Wireless Network Optimization by Perron-Frobenius Theory

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Abstract—A basic question in wireless networking is how to optimize the wireless network resource allocation for utility maximization and interference management. In this paper, we present an overview of a Perron-Frobenius theoretic framework to overcome the notorious non-convexity barriers in wireless utility maximization problems. Through this approach, the optimal value and solution of the optimization problems can be analytically characterized by the spectral property of matrices induced by nonlinear positive mappings. It also provides a systematic way to derive distributed and fast-convergent algorithms and to evaluate the fairness of resource allocation. This approach can even solve several previously open problems in the wireless networking literature, e.g., Kandukuri and Boyd (TWC 2002), Wiesel, Eldar and Shamai (TSP 2006), Krishnan and Luss (WCNC 2011). More generally, this approach links fundamental results in nonnegative matrix theory and (linear and nonlinear) Perron-Frobenius theory with the solvability of non-convex problems. In particular, for seemingly nonconvex problems, e.g., max-min wireless fairness problems, it can solve them optimally; for truly nonconvex problems, e.g., sum rate maximization, it can even be used to identify polynomial-time solvable special cases or to enable convex relaxation for global optimization. To highlight the key aspects, we also present a short survey of our recent efforts in developing the nonlinear Perron-Frobenius theoretic framework to solve wireless network optimization problems with applications in MIMO wireless cellular, heterogeneous small-cell and cognitive radio networks. Key implications arising from these work along with several open issues are discussed.

Keywords: Wireless network, Fairness resource allocation, Cross-layer design, Interference management, MIMO Beamforming, Perron-Frobenius theorem, Nonlinear Perron-Frobenius theory, Nonnegative matrix theory, Mathematical duality.

I. INTRODUCTION

A remarkable feature of the growth in wireless data usage is that many new devices today are operating in the wireless spectrum that are meant to be shared among many different users. Yet, the sharing of the spectrum is far from perfect. Due to the broadcast nature of the wireless medium, interference is a major source of performance impairment. Current systems suffer from deteriorating quality due to fixed resource allocation that fails to consider interference. As wireless networks are becoming more heterogeneous and ubiquitous in our life, they are also becoming more difficult to design and manage. How should these large complex wireless networks be analyzed and designed with clearly-defined fairness and optimality in mind? Wireless network optimization has become an important tool to design resource allocation algorithms to realize the untapped benefits of wireless resources and to manage interference in wireless networks [1], [2], [3], [4],

[5]. Without appropriate resource coordination, the network can become unstable or may operate in a highly inefficient and unfair manner.

Important objectives in wireless networks can be modeled by nonlinear utility functions of wireless link metrics, e.g., signal-to-interference-and-noise ratio (SINR), mean square error (MSE) and the outage probability. The total network utility is then maximized over the joint solution space of all possible wireless link metrics, e.g., powers and interference. This can be used to address issues such as understanding how wireless network algorithms can interact between network layers, e.g., physical and medium access control layers, to achieve provable efficiency for the overall system or understanding how fairness permeates through network layers when interference is dominant. This can open up new opportunities to jointly optimize physical layer innovation and other networking control mechanism, and can lead to optimal wireless network protocols.

The main challenges of solving these wireless utility maximization problems come from the nonlinear dependency of link metrics on channel conditions and powers, as well as possible interference among the users. These are nonconvex problems that are notoriously difficult to solve, and designing scalable algorithms with low complexity to solve them optimally is even harder. In this paper, we will present a new theoretical foundation for solving network optimization problems in wireless networks. The goal is to present a suite of theory and algorithms based on the nonlinear Perron-Frobenius theory[1] to optimize performance metrics in wireless networks and to design efficient algorithms with low complexity that are applicable to a wide range of wireless network applications.

The Perron-Frobenius theory exploits the nonnegativity property that arises naturally in wireless performance metrics and constraints to overcome some of the challenges due to nonlinearity and nonconvexity. The application of the classical (linear) Perron-Frobenius theorem to power control in wireless networks has been widely recognized (see, e.g., [14], [15], [16]), and can be traced back to earlier work on balancing the signal to interference ratio in a wireless network in [17] and downlink beamforming with a single linear constraint [18],

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[1]In nonnegative matrix theory, the classical linear Perron-Frobenius theorem is an important result that concerns the eigenvalue problem of nonnegative matrices, and has many engineering applications [6], [7], [8]. Efforts to extend the linear Perron-Frobenius theorem to nonlinear ones (to study the dynamics of cone-preserving operators) have been intensive, and the nonlinear Perron-Frobenius theory is now emerging to provide a mathematically rigorous and practically relevant technique to engineering problems [9], [10], [11], [12], [13].

[19], [20]. These special case approaches however cannot be used to address the general case optimally or to solve other related nonconvex wireless network optimization problems. A popular approach to tackle nonconvexity in wireless network problems is geometric programming (see [21], [22] for an introduction) and successive convex approximation as used by the authors in [23], [4], [5], [24]. There is indeed an intimate relationship between geometric programming and the Perron-Frobenius theory presented here.

This nonlinear Perron-Frobenius theoretic approach has also been used to tackle several open wireless network optimization problems, for example resolving open questions related to algorithm design in [23] for a worst outage probability problem, the feasibility issue problem in [25] for a total power minimization problem constrained by outage probability specifications, max-min SINR problems for beamforming in [26] and for small cells in [27]. It has also been in [28], [29], [30], [31], [32], [33] to tackle notoriously difficult wireless optimization problems such as the sum rate maximization problem (see for example [34], [35] for other approaches) together with tools in nonnegative matrix theory [36], [37] to develop convex relaxation and useful convex approximation.

This paper is organized as follows. In Section II, we introduce the wireless network model. In Section III and Section IV, we present how the Perron-Frobenius theory is used for solving wireless max-min fairness optimization problems and the general wireless utility optimization problems respectively. In Section V, we discuss several open issues. We conclude the paper in Section VI. The following notation is used. Boldface uppercase letters denote matrices, boldface lowercase letters denote column vectors and italics denote scalars.

II. SYSTEM MODEL

Consider a multiuser communication system with L users (logical transmitter/receiver pairs) sharing a common frequency. Each user employs a singe-user decoder, i.e., treating interference as additive Gaussian noise, and has perfect channel state information at the receiver. Assume a noise power n_l at the lth receiver. At each transmitter, the signal has an average transmit power p_l. The vector $(p_1, \ldots, p_L)^\top$ is the transmit power vector, which is the optimization variable of interest in this paper. The power received from the jth transmitter at lth receiver is given by $G_{lj} p_j$ where G_{lj} represents the nonnegative path gain between the jth transmitter and the lth receiver (it may also encompass antenna gain and coding gain) that is often modeled as proportional to $d_{lj}^{-\alpha}$, where d_{lj} denotes distance and α is the power fall-off factor.

There are several important performance metrics in wireless networks. The signal-to-interference-and-noise ratio (SINR) is an example that measures the quality of service for link transmission. Specifically, in a static channel (or relatively slow channel fading), this metric for the lth user can be given by [38]:

$$\mathsf{SINR}_l(\mathbf{p}) = \frac{G_{ll} p_l}{\sum_{j \neq l} G_{lj} p_j + n_l}. \tag{1}$$

Outage probability is another example for transmission reliability. In fast channel fading scenarios, the SINR is a random variable that depends on the channel fading characteristics. The transmission from the lth transmitter to its receiver is successful if $\mathsf{SINR}_l(\mathbf{p}) \geq \gamma_l$ (zero-outage), where γ_l is a given threshold for reliable communication. An outage occurs at the lth receiver when $\mathsf{SINR}_l(\mathbf{p}) < \gamma_l$, and we denote this outage probability by $P(\mathsf{SINR}_l(\mathbf{p}) < \gamma_l)$. In general, wireless performance metrics that are related to the quality of service requirement are nonlinear functions of the SINR, and can also be dependent on other network parameters, including MIMO beamforming, channel statistical fading (Rayleigh, a Ricean or a Nakagami distribution), the additive background noise, user mobility and the dynamics of power control [38].

In addition to the power resource budget and performance specification constraints present in all wireless networks, there are many other types of nonlinear power constraints in wireless network optimization that are imposed either due to regulatory policy for health consideration or to limit excessive levels of interference to macrocell users. For example, a wireless cognitive radio network has interference temperature to limit interference from secondary users. In heterogeneous wireless networks, small cell network coverage is limited so that macro-cell users' performance are guaranteed [27]. All these constraints can make the problem hard to solve optimally and in a distributed manner.

III. WIRELESS MAX-MIN FAIRNESS OPTIMIZATION

In this section, we introduce a general framework for max-min fairness optimization in wireless networks that can be applied to a variety of wireless network applications. The max-min fairness is an egalitarian fairness guarantee to protect the worst case performance in the network.

A. Problem statement

Let us consider a class of utility functions that satisfy the following assumptions.

Assumption 1 (Competitive Utility Functions):
- *Positivity*: For all i, $u_i(\mathbf{p}) > 0$ if $\mathbf{p} > \mathbf{0}$ and, in addition, $u_i(\mathbf{p}) = 0$ if and only if $p_i = 0$.
- *Competitiveness*: For all i, u_i is strictly increasing with respect to p_i and is strictly decreasing with respect to p_j, for $j \neq i$, when $p_i > 0$.
- *Directional Monotonicity*: For $\lambda > 1$ and $p_i > 0$, $u_i(\lambda \mathbf{p}) > u_i(\mathbf{p})$, for all i.

For the utility functions, the competitiveness assumption models the interaction between users in a wireless network and the directional monotonicity captures the increase in utility as the total power consumption increases.

To optimize max-min utility fairness to all the users, the wireless network optimization can be formulated as

$$\begin{aligned} \text{maximize} \quad & \min_{i=1,\ldots,L} u_i(\mathbf{p}) \\ \text{subject to} \quad & \mathbf{g}(\mathbf{p}) \leqq \bar{\mathbf{g}} \\ \text{variables:} \quad & \mathbf{p}, \end{aligned} \tag{2}$$

where $\mathbf{g}(\mathbf{p})$ is a general performance constraint set, and $\bar{\mathbf{g}} = [\bar{g}_1, \ldots, \bar{g}_K]^T$ is the vector of constraint values. Let the optimal solution to (2) be $\mathbf{p}^*$.

In (2), $u_l(\mathbf{p})$ can be a general performance metric in terms of SINR of the lth link. For example, let $u_l(\mathbf{p}) = \mathsf{SINR}_l(\mathbf{p})/\gamma_l$ (see Section III-D below), which is then the max-min weighted SINR problem [26], [39], [40]. Another example is when the SINR is a random variable and to let $u_l(\mathbf{p}) = 1/\mathrm{Prob}(\mathsf{SINR}_l(\mathbf{p}) < \gamma_l)$, which is the inverse of the outage probability for the event that the SINR of the lth link falls below a given threshold γ_l, and this is the worst outage probability problem [23], [41].

By introducing an auxiliary variable τ, solving (2) is equivalent to solving the following optimization problem:

$$
\begin{aligned}
\text{maximize} \quad & \tau \\
\text{subject to} \quad & u_i(\mathbf{p}) \geq \tau, \quad \text{for } i = 1, \ldots, L, \\
& g_k(\mathbf{p}) \leq \bar{g}_k, \quad \text{for } k = 1, \ldots, K, \\
\text{variables:} \quad & \mathbf{p}, \ \tau.
\end{aligned}
\tag{3}
$$

B. Solution Methodology

Solving (2) is generally difficult and, in the following, we present how to solve (2) with specifically the vector of constraint functions $\mathbf{g}(\mathbf{p}) = [g_1(\mathbf{p}), \ldots, g_K(\mathbf{p})]^T$ and $\bar{\mathbf{g}}$ that describe the set of monotonic constraints satisfying the following assumptions (see [42] for more details).

Assumption 2 (Monotonic Constraints):
- *Strict Monotonicity*: For all k, $g_k(\mathbf{p}_1) > g_k(\mathbf{p}_2)$ if $\mathbf{p}_1 > \mathbf{p}_2$, and $g_k(\mathbf{p}_1) \geqq g_k(\mathbf{p}_2)$ if $\mathbf{p}_1 \geqq \mathbf{p}_2$.
- *Feasibility*: The set $\{\mathbf{p} > \mathbf{0} : \mathbf{g}(\mathbf{p}) \leqq \bar{\mathbf{g}}\}$ is non-empty.
- *Validity*: For any $\mathbf{p} > \mathbf{0}$, there exists $\lambda > 0$ such that $g_k(\lambda \mathbf{p}) \geqq \bar{g}_k$, for some k.

The strict monotonicity captures the increase in cost or resource consumption as $\mathbf{p}$ increases, the feasibility ensures that there exists a positive power vector in the feasible set, and the validity ensures that the set of constraints is meaningful. If the validity condition does not hold, the corresponding constraint can be simply removed without loss of generality.

Lemma 1: For $\{u_i\}_{i=1}^L$, $\{g_k\}_{k=1}^K$, and $\{\bar{g}_k\}_{k=1}^K$ that satisfy Assumptions 1 and 2, the optimal solution $(\tau^*, \mathbf{p}^*)$ in (3) is positive, i.e., $\tau^* > 0$ and $\mathbf{p}^* > \mathbf{0}$, and, at optimality, all the L constraints of $u_i(\mathbf{p}) \geq \tau$ are tight and at least one of the K constraints of $g_k(\mathbf{p}) \leq \bar{g}_k$ is active. That is, $u_i(\mathbf{p}^*) = \tau^*$, for all i and $g_k(\mathbf{p}^*) = \bar{g}_k$, for some k (let us denote this constraint set for $\mathbf{p}$ by $\mathcal{U}$).

By Lemma 1, it follows that

$$
\frac{1}{\tau^*} p_i^* = \frac{1}{u_i(\mathbf{p}^*)} p_i^* \triangleq T_i(\mathbf{p}^*).
\tag{4}
$$

This means that $\mathbf{p}^*$ is a solution to the fixed point equation

$$
\frac{1}{\tau^*} \mathbf{p}^* = [T_1(\mathbf{p}^*), \ldots, T_L(\mathbf{p}^*)]^T \triangleq \mathbf{T}(\mathbf{p}^*).
\tag{5}
$$

Definition 1: The function $\beta : \mathbb{R}_+^L \to \mathbb{R}_+$ of $\mathbf{p}$ (called the *scale* of $\mathbf{p}$) is defined as

$$
\beta(\mathbf{p}) \triangleq \min\{\beta' \geqq 0 : g_k(\mathbf{p}/\beta') \leqq \bar{g}_k, \forall k\}.
\tag{6}
$$

Lemma 2: The scale $\beta : \mathbb{R}_+^L \to \mathbb{R}_+$ defined in Definition 1 satisfies the following properties:
1) β is not identically zero and, in fact, $\beta(\mathbf{p}) > 0$, for all $\mathbf{p} > \mathbf{0}$;
2) $\beta(\lambda \mathbf{p}) = \lambda \beta(\mathbf{p})$ for $\mathbf{p} \geqq \mathbf{0}$ and $\lambda \geqq 0$ (i.e., positively homogeneous);
3) $\mathbf{0} \leqq \mathbf{p} \leqq \mathbf{q}$ implies $\beta(\mathbf{p}) \leqq \beta(\mathbf{q})$ (i.e., monotonic).

It follows that the solution of the optimization problem in (3) (and, thus, (2)) is a solution of a conditional eigenvalue problem where the objective is to find $(\tau^*, \mathbf{p}^*)$ such that [42]:

$$
\frac{1}{\tau^*} \mathbf{p}^* = \mathbf{T}(\mathbf{p}^*), \ \tau^* \in \mathcal{R}, \ \mathbf{p}^* \in \mathcal{U}.
\tag{7}
$$

C. Algorithm

Here, we use $\mathbf{p}(t)$ to represent the power vector obtained in the t-th iteration of the algorithm.

Algorithm 1 (Monotonic Constrained Max-Min Utility):
1) Initialize power vector $\mathbf{p}(0) > \mathbf{0}$.
2) Update power vector $\mathbf{p}(t+1)$:

$$
p_i(t+1) = \frac{p_i(t)}{u_i(\mathbf{p}(t))} \left(\triangleq T_i(\mathbf{p}(t)) \right), \ \forall i.
$$

3) Scale power vector $\mathbf{p}(t+1)$:

$$
\mathbf{p}(t+1) \leftarrow \frac{\mathbf{p}(t+1)}{\beta(\mathbf{p}(t+1))}.
$$

4) Repeat Steps 2 and 3 until convergence.

The following theorem establishes the existence and the uniqueness of the solution of the conditional eigenvalue problem in (7) as well as the convergence of Algorithm 1 [42].

Theorem 1: Suppose that $\mathbf{T} : \mathcal{R}_+^L \to \mathcal{R}_+^L$, defined as $\mathbf{T}(\mathbf{p}) = [T_1(\mathbf{p}), \ldots, T_L(\mathbf{p})]^\top$, where $T_i(\mathbf{p}) = p_i/u_i(\mathbf{p})$, satisfies the following conditions: (i) there exist numbers $a > 0$, $b > 0$, and a vector $\mathbf{e} > \mathbf{0}$ such that $a\mathbf{e} \leqq \mathbf{T}(\mathbf{p}) \leqq b\mathbf{e}$, for all $\mathbf{p} \in \mathcal{U}$; (ii) for any $\mathbf{p}$, $\mathbf{q} \in \mathcal{U}$ and $0 \leq \lambda \leq 1$: If $\lambda \mathbf{p} \leqq \mathbf{q}$, then $\lambda \mathbf{T}(\mathbf{p}) \leqq \mathbf{T}(\mathbf{q})$; and, if $\lambda \mathbf{p} \leq \mathbf{q}$ with $\lambda < 1$, then $\lambda \mathbf{T}(\mathbf{p}) < \mathbf{T}(\mathbf{q})$. Then, the following properties hold:
(a) The conditional eigenvalue problem in (7) has a unique solution $\mathbf{p}^* \in \mathcal{U}$ and $\tau^* > 0$.
(b) The power vector $\mathbf{p}(t)$ in Algorithm 1 converges to $\mathbf{p}^*$ (i.e., the solution of (7) and, thus, (2)) for any initial point $\mathbf{p}(0) \geqq \mathbf{0}$ with $\beta(\mathbf{T}(\mathbf{p}(0))) > 0$.

Starting from any initial point $\mathbf{p}(0)$, $\mathbf{p}(t)$ in Algorithm 1 converges geometrically fast.[2]

A notable special case of Theorem 1 is when $\mathbf{T}(\mathbf{p}) = \mathbf{F}\mathbf{p}$ where $\mathbf{F}$ is an irreducible[3] nonnegative square matrix, and this is the classical linear Perron-Frobenius theory in linear algebra

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[2] Let $\|\cdot\|$ be an arbitrary vector norm. A sequence $\{\mathbf{p}(t)\}$ is said to converge geometrically fast to a fixed point $\mathbf{p}'$ if and only if $\|\mathbf{p}(t) - \mathbf{p}'\|$ converges to zero geometrically fast, i.e., there exists constants $A \geq 0$ and $\eta \in [0, 1)$ such that $\|\mathbf{p}(t) - \mathbf{p}'\| \leq A\eta^t$ for all t [43].

[3] A nonnegative matrix $\mathbf{F}$ is said to be irreducible if there exists a positive integer m such that the matrix $\mathbf{F}^m$ has all entries positive.

[8], [43].[4] Then, $1/\tau^* = \rho(\mathbf{F})$ and $\mathbf{p}^*$ is the right eigenvector of $\rho(\mathbf{F})$ in (7) for this special case.

Another notable special case of Theorem 1 is when $\mathbf{T}(\mathbf{p})$ is *concave* [9], [10], [11] (the *hidden convexity* to be exploited in solving a seemingly nonconvex (2)). See the appendix for other examples of *concave* $\mathbf{T}(\mathbf{p})$ for wireless optimization problems.

D. Example: Max-min Weighted SINR

We illustrate an example of using Theorem 1 and Algorithm 1 to maximize the worst case SINR given as follows:

$$\begin{array}{ll} \text{maximize} & \min_l \dfrac{\mathsf{SINR}_l(\mathbf{p})}{\gamma_l} \\ \text{subject to} & \mathbf{a}^T\mathbf{p} \leq 1,\ \mathbf{p} \geq \mathbf{0} \\ \text{variables:} & \mathbf{p}, \end{array} \quad (8)$$

where $\mathsf{SINR}_l(\mathbf{p})$ is given in (1), and $\mathbf{a}$ is some given positive vector (examples in cellular and heterogeneous femtocell networks can be found in [26], [30], [31] and [27]). There are a few approaches to solving (8). For example, (8) can be reformulated as a geometric program, and be solved numerically using the interior point method [5], [22] or (8) can be solved analytically by using both the Lagrange duality and the Friedland-Karlin inequalities[5] in nonnegative matrix theory [31], [30]. We show how Theorem 1 can solve (8). Since $\mathbf{a}^T\mathbf{p} \leq 1$ is tight at optimality, solving (8) is equivalent to solving for the unknown $(\tau, \mathbf{p})$ in the fixed-point problem:

$$\frac{1}{\tau}\mathbf{p} = \mathbf{F}\mathbf{p} + \mathbf{v}, \quad \mathbf{a}^T\mathbf{p} = 1, \quad (9)$$

where the nonnegative matrix $\mathbf{F}$ has entries $F_{lj} = \gamma_l G_{lj}/G_{ll}$ for $l \neq j$ and 0 otherwise, and $v_l = \gamma_l n_l/G_{ll}$.

Now, by invoking Theorem 1 for $\mathbf{T}(\mathbf{p}) = \mathbf{F}\mathbf{p} + \mathbf{v}$ (notice that this is affine and hence concave [21]), we deduce that the optimal value and solution to (8) is $1/\rho(\mathbf{F} + \mathbf{v}\mathbf{a}^T)$ and the right eigenvector of $\mathbf{F} + \mathbf{v}\mathbf{a}^T$ respectively. Furthermore, the following algorithm solves (8) and this resolves an open issue of algorithm design in [27].

Algorithm 2 (Max-min Weighted SINR):

1) Update power $\mathbf{p}(t+1)$:

$$p_l(t+1) = \left(\frac{\gamma_l}{\mathsf{SINR}_l(\mathbf{p}(t))}\right) p_l(t) \quad \forall l. \quad (10)$$

2) Normalize $\mathbf{p}(t+1)$:

$$\mathbf{p}(t+1) \leftarrow \frac{\mathbf{p}(t+1)}{\mathbf{a}^\top \mathbf{p}(t+1)}. \quad (11)$$

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[4]The Perron-Frobenius eigenvalue of an irreducible nonnegative matrix $\mathbf{F}$ is the spectral radius (eigenvalue with the largest absolute value) of $\mathbf{F}$ denoted by $\rho(\mathbf{F})$, and the Perron (right) and left eigenvector of $\mathbf{F}$ associated with $\rho(\mathbf{F})$ are denoted by $\mathbf{x}(\mathbf{F}) \geq \mathbf{0}$ and $\mathbf{y}(\mathbf{F}) \geq \mathbf{0}$ respectively. Furthermore, $\rho(\mathbf{F})$ is simple and positive, and $\mathbf{x}(\mathbf{F}), \mathbf{y}(\mathbf{F}) > \mathbf{0}$ (cf. [6], [7], [8] and Proposition 6.6 in [43]).

[5]The Friedland-Karlin inequalities are important fundamental results in nonnegative matrix theory related to the linear Perron-Frobenius theorem [36], [37]. See [29] for its generalization and applications to inverse problems in nonnegative matrix theory and also Section IV for further discussions on their applications to nonconvex optimization.

IV. GENERAL WIRELESS UTILITY OPTIMIZATION

If the max-min fairness objective function in (2) is replaced by a more general wireless utility function, its optimal solution can be used to optimize other general wireless performance aspects, e.g., the total throughput or the total energy consumption etc. Some of these wireless optimization problems are well-known to be nonconvex (or even NP-hard) and generally hard to solve. However, there are relatively few techniques that can shed light on achieving global optimality (in polynomial time) or giving convex relaxation. For example, approaches based on Lagrange duality can suffer from the positive duality gap and cannot guarantee finding a global optimal solution.

A. Sum Rate Maximization

We discuss how the nonlinear Perron-Frobenius Theory can be used to tackle the nonconvexity in the sum rate maximization problem given by:

$$\begin{array}{ll} \text{maximize} & \sum_l w_l \log(1 + \mathsf{SINR}_l(\mathbf{p})) \\ \text{subject to} & \mathbf{g}(\mathbf{p}) \leqq \bar{\mathbf{g}} \\ \text{variables:} & p_l \ \ \forall l, \end{array} \quad (12)$$

where $\mathsf{SINR}_l(\mathbf{p})$ is given in (1), and w_l is some given positive weight (to indicate priority for the lth user).

In [30], the authors studied (12) for the single total power constraint (applicable to cellular downlink), and identified polynomial-time solvable special cases (e.g., under low interference scenarios, i.e., when the cross-channel gains in (1) are sufficiently small). By using the Friedland-Karlin inequalities, [30] showed that Algorithm 2 can solve (12) (optimally under special cases when $\mathbf{w}$ takes certain form). In [31], the authors studied (12) for the individual power constraints (applicable to cellular uplink), and provided polynomial-time algorithms (including Algorithm 2) with performance bounds on the suboptimality. In [28], [29], using reformulation and the Friedland-Karlin inequalities, the authors proposed global optimization algorithms to solve (12) for the individual power constraints. In [32], the authors studied (12) for general affine power and interference temperature constraints (applicable to cognitive radio networks), and leverages nonnegative matrix theory to first obtain a convex relaxation of (12) by solving a convex optimization problem over a closed bounded convex set. It also enables the sum-rate optimality to be quantified analytically through the spectrum of specially-crafted nonnegative matrices. Furthermore, polynomial-time verifiable sufficient conditions have been obtained that can identify polynomial-time solvable problem instances. In the general case, the authors in [32] propose a global optimization algorithm by utilizing this convex relaxation and branch-and-bound to compute an ϵ-optimal solution. Extensions to MIMO beamforming case have been studied in [33].

V. OPEN ISSUES

For wireless max-min fairness optimization, it is an open issue to solve (2) for more general constraints, i.e., beyond the monotonic constraints in Assumption 2. Also, exploring the Lagrange duality of (2) is relatively new. For example,

the well-known uplink-downlink duality for total power mini-mization can jointly optimize beamforming and power in a distributed manner [38]. Similar extensions for a cognitive radio network duality in [32] and a MISO-SIMO duality for sum-rate in [33] etc can be fruitful in using the *dual solution* together with the Perron-Frobenius theory for distributed al-gorithm design for problems involving MIMO beamforming, stochastic fading channel models, realistic nontrivial power and interference constraints etc. It is well-known that special cases of (2) can be solved by geometric programming [22], [24], and it is interesting to explore in-depth the connection between Perron-Frobenius theory and geometric programming, for example the Collatz-Wielandt theorem in nonnegative matrix theory (related to the linear Perron-Frobenius theorem) can be proved by geometric programming [36].

For the general wireless utility optimization, it is challeng-ing to solve (12) for more general constraints. For other utility objectives, it is an open issue on how the nonlinear Perron-Frobenius theory, e.g., in [9], [10], can be linked with the nonnegative matrix inequalities such as the Friedland-Karlin inequalities in [36], [37], and this can be fruitful in connecting wireless optimization problems (e.g., connect an easily solved one to a general difficult one, cf. [30], [31], [44], [32], [45]). Reformulation using nonnegative matrix theory, convex relaxation and convex approximation are promising techniques to overcome the nonconvexity barriers. Indeed, quantifying global optimality in polynomial-time (whenever that is pos-sible) and generating tight convex relaxation (to bound the performance of suboptimal algorithm design) can provide new perspectives to solving wireless optimization problems in the large-scale and heterogeneous constraint setting.

VI. CONCLUSION

We have presented an advanced suite of theory and al-gorithms based on the nonlinear Perron-Frobenius theory to solve a class of max-min fairness optimization problems and nonconvex utility maximization problems that find applications for resource allocation and interference management in a wide variety of wireless networks such as cellular networks and cognitive radio networks. For the wireless max-min fairness problems, the nonlinear Perron-Frobenius theory characterizes the optimal solution analytically, and provides a systematic way to derive distributed fast algorithms that enable power control, rate control, antenna beamforming and cross-layer de-sign. For nonconvex wireless network optimization problems, it provides novel methodologies to enable convex relaxation and convex approximations that yield computational-efficient algorithms to overcome the nonconvexity.

APPENDIX: CASE STUDIES

Brief summarizes of some case studies from our recent publications are provided below to illustrate the applications.

MIMO multiple antenna beamforming

The nonconvexity of the MIMO max-min weighted SINR problem is due to the mutual coupling between the transmit and receive beamformers, implying that it is difficult to solve optimally and in a distributed manner. The nonlinear Perron-Frobenius theory approach in [39], [40], [46] can characterize the optimality of the MISO special case leading to new algo-rithms that alternate between the MISO and SIMO problems through fixed point iterations. As a by-product, [40] resolved an open issue in algorithm design in [26]. An extension to large-scale MIMO was studied in [47]. The *concave function* related to the SINR performance in the MISO channel is [40]:

$$T_s(\mathbf{p}) = \frac{\gamma_s}{\tilde{\mathbf{v}}_s^\dagger \mathbf{H}_{k_s} \left(\sum_{\substack{i=1 \\ i \neq s}}^{S} p_i \mathbf{H}_{k_i}^\dagger \tilde{\mathbf{v}}_i \tilde{\mathbf{v}}_i^\dagger \mathbf{H}_{k_i} + \mathbf{I} \right)^{-1} \mathbf{H}_{k_s}^\dagger \tilde{\mathbf{v}}_s} \tag{13}$$

where $\tilde{\mathbf{v}}_s$ and $\mathbf{H}_{k_s}$ are the optimal transmit beamformer and the MISO channel gain respectively.

Reliability optimization against fading

The outage probability max-min fairness problem with either a total or individual power constraints in a multiuser Rayleigh-faded network was studied in [41]. The optimal solution can be analytically characterized using the nonlinear Perron-Frobenius theory and be obtained using a geometrically fast convergent algorithm in a distributed manner. As a by-product, [41] resolved an open issue in algorithm design for the interference-limited case in [23], and also addressed the feasibility issue of a total power minimization problem with outage constraints and its algorithm design for admission control. The *concave function* related to the outage probability for Rayleigh fading is [41]:

$$T_l(\mathbf{p}) = \frac{n_l \gamma_l}{G_{ll}} + \sum_{j \neq l} p_l \log \left(1 + \frac{\gamma_l G_{lj} p_j}{G_{ll} p_l} \right). \tag{14}$$

Cross-layer design

The max-min fairness flow rate allocation problem in a multiuser wireless network was studied in [48] for a large class of link rate functions which includes the Shannon capacity function and the CDMA rate function. The joint flow rate and power control problem can be decoupled into two separate problems on fairness - one at the network layer and one at the link layer, and then to design a cross-layer algorithm using

the nonlinear Perron-Frobenius theory. The *concave function* related to link rate at the physical layer is [48]:

$$T_l(\mathbf{p}) = \left(\frac{(\mathbf{R}\boldsymbol{\nu})_l}{C(\mathsf{SINR}_l(\mathbf{p}))} \right) p_l, \tag{15}$$

where C is a general link rate function in terms of SINR (e.g., Shannon's capacity formula), $\mathbf{R}$ is the routing matrix and $\boldsymbol{\nu}$ is the source end-to-end equilibrium flow at the network layer.

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