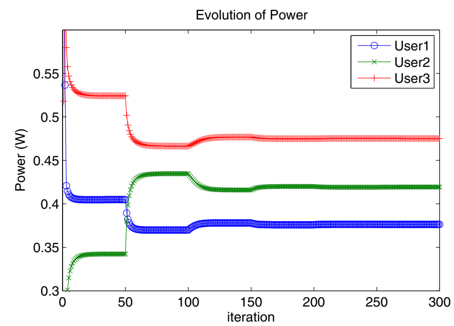
Corollary 3: Algorithm 4 converges geometrically fast to the fixed point $\mathbf{p}^\star$ in Theorem 4 from any initial point $\mathbf{p}(0)$.

In summary, all the three problems studied in Section V have a common structure: 1) They can be reformulated with a polynomial objective function, involving a suitably constructed nonnegative matrix $\mathbf{B}$ or $\mathbf{D}$; 2) A concave self-mapping can be identified to characterize optimality; 3) We can use the nonlinear Perron-Frobenius theory to design fixed-point algorithms to solve these problems optimally.

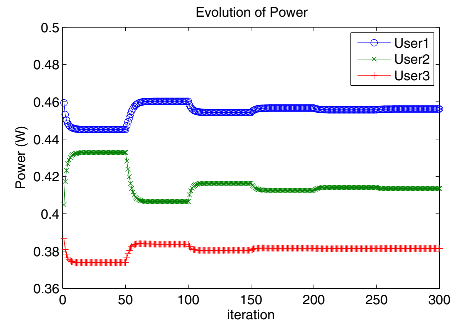
VI. Numerical Examples

In this section, we evaluate the performance of Algorithm 1 numerically to solve (8) for $U(\cdot)$ being the 1-fairness utility function in (6) by replacing Step 2 of Algorithm 1 with the fixed-point algorithms in Section V. Since we connect (15) and (23) (cf. (31)), we compare the convergence of these two different problems that have the same optimal solution. We consider a three-user case, using the following channel gain matrix $\mathbf{G}$: $G_{11} = 0.71$, $G_{12} = 0.13$, $G_{13} = 0.12$, $G_{21} = 0.11$, $G_{22} = 0.73$, $G_{23} = 0.14$, $G_{31} = 0.15$, $G_{32} = 0.16$, $G_{33} = 0.69$, and the following weights for the power constraints: $\mathbf{a}_1 = (0.93\ \ 0.72\ \ 0.74)^\top$, $\mathbf{a}_2 = (0.63\ \ 0.86\ \ 0.93)^\top$ and $\mathbf{a}_3 = (0.98\ \ 0.86\ \ 0.78)^\top$. We set $\bar{\mathbf{p}} = (1.50\ \ 1.00\ \ 1.10)^\top$ W, and the noise power of each user is 1 W.

Figure 2 plots the evolution of the power for three users that run Algorithm 1 with only 6 outer loops. Note that the network utility used in (a) and (b) of Figure 2 are, respectively, in terms of $f_l(\mathbf{p})$ being $\mathsf{SINR}(\mathbf{p})$ in (1) and $\boldsymbol{\Psi}(\mathbf{p})$ in (4). In Figure 2(a), we set the initial power vector to $\mathbf{p}(0) = [0.45\ 0.80\ 0.54]^\top$, and run Algorithm 2 for 50 iterations as an inner loop of



(a)



(b)

Fig. 2. Illustration of the convergence of Algorithm 1 with two different network utility. We plot the power evolution for each 3 of the users that run Algorithm 2 in (a) and Algorithm 4 in (b) respectively for 50 iterations in the inner loop. We can observe that both Algorithm 2 and Algorithm 4 converge in each inner loop, and Algorithm 1 converges in the outer loop.

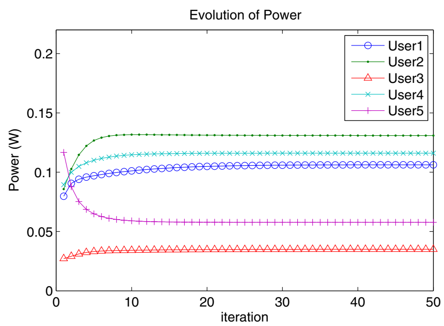
Algorithm 1. The optimal power $\mathbf{p}^\star$ is $[0.38\ \ 0.42\ \ 0.48]^\top$. In Figure 2(b), we set the initial power vector to $\mathbf{p}(0) = [0.50\ 0.38\ 0.42]^\top$ W, and run Algorithm 4 for 50 iterations as an inner loop of Algorithm 1. The optimal power $\mathbf{p}^\star$ is $[0.46\ 0.41\ 0.38]^\top$ W. Furthermore, we also run our fixed-point algorithms (i.e., Algorithm 2, 3 and 4) with a suitably large number of users. Figure 3 plot the evolution of five users out of twenty users, which shows that the inner loop of Algorithm 1, i.e., the fixed-point algorithms for the three problems studied in Section V, converge fast to the optimal point.

Next, we verify the connection between the weighted sum logarithmic SINR maximization problem and the weighted sum inverse SINR minimization problem. Since we know in advance that $\mathbf{p}^\star = [0.42\ \ 0.41\ \ 0.44]^\top$ W, we set $\mathbf{w} = \mathbf{x}(\mathbf{B}(\mathbf{p}^\star)) \circ \mathbf{y}(\mathbf{B}(\mathbf{p}^\star))$ in (15) and (23). Starting from the same initial point $\mathbf{p}(0) = [0.32\ 0.42\ 0.51]^\top$ W, Figure 4 shows that both Algorithm 2 and Algorithm 3 converge geometrically fast to the optimal solution as expected.

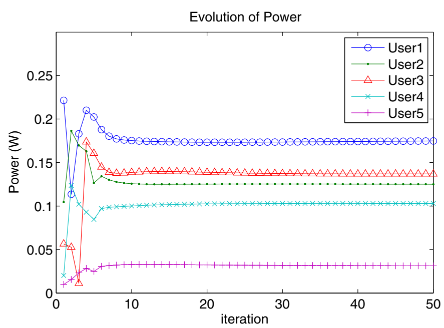
VII. Conclusion

We studied the utility maximization problem for service differentiation in wireless networks. This problem is non-convex, and thus generally hard to solve. Using the Kelly's proportional fairness decomposition and a logarithmic change-of-variable technique, we decomposed the utility maximization problem into separable user subproblems and a network subproblem. Maximizing the weighted sum logarithmic SINR, minimizing the weighted sum inverse SINR and maximizing the weighted sum logarithmic reliability are three special cases

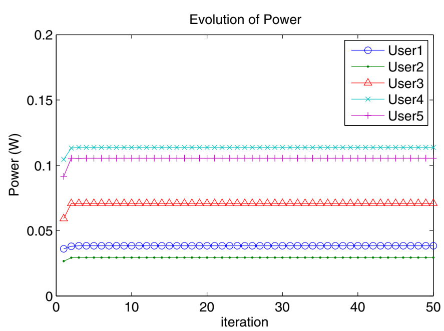
This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.



(a)



(b)



(c)

Fig. 3. Illustration of the convergence of Algorithm 2, 3 and 4 for 20 users. In these figures, we show the power evolution for the first 5 users. (a) Algorithm 2 that solves the weighted sum logarithmic SINR maximization. (b) Algorithm 3 that solves the weighted sum inverse SINR minimization. (c) Algorithm 4 that solves the weighted sum logarithmic reliability maximization.

of the user subproblems solved optimally by our technique. These three special cases hitherto have been solved sub-optimally in the literature. We exploited a common shared optimality feature in these three special cases, which are positive, homogeneous, monotone and quasi-concave self-mappings related to the optimality conditions, and together with the nonlinear Perron-frobenius theory, to design fixed-point optimal algorithms that converge geometrically fast. Numerical examples verified that these optimal algorithms are computationally attractive and converge fast to the optimal solution.

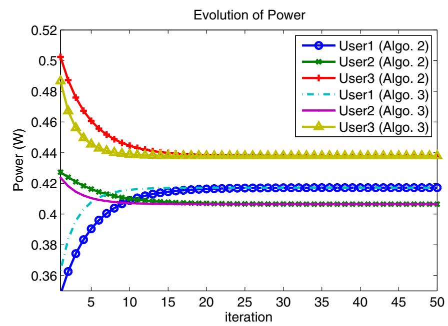


Fig. 4. Performance comparison between Algorithm 2 and Algorithm 3 that solves two different optimization problems that have the same optimal solution via the connection highlighted in Remark 2. Although they are different in the evolution of powers for each of the 3 users, these two algorithms converge geometrically fast.

APPENDIX

A. Proof that a positive, quasi-concave and homogeneous function of degree one is concave

Suppose $f:\mathbb{R}_+^L \rightarrow \mathbb{R}$ is positive, homogeneous of degree one and quasi-concave, then f is concave.

Proof: Given any $\mathbf{s},\mathbf{t}\in\mathbb{R}_+^L$, for constants μ and ν such that $\mu f(\mathbf{t}) = f(\mathbf{s})$ and $\nu f(\mathbf{s}) = f(\mathbf{t})$, we have $f(\mu\mathbf{t}) = f(\mathbf{s})$ and $f(\nu\mathbf{s}) = f(\mathbf{t})$ respectively due to the homogeneity of f. Note that $f(\mu\mathbf{t}) = f(\mathbf{s})$ and $f(\nu\mathbf{s}) = f(\mathbf{t})$ do not mean $\mu\mathbf{t} = \mathbf{s}$ or $\nu\mathbf{s} = \mathbf{t}$. Furthermore, $\mu f(\mathbf{t}) = f(\mathbf{s})$ and $\nu f(\mathbf{s}) = f(\mathbf{t})$ imply that $\nu f(\mathbf{s}) = (1/\mu)f(\mathbf{s})$, i.e., $\mu\nu = 1$. For any constant $\theta\in(0,1)$, we have $\theta f(\mathbf{s}) + (1-\theta)f(\mathbf{t}) = f\big((\theta+(1-\theta)\nu)\mathbf{s}\big)$ and $\theta f(\mathbf{s}) + (1-\theta)f(\mathbf{t}) = f\big((\theta\mu+(1-\theta))\mathbf{t}\big)$.

By defining a constant $\eta = \theta/\big(\theta+(1-\theta)\nu\big)\in(0,1)$, then we have:

$$
\begin{aligned}
\theta f(\mathbf{s}) &+ (1-\theta)f(\mathbf{t})\\
&= \min\Big\{ f\big((\theta+(1-\theta)\nu)\mathbf{s}\big), f\big((\theta\mu+(1-\theta))\mathbf{t}\big)\Big\}\\
&\overset{(a)}{\leq} f\big(\eta(\theta+(1-\theta)\nu)\mathbf{s} + (1-\eta)(\theta\mu+(1-\theta))\mathbf{t}\big)\\
&\overset{(b)}{=} f\big(\theta\mathbf{s}+(1-\theta)\mathbf{t}\big),
\end{aligned}
$$

where (a) is due to quasi-concavity of f, and (b) is due to $\mu\nu = 1$. This completes the proof.

B. Nonlinear Perron-Frobenius Theory [14]

Let $\|\cdot\|$ be a monotone norm on $\mathbb{R}^L$. For a concave mapping $f:\mathbb{R}_+^L\rightarrow\mathbb{R}_+^L$ with $f(\mathbf{z}) > 0$ for $\mathbf{z}\geq\mathbf{0}$, the following statements hold. The conditional eigenvalue problem $f(\mathbf{z}) = \lambda\mathbf{z}$, $\lambda\in\mathbb{R}$, $\mathbf{z}\geq\mathbf{0}$, $\|\mathbf{z}\| = 1$ has a unique solution $(\lambda^*,\mathbf{z}^*)$, where $\lambda^*>0$, $\mathbf{z}^*>\mathbf{0}$. Furthermore, $\lim_{k\rightarrow\infty}\tilde{f}(\mathbf{z}(k))$ converges geometrically fast to $\mathbf{z}^*$, where $\tilde{f}(\mathbf{z}) = f(\mathbf{z})/\|f(\mathbf{z})\|$.

In this paper, we first identify a suitable concave mapping f for each of the different optimization problems, and then use the monotone norm defined on $\mathbb{R}^L$ as $\max\limits_{l=1,\ldots,L}\{\mathbf{a}_l^\top\mathbf{p}/\bar{p}_l\}$ in order to apply the above nonlinear Perron-Frobenius theory in [14].

C. Proof of Lemma 1

It is well-known that $\log\mathsf{SINR}_l(e^{\tilde{\mathbf{p}}})$ is concave in $\tilde{\mathbf{p}}$ for all l [4]. We show below that $\log\Psi_l(e^{\tilde{\mathbf{p}}})$ is concave by inspecting its Hessian. We obtain the Hessian $\nabla^2 \log \Psi_l(e^{\tilde{\mathbf{p}}})$ with entries given by: $(\nabla^2 \log \Psi_l(e^{\tilde{\mathbf{p}}}))_{jk} =$

$$\begin{cases} -v_l\beta_l e^{-\tilde{p}_l} - \sum_{m=1}^{L} \beta_l F_{lm} e^{\tilde{p}_m - \tilde{p}_l}, & \text{if } j = k = l \\ -\beta_l F_{lj} e^{\tilde{p}_j - \tilde{p}_l}, & \text{if } j = k, k \neq l \\ \beta_l F_{lk} e^{\tilde{p}_k - \tilde{p}_l}, & \text{if } j \neq k,\ j = l \\ \beta_l F_{lj} e^{\tilde{p}_j - \tilde{p}_l}, & \text{if } j \neq k,\ k = l \\ 0, & \text{otherwise.} \end{cases}$$

Now, the Hessian $\nabla^2 \log \Psi_l(e^{\tilde{\mathbf{p}}})$ is indeed negative semidefinite: for any real vector $\mathbf{z}$, we have $\mathbf{z}^\top \nabla^2 \log \Psi_l(e^{\tilde{\mathbf{p}}})\mathbf{z} = -v_l\beta_l e^{-\tilde{p}_l} z_l^2 - \sum_{m=1}^{L} \beta_l F_{lm} e^{\tilde{p}_m - \tilde{p}_l}(z_l - z_m)^2 \leq 0$. Hence, $\log \Psi_l(e^{\tilde{\mathbf{p}}})$ is concave in $\tilde{\mathbf{p}}$ for all l.

D. Proof of Lemma 2

Taking the partial derivative of (12) with respect to $\tilde{\sigma}_l$, we have $\nabla_{\tilde{\sigma}_l}\mathcal{L}(\tilde{\mathbf{p}}, \tilde{\boldsymbol{\sigma}}, \boldsymbol{\mu}) = \nabla_{\tilde{\sigma}_l} U_l(e^{\tilde{\sigma}_l}) - \mu_l$, and setting it to zero, we have at optimality $\mu_l^\star = \nabla_{\tilde{\sigma}_l} U_l(e^{\tilde{\sigma}_l^\star})$. In addition, by using the complementary slackness, we know that $\tilde{\sigma}_l^\star = \log f_l(e^{\tilde{\mathbf{p}}^\star})$ from $\mu_l^\star > 0$.

E. Proof of Theorem 1

Under Assumption 1, (11) is a convex optimization problem and thus can be solved by a successive convex approximation technique. In particular, we replace the objective function $U(e^{\tilde{\boldsymbol{\sigma}}})$ of (11) with an approximation by its Taylor series (up to the first order terms): $U(e^{\tilde{\boldsymbol{\sigma}}}) \approx U(e^{\hat{\boldsymbol{\sigma}}}) + \nabla U(e^{\hat{\boldsymbol{\sigma}}})^\top(\tilde{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}})$, where $\hat{\boldsymbol{\sigma}}$ is any feasible point. We then compute a feasible $\tilde{\boldsymbol{\sigma}}(k+1)$ by solving the $(k+1)$th approximation problem:

$$\begin{array}{ll} \text{maximize} & \nabla U(e^{\tilde{\boldsymbol{\sigma}}(k)})^\top(\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}(k)) \\ \text{subject to} & \tilde{\sigma}_l \leq \log f_l(e^{\tilde{\mathbf{p}}}), \quad l = 1, \ldots, L, \\ & e^{\tilde{\mathbf{p}}} \in \mathcal{P}, \\ \text{variables:} & \tilde{\mathbf{p}}, \tilde{\boldsymbol{\sigma}}, \end{array} \tag{40}$$

where $\tilde{\boldsymbol{\sigma}}(k)$ is the optimal solution of the kth approximation problem. Combining (40) with the result stated in Lemma 2, i.e., $\tilde{\boldsymbol{\sigma}}_l^\star = \log f_l(\mathbf{p}^\star)$, and setting $\mathbf{w} = \nabla_{\tilde{\boldsymbol{\sigma}}} U(e^{\tilde{\boldsymbol{\sigma}}(k)})$, this approximation technique converges to the optimal solution.

F. Proof of Theorem 2

From (19), we have $\sum_{l=1}^{L}\sum_{j=1}^{L} \frac{A_{lj}p_l^\star}{\mathbf{a}_j^\top \mathbf{p}^\star}\lambda_j^\star = \sum_{l=1}^{L} w_l - \sum_{l=1}^{L} p_l^\star \sum_{j\neq l} \frac{w_j F_{jl}}{(\mathbf{F}\mathbf{p}^\star + \mathbf{v})_j}$, which implies $\sum_{j=1}^{L}\lambda_j^\star = \sum_{j=1}^{L}\frac{w_j v_j}{(\mathbf{F}\mathbf{p}^\star + \mathbf{v})_j}$. Thus, the optimal dual variable $\boldsymbol{\lambda}^\star$ in (16) satisfies $\boldsymbol{\lambda}^\star = \left(\sum_{j=1}^{L}\frac{w_j v_j}{(\mathbf{F}\mathbf{p}^\star + \mathbf{v})_j}\right)\hat{\boldsymbol{\lambda}}^\star$ with $\hat{\boldsymbol{\lambda}}^\star \in \mathbb{R}_+^L$ being the normalized dual variable (i.e., $\hat{\boldsymbol{\lambda}}^\star = \frac{\boldsymbol{\lambda}^\star}{\mathbf{1}^\top\boldsymbol{\lambda}^\star}$). We then have $\sum_{j=1}^{L}\frac{w_j F_{jl}}{(\mathbf{F}\mathbf{p}^\star + \mathbf{v})_j} + \sum_{k=1}^{L}\frac{\lambda_k^\star A_{lk}}{\mathbf{a}_k^\top \mathbf{p}^\star} = \sum_{j=1}^{L}\frac{w_j(F_{jl} + \sum_{k=1}^{L}\frac{\hat{\lambda}_k^\star}{p_k}v_j A_{lk})}{(\mathbf{F}\mathbf{p}^\star + \mathbf{v})_j}$ for the denominator on the right-hand side of (18). Combining with $\mathbf{F}\mathbf{p}^\star + \mathbf{v} = \mathbf{B}\mathbf{p}^\star$ where $\mathbf{B}$ is given in (20), we have $\sum_{j=1}^{L}\frac{w_j F_{jl}}{(\mathbf{F}\mathbf{p}^\star + \mathbf{v})_j} + \sum_{k=1}^{L}\frac{\lambda_k^\star A_{lk}}{\mathbf{a}_k^\top \mathbf{p}^\star} = \sum_{j=1}^{L}\frac{w_j B_{jl}(\mathbf{p}^\star)}{(\mathbf{B}(\mathbf{p}^\star)\mathbf{p}^\star)_j}$, which can be substituted back to (18) to obtain (22).

G. Proof of Corollary 1

Let $\big(f(\mathbf{p})\big)_l = \frac{w_l}{\sum_{j=1}^{L} w_j B_{jl}/(\mathbf{B}\mathbf{p})_j}$ be the function on the right-hand side of (22), where $\mathbf{B} = \mathbf{B}(\mathbf{p}^\star)$ is a constant matrix. Observe that $f(\mathbf{p})$ is positive, monotone, and homogeneous of degree one. We prove below that $f(\mathbf{p})$ is quasi-concave. Therefore, it is a concave mapping so that the nonlinear Perron-Frobenius theory can be leveraged for the algorithm design (cf. Appendix Sec. A and Sec. B).

For $\alpha \in \mathbb{R}$, the superlevel set of $f(\mathbf{p})$ is given by: $\mathcal{S}_\alpha = \left\{\mathbf{p} \,\middle|\, \frac{w_l}{\sum_{j=1}^{L} w_j B_{jl}/(\mathbf{B}\mathbf{p})_j} \geq \alpha\right\}$. Given any $\mathbf{s}, \mathbf{t} \in \mathcal{S}_\alpha$, we have (due to $\big(f(\mathbf{p})\big)_l \in \mathbb{R}_+$): $w_l \geq \alpha\sum_{j=1}^{L} w_j B_{jl}/(\mathbf{B}\mathbf{s})_j$ and $w_l \geq \alpha\sum_{j=1}^{L} w_j B_{jl}/(\mathbf{B}\mathbf{t})_j$ respectively. Thus, for any θ such that $0 \leq \theta \leq 1$, we can obtain $\frac{w_l}{\sum_{j=1}^{L} w_j B_{jl}/\big(\mathbf{B}(\theta\mathbf{s} + (1-\theta)\mathbf{t})\big)_j} \geq \alpha\frac{\theta\sum_{j=1}^{L} w_j B_{jl}/(\mathbf{B}\mathbf{s})_j + (1-\theta)\sum_{j=1}^{L} w_j B_{jl}/(\mathbf{B}\mathbf{t})_j}{\sum_{j=1}^{L} w_j B_{jl}/\big(\mathbf{B}(\theta\mathbf{s} + (1-\theta)\mathbf{t})\big)_j}$.

According to the arithmetic-geometric mean inequality, we have $\frac{w_j B_{jl}}{\theta(\mathbf{B}\mathbf{s})_j} + \frac{w_j B_{jl}}{(1-\theta)(\mathbf{B}\mathbf{t})_j} \geq 4\frac{w_j B_{jl}}{\theta(\mathbf{B}\mathbf{s})_j + (1-\theta)(\mathbf{B}\mathbf{t})_j}$. Due to that $\frac{\sum_{j=1}^{L} w_j B_{jl}}{\big(\mathbf{B}(\theta\mathbf{s} + (1-\theta)\mathbf{t})\big)_j} \leq \frac{1}{4}\left(\sum_{j=1}^{L}\frac{w_j B_{jl}}{\theta(\mathbf{B}\mathbf{s})_j} + \sum_{j=1}^{L}\frac{w_j B_{jl}}{(1-\theta)(\mathbf{B}\mathbf{t})_j}\right)$, we can thus obtain: $\frac{w_l}{\sum_{j=1}^{L} w_j B_{jl}/\big(\mathbf{B}(\theta\mathbf{s} + (1-\theta)\mathbf{t})\big)_j} \geq 4\alpha\frac{\theta\sum_{j=1}^{L} w_j B_{jl}/(\mathbf{B}\mathbf{s})_j + (1-\theta)\sum_{j=1}^{L} w_j B_{jl}/(\mathbf{B}\mathbf{t})_j}{\frac{1}{\theta}\sum_{j=1}^{L} w_j B_{jl}/(\mathbf{B}\mathbf{s})_j + \frac{1}{1-\theta}\sum_{j=1}^{L} w_j B_{jl}/(\mathbf{B}\mathbf{t})_j} \geq \alpha$. Hence, the superlevel set $\mathcal{S}_\alpha$ of $f(\mathbf{p})$ is convex so that $f(\mathbf{p})$ is quasi-concave.

H. Proof of Convexity of The Optimization Problem in (24)

We first show below that $w_l\frac{1}{\mathsf{SINR}_l(e^{\tilde{\mathbf{p}}})}$ is convex by inspecting its Hessian. We obtain the Hessian $\nabla^2 w_l\frac{1}{\mathsf{SINR}_l(e^{\tilde{\mathbf{p}}})}$ with entries given by $\left(\nabla^2 w_l\frac{1}{\mathsf{SINR}_l(e^{\tilde{\mathbf{p}}})}\right)_{jk} =$

$$\begin{cases} w_l(\mathbf{F}e^{\tilde{\mathbf{p}}} + \mathbf{v})_l e^{-\tilde{p}_l}, & \text{if } j = k = l, \\ w_l F_{lj} e^{\tilde{p}_j - \tilde{p}_l}, & \text{if } j = k, k \neq l, \\ -w_l F_{lk} e^{\tilde{p}_k - \tilde{p}_l}, & \text{if } j \neq k,\ j = l, \\ -w_l F_{lj} e^{\tilde{p}_j - \tilde{p}_l}, & \text{if } j \neq k,\ k = l, \\ 0, & \text{otherwise.} \end{cases}$$

Now, the Hessian $\nabla^2 w_l\frac{1}{\mathsf{SINR}_l(e^{\tilde{\mathbf{p}}})}$ is indeed positive semidefinite: for any real vector $\mathbf{z}$, we have $\mathbf{z}^\top\left(\nabla^2 w_l\frac{1}{\mathsf{SINR}_l(e^{\tilde{\mathbf{p}}})}\right)\mathbf{z} = z_l^2 w_l v_l e^{-\tilde{p}_l} + \sum_{m=1, m\neq l}^{L} w_l F_{lm} e^{\tilde{p}_m - \tilde{p}_l}(z_l - z_m)^2 \geq 0$. Hence, $w_l\frac{1}{\mathsf{SINR}_l(e^{\tilde{\mathbf{p}}})}$ is convex in $\tilde{\mathbf{p}}$ for all l. Since the nonnegative weighted sum of convex functions is also convex, the objective in (24) is thus convex in $\tilde{\mathbf{p}}$. Furthermore, due to the convexity of the *log-sum-exp* function, the constraint set in (24) is also convex. This completes the proof.

I. Proof of Theorem 3

From (27), we have $\sum_{l=1}^{L}\sum_{j=1}^{L}\frac{A_{lj}p_l^\star}{\mathbf{a}_j^\top\mathbf{p}^\star}\lambda_j^\star = \sum_{l=1}^{L}\left(\frac{w_l(\mathbf{F}\mathbf{p}^\star + \mathbf{v})_l}{p_l^\star} - p_l^\star\sum_{j\neq l}\frac{w_j F_{jl}}{p_j^\star}\right)$, which implies $\sum_{j=1}^{L}\lambda_j^\star = \sum_{j=1}^{L}\frac{w_j v_j}{p_j^\star}$. Now, the optimal dual variable $\boldsymbol{\lambda}^\star$ in (24) satisfies $\boldsymbol{\lambda}^\star = \left(\sum_{j=1}^{L}\frac{w_j v_j}{p_j^\star}\right)\hat{\boldsymbol{\lambda}}^\star$ with $\hat{\boldsymbol{\lambda}}^\star \in \mathbb{R}_+^L$ being the normalized dual variable (i.e., $\hat{\boldsymbol{\lambda}}^\star = \frac{\boldsymbol{\lambda}^\star}{\mathbf{1}^\top\boldsymbol{\lambda}^\star}$). We then have $\sum_{j\neq l}^{L}\frac{w_j F_{jl}}{p_j^\star} + \sum_{k=1}^{L}\frac{\lambda_k^\star A_{lk}}{\mathbf{a}_k^\top\mathbf{p}^\star} = \sum_{j=1}^{L}\frac{w_j(F_{jl} + \sum_{k=1}^{L}\frac{\hat{\lambda}_k^\star}{p_k}v_j A_{lk})}{p_j^\star}$. Thus, the denominator on the right-hand side of (26) can be written as $\sum_{j\neq l}^{L} w_j B_{jl}(\mathbf{p}^\star)/p_j^\star$. By substituting $\mathbf{F}\mathbf{p}^\star + \mathbf{v} = \mathbf{B}\mathbf{p}^\star$ and $\sum_{j\neq l}\frac{w_j F_{jl}}{p_j^\star} + \sum_{j=1}^{L}\frac{\lambda_j^\star A_{lj}}{\mathbf{a}_j^\top\mathbf{p}^\star} = \sum_{j\neq l}^{L}\frac{w_j B_{jl}(\mathbf{p}^\star)}{p_j^\star}$ into (26), we then obtain (29).

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

J. Proof of Corollary 2

Similar to the proof of Corollary 1, Corollary 2 can be proved by the positivity, monotonicity, quasi-concavity and homogeneity of the right-hand side of (29) so that it is a concave mapping (cf. Appendix Sec. A). By using the nonlinear Perron-Frobenius theory, the geometrically fast convergence of Algorithm 3 is then proved (cf. Appendix Sec. B).

K. Proof of Theorem 4

From (36), we have $\sum_{l=1}^{L}\big(p_l^\star \sum_{j=1}^{L}\frac{A_{lj}}{\mathbf{a}_j^\top \mathbf{p}^\star}\lambda_j\big) = \sum_{l=1}^{L}\big(\frac{w_l v_l \beta_l}{p_l^\star} + \sum_{j\neq l}\frac{w_l \beta_l F_{lj} p_j^\star}{p_l^\star+\beta_l F_{lj} p_j^\star} - p_l^\star \sum_{j\neq l}\frac{w_j \beta_j F_{jl}}{p_j^\star+\beta_j F_{jl} p_l^\star}\big)$, which implies $\sum_{l=1}^{L}\lambda_j^\star = \sum_{l=1}^{L}\frac{w_l v_l \beta_l}{p_l^\star}$. Then, we can obtain $p_l^{\star 2}\sum_{j=1}^{L}\frac{\lambda_j^\star A_{lj}}{\mathbf{a}_j^\top \mathbf{p}^\star} - w_l v_l \beta_l = p_l^{\star 2}\sum_{j\neq l}^{L}\frac{w_j v_j \beta_j}{p_j^\star}\hat{a}_l - \sum_{j\neq l}^{L} w_l v_l \beta_l p_j \hat{a}_j$, where $\hat{\mathbf{a}} = \sum_{l=1}^{L}(\hat{\lambda}_l^\star/\bar{p}_l)\mathbf{a}_l$. We then substitute $p_l^{\star 2}\sum_{j=1}^{L}\lambda_j^\star A_{lj}/\mathbf{a}_j^\top \mathbf{p}^\star - w_l v_l \beta_l = p_l^{\star 2}\sum_{j\neq l}^{L} w_j v_j \beta_j \hat{a}_l/p_j^\star - \sum_{j\neq l}^{L} w_l v_l \beta_l p_j \hat{a}_j$ into (36) to obtain (39).

L. Proof of Corollary 3

As in the previous (cf. the proof of Corollary 1 and Corollary 2), Corollary 3 can be proved using the nonlinear Perron-Frobenius theory (cf. Appendix Sec. B) by observing that the function on the right-hand side of (39) is positive, monotone, quasi-concave and homogeneous of degree one and is thus a concave mapping (cf. Appendix Sec. A).

References

[1] F. P. Kelly, "Charging and rate control for elastic traffic," *Euro. Trans. Telecommun.*, vol. 8, pp. 33–37, 1997.
[2] F. P. Kelly, A. Maulloo, and D. Tan, "Rate control for communication networks: shadow prices, proportional fairness and stability," *J. Oper. Res. Soc.*, vol. 49, no. 3, pp. 237–252, 1998.
[3] S. H. Low and D. E. Lapsley, "Optimization flow control, I: basic algorithm and convergence," *IEEE/ACM Trans. Netw.*, vol. 7, no. 6, pp. 861–874, 1999.
[4] M. Chiang, "Balancing transport and physical layers in wireless multihop networks: jointly optimal congestion control and power control," *IEEE J. Sel. Areas Commun.*, vol. 23, no. 1, pp. 104–116, 2005.
[5] M. Chiang, "Balancing supply and demand of bandwidth in wireless cellular networks: utility maximization over powers and rates," in *Proc. 2004 IEEE Infocom*.
[6] J. Papandriopoulos, S. Dey, and J. Evans, "Optimal and distributed protocols for cross-layer design of physical and transport layers in MANETs," *IEEE/ACM Trans. Netw.*, vol. 16, no. 6, pp. 1392–1405, 2008.
[7] D. Palomar and M. Chiang, "A tutorial on decomposition methods and distributed network resource allocation," *IEEE J. Sel. Areas Commun.*, vol. 24, no. 8, pp. 1439–1451, 2006.
[8] C. W. Tan, D. P. Palomar, and M. Chiang, "Distributed optimization of coupled systems with applications to network utility maximization," in *Proc. 2006 IEEE ICASSP*.
[9] I.-H. Hou and P. Gupta, "Distributed resource allocation for proportional fairness in multi-band wireless systems," in *Proc. 2011 IEEE ISIT*.
[10] S. C. Borst, M. G. Markakis, and I. Saniee, "Nonconcave utility maximization in locally coupled systems, with applications to wireless and wireline networks," *IEEE/ACM Trans. Netw.*, accepted for publication, 2013.
[11] H. Boche and M.Schubert, "Resource allocation in multiantenna systems-achieving max-min fairness by optimizing a sum of inverse SIR," *IEEE Trans. Signal Process.*, vol. 54, no. 6, pp. 1990–1997, 2006.
[12] T. Alpcan, T. Basar, and S. Dey, "A power control game based on outage probabilities for multicell wireless data networks," *IEEE Trans. Wireless Commun.*, vol. 5, no. 4, pp. 890–899, 2006.
[13] C. S. Chen and F. Baccelli, "Gibbsian method for the self-optimization of cellular networks," Arxiv. Available: http://arxiv.org/abs/1207.3704, 2012.
[14] U. Krause, "Concave Perron-Frobenius theory and applications," *Nonlinear analysis*, vol. 47, pp. 1457–1466, 2001.
[15] C. W. Tan, S. Friedland, and S. H. Low, "Nonnegative matrix inequalities and their application to nonconvex power control optimization," *SIAM J. Matrix Analysis and Applications*, vol. 32, no. 3, pp. 1030–1055, 2011.
[16] C. W. Tan, M. Chiang, and R. Srikant, "Fast algorithms and performance bounds for sum rate maximization in wireless networks," *IEEE/ACM Trans. Netw.*, vol. 21, no. 3, pp. 706–719, 2013.
[17] D. W. H. Cai, C. W. Tan, and S. H. Low, "Optimal max-min fairness rate control in wireless networks: Perron-Frobenius characterization and algorithms," in *Proc. 2012 IEEE Infocom*.
[18] C. W. Tan, M. Chiang, and R. Srikant, "Maximizing sum rate and minimizing MSE on multiuser downlink: optimality, fast algorithms, and equivalence via max-min SINR," *IEEE Trans. Signal Process.*, vol. 59, no. 12, pp. 6127–6143, 2011.
[19] Y.-W. P. Hong, C. W. Tan, L. Zheng, C.-L Hsieh, and C.-H Lee, "A unified framework for wireless max-min utility optimization with general monotonic constraints," in *Proc. 2014 IEEE Infocom*.
[20] L. Zheng and C. W. Tan, "Cognitive radio network duality and algorithms for utility maximization," *IEEE J. Sel. Areas Commun.*, vol. 31, no. 3, pp. 500–513, 2013.
[21] S. Kandukuri and S. Boyd, "Optimal power control in interference-limited fading wireless channels with outage-probability specifications," *IEEE Trans. Wireless Commun.*, vol. 1, no. 1, pp. 46–55, 2002.
[22] J. Mo and J. Walrand, "Fair end-to-end window-based congestion control," *IEEE/ACM Trans. Netw.*, vol. 8, no. 5, pp. 556–567, 2000.
[23] P. Wrycza, M. Bengtsson, and B. Ottersten, "MMSE criteria for downlink beamforming in CDMA wireless systems," in *Proc. 2006 IEEE ICASSP*.
[24] M. Bengtsson and B. Ottersten, *Optimal and Suboptimal Transmit Beamforming*. CRC Press, 2001.
[25] S. Friedland and S. Karlin, "Some inequalities for the spectral radius of non-negative matrices and applications," *Duke Mathematical J.*, vol. 42, no. 3, pp. 459–490, 1975.

Liang Zheng received the bachelor's degree in software engineering from Sichuan University, Chengdu, China, in 2011. She is pursuing her Ph.D. degree at the City University of Hong Kong. Her research interests are in wireless networks, mobile computing, and nonlinear optimization and its applications. She was a Finalist of the Microsoft Research Asia Fellowship in 2013.

Chee Wei Tan (M'08-SM'12) received the M.A. and Ph.D. degrees in electrical engineering from Princeton University, Princeton, NJ, in 2006 and 2008, respectively. He is an Assistant Professor at the City University of Hong Kong. Previously, he was a Postdoctoral Scholar at the California Institute of Technology (Caltech), Pasadena, CA. He was a Visiting Faculty member at Qualcomm R&D, San Diego, CA, in 2011. His research interests are in networks, inference in online large data analytics, and optimization theory and its applications. Dr. Tan currently serves as an Editor for the IEEE TRANSACTIONS ON COMMUNICATIONS. He was the recipient of the 2008 Princeton University Wu Prize for Excellence and was awarded the 2011 IEEE Communications Society AP Outstanding Young Researcher Award. He was a selected participant at the U.S. National Academy of Engineering China-America Frontiers of Engineering Symposium in 2013.