Linear & Quadratic Programming

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Convex Optimization and its Applications to Computer Science

Outline

- Linear programming
- Norm minimization problems
- Dual linear programming
- Algorithms
- Quadratic constrained quadratic programming (QCQP)
- Least-squares
- Second order cone programming (SOCP)
- Dual quadratic programming

Linear Programming

Minimize linear function over linear inequality and equality constraints:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

Variables: $x \in \mathbf{R}^n$.

Standard form LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

Appreciation-Application cycle starting for convex optimization

Transformation To Standard Form

Introduce slack variables s_i for inequality constraints:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx + s = h \\ & Ax = b, \; s \succeq 0 \end{array}$$

Express x as difference between two nonnegative variables $x^+, x^- \succeq 0$: $x = x^+ - x^-$

$$\begin{array}{ll} \text{minimize} & c^T x^+ - c^T x^- \\ \text{subject to} & Gx^+ - Gx^- + s = h \\ & Ax^+ - Ax^- = b \\ & x^+, x^-, s \succeq 0 \end{array}$$

Now in LP standard form with variables x^+, x^-, s

Linear Fractional Programming

Minimize ratio of affine functions over polyhedron:

$$
\begin{array}{ll}
\text{minimize} & \frac{c^T x + d}{e^T x + f} \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

Domain of objective function: $\{x | e^T x + f > 0\}$

Not an LP. But if nonempty feasible set, transformation into an equivalent LP with variables y, z:

$$
\begin{array}{ll}
\text{minimize} & c^T y + dz \\
\text{subject to} & Gy - hz \preceq 0 \\
& Ay - bz = 0 \\
& e^T y + fz = 1 \\
& z \succeq 0
\end{array}
$$

Why: let $y = \frac{x}{e^T x + f}$ and $z = \frac{1}{e^T x + f}$ "Charnes-Cooper" Trick

Norm Minimization Problems

- l_1 norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

  Minimize $\|Ax - b\|_1$ is equivalent to this LP in $x \in \mathbf{R}^n, s \in \mathbf{R}^n$:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T s \\ \text{subject to} & Ax - b \preceq s \\ & Ax - b \succeq -s \end{array}$$

- l_∞ norm: $\|x\|_\infty = \max_i\{|x_i|\}$

  Minimize $\|Ax - b\|_\infty$ is equivalent to this LP in $x \in \mathbf{R}^n, t \in \mathbf{R}$:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \preceq t\mathbf{1} \\ & Ax - b \succeq -t\mathbf{1} \end{array}$$

Dual Linear Programming

1. Primal problem in standard form:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

2. Write down Lagrangian using Lagrange multipliers λ, ν:

$$L(x, \lambda, \nu) = c^T x - \sum_{i=1}^{n} \lambda_i x_i + \nu^T (Ax - b) = -b^T \nu + (c + A^T \nu - \lambda)^T x$$

3. Find Lagrange dual function:

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = -b^T \nu + \inf_x [(c + A^T \nu - \lambda)^T x]$$

Since a linear function is bounded below only if it is identically zero, we have

$$g(\lambda, \nu) = \begin{cases} -b^T\nu & A^T\nu - \lambda + c = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

4. Write down Lagrange dual problem:

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) = \begin{cases} -b^T\nu & A^T\nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases} \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

5. Simplify Lagrange dual problem:

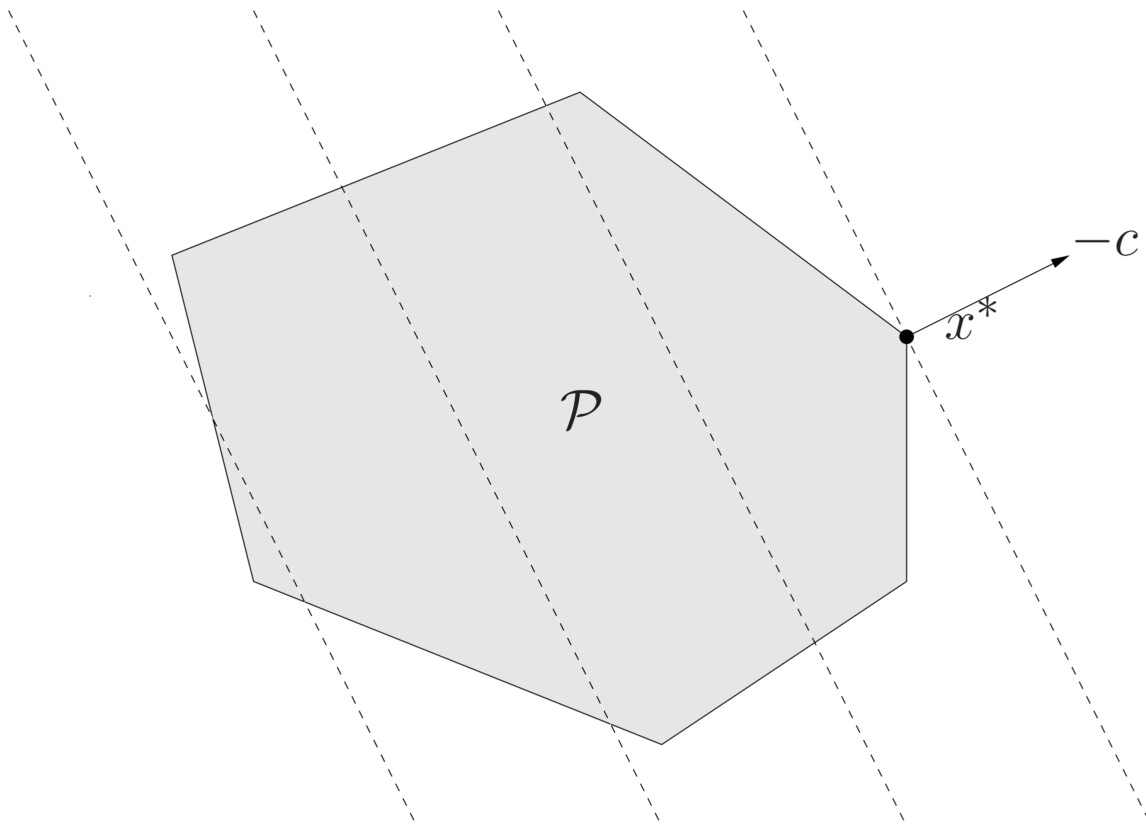
$$\begin{array}{ll} \text{maximize} & -b^T\nu \\ \text{subject to} & A^T\nu + c \succeq 0 \end{array}$$

which is an inequality constrained LP

Basic Properties

Definition: x in polyhedron P is an extreme point if there does not exist two other points $y, z \in P$ such that $x = \theta y + (1-\theta)z$ for some $\theta \in [0,1]$

Theorem: Assume that a LP in standard form is feasible and the optimal objective value is finite. There exists an optimal solution which is an extreme point



Algorithms

- Simplex Method
- Interior-point Method
- Ellipsoid Method
- Cutting-plane Method

Simplex method is very efficient in practice but specialized for LP: move from one vertex to another without enumerating all the vertices

Interior point algorithms are fierce competitors of Simplex since 1984

Convex QCQP

- (Convex) QP (with linear constraints) in x:

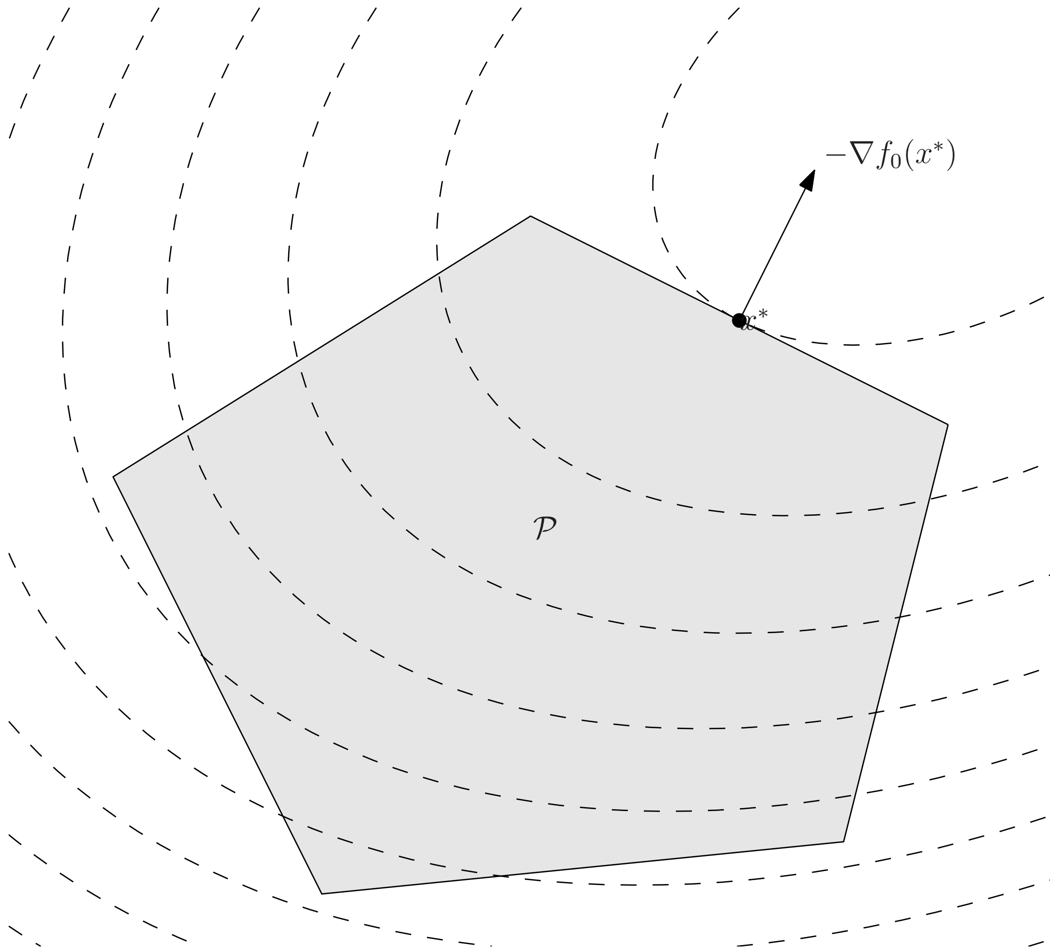
$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

where $P \in \mathbf{S}^n_+, G \in \mathbf{R}^{m \times n}, A \in \mathbf{R}^{p \times n}$

- (Convex) QCQP in x:

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, 2, \ldots, m \\ & Ax = b \end{array}$$

where $P \in \mathbf{S}^n_+$, $\quad i = 0, \dots, m$



Least-squares

- Minimize $\|Ax - b\|_2^2 = x^T A^T Ax - 2b^T Ax + b^T b$ over x. Unconstrained QP, Regression analysis, Least-squares approximation

  Analytic solution: $x^* = A^\dagger b$ where, for $A \in \mathbf{R}^{m\times n}$, $A^\dagger = (A^T A)^{-1} A^T$ if rank of A is n, and $A^\dagger = A^T (AA^T)^{-1}$ if rank of A is m. If not full rank, then by singular value decomposition.

- Constrained least-squares (no general analytic solution). For example:

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & l_i \le x_i \le u_i, \quad i = 1, \ldots, n \end{array}$$

LP with Random Cost

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

Cost $c \in \mathbf{R}^n$ is random, with mean $\bar{c}$ and covariance Ω

Expected cost: $\bar{c}^T x$. Cost variance $x^T \Omega x$

Minimize both expected cost and cost variance (with a weight γ):

$$\begin{array}{ll} \text{minimize} & \bar{c}^T x + \gamma x^T \Omega x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

SOCP

Second Order Cone Programming:

$$
\begin{array}{ll}
\text{minimize} & f^T x \\
\text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \ldots, m \\
 & Fx = g
\end{array}
$$

Variables: $x \in \mathbf{R}^n$. And $A_i \in \mathbf{R}^{n_i \times n}, F \in \mathbf{R}^{p \times n}$

If $c_i = 0$, $\forall i$, SOCP is equivalent to QCQP If $A_i = 0$, $\forall i$, SOCP is equivalent to LP

Robust LP

Consider inequality constrained LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \;\; i = 1, \ldots, m \end{array}$$

Parameters a_i are not accurate. They are only known to lie in given ellipsoids described by $\bar{a}_i$ and $P_i \in \mathbf{R}^{n \times n}$: $a_i \in \mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}$. Since $\sup\{a_i^T x \mid a_i \in \mathcal{E}\} = \bar{a}_i^T x + \|P_i^T x\|_2$,

Robust LP (satisfy constraints for all possible a_i) formulated as SOCP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \;\; i = 1, \ldots, m \end{array}$$

Dual QCQP (Convex Primal QCQP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, 2, \ldots, m \\ & Ax = b \end{array}$$

Lagrangian: $L(x, \lambda) = (1/2)x^T P(\lambda) x + q(\lambda)^T x + r(\lambda)$ where

$$P(\lambda) = P_0 + \sum_{i=1}^{m} \lambda_i P_i, \quad q(\lambda) = q_0 + \sum_{i=1}^{m} \lambda_i q_i, \quad r(\lambda) = r_0 + \sum_{i=1}^{m} \lambda_i r_i$$

Since $\lambda \succeq 0$, we have $P(\lambda) \succ 0$ if $P_0 \succ 0$ and

$$g(\lambda) = \inf_x L(x, \lambda) = -(1/2) q(\lambda)^T P(\lambda)^{-1} q(\lambda) + r(\lambda)$$

Lagrange dual problem:

$$\begin{array}{ll} \text{maximize} & -(1/2) q(\lambda)^T P(\lambda)^{-1} q(\lambda) + r(\lambda) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

KKT Conditions for QP

Primal (convex) QP with linear equality constraints:

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Ax = b \end{array}$$

KKT conditions:

$$Ax^* = b, \quad Px^* + q + A^T \nu^* = 0$$

which can be written in matrix form:

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

Solving a system of linear equations is equivalent to solving equality constrained convex quadratic minimization

Summary

- LP covers a wide range of interesting problems and applications
- The Dual of LP is LP
- First type of nonlinearity: quadratic programming and least-squares
- Nonlinear problems that are or can be converted into convex optimization: QCQP (SOCP). Covers LP as special case
- There are very useful special structures in LP. But most of the important ones (computational efficiency, global optimality, Lagrange duality) can be generalized to convex optimization

Reading assignment: Sections 4.3-4.4 and 6.1-6.2 of textbook.