Max-Min SINR Coordinated Multipoint Downlink Transmission—Duality and Algorithms

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***Abstract*—This paper considers the max-min weighted signal-to-interference-plus-noise ratio (SINR) problem subject to multiple weighted-sum power constraints, where the weights can represent relative power costs of serving different users. First, we study the power control problem. We apply nonlinear Perron-Frobenius theory to derive closed-form expressions for the optimal value and solution and an iterative algorithm which converges geometrically fast to the optimal solution. Then, we use the structure of the closed-form solution to show that the problem can be decoupled into subproblems each involving only one power constraint. Next, we study the multiple-input-single-output (MISO) transmit beamforming and power control problem. We use uplink-downlink duality to show that this problem can be decoupled into subproblems each involving only one power constraint. We apply this decoupling result to derive an iterative subgradient projection algorithm for the problem.**

***Index Terms*—Beamforming, multiple-input-multiple-output (MIMO), uplink-downlink duality.**

I. Introduction

COORDINATED multi-point processing (CoMP) has been proposed as a promising technology for improving spectral efficiency, coverage, and cell-edge throughput for future cellular systems [1]–[3]. The basic idea of CoMP is to allow multiple base stations to cooperate via backhaul links to enhance the reliability of the links between mobile users and the serving base station by managing intercell interference in an efficient manner [4]. In downlink CoMP, signal processing can be performed for coordinated transmission by multiple cells to multiple users [4]–[6]. There are two signal processing categories: coordinated scheduling/coordinated beamforming (CS/CB) and joint processing. The basic idea of CS/CB is to transmit a subframe from one cell to a user, where coordinated beamforming and scheduling are performed among cells to reduce the interference to other cells. Unlike CS/CB, the main idea of the joint processing is to transmit signals from multiple cells to a user using the same time and frequency. On the other hand, the uplink CoMP allows multiple base stations to receive the signals transmitted from a mobile user and it usually involves the exchange of decoded signals or un-decoded signals via backhaul links to the serving BS, which performs some form of joint processing of the aggregate received signals [7]–[9].

This paper considers a wireless network with arbitrary topology, in which each node is equipped with multiple antennas. The goal is to maximize the minimum weighted signal-to-interference-plus-noise ratio (SINR) subject to multiple weighted-sum power constraints (also known as the SINR balancing problem). Multiple power constraints are especially relevant to CoMP where each base station could be subject to its own transmit power constraint or in cognitive radio networks where the secondary user needs to apply interference power constraints to protect multiple primary users [10]. The weights can represent relative power costs of serving different users. While an important challenge in CoMP is to achieve the best performance gains with minimal backhaul burden [11], the emphasis of this paper is on understanding the structure of the problem and on obtaining optimal solutions to the problem.

The earliest work on the max-min weighted SINR problem was [12]. The authors solved the single-input-single-output (SISO) problem with one power constraint by introducing an extended coupling matrix. Their approach was extended to the multiple-input-single-output (MISO) and multiple-input-multiple-output (MIMO) problems in [13]–[17]. Unfortunately, the extended coupling matrix approach is computationally intensive as it requires a centralized power update via an eigenvector computation. Recently in [18], the authors derived an elegant algorithm for the max-min weighted SINR problem which avoided the eigenvector computation. The algorithm was based on nonlinear Perron-Frobenius theory and resembles the distributed power control (DPC) algorithm (for the power minimization problem). Subsequently in [19], the authors extended those results to the MISO case by deriving a joint beamforming and power control algorithm using nonlinear Perron-Frobenius theory.

However, the results for the max-min weighted SINR problem with a single power constraint do not extend in a

Manuscript received July 07, 2011; revised February 12, 2012 and June 17, 2012; accepted June 21, 2012. Date of publication July 13, 2012; date of current version September 11, 2012. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Mathini Sellathurai. The work in this paper is partially supported by grants from the Research Grants Council of Hong Kong, Project No. RGC CityU 125212, Qualcomm Inc., ARO MURI Grant W911NF-08-1-0233, NSF NetSE Grant CNS 0911041, SRG ISTD 2012 037, CAS Fellowship for Young International Scientist, and Bell Labs Lucent-Alcatel. The material in this paper was presented in part at the IEEE International Conference on Communications (ICC), Kyoto, Japan, June 2011, and at the Ninth International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOpt), Princeton, NJ, May 2011.

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Digital Object Identifier 10.1109/TSP.2012.2208631

straightforward manner to the case of multiple power constraints. The main challenge lies in identifying which power constraint is tight at the optimal solution. This is necessary for both the extended coupling matrix approach and the nonlinear Perron-Frobenius approach. To our knowledge, the only work on the max-min weighted SINR problem with multiple power constraints was [20], [21]. In [20], the authors considered the SIMO multiple access channel (MAC). They used the extended coupling matrix approach to show that the original problem can be decoupled into subproblems each involving one power constraint. The proposed algorithm involves solving for the solution to each subproblem separately, and choosing the solution which gives the smallest value for the objective. Because [20] relies on the extended coupling matrix approach, their algorithm requires a centralized network controller to solve each subproblem. Moreover, the proof in [20] that shows one can decouple multiple power constraints into subproblems each involving one power constraint relies on induction, which does not give insight into the problem structure. In our paper, we derive a novel algorithm for the max-min weighted SINR problem which directly solves for the optimal powers of the original problem without solving each subproblem separately.

Another commonly used approach for tackling problems with multiple power constraints is to form a relaxed problem by taking a parameterized sum of the original power constraints. This strategy was applied to the MISO sum-rate maximization problem in [21], [22], where it turned out that the solution to the relaxed problem with a single power constraint is also a solution to the original problem with multiple power constraints. Moreover, the relaxed problem can be solved using an inner-outer subgradient algorithm. In [21], the authors used that strategy to tackle the MISO max-min weighted SINR problem with multiple linear transmit covariance constraints. However, the authors were unable to extend their results for the sum-rate maximization problem to the max-min weighted SINR problem due to the lack of a convex reformulation of the relaxed problem. Nevertheless, the authors observed that, numerically, the optimal solution to the relaxed problem always satisfies the constraints in the original problem. In our paper, we prove that the relaxation is tight for the MISO max-min weighted SINR problem with multiple sum power constraints.

The first contribution of this paper is extending the application of nonlinear Perron-Frobenius theory to the SISO max-min weighted SINR problem with multiple sum power constraints. By reformulating the multiple power constraints as a single weighted norm constraint, we derive closed-form expressions for the optimal value and powers, and an iterative algorithm which converges geometrically fast to the optimal powers. Compared to the algorithm derived in [20], our algorithm has computational advantages as it solves the multiple-constrained problem directly and does not solve for the optimal powers of the subproblems separately. Furthermore, the closed-form expressions for the optimal value and solution provide the insight that the multiple-constrained problem can be decoupled into its single-constrained subproblems. We also connect the closed-form expressions with uplink-downlink duality theory and convex optimization.

The second contribution of this paper is a novel derivation for the decoupling of the MISO max-min weighted SINR problem with multiple power constraints into subproblems each involving one power constraint. Our derivation is significant in that it is based only on uplink-downlink duality theory and provides insight into the fundamental structure enabling decoupling. Uplink-downlink duality [21], [23]–[26] is often used to convert transmit beamforming problems into equivalent uplink receive beamforming problems [14]–[19], [21], [22], [24], [25]. As our proof relies on only uplink-downlink duality theory, it can be extended to the MIMO problem where both the transmit and receive beamformers are being optimized.

This paper is organized as follows. We begin in Section II by introducing the system model. Next, in Section III, we study the max-min weighted SINR power control problem. In Section IV, we study the joint transmit beamforming and power control problem. In Section V, we evaluate the convergence and performance of our algorithms. We conclude the paper in Section VI.

The following notations are used.

- Boldface upper-case letters denote matrices, boldface lowercase letters denote column vectors, italics denote scalars.
- For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L$, let $\mathbf{x} \leq \mathbf{y}$ denote $x_i \leq y_i$ for $1 \leq i \leq L$, let $\mathbf{x} < \mathbf{y}$ denote $x_i < y_i$ for $1 \leq i \leq L$, and let $\mathbf{x} \lneq \mathbf{y}$ denote $x_i \leq y_i$ and $\mathbf{x} \neq \mathbf{y}$.
- $\mathbf{u} \circ \mathbf{v}$ denotes the Schur product between $\mathbf{u}$ and $\mathbf{v}$.
- The Perron-Frobenius eigenvalue of a nonnegative matrix $\mathbf{F}$ is denoted by $\rho(\mathbf{F})$, and the associated right and left eigenvectors of $\mathbf{F}$ are denoted by $\mathbf{x}(\mathbf{F})$ and $\mathbf{y}(\mathbf{F})$ respectively.
- $\mathrm{diag}(\mathbf{v})$ denotes the diagonal matrix formed by the components of a vector $\mathbf{v}$.
- The super-scripts $(\cdot)^\top$ and $(\cdot)^\dagger$ denote transpose and complex conjugate transpose respectively.

II. System Model

We consider a MIMO network where L independent data streams are transmitted over a common frequency band. The transmitter and receiver of the lth stream, denoted by t_l and r_l respectively, are equipped with N_{t_l} and N_{r_l} antennas respectively. We model the downlink channel as a Gaussian broadcast channel given by

$$\mathbf{y}_l := \sum_{i=1}^{L} \mathbf{H}_{li}\mathbf{x}_i + \mathbf{z}_l, \quad l = 1, \ldots, L \tag{1}$$

where $\mathbf{y}_l \in \mathbb{C}^{N_{r_l}}$ is the received signal at receiver r_l, $\mathbf{x}_i \in \mathbb{C}^{N_{t_i}}$ is the transmitted signal vector of the ith stream, $\mathbf{H}_{li} \in \mathbb{C}^{N_{r_l} \times N_{t_i}}$ is the channel vector between t_i and r_l, and $\mathbf{z}_l \sim \mathcal{CN}(\mathbf{0}, n_l\mathbf{I})$ is the circularly symmetric Gaussian noise vector with covariance $n_l\mathbf{I}$ at r_l, where $n_l \in \mathbb{R}_{>0}$.

We assume linear transmit and receive beamforming. The transmitted signal vector of the ith stream can be written as $\mathbf{x}_i := d_i\sqrt{p_i}\mathbf{u}_i$, where $\mathbf{u}_i \in \mathbb{C}^{N_{t_i}}$ is the normalized transmit beamformer, and d_i and $p_i \in \mathbb{R}_{\geq 0}$ are the information signal and transmit power respectively for that stream. The lth stream is decoded using a fixed normalized receive beamformer $\mathbf{v}_l \in \mathbb{C}^{N_{r_l}}$. We denote by $\mathbf{p} = [p_1, \ldots, p_L]^\top$ the power vector and by $\mathbf{n} = [n_1, \ldots, n_L]^\top$ the noise vector. We also denote by $\mathbb{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_L)$ the tuples of transmit beamformers. Let the matrix $\mathbf{G} \in \mathbb{R}_{\geq 0}^{L \times L}$ be defined by $G_{li} := |\mathbf{v}_l^\dagger \mathbf{H}_{li}\mathbf{u}_i|^2$, i.e., G_{li}

is the effective link gain between t_i and r_l. The SINR of the lth stream in the downlink can be expressed as

$$\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p},\mathbb{U}) := \frac{p_l G_{ll}}{\sum_{i\neq l} p_i G_{li} + n_l}. \tag{2}$$

We are interested in the max-min weighted SINR transmit beamforming and power control problem given by

$$\mathcal{S} := \begin{cases} \max\limits_{\mathbf{p},\mathbb{U}} & \min\limits_{l} \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p},\mathbb{U})}{\beta_l} \\ \text{subject to} & \mathbf{w}_j^{\mathsf{T}}\mathbf{p} \leq \bar{P}_j, \quad j=1,\ldots,J \\ & \mathbf{p} > \mathbf{0} \\ & \|\mathbf{u}_l\| = 1, \quad l=1,\ldots,L \end{cases} \tag{3}$$

where $\bar{P}_j \in \mathbb{R}_{>0}$ is the given jth power constraint, $\boldsymbol{\beta} = [\beta_1,\ldots,\beta_L]^{\mathsf{T}}$ is the priority vector where $\beta_l \in \mathbb{R}_{>0}$ is the priority assigned by the network to the lth stream, and $\mathbf{w}_j = [w_{j1},\ldots,w_{jL}]^{\mathsf{T}}$ is the weight vector such that $w_{jl} \in \mathbb{R}_{>0}$ is the weight associated with p_l in the jth power constraint. The restriction to strictly positive weights is only for technical purposes. In order to exclude a certain stream from a power constraint, one could choose an infinitesimally small value for the weight on that stream.

III. Power Control Optimization

In this section, we fix the transmit beamformers $\mathbb{U}$ inside problem $\mathcal{S}$ and study the power control problem given by:

$$\mathcal{S}_{\mathbb{U}} := \begin{cases} \max\limits_{\mathbf{p}} & \min\limits_{l} \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p})}{\beta_l} \\ \text{subject to} & \mathbf{w}_j^{\mathsf{T}}\mathbf{p} \leq \bar{P}_j, \quad j=1,\ldots,J \\ & \mathbf{p} > \mathbf{0}. \end{cases} \tag{4}$$

For notational brevity, we omitted the dependence of $\mathsf{SINR}_l^{\mathsf{DL}}$ on $\mathbb{U}$. Notice that at the optimal solution to the problem $\mathcal{S}_{\mathbb{U}}$, at least one power constraint must be active. Moreover, we will assume that $\mathbb{U}$ was chosen such that $\mathbf{G}$ is irreducible. This is a valid assumption because a user will typically have interference coming from at least one other user. This assumption implies that at the optimal solution to $\mathcal{S}_{\mathbb{U}}$, all the weighted SINR's are equal.

It is well-known that $\mathcal{S}_{\mathbb{U}}$ can be reformulated as a Geometric Program (GP) and solved using standard convex solvers. However, our main contribution is in deriving closed-form expressions for the optimal value and solution. These expressions also provide insight into the structure of the optimal solution. We begin in Section III.A by using nonlinear Perron-Frobenius theory to derive closed-form expressions for the solution and optimal value of $\mathcal{S}_{\mathbb{U}}$, and an iterative algorithm to compute the solution. Then, in Section III.B, we exploit the structure of the closed-form expression to show that $\mathcal{S}_{\mathbb{U}}$ can be decoupled into subproblems each involving only one power constraint. In Section III.C, we give insights on the relationship between uplink-downlink duality and the solution structure of $\mathcal{S}_{\mathbb{U}}$. Finally, in Section III.D, we use nonlinear Perron-Frobenius theory to analyze the GP reformulation of $\mathcal{S}_{\mathbb{U}}$.

A. Optimal Solution and Algorithm

In this section, we use nonlinear Perron-Frobenius theory to derive closed-form expressions for the solution and optimal value of $\mathcal{S}_{\mathbb{U}}$, and an iterative algorithm to compute the solution. We review in Appendix A the main Perron-Frobenius theorems that will be used.

For ease of notation, we define the (cross channel interference) matrix $\mathbf{F} \in \mathbb{R}_{\geq 0}^{L\times L}$ and the normalized priority vector $\hat{\boldsymbol{\beta}} \in \mathbb{R}_{>0}^{L}$ with the following entries:

$$F_{li} := \begin{cases} 0, & \text{if } l = i \\ G_{li}, & \text{if } l \neq i \end{cases} \tag{5}$$

$$\hat{\boldsymbol{\beta}} := \left(\frac{\beta_1}{G_{11}},\ldots,\frac{\beta_L}{G_{LL}}\right)^{\mathsf{T}}. \tag{6}$$

Hence, $\mathcal{S}_{\mathbb{U}}$ can be rewritten as

$$\mathcal{S}_{\mathbb{U}} = \begin{cases} \max\limits_{\mathbf{p}} & \min\limits_{l} \frac{p_l}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}+\mathbf{n}))_l} \\ \text{subject to} & \mathbf{w}_j^{\mathsf{T}}\mathbf{p} \leq \bar{P}_j, \quad j=1,\ldots,J \\ & \mathbf{p} > \mathbf{0}. \end{cases} \tag{7}$$

We state our main results for $\mathcal{S}_{\mathbb{U}}$ in the following three theorems.

Theorem 1: Let $\mathbf{B}_j = \mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F} + (1/\bar{P}_j)\mathbf{n}\mathbf{w}_j^{\mathsf{T}})$ and $k = \arg\min\{(1/\rho(\mathbf{B}_j)) : j = 1,\ldots,J\}$. The optimal value and unique solution of $\mathcal{S}_{\mathbb{U}}$ are given by

$$\mathcal{S}_{\mathbb{U}} = 1/\rho(\mathbf{B}_k) \tag{8}$$

$$\mathbf{p}_* = \left(\bar{P}_k/\mathbf{w}_k^{\mathsf{T}}\mathbf{x}(\mathbf{B}_k)\right)\mathbf{x}(\mathbf{B}_k) \tag{9}$$

where $\mathbf{x}(\mathbf{B}_k)$ denotes the Perron (right) eigenvector of $\mathbf{B}_k$ associated with its largest eigenvalue $\rho(\mathbf{B}_k)$.

Proof: Refer to Appendix B.

Algorithm 1: Max-min weighted SINR coordinated power control

1) Compute auxiliary variables:

$$p_l^{(n+1)} = \left(\frac{\beta_l}{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}^{(n)})}\right) p_l^{(n)}, \quad l=1,\ldots,L.$$

2) Update downlink power:

$$\mathbf{p}^{(n+1)} \leftarrow \left(\min_j \frac{\bar{P}_j}{\mathbf{w}_j^{\mathsf{T}}\mathbf{p}^{(n+1)}}\right)\mathbf{p}^{(n+1)}.$$

Theorem 2: Starting from any initial point $\mathbf{p}^{(0)} > \mathbf{0}$, $\mathbf{p}^{(n)}$ in Algorithm 1 converges geometrically fast to the unique solution of $\mathcal{S}_{\mathbb{U}}$ given in Theorem 1.

Proof: Refer to Appendix C. □

Theorem 3: In Algorithm 1, $\min_l(\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}^{(n)})/\beta_l)$ and $\max_l(\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}^{(n)})/\beta_l)$ increases and decreases monotonically respectively.

Proof: Refer to Appendix D. □

Theorem 1 and 2 give two approaches to compute the optimal power of $\mathcal{S}_{\mathbb{U}}$. The first approach is to directly compute the largest eigenvector of the matrix as given in Theorem 1. The second approach is to compute the powers iteratively via Algorithm 1. We briefly discuss the tradeoffs between the two approaches.

Computing the largest eigenvector directly has the disadvantage that it requires a centralized controller. However, it could be computationally fast under certain scenarios. With centralized computation of the eigenvector, the computational complexity would typically depend on the size of the matrix $\mathbf{B}_j$ and the number of power constraints. For instance, if QR factorization is used, the computational complexity would be $O(L^3 J)$. Hence, centralized computation might be appropriate when there are only a few data streams and a few power constraints.

On the other hand, although Algorithm 1 has the advantage that it is distributed and requires minimal coordination, geometric convergence could be arbitrarily slow. Consider the particular setup where there is only a sum-power constraint (i.e., $J = 1$, $\mathbf{w}_1 = \mathbf{1}$) and all the data streams are subject to noise with unit variance (i.e., $\mathbf{n} = \mathbf{1}$), Algorithm 1 reduces to the max-min weighted SINR algorithm proved in [18]. In this special case, the convergence rate can be easily computed. Let the norm $\|\cdot\|_{\bar{P}_1}$ be defined by $\|\mathbf{p}\|_{\bar{P}_1} = (1/\bar{P}_1)\sum_l w_{1l}|p_l|$. Algorithm 1 can be rewritten as

$$\mathbf{p}^{(n+1)} = \frac{\left(\mathrm{diag}(\hat{\boldsymbol{\beta}})\left(\mathbf{F} + \mathbf{n}\mathbf{w}_1^\top\right)\right)\mathbf{p}^{(n)}}{\left\|\left(\mathrm{diag}(\hat{\boldsymbol{\beta}})\left(\mathbf{F} + \mathbf{n}\mathbf{w}_1^\top\right)\right)\mathbf{p}^{(n)}\right\|_{\bar{P}_1}} \tag{10}$$

which is the power method applied to the matrix $\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F} + \mathbf{n}\mathbf{w}_1^\top)$. Hence, the rate of convergence of Algorithm 1 is given by $|\lambda_2|/\rho(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}+\mathbf{n}\mathbf{w}_1^\top))$, where $|\lambda_2|$ is the second largest eigenvalue of $\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}+\mathbf{n}\mathbf{w}_1^\top)$ [27]. Hence, Algorithm 1 will converge slowly if the second largest eigenvalue is close in magnitude to the largest eigenvalue. This scenario could occur regardless of the number of data streams.

Observe that step 1 of Algorithm 1 resembles the distributed power control (DPC) algorithm and step 2 simply normalizes the auxiliary variables such that all the power constraints are satisfied. It turns out that, in certain scenarios, this normalization has an intuitive interpretation. Consider a cellular network with J cells where each cell serves a different set of users. Denote the base station of the jth cell by b_j. It is reasonable to suppose that b_j is subject to a sum power constraint $\bar{P}_j$. Hence, we can associate with b_j a power constraint $\mathbf{w}_j^\top\mathbf{p} \le \bar{P}_j$, where the entries of $\mathbf{w}_j$ are in $\{0,1\}$, and the indexes of the nonzero entries of $\mathbf{w}_j$ correspond to the links associated with b_j. Let the set $\mathcal{I}_j$ contain the indexes of the nonzero entries of $\mathbf{w}_j$. Then, step 2 of the algorithm can be rewritten as

$$\mathbf{p}^{(n+1)} \leftarrow \left(\min_j \frac{\bar{P}_j}{\sum_{l\in\mathcal{I}_j} p_l^{(n+1)}}\right)\mathbf{p}^{(n+1)}. \tag{11}$$

Observe that the term $\bar{P}_j/\sum_{l\in\mathcal{I}_j} p_l^{(n+1)}$ can be completely computed at b_j. Moreover, it has the intuitive interpretation as b_j's normalization factor ignoring the presence of other base stations. Larger normalization factors indicate relatively more available transmit power.

In the algorithm, only these normalization factors need to be shared between base stations at each time step, and all base stations would normalize their transmit powers by the smallest normalization factor. This coordination could be carried out using high-speed backhaul links and gossip algorithms. Furthermore, for situations in which optimality is not absolutely important, Algorithm 1 can be easily modified to reduce the backhaul burden at the expense of performance. For instance, a base station b_j could send its normalization factor to only a subset of base stations that have the strongest interference on b_j (these base stations are also likely to be physically closest to b_j).

B. Decoupling into Single-Constrained Subproblems

In this section, we use the closed-form expressions for the solution to $\mathcal{S}_\cup$ derived in Theorem 1 to establish the relationship between $\mathcal{S}_\cup$ and its single-constrained subproblems given by

$$\mathcal{S}_\cup^j := \begin{cases} \max\limits_{\mathbf{p}} & \min\limits_l \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p})}{\beta_l} \\ \text{subject to} & \mathbf{w}_j^\top\mathbf{p} \le \bar{P}_j, \quad \mathbf{p} > \mathbf{0} \end{cases} \tag{12}$$

for $j = 1,\ldots,J$. The following corollary states that the multiple-constrained problem can be decoupled into its subproblems.

Corollary 1: Let $\mathbf{p}_*^j$ denote the unique solution to $\mathcal{S}_\cup^j$ and let $k = \arg\min\{\mathcal{S}_\cup^j : j = 1,\ldots,J\}$. The optimal value and unique solution of $\mathcal{S}_\cup$ are given by $\mathcal{S}_\cup = \mathcal{S}_\cup^k$ and $\mathbf{p}_* = \mathbf{p}_*^k$ respectively.

Proof: From Theorem 1, the optimal value and unique solution of $\mathcal{S}_\cup^j$ are given by $1/\rho(\mathbf{B}_j)$ and $(\bar{P}_j/\mathbf{w}_j^\top\mathbf{x}(\mathbf{B}_j))\mathbf{x}(\mathbf{B}_j)$ respectively. Corollary 1 follows from identifying these expressions with those for the optimal value and solution of $\mathcal{S}_\cup$. □

C. Revisiting Uplink-Downlink Duality

In this section, we revisit the uplink-downlink duality theory using the closed-form expressions for the solution and optimal value of $\mathcal{S}_\cup$ given by Theorem 1. In [18], the authors studied the max-min weighted SINR problem subject to a total power constraint and provided a simple derivation of uplink-downlink duality by applying spectral radius relations. A similar line of analysis applies to $\mathcal{S}_\cup$. Let k be defined as in Theorem 1. Using the fact that $\rho(\mathbf{A}) = \rho(\mathbf{A}^\top)$ and $\rho(\mathbf{AC}) = \rho(\mathbf{CA})$ for any irreducible nonnegative matrices $\mathbf{A}$ and $\mathbf{C}$, we have

$$\rho(\mathbf{B}_k) = \rho\left(\mathrm{diag}(\hat{\boldsymbol{\beta}})\left(\mathbf{F} + (1/\bar{P}_k)\mathbf{n}\mathbf{w}_k^\top\right)\right) \tag{13}$$

$$= \rho\left(\mathrm{diag}(\hat{\boldsymbol{\beta}})\left(\mathbf{F}^\top + (1/\bar{P}_k)\mathbf{w}_k\mathbf{n}^\top\right)\right). \tag{14}$$

These equations imply that an uplink channel with noise vector $\mathbf{w}_k$ and power constraint $\mathbf{n}^\top\mathbf{q} \le \bar{P}_k$ achieves the same max-min weighted SINR as problem $\mathcal{S}_\cup$. This is illustrated in Fig. 1.

In Appendix E, we re-derive uplink-downlink duality by reformulating the single-constrained max-min weighted SINR problem as a Geometric Program and we state the result formally in Theorem 7. Here, we show how to reconcile Theorem 7 with the spectral radius equations in (13)–(14). Observe that $\mathbf{F}^\top$ is the cross-channel interference matrix of the dual uplink network. Hence, by applying Theorem 1 on (14), we conclude that the optimal value of $\mathcal{S}_\cup$ is equal to the optimal value of the following uplink problem

$$\begin{array}{ll} \max\limits_{\mathbf{q}} & \min\limits_l \dfrac{\mathsf{SINR}_l^{\mathsf{UL},k}(\mathbf{q})}{\beta_l} \\ \text{subject to} & \mathbf{n}^\top\mathbf{q} \le \bar{P}_k, \quad \mathbf{q} > \mathbf{0} \end{array} \tag{15}$$

where

$$\mathsf{SINR}_l^{\mathsf{UL},k}(\mathbf{q}) := \frac{q_l G_{ll}}{\sum_{i\ne l} q_i G_{il} + w_{kl}} \tag{16}$$

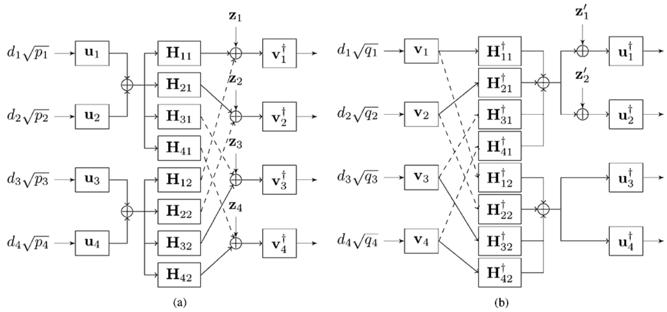

Fig. 1. Block diagrams of a two-cell MIMO downlink system and its dual uplink system (assuming $\mathbf{w}_1^\top \mathbf{p} = \bar{P}_1$ at optimality). Each cell serves two users. The per-cell power constraints in the downlink are given by $\mathbf{w}_1^\top \mathbf{p} \leq \bar{P}_1$ and $\mathbf{w}_2^\top \mathbf{p} \leq \bar{P}_2$, where $\mathbf{w}_1 = [w_{11}\ w_{12}\ 0\ 0]^\top$ and $\mathbf{w}_2 = [0\ 0\ w_{23}\ w_{24}]^\top$. The dual uplink system has a total power constraint given by $\mathbf{n}^\top \mathbf{q} \leq \bar{P}_1$. Dotted lines denote the inter-cell interfering paths. (a) Downlink, $\mathbf{z}_l \sim \mathcal{CN}(\mathbf{0}, n_l \mathbf{I})$. (b) Dual uplink (assuming $\mathbf{w}_1^\top \mathbf{p} = \bar{P}_1$ at optimality), $\mathbf{z}'_l \sim \mathcal{CN}(\mathbf{0}, w_{1l}\mathbf{I})$.

is the dual uplink SINR. This hints at one potential challenge in solving $\mathcal{S}_{\mathbb{U}}$ in the dual uplink domain. Specifically, to establish the noise vector and power constraint in the dual uplink problem, we must first know which power constraint is tight in the optimal solution to the downlink problem. This challenge will be evident later when we consider the joint beamforming and power control problem.

From Corollary 1 and (13) and (14), it is clear that $\mathcal{S}_{\mathbb{U}}$ can also be decoupled into separate uplink problems. In fact, when we prove the decoupling of the MISO problem in Theorem 4, it will be evident that this decoupling is a fundamental consequence of uplink-downlink duality.

D. Relationship With Convex Optimization

A key step in the proof of Theorem 1 is the reformulation of the multiple power constraints as a single norm constraint. This reformulation can also be used to establish the relationship between the optimal primal and dual variables of the GP reformulation.

We refer the reader to Appendix E for the steps involved in reformulating $\mathcal{S}_{\mathbb{U}}$ as a GP. From the following convex reformulation

$$\begin{aligned} &\min_{\tilde{\tau}, \tilde{\mathbf{p}}} \quad -\tilde{\tau} \\ &\text{subject to} \quad \log\left(\frac{e^{\tilde{\tau}}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}e^{\tilde{\mathbf{p}}} + \mathbf{n}))_l}{(e^{\tilde{\mathbf{p}}})_l}\right) \leq 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad l = 1, \ldots, L \\ &\qquad\qquad \log\left(\frac{1}{\bar{P}_j}\mathbf{w}_j^\top e^{\tilde{\mathbf{p}}}\right) \leq 0, \quad j = 1, \ldots, J \end{aligned} \tag{17}$$

we write the partial Lagrangian with respect to the SINR constraints and optimize with respect to the auxiliary variable τ to obtain the partial Lagrange dual problem given by

$$\max_{\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^\top \boldsymbol{\lambda} = 1} \min_{\|\mathbf{p}\|_\xi = 1} \sum_{l=1}^{L} \lambda_l \log\left(\frac{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p} + \mathbf{n}))_l}{p_l}\right) \tag{18}$$

where $\|\cdot\|_\xi$ is the norm that was used in the proof of Theorem 1 (cf. Appendix B). Applying the nonlinear Perron-Frobenius minimax characterization (Lemma 2 in [18]), the optimal value of (18) is given by $\log \rho(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F} + \mathbf{n}\mathbf{c}_*^\top)$ where

$$\begin{aligned} \mathbf{c}_* &= \arg \max_{\|\mathbf{c}\|_\xi^{\mathrm{D}} = 1} \log \rho(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F} + \mathbf{n}\mathbf{c}^\top)) & (19) \\ &= \arg \max_j \rho(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F} + (1/\bar{P}_j)\mathbf{n}\mathbf{w}_j^\top)). & (20) \end{aligned}$$

Furthermore, the optimal downlink power is given by $\mathbf{p}_* = \mathbf{x}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F} + \mathbf{n}\mathbf{c}_*^\top))$ normalized such that $\|\mathbf{p}_*\|_\xi = 1$, and the optimal $\boldsymbol{\lambda}$ is given by $\boldsymbol{\lambda}_* = \mathbf{x}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F} + \mathbf{n}\mathbf{c}_*^\top)) \circ \mathbf{y}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F} + \mathbf{n}\mathbf{c}_*^\top))$ normalized such that $\mathbf{1}^\top \boldsymbol{\lambda}_* = 1$. Finally, the optimal dual uplink power is given by $\mathbf{q}_* = \mathrm{diag}(\boldsymbol{\beta})\mathbf{y}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F} + \mathbf{n}\mathbf{c}_*^\top))$ up to a scaling constant (Remark 2 in [18]). Hence, $\boldsymbol{\lambda}_*$ can also be written as

$$\boldsymbol{\lambda}_* = \mathbf{p}_* \circ \mathrm{diag}(\hat{\boldsymbol{\beta}})^{-1}\mathbf{q}_* \tag{21}$$

up to a scaling constant. This agrees with the analysis of the GP reformulation in Appendix E.

IV. Joint Transmit Beamforming and Power Control Optimization

In this section, we allow the transmit beamformers to be optimized and we consider the joint beamforming and power control problem $\mathcal{S}$ which we repeat here:

$$\mathcal{S} := \begin{cases} \max\limits_{\mathbf{p}, \mathbb{U}} & \min\limits_{l} \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}, \mathbb{U})}{\beta_l} \\ \text{subject to} & \mathbf{w}_j^\top \mathbf{p} \leq \bar{P}_j, \quad j = 1, \ldots, J \\ & \mathbf{p} > \mathbf{0} \\ & \|\mathbf{u}_l\| = 1, \quad l = 1, \ldots, L. \end{cases} \tag{22}$$

In Section IV.A, we use uplink-downlink duality to prove that $\mathcal{S}$ can be decoupled into its single-constrained subproblems. Then, in Section IV.B, we derive an inner-outer subgradient algorithm for solving $\mathcal{S}$.

A. Relationship With Single-Constrained Subproblem

In this section, we use uplink-downlink duality to prove that $\mathcal{S}$ can be decoupled into its single-constrained subproblems given by

$$\mathcal{S}^j := \begin{cases} \max\limits_{\mathbf{p}, \mathbb{U}} & \min\limits_{l} \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}, \mathbb{U})}{\beta_l} \\ \text{subject to} & \mathbf{w}_j^\top \mathbf{p} \leq \bar{P}_j, \quad \mathbf{p} > \mathbf{0} \\ & \|\mathbf{u}_l\| = 1, \quad l = 1, \ldots, L \end{cases} \tag{23}$$

for $j = 1, \ldots, J$. We state our main result in the following theorem.

Theorem 4: Let $\mathbf{p}_*^j$ and $\mathbb{U}_*^j$ denote the optimal solution to $\mathcal{S}^j$ and let $k = \arg\min\{\mathcal{S}^j : j = 1, \ldots, J\}$. The optimal value and unique solution of $\mathcal{S}$ are given by $\mathcal{S} = \mathcal{S}^k$, $\mathbf{p}_* = \mathbf{p}_*^k$, and $\mathbb{U}_* = \mathbb{U}_*^k$.

Proof: Refer to Appendix F.

Theorem 4 implies that $\mathcal{S}$ can be solved by separately solving J downlink problems. By uplink-downlink duality (cf. Theorem 7), it follows that $\mathcal{S}$ can also be solved by separately solving J associated uplink problems.

B. Algorithm

In this section, we derive an iterative algorithm which solves $\mathcal{S}$ jointly without resorting to solving each subproblem separately. Let $g(\boldsymbol{\theta})$ be defined by:

$$g(\boldsymbol{\theta}) := \begin{cases} \max\limits_{\mathbf{p},\mathbb{U}} & \min\limits_{l} \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p},\mathbb{U})}{\beta_l} \\ \text{subject to} & \sum_j \theta_j \mathbf{w}_j^\mathsf{T}\mathbf{p} \leq \sum_j \theta_j \bar{P}_j \\ & \mathbf{p} > \mathbf{0} \\ & \|\mathbf{u}_l\| = 1, \quad l = 1,\ldots,L. \end{cases} \tag{24}$$

The follow theorem relates $\mathcal{S}$ to $g(\boldsymbol{\theta})$.

Corollary 2: We have that

$$\mathcal{S} = \min\{g(\boldsymbol{\theta}) : \boldsymbol{\theta} \geq \mathbf{0}\}. \tag{25}$$

Proof: Refer to Appendix G. □

Notice that (24) is a relaxation of $\mathcal{S}^j$. Furthermore, $\boldsymbol{\theta}$ can be interpreted as the weights on the individual power constraints in the relaxed problem. Corollary 2 states that the relaxation is tight; hence, Corollary 2 proves the conjecture in [21] for the special case of linear sum power constraints.

In Appendix H, we prove that $g(\boldsymbol{\theta})$ is quasiconvex. Hence, (25) can be solved via a subgradient projection method [21], [22]. For any fixed $\boldsymbol{\theta}$, Algorithm 1 in [19] can be used to compute $g(\boldsymbol{\theta})$. The subgradient of $g(\boldsymbol{\theta})$ can be easily computed to be $[\bar{P}_1 - \mathbf{w}_1^\mathsf{T}\mathbf{p}_*(\boldsymbol{\theta}),\ldots,\bar{P}_J - \mathbf{w}_J^\mathsf{T}\mathbf{p}_*(\boldsymbol{\theta})]^\mathsf{T}$ where $\mathbf{p}_*(\boldsymbol{\theta})$ is the optimal downlink power of $g(\boldsymbol{\theta})$. This iterative algorithm based on the subgradient projection method is given as follows.

Algorithm 2: Max-min weighted SINR joint transmit beamforming and power control (in the downlink)

- Define: $\mathcal{C} = \{\boldsymbol{\theta} \in \mathbb{R}_{\geq 0}^J : \boldsymbol{\theta} \gneq \mathbf{0}\}$.
- Parameters: threshold $\epsilon > 0$ for Algorithm 1, step size $\alpha_n > 0$ for subgradient update.
- Initialize: arbitrary $\boldsymbol{\theta}^{(0)} \in \mathcal{C}$, $\mathbf{q}^{(0)} > \mathbf{0}$, $\mathbf{p}^{(0)} > \mathbf{0}$, $\{\mathbb{U}^{(0)} : \|\mathbf{u}_l^{(0)}\| = 1\,\forall l\}$.

1) Fix $\boldsymbol{\theta}^{(n)}$ and fix the power constraint:

$$\sum_j \theta_j^{(n)} \mathbf{w}_j^\mathsf{T}\mathbf{p}^{(n)} \leq \sum_j \theta_j^{(n)} \bar{P}_j.$$

Use Algorithm 1 in [19] to find the optimal $\mathbf{p}^{(n)}$, $\mathbf{q}^{(n)}$, and $\mathbb{U}^{(n)}$ subject to the above power constraint until

$$\left\|\mathbf{q}^{(n+1)} - \mathbf{q}^{(n)}\right\| < \epsilon.$$

2) Update $\boldsymbol{\theta}^{(n+1)}$ using the subgradient projection method:

$$\boldsymbol{\theta}^{(n+1)} = \mathcal{P}_\mathcal{C}\left\{\boldsymbol{\theta}^{(n)} - \alpha_n \hat{\mathbf{g}}\left(\mathbf{p}^{(n)}\right)\right\}$$

where $\hat{\mathbf{g}}(\mathbf{p}^{(n)}) = [\bar{P}_1 - \mathbf{w}_1^\mathsf{T}\mathbf{p}^{(n)},\ldots,\bar{P}_J - \mathbf{w}_J^\mathsf{T}\mathbf{p}^{(n)}]^\mathsf{T}$ and $\mathcal{P}_\mathcal{C}$ is the projection operator onto $\mathcal{C}$.

V. Numerical Examples

A. Convergence of Algorithm 1

In this section, we demonstrate the convergence of Algorithm 1 (the power control algorithm) in a 7-cell network with 2 randomly located users per cell, as illustrated in Fig. 2. The cell radius is 1.4 km. Each base station is assumed to have a total power constraint of 10 W and each user is served one independent data stream from its base station. Every data stream has the same priority, i.e., $\boldsymbol{\beta} = \mathbf{1}$. The base stations are equipped with $N_{t_l} = 4$ antennas and each user is equipped with $N_{r_l} = 2$ antennas. Both the base station and the users are assumed to carry out Maximum Ratio Transmission (MRT) and Maximum Ratio Combining (MRC) beamforming. The noise power spectral density is set to -162 dBm/Hz. Each user communicates with the base station over independent MIMO rayleigh fading channels, with a path loss of $L = 128.1 + 37.6\log_{10}(d)$ dB, where d is the distance in kilometers. Fig. 3 shows the minimum and maximum SINR as a function of the iteration index n. As stated in Theorems 2 and 3, both the minimum and maximum SINRs in the network converge monotonically and rapidly to the optimal value computed using Theorem 1.

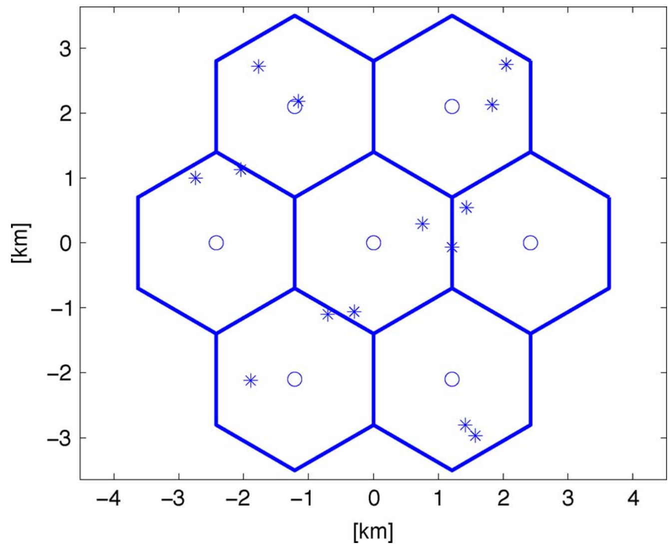

Fig. 2. Layout of 7-cell network used for convergence analysis. Each base station serves two users and is subject to a total power constraint of 10 W.

B. Convergence of Algorithm 2

We assume the same network configuration as in the previous section and allow the base stations to optimize the transmit beamformers using Algorithm 2. Fig. 4 shows the minimum SINR of Algorithm 2 as a function of the iteration index n. In each monotone increasing segment, the network is optimized subject to a fixed single power constraint determined by $\boldsymbol{\theta}^{(n)}$ (i.e., step 1). Each dip in the minimum SINR corresponds to a subgradient update of $\boldsymbol{\theta}^{(n)}$ (i.e., step 2). As a result of the subgradient update, each monotone segment converges to a value closer to the optimal solution, which is computed using Theorem 4. Although $\boldsymbol{\theta}^{(0)}$ could have been arbitrarily initialized, in this simulation, we chose $\boldsymbol{\theta}^{(0)} = \mathbf{1}$. Comparing Figs. 4 with 3, one can see that transmit beamforming provides a gain of about 10 dB over MRT beamforming.

C. Performance

Next, we investigate the benefits of optimizing the transmit power and beamformers of all base stations in a coordinated scheme versus an uncoordinated scheme (where each base station seeks to satisfy its own power constraint only). We conduct

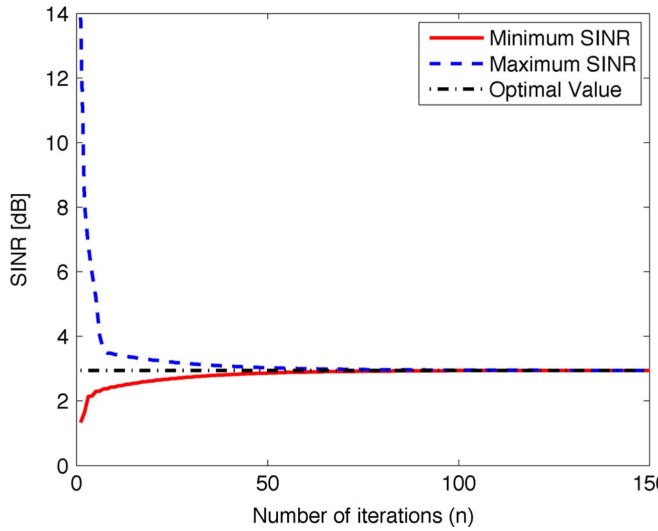

Fig. 3. Convergence of maximum SINR and minimum SINR values in Algorithm 1. Both the minimum and maximum SINRs in the network converge to the optimal value computed using Theorem 1.

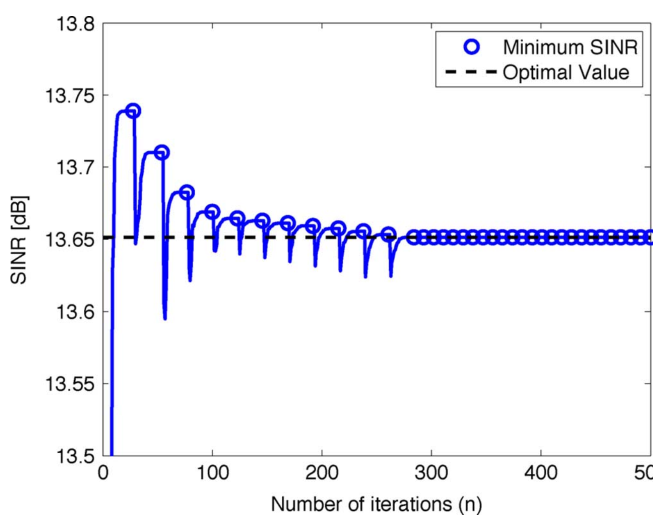

Fig. 4. Convergence of minimum SINR in Algorithm 2. The optimal value is computed using Theorem 4.

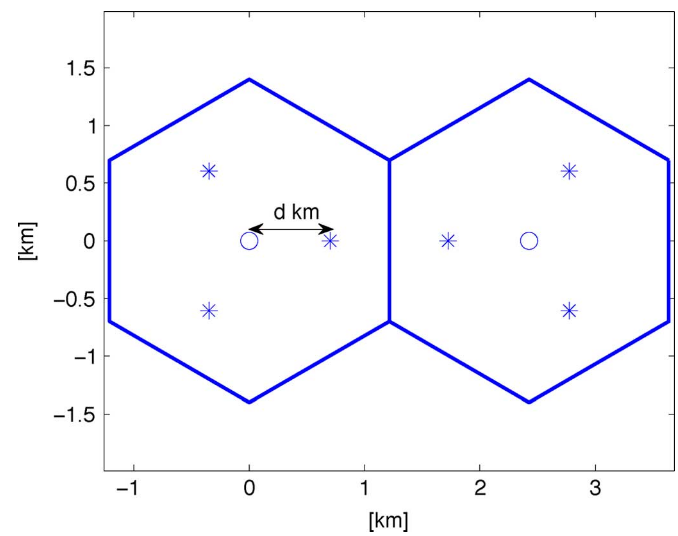

Fig. 5. Layout of 2-cell network used in performance analysis.

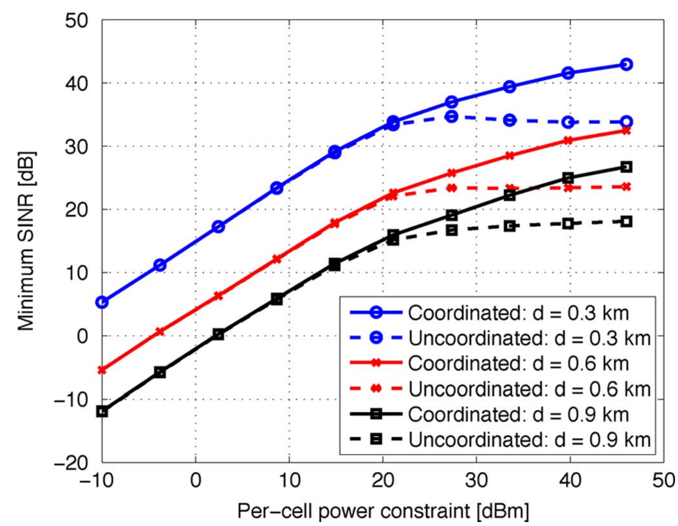

Fig. 6. Performance comparison of coordinated optimization versus uncoordinated optimization.

this study in a 2-cell network with 3 users per cell, as illustrated in Fig. 5. Each user is located d km away from its respective base station. The rest of the cellular parameters are kept fixed from the previous section. For the coordinated scheme, we apply Theorem 4 and compute the optimal minimum SINR by decoupling the problem into two subproblems and solving each subproblem using Algorithm 1 from [19]. For the uncoordinated scheme, both base stations run Algorithm 1 from [19] simultaneously.

Fig. 6 shows the performance of the coordinated and uncoordinated schemes as a function of the per-cell transmit power constraint $\bar{P}_j$ for three values of d. When the power constraint is small, the interference between users in neighboring cells is small so there is no observable difference in performance between the coordinated and uncoordinated schemes. However, as the power constraint increases, the interference between users in neighboring cells has significant impact on the performance of the users. As expected, when the users move further away from their serving base stations (i.e., they move closer to neighboring base stations), the performance of the whole network suffers due to the increased inter-cell interference.

VI. CONCLUSION

We exploited nonlinear Perron-Frobenius theory and uplink-downlink duality to analyze the max-min weighted SINR problem in the downlink subject to multiple weighted-sum power constraints. First, we used nonlinear Perron-Frobenius theory to derive closed-form expressions for the solution and optimal value of the SISO power control problem, and an algorithm which converges geometrically fast to the solution. The key insight is the reformulation of multiple weighted-sum power constraints as a single norm constraint. Then, we made use of the structure of the closed-form solution to show that the power control problem can be decoupled into its single-constrained downlink or uplink subproblems. Next, we extended this analysis to the MISO beamforming and power control problem. Specifically, we used uplink-downlink duality to show that the MISO problem can in fact be decoupled into its single-constrained subproblems and we used this decoupling to derive a subgradient algorithm for the original problem.

While both Algorithm 1 and Algorithm 2 are optimal, there are still many issues that can be explored with regards to both algorithms. In practice, having all base stations fully coordinate their power updates could lead to high latencies. Hence, an interesting issue is a rigorous analysis of suboptimal modifications of Algorithm 1 and Algorithm 2, and a characterization of the loss in performance. Another issue to investigate is the convergence of Algorithm 1 and Algorithm 2 under asynchronous coordination.

APPENDIX

A. Review of Nonlinear Perron-Frobenius Theorem

Let $\mathbf{A} \in \mathbb{R}_{\geq 0}^{L \times L}$ be a nonnegative matrix and $\mathbf{b} \in \mathbb{R}_{\geq 0}^{L}$ be a nonnegative vector. Let $\|\cdot\|$ be a monotone vector norm and let $\|\cdot\|^{\mathsf{D}}$ be its dual norm. We consider the problem of finding $(\lambda, \mathbf{p})$ such that

$$\lambda \mathbf{p} = \mathbf{A}\mathbf{p} + \mathbf{b}, \quad \lambda \in \mathbb{R}, \quad \mathbf{p} \gneq \mathbf{0}, \quad \|\mathbf{p}\| = 1. \tag{26}$$

In [28], [29], this problem was referred to as the conditional eigenvalue problem. In this Appendix, we review two main results pertaining to (26) which form the basis for our main theorems on the power control problem $\mathcal{S}_{\mathrm{U}}$. The first result states that (26) has a unique solution and gives a closed-form expression for the optimal value and solution. The second result gives an iterative algorithm for computing the solution.

Theorem 5 ([29]): If $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{A}\mathbf{p} + \mathbf{b} > \mathbf{0}$ for any $\mathbf{p} \gneq \mathbf{0}$, then (26) has a unique solution $(\lambda_*, \mathbf{p}_*)$, where

$$\lambda_* = \rho\left(\mathbf{A} + \mathbf{b}\mathbf{c}_*^{\mathsf{T}}\right) = \max_{\|\mathbf{c}\|^{\mathsf{D}}=1} \rho(\mathbf{A} + \mathbf{b}\mathbf{c}^{\mathsf{T}}) \tag{27}$$

with $\mathbf{c}_* \geq \mathbf{0}$, and

$$\mathbf{p}_* = \mathbf{x}\left(\mathbf{A} + \mathbf{b}\mathbf{c}_*^{\mathsf{T}}\right) \tag{28}$$

where $\mathbf{x}(\mathbf{A} + \mathbf{b}\mathbf{c}_*^{\mathsf{T}})$ is the unique normalized Perron vector of $\mathbf{A} + \mathbf{b}\mathbf{c}_*^{\mathsf{T}}$. Moreover, $(\mathbf{p}_*, \mathbf{c}_*)$ is a dual pair with respect to $\|\cdot\|$.

Theorem 6 ([28]): If $\mathbf{A}\mathbf{p} + \mathbf{b} > \mathbf{0}$ for any $\mathbf{p} \gneq \mathbf{0}$, then (26) has a unique solution $(\lambda_*, \mathbf{p}_*)$ with $\lambda_* > 0$ and $\mathbf{p}_* > \mathbf{0}$. Moreover, let $\tilde{\mathbf{f}} : \mathbb{R}_{\geq 0}^{L} \rightarrow \mathbb{R}_{\geq 0}^{L}$ denote the normalized mapping:

$$\tilde{\mathbf{f}}(\mathbf{p}) = \left(\frac{1}{\|\mathbf{A}\mathbf{p} + \mathbf{b}\|}\right)(\mathbf{A}\mathbf{p} + \mathbf{b}). \tag{29}$$

Then $\tilde{\mathbf{f}}^k(\mathbf{p})$ converges geometrically fast to $\mathbf{p}_*$ for all $\mathbf{p} \gneq \mathbf{0}$.

B. Proof of Theorem 1

Proof: We will cast $\mathcal{S}_{\mathrm{U}}$ as a conditional eigenvalue problem and apply Theorem 5. Let us denote by τ_* and $\mathbf{p}_*$ the optimal value and solution of $\mathcal{S}_{\mathrm{U}}$. Recall that $\mathbf{G}$ is irreducible so all the weighted SINR's are equal. Hence, $(\tau_*, \mathbf{p}_*)$ must satisfy

$$(1/\tau_*)\mathbf{p}_* = \mathrm{diag}(\hat{\boldsymbol{\beta}})\mathbf{F}\mathbf{p}_* + \mathrm{diag}(\hat{\boldsymbol{\beta}})\mathbf{n}. \tag{30}$$

Next, we rewrite the power constraints in $\mathcal{S}_{\mathrm{U}}$ as a norm constraint. Define the following weighted maximum norm $\|\cdot\|_{\xi}$ on $\mathbb{R}^L$ and its dual norm $\|\cdot\|_{\xi}^{\mathsf{D}}$:

$$\|\mathbf{p}\|_{\xi} := \max_{j} (1/\bar{P}_j) \sum_{l=1}^{L} w_{jl}|p_l|, \tag{31}$$

$$\|\mathbf{p}\|_{\xi}^{\mathsf{D}} := \max_{\|\mathbf{x}\|_{\xi}=1} |\mathbf{p}^{\mathsf{T}}\mathbf{x}|. \tag{32}$$

One can check that $\|\cdot\|_{\xi}$ satisfies the properties of a monotone norm [30]. We can now cast $\mathcal{S}_{\mathrm{U}}$ as a conditional eigenvalue problem. Since at the optimal solution at least one power constraint must be tight, we must have $\|\mathbf{p}_*\|_{\xi} = 1$. Hence, $\mathcal{S}_{\mathrm{U}}$ is equivalent to finding $(\tau_*, \mathbf{p}_*)$ which satisfies (30) with $\mathbf{p}_* \gneq \mathbf{0}$ and $\|\mathbf{p}_*\|_{\xi} = 1$. Clearly, $\mathrm{diag}(\hat{\boldsymbol{\beta}})\mathbf{n} \neq \mathbf{0}$ and $\mathrm{diag}(\hat{\boldsymbol{\beta}})\mathbf{F}\mathbf{p} + \mathrm{diag}(\hat{\boldsymbol{\beta}})\mathbf{n} > \mathbf{0}$ for any $\mathbf{p} \gneq \mathbf{0}$ so the conditions in Theorem 5 are satisfied. Hence, we conclude that $\mathbf{p}_*$ is unique.

To complete this proof, it remains to show that

$$\max_{\|\mathbf{w}\|_{\xi}^{\mathsf{D}}=1} \rho\left(\mathrm{diag}(\hat{\boldsymbol{\beta}})\left(\mathbf{F} + \mathbf{n}\mathbf{w}^{\mathsf{T}}\right)\right) = \max_{j} \rho\left(\mathrm{diag}(\hat{\boldsymbol{\beta}})\left(\mathbf{F} + \left(1/\bar{P}_j\right)\mathbf{n}\mathbf{w}_j^{\mathsf{T}}\right)\right). \tag{33}$$

Let $\mathbf{w}_*$ denote the optimal solution to the L.H.S. of (33). From Theorem 5, we know that $(\mathbf{p}_*, \mathbf{w}_*)$ is a dual pair, and since $\|\mathbf{p}_*\|_{\xi} = 1$, it follows that $\mathbf{w}_*^{\mathsf{T}}\mathbf{p}_* = 1$. Next, we state two lemmas from [29]:

Lemma 1 ([29]): Let $\mathbf{A}$ be a nonnegative matrix and $\mathbf{b}, \mathbf{c}$ be nonnegative vectors. If $\rho(\mathbf{A} + \mathbf{b}\mathbf{c}^{\mathsf{T}}) > \rho(\mathbf{A})$, then, for any nonnegative vector $\mathbf{d}$,

$$\mathrm{sign}\left(\rho(\mathbf{A} + \mathbf{b}\mathbf{c}^{\mathsf{T}}) - \rho(\mathbf{A} + \mathbf{b}\mathbf{d}^{\mathsf{T}})\right) = \mathrm{sign}(\mathbf{c}^{\mathsf{T}}\mathbf{u} - \mathbf{d}^{\mathsf{T}}\mathbf{u}) \tag{34}$$

where $\mathbf{u} = \mathbf{x}(\mathbf{A} + \mathbf{b}\mathbf{c}^{\mathsf{T}})$ which is the Perron vector of $\mathbf{A} + \mathbf{b}\mathbf{c}^{\mathsf{T}}$.

Lemma 2 ([29]): Let $\mathbf{A}$ be a nonnegative irreducible matrix, and $\mathbf{b}, \mathbf{c} \gneq \mathbf{0}$ be two nonnegative vectors. Then $\rho(\mathbf{A} + \mathbf{b}\mathbf{c}^{\mathsf{T}}) > \rho(\mathbf{A})$.

From the above two lemmas, it follows that

$$\begin{aligned}
&\mathrm{sign}\left(\rho\left(\mathrm{diag}(\hat{\boldsymbol{\beta}})\left(\mathbf{F} + \mathbf{n}\mathbf{w}_*^{\mathsf{T}}\right)\right)\right. \\
&\qquad \left. -\rho\left(\mathrm{diag}(\hat{\boldsymbol{\beta}})\left(\mathbf{F} + (1/\bar{P}_j)\mathbf{n}\mathbf{w}_j^{\mathsf{T}}\right)\right)\right) \\
&\qquad = \mathrm{sign}\left(\mathbf{w}_*^{\mathsf{T}}\mathbf{p}_* - (1/\bar{P}_j)\mathbf{w}_j^{\mathsf{T}}\mathbf{p}_*\right) && (35) \\
&\qquad = \mathrm{sign}\left(1 - (1/\bar{P}_j)\mathbf{w}_j^{\mathsf{T}}\mathbf{p}_*\right). && (36)
\end{aligned}$$

Recall that at the optimal solution at least one power constraint must be tight. Hence, there exists some $k \in \{1, \ldots, J\}$ such that $(1/\bar{P}_k)\mathbf{w}_k^{\mathsf{T}}\mathbf{p}_* = 1$. From (36), we have that

$$\rho\left(\mathrm{diag}(\hat{\boldsymbol{\beta}})\left(\mathbf{F} + \mathbf{n}\mathbf{w}_*^{\mathsf{T}}\right)\right) = \rho\left(\mathrm{diag}(\hat{\boldsymbol{\beta}})\left(\mathbf{F} + (1/\bar{P}_k)\mathbf{n}\mathbf{w}_k^{\mathsf{T}}\right)\right) \tag{37}$$

and hence (33) holds. $\square$

C. Proof of Theorem 2

Proof: The proof follows directly by casting $\mathcal{S}_{\mathsf{U}}$ as a conditional eigenvalue problem (cf. Appendix B) and applying Theorem 6. ☐

D. Proof of Theorem 3

Proof: We give the proof for the result that the minimum weighted SINR increases. The proof that the maximum weighted SINR decreases takes a similar approach.

Let $\mathbf{f}:\mathbb{R}^L_{\geq 0}\rightarrow\mathbb{R}^L_{\geq 0}$ be defined by $\mathbf{f}(\mathbf{p})=\operatorname{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}+\mathbf{n})$ and $\|\cdot\|_\xi$ be defined by (31). Then Algorithm 1 can be written as

$$\mathbf{p}^{(n+1)}=\left(1\bigg/\left\|\mathbf{f}\left(\mathbf{p}^{(n)}\right)\right\|_\xi\right)\mathbf{f}(\mathbf{p}^{(n)}) \tag{38}$$

and the minimum weighted SINR at step n is given by $\min_l(p_l^{(n)}/f_l(\mathbf{p}^{(n)}))$. The result that this is strictly increasing (unless the algorithm has converged) follows directly from the next lemma.

Lemma 3: Let $\mathbf{f}:\mathbb{R}^L_{\geq 0}\rightarrow\mathbb{R}^L_{\geq 0}$ be a monotone and concave mapping, i.e., $\mathbf{0}\leq\mathbf{x}\leq\mathbf{y}$ implies $\mathbf{0}\leq\mathbf{f}(\mathbf{x})\leq\mathbf{f}(\mathbf{y})$ and $\mathbf{f}(\alpha\mathbf{x}+(1-\alpha)\mathbf{y})\geq\alpha\mathbf{f}(\mathbf{x})+(1-\alpha)\mathbf{f}(\mathbf{y})$ for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^L_{\geq 0}$ and all $\alpha\in[0,1]$. Moreover, suppose $\mathbf{f}(\mathbf{0})>\mathbf{0}$. Let $\|\cdot\|$ be a monotone norm on $\mathbb{R}^L$. Consider the iteration defined by

$$\mathbf{p}^{(n+1)}=\left(1\bigg/\left\|\mathbf{f}\left(\mathbf{p}^{(n)}\right)\right\|\right)\mathbf{f}(\mathbf{p}^{(n)}) \tag{39}$$

and let

$$\tau^{(n)}=\min_l\frac{p_l^{(n)}}{f_l\left(\mathbf{p}^{(n)}\right)}. \tag{40}$$

Then, for all n, we have that

$$\tau^{(n+1)}\geq\tau^{(n)} \tag{41}$$

with equality if and only if $\tau^{(n)}=p_l^{(n)}/f_l(\mathbf{p}^{(n)})$ for all l.

Proof: Consider first the case where $\tau^{(n)}<p_l^{(n)}/f_l(\mathbf{p}^{(n)})$ for some l. We shall denote by $\mathbf{x}\geqq\mathbf{y}$ the component-wise inequality between two vectors $\mathbf{x}\in\mathbb{R}^L$ and $\mathbf{y}\in\mathbb{R}^L$ such that equality holds in at least one component. From the definition of $\tau^{(n)}$ and $\mathbf{p}^{(n+1)}$, we have that

$$\begin{aligned}\mathbf{p}^{(n)}&\geqq\tau^{(n)}\mathbf{f}(\mathbf{p}^{(n)}) &(42)\\ &=\tau^{(n)}\left\|\mathbf{f}\left(\mathbf{p}^{(n)}\right)\right\|\mathbf{p}^{(n+1)}. &(43)\end{aligned}$$

Taking the norm on both sides of (43), we have that $\tau^{(n)}\|\mathbf{f}(\mathbf{p}^{(n)})\|<1$. We can write

$$\begin{aligned}\mathbf{p}^{(n+1)}&=\left(1\bigg/\left\|\mathbf{f}\left(\mathbf{p}^{(n)}\right)\right\|\right)\mathbf{f}\left(\mathbf{p}^{(n)}\right) &(44)\\ &\geq\left(1\bigg/\left\|\mathbf{f}\left(\mathbf{p}^{(n)}\right)\right\|\right)\mathbf{f}\left(\tau^{(n)}\left\|\mathbf{f}\left(\mathbf{p}^{(n)}\right)\right\|\mathbf{p}^{(n+1)}\right) &(45)\\ &>\tau^{(n)}\mathbf{f}\left(\mathbf{p}^{(n+1)}\right). &(46)\end{aligned}$$

Here, the first inequality follows from the definition of $\mathbf{p}^{(n+1)}$. The second inequality follows from (43) and the monotonicity of $\mathbf{f}$. The third inequality follows from the concavity of $\mathbf{f}$, the fact that $\tau^{(n)}\|\mathbf{f}(\mathbf{p}^{(n)})\|<1$, and the fact that $\mathbf{f}(\mathbf{0})>\mathbf{0}$. Now, further substituting the definition of $\mathbf{p}^{(n+2)}$, we have

$$\mathbf{p}^{(n+1)}>\tau^{(n)}\left\|\mathbf{f}\left(\mathbf{p}^{(n+1)}\right)\right\|\mathbf{p}^{(n+2)}. \tag{47}$$

Observe that (43) rewritten at the $(n+1)$th step gives

$$\mathbf{p}^{(n+1)}\geqq\tau^{(n+1)}\left\|f\left(\mathbf{p}^{(n+1)}\right)\right\|\mathbf{p}^{(n+2)}. \tag{48}$$

Equations (47) and (48) imply that we must have $\tau^{(n+1)}>\tau^{(n)}$ which is our desired result.

The case for equality in (41) can be shown by applying the same arguments where $\tau^{(n)}=p_l^{(n)}/f_l(\mathbf{p}^{(n)})$ for all l. ☐

Since $\mathbf{f}(\mathbf{p})=\operatorname{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}+\mathbf{n})$ is a monotone and concave mapping with $\mathbf{f}(\mathbf{0})>\mathbf{0}$ and $\|\cdot\|_\xi$ is a monotone norm on $\mathbb{R}^L$, our result follows directly from Lemma 3. ☐

E. Uplink-Downlink Duality via Geometric Program (GP)

In this section, we show that uplink-downlink duality can be derived from the GP reformulation of the max-min weighted SINR problem. Formally, we state uplink-downlink duality for the context of the max-min weighted SINR problem in the following theorem.

Theorem 7 ([16], [21], [23]–[26]): Consider the following downlink power control problem with one power constraint:

$$\mathcal{S}_{\mathsf{DL}}=\begin{cases}\max\limits_{\mathbf{p}} & \min\limits_{l}\frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p})}{\beta_l}\\ \text{subject to} & \mathbf{w}^{\mathsf{T}}\mathbf{p}\leq\bar{P},\quad\mathbf{p}>\mathbf{0}\end{cases} \tag{49}$$

where

$$\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p})=\frac{p_lG_{ll}}{\sum_{i\neq l}q_iG_{li}+n_l} \tag{50}$$

is the downlink SINR. The optimal value $\mathcal{S}_{\mathsf{DL}}$ is equal to the optimal value of an uplink power control problem given by

$$\mathcal{S}_{\mathsf{UL}}=\begin{cases}\max\limits_{\mathbf{q}} & \min\limits_{l}\frac{\mathsf{SINR}_l^{\mathsf{UL}}(\mathbf{q})}{\beta_l}\\ \text{subject to} & \mathbf{n}^{\mathsf{T}}\mathbf{q}\leq\bar{P},\quad\mathbf{q}>\mathbf{0}\end{cases} \tag{51}$$

where

$$\mathsf{SINR}_l^{\mathsf{UL}}(\mathbf{q})=\frac{q_lG_{ll}}{\sum_{i\neq l}q_iG_{il}+w_l} \tag{52}$$

is the uplink SINR.

Proof: First, we obtain necessary and sufficient conditions for the optimal solution to $\mathcal{S}_{\mathsf{DL}}$. These conditions were previously derived in [19]. However, we include the derivation for completeness. Introducing an auxiliary variable τ, we can rewrite $\mathcal{S}_{\mathsf{DL}}$ in epigraph form as

$$\mathcal{S}_{\mathsf{DL}}=\begin{cases}\max\limits_{\tau,\mathbf{p}} & \tau\\ \text{subject to} & \tau\leq\frac{p_l}{(\operatorname{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}+\mathbf{n}))_l}\\ & \mathbf{w}^{\mathsf{T}}\mathbf{p}\leq\bar{P},\quad\mathbf{p}>\mathbf{0}.\end{cases} \tag{53}$$

By making a change of variables $\tilde{\tau} = \log \tau$ and $\tilde{p}_l = \log p_l$ for all l, we arrive at the following equivalent convex problem:

$$\begin{array}{ll} \min\limits_{\tilde{\tau}, \tilde{\mathbf{p}}} & -\tilde{\tau} \\ \text{subject to} & \log\left(\dfrac{e^{\tilde{\tau}}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}e^{\tilde{\mathbf{p}}}+\mathbf{n}))_l}{(e^{\tilde{\mathbf{p}}})_l}\right) \leq 0, \quad l=1,\ldots,L \\ & \log\left(\dfrac{1}{\bar{P}}\mathbf{w}^{\mathsf{T}}e^{\tilde{\mathbf{p}}}\right) \leq 0. \end{array} \tag{54}$$

The Lagrangian associated with (54) is

$$\mathcal{L}(\tilde{\tau}, \tilde{\mathbf{p}}, \boldsymbol{\lambda}, \mu) := -\tilde{\tau} + \sum_{l=1}^{L} \lambda_l \log\left(\frac{e^{\tilde{\tau}}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}e^{\tilde{\mathbf{p}}}+\mathbf{n}))_l}{(e^{\tilde{\mathbf{p}}})_l}\right) + \mu \log\left(\frac{1}{\bar{P}}\mathbf{w}^{\mathsf{T}}e^{\tilde{\mathbf{p}}}\right) \tag{55}$$

where λ_l and μ are the nonnegative Lagrange dual variables.

It is easy to check that the convex problem given by (54) satisfies Slater's condition. Hence, the Karush-Kuhn Tucker (KKT) conditions of (54) are necessary and sufficient conditions for the optimal solution. Recall that, at the optimal solution to (54), all the inequality constraints must be active. Hence, we can replace the inequality constraints in the primal feasibility conditions with equality, and drop the complementary slackness conditions. Making a change of variables back into τ and $\mathbf{p}$, we arrive at the following necessary and sufficient conditions for the optimal solution to $\mathcal{S}_{\mathsf{DL}}$:

$$\tau_* = \frac{p_{*l}}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}_* + \mathbf{n}))_l}, \quad l = 1, \ldots, L \tag{56}$$

$$\mathbf{w}^{\mathsf{T}}\mathbf{p}_* = \bar{P} \tag{57}$$

$$\tau_* = \frac{q_{*l}}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}^{\mathsf{T}}\mathbf{q}_* + \mathbf{w}))_l}, \quad l = 1, \ldots, L \tag{58}$$

$$\mathbf{1}^{\mathsf{T}}\boldsymbol{\lambda}_* = 1 \tag{59}$$

$$\mathbf{q}_* = \frac{\tau_* \bar{P}}{\mu_*} \cdot \left(\frac{\lambda_{*1}\hat{\beta}_1}{p_{*1}}, \ldots, \frac{\lambda_{*L}\hat{\beta}_L}{p_{*L}}\right)^{\mathsf{T}} \tag{60}$$

$$\lambda_{*l} > 0, \quad l = 1, \ldots, L \tag{61}$$

$$\mu_* > 0. \tag{62}$$

Here, $(\tau_*, \mathbf{p}_*)$ and $(\boldsymbol{\lambda}_*, \mu_*)$ are the optimal primal and dual variables. Here, (58) follows from $\partial\mathcal{L}/\partial\tilde{p}_l = 0$, (59) follows from $\partial\mathcal{L}/\partial\tilde{\tau} = 0$, and (61)–(62) follow from the fact that λ_{*l} and μ_* must be strictly positive in order for $\tilde{p}_{*l}$ and $\mathcal{L}(\tilde{\tau}_*, \tilde{\mathbf{p}}_*, \boldsymbol{\lambda}_*, \mu_*)$ to be bounded from below. Hence, the auxiliary variable $\mathbf{q}_*$ is strictly positive.

Observe that $\mathbf{F}^{\mathsf{T}}$ is the channel matrix in the dual uplink network. Hence, from (58), we conclude that $\mathbf{q}_*$ is the optimal dual uplink power and $\mathbf{w}$ is the noise vector in the dual uplink network [23], [25]. To obtain the equivalent power constraint in the dual uplink network, we first rewrite (58) in vector form as

$$\mathbf{F}^{\mathsf{T}}\mathbf{q}_* + \mathbf{w} = \frac{1}{\tau_*}\mathrm{diag}(\hat{\boldsymbol{\beta}})^{-1}\mathbf{q}_*. \tag{63}$$

Then, taking the inner product with $\mathbf{p}_*$, and substituting for $\mathbf{F}\mathbf{p}_*$ and $\mathbf{w}^{\mathsf{T}}\mathbf{p}_*$ from (56) and (57) respectively, we have that

$$\mathbf{n}^{\mathsf{T}}\mathbf{q}_* = \bar{P}. \tag{64}$$

Hence, (56)–(62) are equivalent to the following conditions

$$\tau_* = \frac{q_{*l}}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}^{\mathsf{T}}\mathbf{q}_* + \mathbf{w}))_l}, \quad l = 1, \ldots, L \tag{65}$$

$$\mathbf{n}^{\mathsf{T}}\mathbf{q}_* = \bar{P} \tag{66}$$

$$\tau_* = \frac{p_{*l}}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}_* + \mathbf{n}))_l}, \quad l = 1, \ldots, L \tag{67}$$

$$\mathbf{1}^{\mathsf{T}}\boldsymbol{\lambda}_* = 1 \tag{68}$$

$$\mathbf{p}_* = \frac{\tau_* \bar{P}}{\mu_*} \cdot \left(\frac{\lambda_{*1}\hat{\beta}_1}{q_{*1}}, \ldots, \frac{\lambda_{*L}\hat{\beta}_L}{q_{*L}}\right)^{\mathsf{T}} \tag{69}$$

$$\lambda_{*l} > 0, \quad l = 1, \ldots, L \tag{70}$$

$$\mu_* > 0. \tag{71}$$

Here, we replaced the downlink power constraint (57) with the uplink power constraint (66). Now, observe that (65)–(69) can be obtained from (56)–(60) via the following substitutions:

$$\begin{array}{c}\text{Uplink}-\text{Downlink} \\ \text{Duality Mapping}\end{array} : \begin{cases} \mathbf{p} \longleftrightarrow \mathbf{q} \\ \mathbf{w} \longleftrightarrow \mathbf{n} \\ \mathbf{n} \longleftrightarrow \mathbf{w} \\ \mathbf{F} \longleftrightarrow \mathbf{F}^{\mathsf{T}} \\ \bar{P} \longleftrightarrow \bar{P}. \end{cases} \tag{72}$$

Hence, (65)–(71) are necessary and sufficient conditions for the optimal solution to $\mathcal{S}_{\mathsf{UL}}$ where the parameters and variables in $\mathcal{S}_{\mathsf{UL}}$ are defined by the mappings in (72). □

Observe that the optimal dual variables in the downlink and uplink problems have the same value. Moreover, the dual and primal variables are related by

$$\boldsymbol{\lambda}_* = \frac{\mu_*}{\tau_* \bar{P}_j}\mathbf{p}_* \circ \mathrm{diag}(\hat{\boldsymbol{\beta}})^{-1}\mathbf{q}_* \tag{73}$$

which agrees with (21) derived from nonlinear Perron-Frobenius minimax characterization.

F. Proof of Theorem 4

Proof: First, consider the set of feasible values for the objective in $\mathcal{S}$:

$$\mathcal{T}^{\mathsf{DL}} = \left\{ \min_l \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}, \mathbb{U})}{\beta_l} \;\middle|\; \begin{array}{c}\mathbf{p} \geq \mathbf{0},\ \mathbf{w}_j^{\mathsf{T}}\mathbf{p} \leq \bar{P}_j\, \forall\, j, \\ \|\mathbf{u}_l\| \leq 1\, \forall\, l\end{array} \right\}. \tag{74}$$

Notice that we have expanded the set of feasible values to include $p_l = 0$. This is allowed because $p_l = 0$ implies that $\mathsf{SINR}_l(\mathbf{p}, \mathbb{U}) = 0$ which can never be an optimal solution. Also notice that the equality norm constraint on the beamformers in $\mathcal{S}$ has been replaced by an inequality constraint. This is allowed since any set of SINR's which are achievable using beamformers with less than unit norm can also be achieved by scaling the beamformers up to unit norm and scaling the transmit powers down accordingly. The latter transmit powers will always satisfy the original power constraint. Since the feasible set is compact, and $\min_l(\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p},\mathbb{U})/\beta_l)$ is continuous in $\mathbf{p}$ and $\mathbb{U}$, it follows that $\mathcal{S} = \max \mathcal{T}^{\mathsf{DL}}$ is well-defined.

Now, let $\mathcal{T}^{\mathsf{UL},j}$ denote the set of feasible values for the objective of an uplink problem with noise vector $\mathbf{w}_j$, power constraint $\mathbf{n}^\mathsf{T}\mathbf{q} \le \bar{P}_j$, and beamforming constraint $\|\mathbf{u}_l\| \le 1$:

$$\mathcal{T}^{\mathsf{UL},j} = \left\{ \min_l \frac{\mathsf{SINR}_l^{\mathsf{UL},j}(\mathbf{q},\mathbb{U})}{\beta_l} \;\middle|\; \begin{array}{c} \mathbf{q} \ge \mathbf{0},\, \mathbf{n}^\mathsf{T}\mathbf{q} \le \bar{P}_j, \\ \|\mathbf{u}_l\| \le 1\,\forall\, l \end{array} \right\} \tag{75}$$

where

$$\mathsf{SINR}_l^{\mathsf{UL},j}(\mathbf{q},\mathbb{U}) = \frac{q_l G_{ll}}{\sum_{i\ne l} q_i G_{il} + w_{jl}} \tag{76}$$

is the uplink SINR. By uplink-downlink duality (cf. Theorem 7), we have that $\mathcal{T}^{\mathsf{DL}} = \cap_{j=1}^J \mathcal{T}^{\mathsf{UL},j}$. Let $\tau_j = \max \mathcal{T}^{\mathsf{UL},j}$. Observe that $\mathcal{T}^{\mathsf{UL},j} = [0,\tau_j]$. Hence,

$$\begin{aligned} \mathcal{S} &= \max \cap_{j=1}^J [0,\tau_j] && (77)\\ &= \min\{\tau_j : j = 1,\ldots,J\} && (78)\\ &= \min_j \max \mathcal{T}^{\mathsf{UL},j}. && (79) \end{aligned}$$

This last equation shows that $\mathcal{S}$ can be decoupled into separate uplink problems. Finally, by applying uplink-downlink duality in the other direction, we have that $\mathcal{S} = \min\{\mathcal{S}^j : j = 1,\ldots,J\}$ so $\mathcal{S}$ can be decoupled into separate downlink problems each involving one of its J power constraints.

The uniqueness of the solution $\mathbf{p}_*$ and $\mathbb{U}_*$ follow from the uniqueness of the solution to $\mathcal{S}^j$ (Theorem 1 in [19]). □

G. Proof of Corollary 2

Proof: Observe that the solution to $\mathcal{S}$ is always in the feasible set of the max-min problem defined in $g(\boldsymbol{\theta})$. Hence, $g(\boldsymbol{\theta})$ is an upper-bound to the optimal value of $\mathcal{S}$. From Theorem 4, it follows that $\mathcal{S} = g(\mathbf{e}_k)$ where $\mathbf{e}_k \in \mathbb{R}^J$ denotes the unit vector with a 1 in its kth entry and 0 everywhere else. Hence, the upper-bound is tight and so $\mathcal{S} = \min\{g(\boldsymbol{\theta}) : \boldsymbol{\theta} \ge \mathbf{0}\}$. □

H. Proof That $g(\boldsymbol{\theta})$ Is Quasiconvex

In this Appendix, we prove that $g(\boldsymbol{\theta})$ is quasiconvex. Given $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathbb{R}^J$ and $\lambda \in \mathbb{R}$ such that $0 < \lambda < 1$, we show that

$$g(\lambda\boldsymbol{\theta}_1 + (1-\lambda)\boldsymbol{\theta}_2) \le \max\{g(\boldsymbol{\theta}_1), g(\boldsymbol{\theta}_2)\}. \tag{80}$$

Let $\mathbf{p}_*$ and $\mathbb{U}_*$ denote the optimal power and beamformers in $g(\lambda\boldsymbol{\theta}_1 + (1-\lambda)\boldsymbol{\theta}_2)$. Note that

$$\left(\lambda \sum_{j=1}^J \theta_{1j}\mathbf{w}_j^\mathsf{T} + (1-\lambda)\sum_{j=1}^J \theta_{2j}\mathbf{w}_j^\mathsf{T}\right)\mathbf{p}_* \le \lambda \sum_{j=1}^J \theta_{1j}\bar{P}_j + (1-\lambda)\sum_{j=1}^J \theta_{2j}\bar{P}_j. \tag{81}$$

Hence, at least one of the two inequalities

$$\sum_{j=1}^J \theta_{1j}\mathbf{w}_j^\mathsf{T}\mathbf{p}_* \le \sum_{j=1}^J \theta_{1j}\bar{P}_j \tag{82}$$

or

$$\sum_{j=1}^J \theta_{2j}\mathbf{w}_j^\mathsf{T}\mathbf{p}_* \le \sum_{j=1}^J \theta_{2j}\bar{P}_j \tag{83}$$

is true. In other words, $\mathbf{p}_*$ is a feasible power vector for one of the problems $g(\boldsymbol{\theta}_1)$ or $g(\boldsymbol{\theta}_2)$. Without loss of generality, suppose that (82) is true so $\mathbf{p}_*$ is feasible for $g(\boldsymbol{\theta}_1)$. Define

$$\tilde{\mathbf{p}} = \left(\frac{\sum_{j=1}^J \theta_{1j}\bar{P}_j}{\sum_{j=1} \theta_{1j}\mathbf{w}_j^\mathsf{T}\mathbf{p}_*}\right)\mathbf{p}_*. \tag{84}$$

Note that $\tilde{\mathbf{p}}$ is also feasible for $g(\boldsymbol{\theta}_1)$. We have that

$$\begin{aligned} g(\lambda\boldsymbol{\theta}_1 + (1-\lambda)\boldsymbol{\theta}_2) &= \min_l \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}_*,\mathbb{U}_*)}{\beta_l} && (85)\\ &\le \min_l \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\tilde{\mathbf{p}},\mathbb{U}_*)}{\beta_l} && (86)\\ &\le g(\boldsymbol{\theta}_1). && (87) \end{aligned}$$

Here, the second inequality follows from the fact that $\tilde{\mathbf{p}} = \alpha\mathbf{p}_*$ where $\alpha \ge 1$ and the third inequality follows from the problem definition of $g(\boldsymbol{\theta}_1)$ and the fact that $\tilde{\mathbf{p}}$ is feasible for $g(\boldsymbol{\theta}_1)$.

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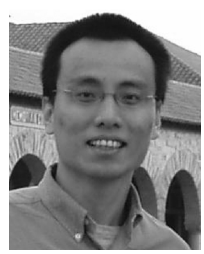

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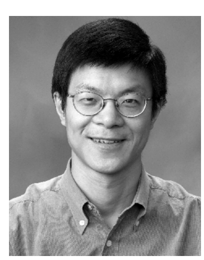

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