Fast Algorithms and Performance Bounds for Sum Rate Maximization in Wireless Networks

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***Abstract*—In this paper, we consider a wireless network where interference is treated as noise, and we study the nonconvex problem of sum rate maximization by power control. We focus on finding approximately optimal solutions that can be efficiently computed to this NP-hard problem by studying the solutions to two related problems, the sum rate maximization using a signal-to-interference-plus-noise ratio (SINR) approximation and the max-min weighted SINR optimization. We show that these two problems are intimately connected, can be solved efficiently by algorithms with fast convergence and minimal parameter configuration, and can yield high-quality approximately optimal solutions to sum rate maximization in the low interference regime. As an application of these results, we analyze the connection-level stability of cross-layer utility maximization in the wireless network, where users arrive and depart randomly and are subject to congestion control, and the queue service rates at all the links are determined by the sum rate maximization problem. In particular, we determine the stability region when all the links solve the max-min weighted SINR problem, using instantaneous queue sizes as weights.**

***Index Terms*—Distributed optimization, duality, nonnegative matrix theory, power control, wireless networks.**

I. Introduction

CROSS-LAYER resource allocation problems in wireless networks often can be formulated as optimization problems whose objective functions, together with suitably formulated constraint sets, capture the dependence on competing network resource under interference. For example, in wireless code division multiple access (CDMA) networks, communication links adjust their transmission powers over a shared wireless medium at the physical layer [2]. However, the objectives and constraint sets in the wireless power control optimization problems are often nonconvex, thus making it hard to solve efficiently for the global optimal solution [2]–[10].

Manuscript received May 31, 2011; revised March 03, 2012 and June 14, 2012; accepted June 30, 2012; approved by IEEE/ACM TRANSACTIONS ON NETWORKING Editor M. Liu. Date of publication August 02, 2012; date of current version June 12, 2013. This work was supported in part by the AFOSR under Grant FA9550-09-1-0643, the ONR under Grant N00014-07-1-0864, the NSF under Grant CNS 0720570, the ARO under MURI Award W911NF-08-1-0233, the Research Grants Council of Hong Kong under Project No. RGC CityU 125212, and Qualcomm, Inc. The material in this paper was presented in part at the IEEE International Conference on Computer Communications (INFOCOM), Rio de Janeiro, Brazil, April 19–25, 2009.

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Digital Object Identifier 10.1109/TNET.2012.2210240

The following problem of sum rate maximization through power control has been extensively studied in wireless and digital-subscriber-line wireline network design (e.g., a partial list of recent work includes [2]–[10])

$$\begin{array}{ll} \text{maximize} & \sum_l w_l \log(1+\mathsf{SINR}_l(\mathbf{p})) \\ \text{subject to} & 0 \le p_l \le \bar{p}_l \qquad \forall\, l \\ \text{variables}: & p_l \qquad \forall\, l \end{array} \tag{1}$$

where p_l is the transmit power, w_l is some positive weight assigned by the network to the lth link (to reflect some long-term priority), and $\mathsf{SINR}_l(\mathbf{p})$ is the signal-to-interference-plus-noise ratio (SINR), which is a linear-fractional function of $\mathbf{p}$ [cf. (2) in the paper]. Without loss of generality, we assume that $\mathbf{w}$ is a probability vector. Let us denote the optimal solution to (1) by $\mathbf{p}^\star$. An achievable data rate r_l for the lth link is then given by $r_l = \log(1+\mathsf{SINR}_l(\mathbf{p}^\star))$.

This optimization problem is difficult to solve: (1) is nonconvex and NP-hard [9]. Moreover, (1) may even be hard to approximate [9]. Finding $\mathbf{p}^\star$ in a distributed fashion is also difficult. Most existing algorithms cannot guarantee the suboptimality of their solution. For example, algorithms based on Lagrange duality suffers from the positive duality gap (finding the optimal Lagrange dual and a feasible primal solution from a feasible dual are both hard). In [3] and [8], an approximate solution to (1) can be computed efficiently using geometric programming (GP). In [4] and [5], binary power allocation is proposed as an approximate solution. However, these approximate solutions' performance can be hard to quantify. It is therefore interesting to identify good-quality approximately optimal solutions that can be computed efficiently and in a distributed manner, especially for large-scale problems of (1).

A closely related problem is the characterization of the feasible rate region, i.e., the r_l for all l (or the SINR region) over the feasible powers. The resulting Pareto-optimal rate region (obtained by varying $\mathbf{w}$ and solving for $\mathbf{p}^\star$ accordingly) is in general nonconvex. Recently, the authors in [11] showed that the feasible SINR region can be characterized through solving an (infinite number of) max-min weighted SINR problems, each of which can be solved efficiently by GP [3]. An analytical form for the Pareto rate region was derived in [12]. It is, however, unclear how these optimization problems—i.e., (1), the max-min weighted SINR problem, and the GP approximation in [3] and [8]—are related. This paper investigates these connections and leverages them to find computationally fast algorithms to solve (1).

This paper aims at answering the following questions, related to solving (1) as well as its application to understanding cross-layer optimization involving transmit powers.

- Can some related and efficiently solvable problems provide high-quality approximately optimal solutions to (1)? We show how to solve (1) by: 1) approximating the function that describes rate as a function of SINR; and 2) by solving the max-min weighted SINR problem in Section III (Theorems 1, 3, and 4). We derive fixed-point algorithms that are configuration-free and converge quickly. We then quantify their suboptimality performance with respect to $\mathbf{p}^\star$ of (1) in Section IV (Theorems 5 and 6).
- Cross-layer algorithms involving congestion control and maxweight optimal power control have been proposed in [8] and [13]–[18] to provide fair resource allocation in wireless networks. Therefore, it is interesting to assess the impact of suboptimal power control on the stability region in cross-layer optimization, i.e., how stable are cross-layer algorithms when the links adjust their powers by solving (1)? We extend our recent results characterizing the feasible rate region in [1] to answer this question in the regime when interference cross-channel gains or the signal-to-noise ratio (SNR) are sufficiently small.

Overall, the contributions of the paper are as follows.

1) We start with the weighted sum rate maximization problem. Then, as in past work based on GP [3], we use convex approximation by approximating $\log(1+\mathsf{SINR})$ by $\log \mathsf{SINR}$, followed by a change of variables. At this point, we make a departure and propose an asynchronous iterative algorithm to solve this SINR approximation power control (SAPC) problem.
2) Then, we consider the max-min weighted SINR problem that we solve analytically. We connect it to the SAPC problem, thus motivating a fast two-timescale algorithm. We also derive a synchronous algorithm with geometrically fast convergence that solves the max-min weighted SINR problem by exploiting a connection to the nonlinear Perron–Frobenius theory.
3) We apply our results to analyze the connection-level stability of flows for utility maximization in a wireless network when the sending rates are subject to congestion control. This is achieved by an imperfect scheduling policy that suboptimally solves the weighted sum rate maximization by using instantaneous queue lengths as weights in our max-min weighted SINR algorithm.

This paper is organized as follows. We present the system model in Section II. We look at two power control problems and derive fast algorithms to solve them in Section III. In Section IV, we sharpen and apply a variety of tools from nonnegative matrix theory to derive performance guarantees of our algorithms in maximizing sum rates. In Section V, we apply our results and algorithms to investigate the connection-level stability of a utility maximization problem with congestion control. We conclude with a summary in Section VI. All the proofs are found in the Appendix.

The following notation is used. Boldface uppercase letters denote matrices, boldface lowercase letters denote column vectors, and $\mathbf{u} \geq \mathbf{v}$ denotes componentwise inequality between vectors $\mathbf{u}$ and $\mathbf{v}$. We also let $(\mathbf{By})_l$ denote the lth element of $\mathbf{By}$. Let $\mathbf{x} \circ \mathbf{y}$ denote the Schur product of the vectors $\mathbf{x}$ and $\mathbf{y}$, i.e., $\mathbf{x} \circ \mathbf{y} = [x_1 y_1, \ldots, x_L y_L]^\top$. Let $\|\mathbf{w}\|_\infty^{\mathbf{x}}$ be the weighted maximum norm of the vector $\mathbf{w}$ with respect to the weight $\mathbf{x}$, i.e., $\|\mathbf{w}\|_\infty^{\mathbf{x}} = \max_l w_l / x_l, \mathbf{x} > \mathbf{0}$. We write $\mathbf{B} \geq \mathbf{F}$ if $B_{ij} \geq F_{ij}$ for all i, j. The Perron–Frobenius eigenvalue of a nonnegative matrix $\mathbf{F}$ is denoted as $\rho(\mathbf{F})$, and the Perron right and left eigenvector of $\mathbf{F}$ associated with $\rho(\mathbf{F})$ are denoted by $\mathbf{x}(\mathbf{F}) \geq \mathbf{0}$ and $\mathbf{y}(\mathbf{F}) \geq \mathbf{0}$ (or simply $\mathbf{x}$ and $\mathbf{y}$ when the context is clear), respectively. Recall that the Perron–Frobenius eigenvalue of $\mathbf{F}$ is the eigenvalue with the largest absolute value. Assume that $\mathbf{F}$ is a nonnegative irreducible matrix. Then, $\rho(\mathbf{F})$ is simple and positive, and $\mathbf{x}(\mathbf{F}), \mathbf{y}(\mathbf{F}) > \mathbf{0}$ [19]. We will assume the normalization $\mathbf{x}(\mathbf{F}) \circ \mathbf{y}(\mathbf{F})$ is a probability vector. The superscript $(\cdot)^\top$ denotes transpose. We denote $\mathbf{e}_l$ as the lth unit coordinate vector and $\mathbf{I}$ as the identity matrix.

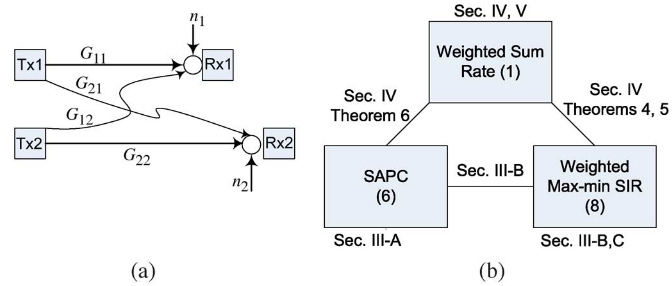

Fig. 1. (a) Interference channel model for two links. (b) Overview of the connection between the three main optimization problems in the respective sections: i) weighted sum rate maximization in (1); ii) SAPC in (6); iii) max-min weighted SINR in (8). The respective sections and theorems connecting the optimization problems are also indicated.

II. Review of a Standard System Model

We consider a wireless network modeled by the Gaussian interference channel, and all the L links (equivalently, transceiver pairs) treat multiuser interference as noise, i.e., no interference cancellation. This is a commonly used model (in, e.g., [2], [3], [5], [7], [8], and [20]) for wireless CDMA cellular and ad hoc networks. Fig. 1(a) shows the interference channel model for two links.

The transmit power for the lth link is denoted by p_l for all l. Assuming a matched-filter receiver, the SINR for the lth receiver is given by

$$\mathsf{SINR}_l(\mathbf{p}) = \frac{G_{ll} p_l}{\sum_{j \neq l} G_{lj} p_j + n_l} \tag{2}$$

where G_{lj} are the channel gains from transmitter j to receiver l and n_l is the additive white Gaussian noise (AWGN) power for the lth receiver. The channel gain matrix $\mathbf{G}$ takes into account propagation loss, spreading loss and other transmission modulation factors. Assuming a fixed bit error rate (BER) at the receiver, the Shannon capacity formula can be used to deduce the achievable data rate of the lth link as [8], [18]

$$\log\left(1 + \frac{\mathsf{SINR}_l(\mathbf{p})}{\Gamma}\right) \quad \text{nats/symbol} \tag{3}$$

where Γ is the gap to capacity, which is always greater than 1 and models the nonideal coding and modulation schemes used. We absorb $(1/\Gamma)$ into G_{ll} for all l and write the achievable data rate as $\log(1+\mathsf{SINR}_l(\mathbf{p}))$ nats/symbol.

Next, we define a nonnegative matrix $\mathbf{F}$ with entries

$$F_{lj} = \begin{cases} 0, & \text{if } l = j \\ \frac{G_{lj}}{G_{ll}}, & \text{if } l \neq j \end{cases} \tag{4}$$

and

$$\mathbf{v} = \left(\frac{n_1}{G_{11}}, \frac{n_2}{G_{22}}, \ldots, \frac{n_L}{G_{LL}}\right)^\top. \tag{5}$$

Moreover, we assume that $\mathbf{F}$ is irreducible, i.e., each link has at least an interferer. Using this matrix notation, the SINR for the lth link can be expressed as

$$\mathsf{SINR}_l(\mathbf{p}) = \frac{p_l}{(\mathbf{Fp})_l + v_l}.$$

In addition, the lth link has a maximum power constraint $\bar{p}_l$ and a maximum constant SNR of $\bar{p}_l / v_l$.

III. Fast Power Control Algorithms

In this section, we consider two widely studied power control problems. The first one we consider is a direct approximation to (1), i.e., maximize a weighted sum of link rates by first approximating the link rate expression $\log(1 + \mathsf{SINR})$ by $\log(\mathsf{SINR})$, perform a change of variables to convexify the problem, and then use a dual iterative method to solve the problem [3], [8]. We call such an approximation method SAPC. Here, we show that the dual iterative method can be replaced by an alternative procedure that leads to faster convergence. Specifically, we show that the approximation to the link rate expression can be viewed in terms of standard interference function introduced in [21], which allows us to use *greedy* algorithm. Both theoretical analysis and numerical results confirm that this approach leads to much faster convergence than the dual iterative method.

Another problem we consider in this section is the widely studied max-min weighted SINR power control problem where the goal is to maximize the minimum weighted SINR at any receiver in the network. By studying its Lagrangian dual problem, we observe an interesting connection between the max-min weighted SINR problem and SAPC, which allows us to use the interference function approach mentioned earlier as an intermediate step to solve the Lagrange dual of the max-min weighted SINR problem under asynchronous updates. The asynchronous iteration allows some links to execute more iterations than others and allows links to perform these updates using outdated information from the other links. We also propose a synchronous algorithm (where all updates are performed synchronously in steps by all links) when the maximum powers of all the users are equal.

In addition, we derive theoretical results that are of interest in their own right. We first derive a closed-form expression for the max-min weighted power control problem, and then derive conditions under which the max-min SINR problem and SAPC have the same solution. These results are shown by exploiting a connection between power-control problems and nonnegative matrix theory. Fig. 1(b) overviews the connections between these main optimization problems in the respective sections.

A. SAPC

In this section, we consider the following SINR approximation to (1):

$$\begin{array}{ll} \text{maximize} & \sum_l w_l \log \mathsf{SINR}_l(\mathbf{p}) \\ \text{subject to} & 0 \le p_l \le \bar{p}_l \quad \forall\, l \\ \text{variables :} & p_l \quad \forall\, l. \end{array} \tag{6}$$

It is well known that problem (6) can be turned into a GP [3], [8]. We denote $\mathbf{p}'$ as the optimal solution to (6). In particular, by making a change of variable, i.e., $\tilde{p}_l = \log p_l$ for all l, (6) is convex in $\tilde{\mathbf{p}}$ and thus $\tilde{p}'_l = \log p'_l$ for all l [8]. To solve (6) optimally, there are dual iterative algorithms (requiring step-size tuning) in [8], which are based on the dynamical system $\partial \mathbf{p}/\partial t = f(\mathbf{p})$ for some function f after changing the variables in $\tilde{\mathbf{p}}$ to $\mathbf{p}$.

A different approach is to derive a fixed-point iteration, i.e., $\mathbf{p} = g(\mathbf{p})$ for some function g based on the Karush–Kuhn–Tucker (KKT) optimality conditions (see [22]). Now, it can be verified (as shown in the Appendix) that the KKT conditions of the convex form of (6) in the variables $\tilde{\mathbf{p}}$ can be rewritten in terms of $\mathbf{p}$ as $\mathbf{p} = \min(I(\mathbf{p}), \bar{\mathbf{p}})$ for some vector function I. Furthermore, it can be shown that $I(\mathbf{p})$ is a standard interference function [21]. By leveraging the interference function results in [21], this leads us to propose the following (step-size-free) algorithm that converges to the optimal solution of (6). This algorithm is also used as a building block for the max-min weighted SINR power control later.

Algorithm 1 (SAPC Algorithm):

1) Update $\mathbf{p}(k+1)$

$$p_l(k+1) = \min\left\{ w_l \Big/ \left(\sum_{j \neq l} \frac{w_j F_{jl} \mathsf{SINR}_j(\mathbf{p}(k))}{p_j(k)} \right), \bar{p}_l \right\} \tag{7}$$

for all l, where k indexes discrete time-slots.

Remark 1: The information required for computation in (7) can be obtained by distributed message passing: For $j \neq l$, the jth link first computes $w_j \mathsf{SINR}_j(\mathbf{p}(k))/(G_{jj} p_j)$ and measures G_{jl} by a pilot signal transmitted from the lth link. Then, $G_{jl} w_j \mathsf{SINR}_j(\mathbf{p}(k))/(G_{jj} p_j)$ is broadcasted to the lth link for computation.

Theorem 1: Starting from any initial point $\mathbf{p}(0)$, $\mathbf{p}(k)$ in Algorithm SAPC converges to $\mathbf{p}'$ asymptotically under asynchronous update.

Remark 2: The proof of Theorem 1 uses the standard interference function approach [21], and the key is to show that (7) is a standard interference function. A good initial point is to set $\mathbf{p}(0) = \bar{\mathbf{p}}$ because it can be shown that, at optimality of (6), some links transmit at maximum power. Hence, some links will not need further power updates since they are already optimal at the initial step.

Example 1: We evaluate the performance of Algorithm SAPC and the dual iterative algorithm in [8]. We adopt the path loss model with a path loss exponent of 3.7, log-normal shadowing with standard deviation of 8.9 dB, and we assume slow fading. Ten links are distributed uniformly in a cell. All links have the same maximum power of 33 mW. There are many standard choices of step-size $\alpha(k)$ used in the dual iterative method [23]. We use a step-size configuration $\alpha(k) = (a+1)/(a+k)$ for $a = 1$, 20, 500, and 1000 [23] to illustrate the convergence for the dual iterative algorithm in [8]. Fig. 2 shows the evolution of the objective value in (6), and Table I records the running time for the algorithm in [8]

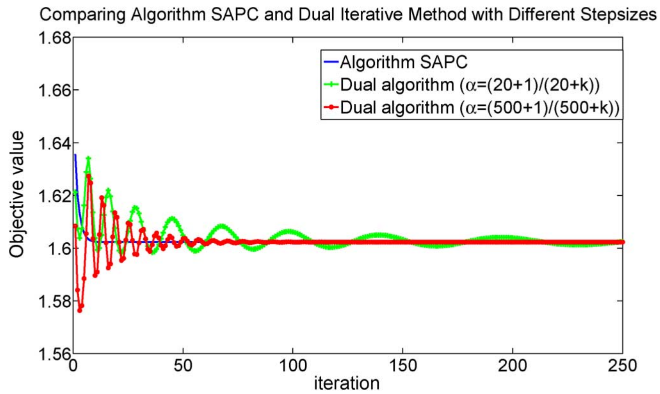

Fig. 2. Example 1. Performance comparison of Algorithm SAPC and the dual iterative method in [8] that uses different step-sizes α of: 1) $\alpha(k) = (20+1)/(20+k)$; and 2) $\alpha(k) = (500+1)/(500+k)$. The evolution of the objective value in (6), i.e., $\sum_l w_l \log \mathsf{SINR}_l(\mathbf{p}(k))$ is shown.

TABLE I
EXAMPLE 1. TABULATING THE RUNNING TIME OF THE ALGORITHM IN [8] TO CONVERGE WITHIN 1% AND 5% OF THE OPTIMAL VALUE WITH DIFFERENT STEP-SIZES $\alpha(k) = (a+1)/(a+k)$ FOR $a = 1, 20, 500, 1000$. THE RUNNING TIMES FOR ALGORITHM SAPC TO CONVERGE WITHIN 1% AND 5% ARE THREE AND FIVE ITERATIONS, RESPECTIVELY. THIS DEMONSTRATES THAT ALGORITHM SAPC IS ORDERS OF MAGNITUDE FASTER THAN THE DUAL METHOD

Parameter a	Running time (1%)	Running time (5%)
1	780	490
20	260	220
500	90	75
1000	65	40

using different a to converge within 1% and 5% of the optimal value. This illustrates that Algorithm SAPC converges much faster than the dual iterative algorithm in [8], regardless of the step-sizes. In most of our simulations, the convergence speed of Algorithm SAPC is several orders magnitude faster than the dual iterative algorithm.

B. Max-Min Weighted SINR *Power Control*

In this section, we study the constrained max-min weighted SINR power control. Let $\boldsymbol{\beta}$ be a positive vector similar to the use of $\mathbf{w}$ in (6) to indicate priority. Consider the following problem:

$$\begin{array}{ll} \text{maximize} & \min\limits_l \dfrac{\mathsf{SINR}_l(\mathbf{p})}{\beta_l} \\ \text{subject to} & \mathbf{0} \le \mathbf{p} \le \bar{\mathbf{p}} \\ \text{variables :} & \mathbf{p}. \end{array} \tag{8}$$

By exploiting a connection between nonnegative matrix theory and the algebraic structure of max-min SINR power control, we can give a closed-form solution to (8).

Theorem 2: An optimal solution to (8) is such that the weighted SINR for all links are equal. This weighted SINR is given by

$$\gamma^* = \frac{1}{\rho\left(\mathrm{diag}(\boldsymbol{\beta})\left(\mathbf{F} + \left(\frac{1}{\bar{p}_i}\right)\mathbf{v}\mathbf{e}_i^\top\right)\right)} \tag{9}$$

where

$$i = \arg\min_l \frac{1}{\rho\left(\mathrm{diag}(\boldsymbol{\beta})\left(\mathbf{F} + \left(\frac{1}{\bar{p}_l}\right)\mathbf{v}\mathbf{e}_l^\top\right)\right)}. \tag{10}$$

Furthermore, all links i that achieve the minimum in (10) transmit at maximum power $\bar{p}_i$ and the rest do not. The optimal $\mathbf{p}$, denoted by $\mathbf{p}^*$, is $t\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$ for a constant $t = \bar{p}_i/x_i$.

Remark 3: Note that γ^*, as the optimal value to (8), was previously derived using a different approach in [11]. The analytical optimal solution to (8) given here in Theorem 2 has, however, not been previously derived. Suppose $\boldsymbol{\beta} = \mathbf{1}$ and we let $\bar{p}_l \to \infty$ for all l or let $\mathbf{v} = \mathbf{0}$ in Theorem 2, we obtain as a special case the result in [24], where the additive white Gaussian noise and the maximum power constraints are assumed to be absent.

While the closed-form solution in Theorem 2 is only useful when the eigenvector $\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$ can be computed centrally, we next discuss how to solve (8) for any number of links distributively. The proof of Theorem 2 computes an optimal dual function that has a form similar to the SAPC problem in (6) [cf. (37) and (38) in the Appendix], which allows us to evaluate this dual function, possibly asynchronously, with fast convergence (cf. Theorem 1). This leads us to solve (8) using the following max-min weighted SINR Algorithm.

Algorithm 2 (Max-Min Weighted SINR *Algorithm):*

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1) Initialize an arbitrarily positive $\mathbf{w}(t)$ and small $\epsilon, \alpha(1)$.
2) Set $\mathbf{p}(0) = \bar{\mathbf{p}}$. Repeat
$$p_l(k+1) = \min\left\{\frac{w_l(t)}{\sum_{j\ne l}\frac{w_j(t)F_{jl}\mathsf{SINR}_j(\mathbf{p}(k))}{p_j(k)}}, \bar{p}_l\right\}$$
until $\|\mathbf{p}(k+1) - \mathbf{p}(k)\| \le \epsilon$.
3) Compute
$$w_l(t+1) = \max\Bigg\{0, w_l(t) + \alpha(t) \times\left(\sum_j w_j(t)\log\left(\frac{\mathsf{SINR}_j(\mathbf{p}(k))}{\beta_j}\right) - \log\left(\frac{\mathsf{SINR}_l(\mathbf{p}(k))}{\beta_l}\right)\right)\Bigg\}$$
for all l, where t indexes discrete time-slots much larger than k.
4) Normalize $\mathbf{w}(t+1)$ so that $\mathbf{1}^\top\mathbf{w}(t+1) = 1$.
5) Update $\alpha(t)$ such that $\{\alpha(t)\}_{k=1}^\infty$ satisfies $\lim_{k\to\infty}\alpha(t) = 0$ and $\sum_{k=1}^\infty \alpha(t) = \infty$. Go to Step 2.

Theorem 3: Starting from any initial point $\mathbf{w}(0)$ and small ϵ, if the positive step-size $\alpha(t)$ is strictly less than

$$2\left(-\log\rho\left(\mathrm{diag}(\boldsymbol{\beta})\left(\mathbf{F} + (1/\bar{p}_i)\right)\mathbf{v}\mathbf{e}_i^\top\right) - \sum_l w_l(t)\log\left(\mathsf{SINR}_l(\mathbf{p}(k))/\beta_l\right)\right) / \|\mathbf{g}(t)\|^2$$

where i is given in (10) and

$$(\mathbf{g}(t))_l = \sum_j w_j(t) \log\left(\frac{\mathsf{SINR}_j(\mathbf{p}(k))}{\beta_j}\right) - \log\left(\frac{\mathsf{SINR}_l(\mathbf{p}(k))}{\beta_l}\right)$$

then $\mathbf{p}(k)$ in Algorithm 2 converges, synchronously or asynchronously, to

$$\mathbf{x}\left(\mathrm{diag}(\boldsymbol{\beta})\left(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top\right)\right)$$

(unique up to a scaling constant), and $\mathbf{w}(t)$ converges to

$$\mathbf{x}\left(\mathrm{diag}(\boldsymbol{\beta})\left(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top\right)\right) \circ \mathbf{y}\left(\mathrm{diag}(\boldsymbol{\beta})\left(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top\right)\right).$$

Remark 4: For a given $\mathbf{w}(t)$ at Step 2 of Algorithm 2, Algorithm SAPC is used as an intermediate iterative method. Besides the step-size condition in Theorem 3, there are other standard convergence results of the subgradient method, where the step-size can be made simpler (cf. [23, Sec. 6.3]). For example, Algorithm 2 can also converge for a sufficiently small constant step-size that does not depend on the problem parameters and permits a simpler distributed implementation with convergence to the neighborhood of the optimal solution [23].

Now, we connect Algorithm SAPC and Algorithm 2. The following result illustrates that if the weight vector $\mathbf{w}$ in (6) is chosen in a particular form as a function of $\boldsymbol{\beta}$, the solution obtained by Algorithm SAPC is equivalent to that of (8).

Corollary 1 (Connect Max-Min Weighted SINR *and SAPC):* Let $\mathbf{x}$ and $\mathbf{y}$ be the Perron right and left eigenvectors of $\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$, respectively, where i is defined in (10). Recall that $t\mathbf{x}$ is also the optimal power vector for the Max-min SINR problem for some $t > 0$. Now consider the SAPC with $\mathbf{w} = \mathbf{x} \circ \mathbf{y}$. Then, $t\mathbf{x}$ is also the optimal power vector for the SAPC.

From Corollary 1, we deduce that the optimal SINR allocation in (6) is a weighted geometric mean of the optimal SINR in (8), where the weights are the Schur product of the Perron right and left eigenvectors of $\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$

$$\prod_l \left(\mathsf{SINR}_l(t\mathbf{x})\right)^{x_l y_l} = 1/\rho\left(\mathrm{diag}(\boldsymbol{\beta})\left(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top\right)\right) \quad (11)$$

where i is given in (10) and $t = \bar{p}_i / x_i$.

C. Synchronous Max-Min Weighted SINR Algorithm

Algorithm 2 is a (two-timescale) primal-dual algorithm that solves the Lagrange dual of (8), but requires a sufficiently small step-size that is adapted iteratively. In the special case of synchronous updates and equal maximum power constraint, we give the following (single-timescale and step-size-free) algorithm to solve (8), also with the added advantage of backward compatibility with existing CDMA power control.

Algorithm 3 (Synchronous Max-Min Weighted SINR*):*

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1) Update power $\mathbf{p}(k+1)$:

$$p_l(k+1) = \frac{\beta_l}{\mathsf{SINR}_l(\mathbf{p}(k))} p_l(k) \qquad \forall\, l.$$

2) Normalize $\mathbf{p}(k+1)$:

$$p_l(k+1) \leftarrow p_l(k+1) \cdot \bar{p} / \max_j p_j(k+1) \qquad \forall\, l.$$

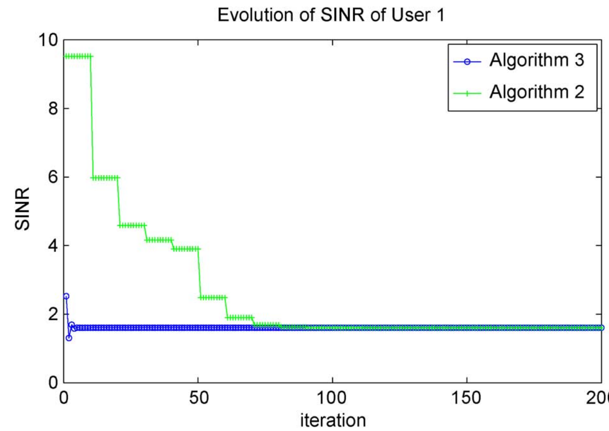

Fig. 3. Performance comparison of the max-min SINR Algorithms 2 and 3 in a typical numerical example of 10 links with equal power constraint. The evolution of SINR_1 is shown.

Theorem 4: Starting from any initial point $\mathbf{p}(0)$, $\mathbf{p}(k)$ in Algorithm 3 converges geometrically fast[1] to $\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$ (unique up to a scaling constant).

Remark 5: Interestingly, Step 1 of Algorithm 3 is simply the well-known Foschini–Miljanic power control update in [25], where the lth link has a virtual SINR threshold of β_l. This update in [25] can be implemented using a local feedback from the lth receiver to the lth transmitter after a local SINR measurement and is widely used in existing CDMA power control [2]. The only global coordination is for computing $\max_j p_j(k+1)$ in Step 2.

Using the numerical simulation model in Example 1, under synchronous updates, Fig. 3 shows that Algorithm 3 can converge much faster than Algorithm 2. Algorithm 2, however, can be applied to the case where the power updates are assumed to be asynchronous and the maximum power constraints are different.

IV. Approximability and Performance Guarantees

The two problems (6) and (8) in Section III can be solved efficiently and in a distributed manner. Yet back to our original question, how well do they approximate the difficult problem of sum rate maximization (1)? In this section, we will build on related results in [26] to assess the approximability of Algorithm 1 and the max-min weighted SINR power control (using Algorithm 2 or 3) in the special cases of sufficiently weak cross-channel gains or low SNR. In addition, conditions under which the fast algorithms in Section III solve (1) optimally are also given.

A. Approximability Under Weak Interference

Consider the following relaxed (but still nonconvex) problem to (1):

$$\begin{array}{ll} \text{maximize} & \sum_l w_l \log\left(1 + \mathsf{SINR}_l(\mathbf{p})\right) \\ \text{subject to} & \sum_l p_l \le \sum_l \bar{p}_l, \qquad p_l \ge 0\ \forall\, l \\ \text{variables}: & p_l \qquad \forall\, l. \end{array} \quad (12)$$

[1]Convergence to a point is geometrically fast when $e(k+1) = e(k)/\psi$, where $e(k)$ is the error of the ith iterate, and ψ is a constant greater than 1.

The optimal objective of (12) is larger than that of (1) as the constraint set of (12) contains the one in (1). Despite its nonconvexity, the optimal solution to (12) can be obtained analytically under a sufficient condition on the problem parameters (when the cross-channel gains are weak) [26]. We describe this condition in the following.

Let $\mathbf{B}$ be a nonnegative matrix with entries

$$\mathbf{B} = \mathbf{F} + \sum_l \frac{1}{\mathbf{1}^\top \bar{\mathbf{p}}} \mathbf{v}\mathbf{e}_l^\top. \tag{13}$$

Following [26], we introduce the notion of quasi-invertibility of a nonnegative matrix in [27], particularly of $\mathbf{B}$ in (13), which will be useful in solving (12) optimally.

Definition 1 (Quasi-Invertibility): A square nonnegative matrix $\mathbf{B}$ is a quasi-inverse of a square nonnegative matrix $\tilde{\mathbf{B}}$ if $\mathbf{B} - \tilde{\mathbf{B}} = \mathbf{B}\tilde{\mathbf{B}} = \tilde{\mathbf{B}}\mathbf{B}$. Furthermore, $(\mathbf{I} - \tilde{\mathbf{B}})^{-1} = \mathbf{I} + \mathbf{B}$ [27].

For a given square nonnegative matrix $\mathbf{B}$, we first compute either $\mathbf{B}(\mathbf{I}+\mathbf{B})^{-1}$ or $(\mathbf{I}+\mathbf{B})^{-1}\mathbf{B}$. If this computed matrix is nonnegative, we say that $\tilde{\mathbf{B}}$ exists (as from Definition 1, this computed matrix is $\tilde{\mathbf{B}} \geq \mathbf{0}$). The existence of $\tilde{\mathbf{B}}$ can be related to the different SNR regimes. Roughly speaking, in the high-SNR regime or when the cross-channel gains (off-diagonals of $\mathbf{F}$) are very large, $\tilde{\mathbf{B}}$ does not exist. On the other hand, in the sufficiently low-SNR regime or when the cross-channel gains are small, $\tilde{\mathbf{B}}$ exists. In the following, we assume that $\tilde{\mathbf{B}}$ exists.

We give the following results that give a useful upper bound to (12) in terms of the problem parameters, which in turn upper-bounds (1). They also give insight into the connection between (1) and the max-min weighted SINR problem. First, building on a result from [26], we have the following lemma.

Lemma 1: If $\tilde{\mathbf{B}}$ exists, then

$$\sum_l w_l \log\left(1 + \mathsf{SINR}_l(\mathbf{p}^\star)\right) \leq \|\mathbf{w}\|_\infty^{\mathbf{x}\circ\mathbf{y}} \log\left(1 + 1/\rho(\mathbf{B})\right) \tag{14}$$

where $\mathbf{x}, \mathbf{y}$ are the Perron right and left eigenvectors of $\mathbf{B}$, respectively.

We state when the upper bound in (14) can be achieved for (1).

Lemma 2: Equality is achieved in (14) if and only if $\mathbf{w} = \mathbf{x}\circ\mathbf{y}$ and $\rho(\mathbf{B}) = \rho(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$, where i is given in (10). In this case, $\mathbf{p}^\star = \mathbf{x}(\mathbf{B})$ (unique up to a scaling constant).

Remark 6: Lemma 2 illustrates a sufficient condition under which (8) (with $\boldsymbol{\beta} = \mathbf{1}$) optimally solves (1). For example, when $\mathbf{F}$ is symmetric, $\bar{p}_l = \bar{p}_j, v_l = v_j$ for all l, j, then $\rho(\mathbf{B}) = \rho(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$, where i is given in (10). Thus, when $\mathbf{w} = \mathbf{x}\circ\mathbf{y}$, $\mathbf{p}^\star = \mathbf{x}(\mathbf{B})$ implying that solving (8) (with $\boldsymbol{\beta} = \mathbf{1}$) is equivalent to solving (1) optimally.

Based on Lemma 1, we now quantify the performance of the approximately optimal solution to solve (1) that is obtained from the max-min weighted SINR problem (with $\boldsymbol{\beta} = \mathbf{w}$) using Algorithm 2.

Theorem 5: Suppose $\tilde{\mathbf{B}}$ exists. Then

$$\frac{\sum_l w_l \log\left(1 + w_l/\rho\left(\mathrm{diag}(\mathbf{w})\left(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top\right)\right)\right)}{\|\mathbf{w}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})} \log\left(1 + 1/\rho(\mathbf{B})\right)} \tag{15}$$

where i is given in (10), bounds the ratio of the optimal value of (1) and the value of the sum rates evaluated at the solution of (8).

Similarly, the performance of Algorithm SAPC in solving (1) can be evaluated as follows.

Theorem 6: Suppose $\tilde{\mathbf{B}}$ exists. Then

$$\frac{\sum_l w_l \log\left(1 + \mathsf{SINR}(\mathbf{p}')\right)}{\|\mathbf{w}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})} \log\left(1 + 1/\rho(\mathbf{B})\right)} \tag{16}$$

where $\mathbf{p}'$ is the optimal solution to (6), bounds the ratio of the optimal value of (1) and the value of the sum rates evaluated at the solution of (6).

B. General Approximability

The results in Section IV-A depend on the existence of a nonnegative $\tilde{\mathbf{B}}$. If this sufficient condition is not satisfied, we show how the results in Section IV-A can still be used to construct useful upper bounds to (1). First, we define a transmission configuration as a set of links $\mathcal{C} = \{l | l = 1, \ldots, L\}$ with $|\mathcal{C}| \leq L$. Links in $\mathcal{C}$ transmit with positive power. In general, a transmission configuration can be used to construct a scheduling policy in which links that belong to $\mathcal{C}$ solve (1) using power control, whereas links that belong to $\bar{\mathcal{C}}$ transmit one at a time. Clearly, $|\mathcal{C}| + |\bar{\mathcal{C}}| = L$. For example, when $\tilde{\mathbf{B}}$ exists and all links have positive optimal power, we need to only consider $\mathcal{C}$ such that $|\mathcal{C}| = L$. A scheduling policy determines how links in $\mathcal{C}$ and $\bar{\mathcal{C}}$ are time-shared in time-division multiple access. When there are L links, there are

$$\sum_{l=1}^{L-2} \binom{L}{l} + 2 = 2^L - L \tag{17}$$

possible transmission configurations. Now, for any transmission configuration $\mathcal{C}$ and any $\mathbf{p}$, we have

$$\sum_{l=1}^{L} w_l \log\left(1 + \mathsf{SINR}_l(\mathbf{p})\right) \leq \sum_{l\in\mathcal{C}} w_l \log\left(1 + \mathsf{SINR}_l(\mathbf{p})\right) + \sum_{l\in\bar{\mathcal{C}}} w_l \log(1 + p_l/v_l) \tag{18}$$

where $\mathsf{SINR}_l(\mathbf{p}), l \in \mathcal{C}$, in the first summand on the right-hand side of (18) contains only interference terms coming from links in $\mathcal{C}$. Therefore, it can be deduced that

$$\sum_{l=1}^{L} w_l \log\left(1 + \mathsf{SINR}_l(\mathbf{p}^\star)\right) \leq \max_{\mathbf{0}\leq\mathbf{p}\leq\bar{\mathbf{p}}} \sum_{l\in\mathcal{C}} w_l \log\left(1 + \mathsf{SINR}_l(\mathbf{p})\right) + \sum_{l\in\bar{\mathcal{C}}} w_l \log(1 + \bar{p}_l/v_l). \tag{19}$$

Let $\mathbf{B}_\mathcal{C}$ denote a submatrix obtained from $\mathbf{B}$ by deleting those rows and columns whose indices belong to $\bar{\mathcal{C}}$. If $\mathbf{B}_\mathcal{C}$ is a quasi-inverse of a nonnegative matrix, we denote that matrix as $\tilde{\mathbf{B}}_\mathcal{C}$.

TABLE II
TYPICAL NUMERICAL EXAMPLE IN A 10-LINK NETWORK WITH TWO MAXIMUM POWER CONSTRAINT SETTINGS: $\bar{p}_l = 33$ mW OR $\bar{p}_l = 1$ W FOR ALL l. THE PERCENTAGE OF INSTANCES WHERE $\tilde{\mathbf{B}}$ EXISTS IS RECORDED

$\bar{p}_l$	$\tilde{\mathbf{B}} \geq \mathbf{0}$	SAPC ratio	Max-min SINR ratio	On-off sched. ratio
33mW	99.3%	0.80 [0.47-0.98] (0.96 [0.87-0.99])	0.82 [0.52-0.97] (0.96 [0.88-0.99])	0.74 [0.44-0.87] (0.91 [0.83-0.96])
1W	99.2%	0.92 [0.52-0.97] (0.95 [0.82-0.99])	0.91 [0.47-0.97] (0.95 [0.88-0.99])	0.82 [0.41-0.92] (0.87 [0.52-0.97])

Hence, for any transmission configuration $\mathcal{C}$, if $\tilde{\mathbf{B}}_{\mathcal{C}}$ exists, we can use the previous results to deduce the bound

$$\sum_{l=1}^{L} w_l \log\left(1 + \mathsf{SINR}_l(\mathbf{p}^\star)\right) \leq \max_{l \in \mathcal{C}} \frac{w_l}{(\mathbf{x}(\mathbf{B}_{\mathcal{C}}) \circ \mathbf{y}(\mathbf{B}_{\mathcal{C}}))_l} \log\left(1 + \frac{1}{\rho(\mathbf{B}_{\mathcal{C}})}\right) + \sum_{l \in \bar{\mathcal{C}}} w_l \log(1 + G_{ll}\bar{p}_l/n_l). \tag{20}$$

Subject to the existence of quasi-inverses, the tightest upper bounds to (1) can be found by searching all $2^L - L$ transmission configurations. Note that if $\tilde{\mathbf{B}}$ exists, then $\tilde{\mathbf{B}}_{\mathcal{C}}$ exists for any $\mathcal{C}$. On the other hand, as $|\mathcal{C}|$ gets smaller, then it is more likely that $\tilde{\mathbf{B}}_{\mathcal{C}}$ exists because the second summand in a submatrix of (13) tends to dominate the first summand.

Example 2: We now examine the implications of our results for practical wireless networks such as the IEEE 802.11b ad hoc network using a numerical example. We also consider the ON–OFF scheduling algorithm, which finds the power vector that maximizes the sum rates in which links transmit at either the maximum or zero power, as a baseline for comparison to SAPC and max-min SINR. Table II records a typical numerical example for a 10-link network, where the maximum power is set as 33 mW with SNR of 7 dB and 1 W (the largest possible value allowed in IEEE 802.11b) with SNR of 40 dB. We set $\mathbf{w} = \mathbf{x} \circ \mathbf{y}$.

For each maximum power constraint, we average the percentage of instances where $\tilde{\mathbf{B}}$ exists, the sum rates, and the approximation ratios over 10 000 random instances. Particularly, each problem instance is created randomly by generating the cross channel gain G_{lj} for all $l \neq j$ uniformly between 0.01 and 0.1, and the direct channel gain G_{ll} is uniformly generated between 0.9 and 1.5 for all l. In the case where $\tilde{\mathbf{B}}$ does not exist, the tightest upper bound using the general approximability technique in Section IV-B, i.e., searching all $2^{10} - 10$ transmission configurations for the one that has the smallest upper bound in (20), is computed, and the approximation ratios of SAPC, max-min weighted SINR (with $\boldsymbol{\beta} = \mathbf{w}$), and ON–OFF scheduling are computed. Recorded without the parentheses, the average approximation ratios for max-min weighted SINR (with $\boldsymbol{\beta} = \mathbf{w}$) and SAPC are computed using Theorem 5 and 6, respectively, and the approximation ratio of ON–OFF scheduling is computed by searching through all $2^{10} - 1$ possible transmission configurations. Also shown with the parentheses in each cell is the average of the actual ratios of the achieved rates by the respective algorithms to the global optimal sum rates (obtained using the global optimization algorithm in [12]). We also record the maximum and the minimum ratios listed in the square brackets beside each of the average ratio. As shown in Table II, the sufficient condition for the existence of $\tilde{\mathbf{B}}$ holds over a large proportion of time, and the relatively large approximation ratios using SAPC and max-min SINR indicate the usefulness of using power control to maximize sum rates in a typical IEEE 802.11b network.

V. APPLICATION TO CONNECTION-LEVEL STABILITY WITH END-TO-END CONGESTION CONTROL

So far, we have assumed that each link has an infinite backlog of data to transmit rather than a dynamic arrival model of link for each connection. In this section, we illustrate how our previous results and algorithms can be applied to analyze the connection-level stability of end-to-end flows with congestion control. We will propose a distributed congestion controller to perform joint congestion control and power allocation. This Algorithm 4 leverages the distributed power control updates in Section III, and its performance can be quantified using the results in Section IV.

For technical simplicity, we focus only on the max-min weighted SINR algorithm (Algorithm 3). We consider a network with L links and S classes of users. Let $H_{ls} = 1$ if the path of users of class s uses link l, and $H_{ls} = 0$, otherwise. We are interested in the connection-level stability as flows (the sth flow sends with an average rate m_s and an instantaneous rate $m_s(t)$ at time t) arrive stochastically to the network and their rates are regulated by congestion control. We assume that users of class s arrive to the network according to a Poisson process with rate λ_s, and that each user transmits a file whose size is exponentially distributed with mean $1/\mu_s$. The load brought by users of class s is then $\varrho_s = \lambda_s/\mu_s$. Assuming Poisson arrivals and exponential file size distribution, the evolution of $\mathbf{n}(t)$ will be governed by a Markov process, whose transition rates are given by

$$\begin{aligned} n_s(t) &\to n_s(t) + 1, \quad \text{with rate } \lambda_s \\ n_s(t) &\to n_s(t) - 1, \quad \text{with rate } \mu_s n_s(t) m_s(t) \text{ if } n_s(t) > 0. \end{aligned}$$

We follow the optimization-based congestion control model in [13]–[15], where we assume that time in the system is divided into time-slots of length T, and also assume that the lth link has an implicit cost q_l that corresponds to the congestion cost (in fact, q_l is a scalar multiple of the actual queue length at the lth link) [14], [15]. These implicit costs are updated at the end of each time-slot. However, users may arrive and depart at any point within a single time-slot. Let $\mathbf{q}(kT)$ denote the implicit costs at time-slot k. The definition of connection-level stability used in this section is a standard one.

Definition 2: The Markov process $\{\mathbf{n}(t), \mathbf{q}(t)\}$ is stable-in-the-mean, or simply stable, if

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{E}\left[\sum_{s=1}^{S} n_s(t) + \sum_{l=1}^{L} q_l(t)\right] dt < \infty.$$

We define the stability region Λ of the system under a given congestion controller to be the set of offered load $\boldsymbol{\varrho}$ such that the system is stable for any $\boldsymbol{\varrho} \in \Lambda$. Here, by stability, we mean that the number of active users in the system and the queue lengths at each link in the network remain finite. For any offered load, the largest possible stability region is one that can be stabilized by an optimal congestion controller [13]–[15]. The goal of the optimal congestion controller is to select a rate r_l for the lth link from the Pareto rate region $\mathcal{R}$ [realized by solving for the powers in (1)] given by [1], [12]

$$\left\{\mathbf{r} \in R_+^L : \rho\left(\mathrm{diag}\left(\exp(\mathbf{r}) - \mathbf{1}\right)\left(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top\right)\right) \le 1\ \forall l\right\}. \tag{21}$$

Note that the achievable rate region $\mathcal{R}$ using power control and time-sharing is given by the convex hull of $\mathcal{R}$, i.e., $\mathrm{Co}(\mathcal{R})$.

We next study an optimization-based congestion controller that allocates rate from the rate region $\mathcal{R}$ in (21). Assume that each user of class s has a maximum data-rate limit of M_s. The optimization-based rate allocation for congestion control solves the following problem [13]–[15].

- Find the flow rate vector $\mathbf{m}$ that achieves weighted proportional fairness for all users (with weights $\boldsymbol{\phi}$), i.e.,

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{s=1}^{S} n_s\phi_s \log m_s \\ \text{subject to} & \mathbf{H}(\mathbf{n}\circ\mathbf{m}) \le \mathbf{r}(\mathbf{p}) \\ & \mathbf{r}(\mathbf{p}) \in \mathrm{co}(\mathcal{R}) \\ & 0 \le p_l \le \bar{p}, \qquad l = 1,\ldots,L \\ & 0 \le m_s \le M_s, \qquad s = 1,\ldots,n_s \\ \text{variables}: & \mathbf{m}, \mathbf{p}. \end{array} \tag{22}$$

- Find the associated scheduling policy that stabilizes the system, i.e., each file is served in finite time and the queue at each link is bounded.

The optimization-based congestion controller can be viewed as an iterative solution that solves (22) and is based on the "dual solution" of (22) [28]. More precisely, the optimization-based congestion controller is a dual algorithm with an appropriately chosen step-size.

Now, finding the optimal scheduling policy to stabilize the system is difficult since $\mathcal{R}$ in (21) is nonconvex. Instead of finding the optimal rate allocation at each time-slot kT, we use the results established in Section IV to determine a sub-optimal schedule that provides an overall stability guarantee to the system (also known as imperfect scheduling first proposed in [14]). More precisely, at time-slot kT, we compute a schedule $\mathbf{r} \in \mathcal{R}$ such that it satisfies the following definition.

Definition 3 (Slot-Dependent Imperfect Scheduling): At time-slot kT, a schedule $\mathbf{r} \in \mathcal{R}$ is called *Slot-Dependent Imperfect Scheduling* if it satisfies

$$\sum_l q_l(kT)r_l(kT) \ge \eta(kT)\cdot\max_{\mathbf{r}\in\mathcal{R}}\sum_l q_l(kT)r_l \tag{23}$$

where $\eta(kT) \in (0, 1]$.

When $\eta(kT)$ is a constant independent of k, this corresponds to the case studied in [14]. We say the schedule given by (23) is imperfect because it attains only a factor $\eta(kT)$ of the achievable rate region at time-slot kT.[2] We now investigate the case when $\eta(kT)$ in (23) is a function of the queue size $\mathbf{q}(kT)$, and $\mathbf{r}(kT)$ is given by (3). Note that the maximization on the right-hand side of (23) is performed over $\mathcal{R}$ only at time-slot kT.

We propose the following congestion-control dual algorithm that finds a schedule given by (23).

Algorithm 4:

---

1) At the beginning of each time-slot kT, the lth link that has $q_l(kT) > 0$ uses Algorithm 3 to transmit at a rate

$$r_l(kT) = \log\left(1 + \frac{q_l(kT)}{\rho\left(\mathrm{diag}\left(\mathbf{q}(kT)\right)\left(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_{j^k}^\top\right)\right)}\right) \tag{24}$$

for all l, where $j^k = \arg\max_l \rho(\mathrm{diag}(\mathbf{q}(kT))(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_l))$ (Dimensions of $\mathbf{F}$ and $\mathbf{v}$ include only links with positive $q(kT)$.)

2) At any time t in time-slot kT for $kT \le t \le (k+1)T$, the data rate of users of class s is given by

$$m_s(t) = m_s(kT) = \min\left\{\frac{\phi_s}{\left(\mathbf{H}^\top\mathbf{q}(kT)\right)_s}, M_s\right\}. \tag{25}$$

3) At the end of each time-slot kT, the implicit costs are updated by each link as

$$q_l\left((k+1)T\right) = \max\left\{0, q_l(kT) + \alpha_l\left(\sum_{s=1}^{S} H_{ls}\int_{kT}^{(k+1)T} n_s(t)m_s(kT)dt - Tr_l(kT)\right)\right\}.$$

Let $\bar{\mathcal{S}} = \max_l \sum_{s=1}^{S} H_{ls}$ denote the maximum number of classes using any link, and let $\bar{\mathcal{L}} = \max_l \sum_{l=1}^{L} H_{ls}$ denote the maximum number of links used by any class.

We now apply Theorem 5 to show the stability performance guarantee of Algorithm 4 where all the links solve the max-min weighted SINR problem at each time-slot kT (can be done distributively using Algorithm 3).

[2]Under a node-exclusive interference model, e.g., in [14], when $\eta(kT) = \eta$, $\forall t$, η can be viewed as a parameter indicating the degree of complexity to approximate maximum weight matching: The complexity of finding a schedule satisfying (23) decreases for smaller η.

Theorem 7: Let $i = \arg\max_l \lim_{k\to\infty} \rho(\mathrm{diag}(\mathbf{q}(kT))(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_l))$ and $\mathbf{B} = \mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{1}$. Suppose that $\tilde{\mathbf{B}} \geq \mathbf{0}$. If $\max_l \alpha_l \leq (16T\bar{\mathcal{S}}\bar{\mathcal{L}})^{-1} \min_s \phi_s/(\varrho_s M_s)$, then for any offered load $\boldsymbol{\varrho}$ that resides strictly inside

$$\frac{\min_l (\mathbf{x}(\mathbf{B}) \circ \mathbf{y}(\mathbf{B}))_l \log(1 + 1/\rho(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_i))}{\log(1 + 1/\rho(\mathbf{B}))}\Lambda \quad (26)$$

the system described by the Markov process $\{\mathbf{n}(kT), \mathbf{q}(kT)\}$ with the link rates scheduled by Algorithm 4 is stable in the sense of Definition 2.

Remark 7: Theorem 7 states that if the offered load lies strictly in a given subset (which can be *a priori* determined based only on the channel parameters and maximum power) of the stability region, Algorithm 4 (with link transmission rates given by Algorithm 3) stabilizes the queue size and the number of flows for an appropriately chosen step-size at each link (chosen independently of the offered load).

This average fraction of the largest possible stability region in Theorem 7 can be decomposed into the Perron–Frobenius eigenvalues and eigenvectors of the nonnegative matrices given in (10) and (13), which are the properties that reflect the *interference level* and characterize the resource allocation at the physical layer. These physical-layer properties in turn affect the connection-level stability at the higher layers of the network protocol stack. Finding a useful lower bound to this average fraction (to take care of the worst-case problem instance) can be challenging and deserves further investigation in future work.

A. Numerical Example

In this section, we present simulation results to illustrate the use of Algorithm 4 for imperfect scheduling. We consider an example for three users (classes) to demonstrate the flow performance of using Algorithm 4 in a simple one-hop network ($\mathbf{H} = \mathbf{I}$). The maximum power is set to 1 W for all links. Algorithm 3 is used in Step 1 of Algorithm 4. We use the global optimization algorithm in [12] to solve for the optimal power control allocation on the right-hand side of (23).

We simulate the case when there are dynamic arrivals and departures of the users, whereby users of each class arrive to the network according to a Poisson process with rate λ. Each user brings with it a file to transfer, whose size is exponentially distributed with mean $1/\mu = 1$ unit. We set the arrival rate $\lambda = 0.12$ for each class. Fig. 4 shows the trajectory of the number of flows in each class. As observed from Fig. 4, the penalty of imperfect scheduling is the increase in the number of flows per class (hence the increase in sojourn time). As compared to the computation time of the optimal power control allocation in [12], the fast computation time of Algorithm 4 (orders of magnitude faster) outweighs the penalty in increased sojourn time in the system.

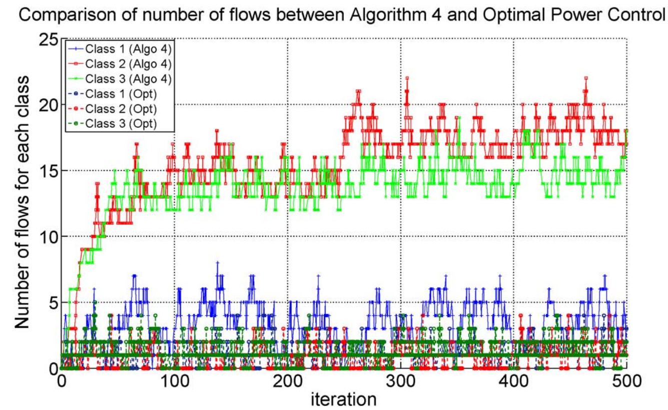

Fig. 4. Comparison of the number of flows in the three classes when Algorithm 4 and the optimal power control allocation are used.

VI. Conclusion

Sum rate maximization using power control is an NP-hard problem. We show that good quality approximately optimal solutions can be obtained through two related problems: SINR approximation power control (SAPC) and max-min weighted SINR optimization. These two problems have also been studied before, but now we have faster algorithms, often independent of parameter tuning and network configuration. These results are derived based on new techniques from nonnegative matrix theory. We then apply our results and algorithms to analyze the connection-level stability of utility maximization in a network, where users arrive and depart randomly and are subject to congestion control. In particular, we determine the stability region when all the links solve the max-min weighted SINR problem using instantaneous queue size as weights and the users adjust the source sending rates by congestion control to solve a system-wide utility maximization problem. An important and challenging direction for future work is to characterize a lower bound on the achievable fraction of the stability region that is independent of the problem instance parameters.

Appendix

A. Proof of Theorem 1

The objective function in (6) can be made convex by a change-of-variable technique: Let $\tilde{p}_l = \log p_l$ for all l. First, ignoring the maximum power constraints in (6), we let $\partial(\sum_l w_l \log \mathsf{SINR}_l(\tilde{\mathbf{p}}))/\partial\tilde{p}_l = 0$ for all l to deduce

$$e^{\tilde{p}_l} = w_l \Big/ \left(\sum_{j\neq l} \frac{w_j F_{jl}}{\sum_{i\neq j} F_{ij} e^{\tilde{p}} + v_j} \right) \qquad \forall l. \quad (27)$$

Clearly, $\tilde{p}'_l = \log p'_l$ for all l. Since $\mathbf{p}' \leq \bar{\mathbf{p}}$, we state the following lemma to show that the KKT conditions of (6) are equivalent to projecting (27) to $(-\infty, \log \bar{p}_l]$ for all l.

Lemma 3 (Box-Constrained Projection [29, p. 520]): Consider the problem $\min_{\underline{\mathbf{z}}\leq\mathbf{z}\leq\bar{\mathbf{z}}} f(\mathbf{z})$. The KKT conditions are equivalent to the condition $P_{[\underline{\mathbf{z}},\bar{\mathbf{z}}]}\partial f(\mathbf{z})/\partial\mathbf{z} = \mathbf{0}$, where $P_{[\underline{\mathbf{z}},\bar{\mathbf{z}}]}\mathbf{g}$ is the projection of the vector $\mathbf{g}$ onto the box $[\underline{\mathbf{z}}, \bar{\mathbf{z}}]$ defined by $(P_{[\underline{\mathbf{z}},\bar{\mathbf{z}}]}\mathbf{g})_l = \min\{0, g_l\}$ if $z_l = \underline{z}_l$, $(P_{[\underline{\mathbf{z}},\bar{\mathbf{z}}]}\mathbf{g})_l = g_l$ if $z_l \in (\underline{z}, \bar{z}_l)$ and $(P_{[\underline{\mathbf{z}},\bar{\mathbf{z}}]}\mathbf{g})_l = \max\{0, g_l\}$ if $z_l = \bar{z}_l$.

Applying Lemma 3 to (6) in the $\tilde{\mathbf{p}}$ domain (let $\underline{\mathbf{z}} = -\infty, \bar{\mathbf{z}} = \log\bar{\mathbf{p}}$ in Lemma 3) and then converting it back to the $\mathbf{p}$ domain, we deduce

$$p'_l = \min\left\{ w_l \Big/ \left(\sum_{j\neq l} \frac{w_j F_{jl}}{(\mathbf{F}\mathbf{p}')_j + v_j} \right), \bar{p}_l \right\} \quad (28)$$

where p'_l is unique since $\sum_l w_l \log \mathsf{SINR}_l(\tilde{\mathbf{p}})$ is strictly convex in $\tilde{\mathbf{p}}$ and the transformation $\tilde{p}_l = \log p_l$ is one-to-one.

Next, we establish the convergence of Algorithm SAPC. The convergence proof of (7) is based on the standard interference function approach [21], which is summarized as follows.

Definition 4 (Standard Interference Function): $I(\mathbf{p})$ is a standard interference function if it satisfies the following [21].

1) (monotonicity) $I(\mathbf{p}') > I(\mathbf{p})$ if $\mathbf{p}' > \mathbf{p}$.
2) (scalability) Suppose $\beta > 1$. Then, $\beta I(\mathbf{p}) > I(\beta\mathbf{p})$.

Definition 5: A standard interference function $I(\mathbf{p})$ is feasible if $\mathbf{p} \geq I(\mathbf{p})$ has a feasible solution [21].

Lemma 4 ([21, Theorems 2 and 4]): If $I(\mathbf{p})$ is feasible, then $\mathbf{p}(k+1) = I(\mathbf{p}(k))$ converges synchronously and totally asynchronously to the unique fixed point from any initial point $\mathbf{p}(0)$.

First, it can be easily verified that

$$(I(\mathbf{p}))_l = w_l / \left(\sum_{j \neq l} \frac{w_j F_{jl}}{(\mathbf{F}\mathbf{p}(k))_j + v_j} \right) \tag{29}$$

is standard for all l. Furthermore, if $I(\mathbf{p})$ is standard, then $\min\{(I(\mathbf{p}))_l, \bar{p}_l\}$ is standard (see [21, Theorem 7]). Hence, the iterative equation in (7) is standard. Clearly, from the KKT conditions, $\mathbf{p} = I(\mathbf{p})$ has a feasible solution, thus $\min(I(\mathbf{p}), \bar{\mathbf{p}})$ is feasible. Therefore, by Lemma 4, $\mathbf{p}(k+1)$ converges to the fixed point of (28) and hence the optimal solution of (6).

B. Proof of Theorem 2

Introducing an auxiliary variable τ to (8), we rewrite (8) as

$$\begin{aligned} &\text{maximize} && \tau \\ &\text{subject to} && \tau \leq \frac{\mathsf{SINR}_l(\mathbf{p})}{\beta_l} \ \forall l \\ &&& \mathbf{p} \leq \bar{\mathbf{p}}, \quad \tau > 0. \end{aligned} \tag{30}$$

Making a change of variables: $\tilde{\tau} = \log\tau$ and $\tilde{p}_l = \log p_l$ for all l, (30) is equivalent to the following convex problem:

$$\begin{aligned} &\text{maximize} && \tilde{\tau} \\ &\text{subject to} && \tilde{\tau} \leq \log\left(\frac{\mathsf{SINR}_l(\tilde{\mathbf{p}})}{\beta_l}\right) \qquad \forall l \\ &&& \tilde{p}_l \leq \log \bar{p}_l \qquad \forall l. \end{aligned} \tag{31}$$

Next, introducing the dual variable $\boldsymbol{\lambda}$, the partial Lagrangian of (31) is given by

$$L(\{\tilde{\tau}, \tilde{\mathbf{p}}\}, \{\boldsymbol{\lambda}\}) = \tilde{\tau}\left(1 - \sum_l \lambda_l\right) + \sum_l \lambda_l \log\left(\mathsf{SINR}_l(\tilde{\mathbf{p}})/\beta_l\right). \tag{32}$$

In order for (32) to be bounded, we must have $\sum_l \lambda_l = 1$. Hence, the dual problem of (31) is given by

$$\begin{aligned} &\text{minimize} && \max_{\tilde{\tau}, \tilde{p}_l \leq \log \bar{p}_l \ \forall l} L(\{\tilde{\tau}, \tilde{\mathbf{p}}\}, \{\boldsymbol{\lambda}\}) \\ &\text{subject to} && \mathbf{1}^\top \boldsymbol{\lambda} = 1, \quad \boldsymbol{\lambda} \geq \mathbf{0}. \end{aligned} \tag{33}$$

At optimality, the optimal objective to (31), i.e., $\tilde{\tau}^\star$, is equal to the optimal dual function of (33) given by

$$\max_{\tilde{p}_l \leq \log \bar{p}_l \ \forall l} L\left(\{\tilde{\tau}, \tilde{\mathbf{p}}\}, \{\boldsymbol{\lambda}^\star\}\right) = \sum_l \lambda_l^\star \log\left(\mathsf{SINR}_l\left(\tilde{\mathbf{p}}(\boldsymbol{\lambda}^\star)\right)/\beta_l\right) \tag{34}$$

where $\boldsymbol{\lambda}^\star$ is the optimal dual variable in (33) and

$$\tilde{\mathbf{p}}(\boldsymbol{\lambda}) = \arg \max_{\tilde{p}_l \leq \log \bar{p}_l \ \forall l} L(\{\tilde{\tau}, \tilde{\mathbf{p}}\}, \{\boldsymbol{\lambda}\}) \tag{35}$$

for a feasible $\boldsymbol{\lambda}$ in (33). Next, we compute an upper bound to $\tilde{\tau}$ in (30). For any feasible $\boldsymbol{\lambda}$ in (34), we have

$$\tilde{\tau} \leq \tilde{\tau}^\star \leq \sum_l \lambda_l \log\left(\mathsf{SINR}_l\left(\tilde{\mathbf{p}}(\boldsymbol{\lambda})\right)/\beta_l\right). \tag{36}$$

First, we prove the following result.

Lemma 5: Let i be given in (10). Let $\mathbf{x}$ and $\mathbf{y}$ be the Perron right and left eigenvector of $\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$. For any power vector $\mathbf{p}$

$$\sum_l x_l y_l \log\left(\mathsf{SINR}_l(\mathbf{p})/\beta_l\right) \leq -\log \rho\left(\mathrm{diag}(\boldsymbol{\beta})\left(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top\right)\right). \tag{37}$$

Equality is achieved in (37) if and only if $\mathbf{p} = a\mathbf{x}$, where $a = \bar{p}_i / x_i$. In this case, $p_i = \bar{p}_i$.

First, let i be given as in (10), and we state the following key result in [30].

Theorem 8 ([30, Theorem 3.1]): For any irreducible nonnegative matrix $\mathbf{A}$

$$\prod_l \left((\mathbf{A}\mathbf{z})_l / z_l\right)^{x_l y_l} \geq \rho(\mathbf{A}) \tag{38}$$

for all strictly positive $\mathbf{z}$, where $\mathbf{x}$ and $\mathbf{y}$ are the Perron right and left eigenvectors of $\mathbf{A}$, respectively. Equality holds in (38) if and only if $\mathbf{z} = a\mathbf{x}$ for some positive a.

Theorem 8 leads to the following result.

Lemma 6: If $\mathbf{z} = \mathbf{x}$ in (38), then $(\mathbf{A}\mathbf{z})_l / z_l = (\mathbf{A}\mathbf{z})_j / z_j \ \forall j \neq l$.

Since $\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$ is irreducible, we let $\mathbf{x}$ and $\mathbf{y}$ be the Perron right and left eigenvectors of $\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$, respectively, and write, for all $\mathbf{p}$ in (6)

$$\begin{aligned} &\prod_l \left(\mathsf{SINR}(\mathbf{p})/\beta_l\right)^{x_l y_l} \\ &\quad = \prod_l \left(\frac{p_l}{\beta_l(\mathbf{F}\mathbf{p} + \mathbf{v})_l}\right)^{x_l y_l} \\ &\quad \overset{(a)}{\leq} \prod_l \left(\frac{p_l}{\left(\mathrm{diag}(\boldsymbol{\beta})\left(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top\right)\mathbf{p}\right)_l}\right)^{x_l y_l} \\ &\quad \overset{(b)}{\leq} 1/\rho\left(\mathrm{diag}(\boldsymbol{\beta})\left(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top\right)\right) \end{aligned} \tag{39}$$

where, in (39), (a) is due to $p_i \leq \bar{p}_i$ and (b) is due to letting $\mathbf{A} = \mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$ in (38). Thus, we deduce (37).

Lemma 5 is proved by showing that the inequality in (37) becomes an equality if and only if $\mathbf{p} = \mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$ (unique up to a scaling constant) as follows. First, we note that (b) in (39) becomes an equality if and only if $\mathbf{p} = \mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$ (unique up to a scaling constant) using Theorem 8. Equality of (a) in (39) follows from the fact that for the max-min SINR problem, $p_i = \bar{p}_i$ and thus $p_i/\bar{p}_i = 1$.

Letting $\boldsymbol{\lambda} = \mathbf{x} \circ \mathbf{y}$ in (36) (which is clearly feasible in (33)), we deduce that

$$\begin{aligned}\tilde{\tau}^\star &\leq \sum_l x_l y_l \log\left(\mathsf{SINR}_l\left(\mathbf{p}(\mathbf{x}\circ\mathbf{y})\right)/\beta_l\right) \\ &\overset{(a)}{\leq} -\log\rho\left(\mathrm{diag}(\boldsymbol{\beta})\left(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top\right)\right) \end{aligned} \tag{40}$$

where (a) in (40) is due to Lemma 5, thus proving that $\tau \leq 1/\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$. Furthermore, from Lemma 5, equality of (a) in (40) is achieved if and only if $\mathbf{p} = a\mathbf{x}$ for some positive a (such that $p_i = \bar{p}_i$)

$$\begin{aligned}
&\max_{\tilde{\tau},\tilde{p}_l\leq\log\bar{p}_l\ \forall l} L\left(\{\tilde{\tau},\tilde{\mathbf{p}}\},\{\boldsymbol{\lambda}\}\right) \\
&= \max_{\tilde{\tau},\tilde{p}_l\leq\log\bar{p}_l\ \forall l} \tilde{\tau}(1-\sum_l \lambda_l) + \sum_l \lambda_l \log\left(\mathsf{SINR}_l(\tilde{\mathbf{p}})/\beta_l\right) \\
&\overset{(a)}{\geq} \tilde{\tau}\left(1-\sum_l \lambda_l\right) + \sum_l \lambda_l \log\left(\mathsf{SINR}_l\left(\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}})\right)\beta_l\right) \\
&= \tilde{\tau}(1-\sum_l \widehat{\lambda}_l) - \sum_l(\lambda_l - \widehat{\lambda}_l)\tilde{\tau} \\
&\quad + \sum_l(\lambda_l - \widehat{\lambda}_l)\log\left(\mathsf{SINR}_l\left(\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}})\right)\beta_l\right) \\
&\quad + \sum_l \widehat{\lambda}_l \log\left(\mathsf{SINR}_l\left(\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}})\right)\beta_l\right) \\
&\overset{(b)}{\geq} \tilde{\tau}\left(1-\sum_l \widehat{\lambda}_l\right) + \sum_l \widehat{\lambda}_l \log\left(\mathsf{SINR}_l\left(\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}})\right)\beta_l\right) \\
&\quad - \sum_l(\lambda_l - \widehat{\lambda}_l)\left(\sum_j \widehat{\lambda}_j \log\left(\mathsf{SINR}_j\left(\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}})\right)\beta_j\right)\right. \\
&\qquad\qquad \left. - \log\left(\mathsf{SINR}_l\left(\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}})\right)\beta_l\right)\right) \\
&\overset{(c)}{=} \max_{\tilde{\tau},\tilde{p}_l\leq\log\bar{p}_l\ \forall l} L\left(\{\tilde{\tau},\tilde{\mathbf{p}}\},\{\widehat{\boldsymbol{\lambda}}\}\right) \\
&\quad - \sum_l(\lambda_l - \widehat{\lambda}_l)\left(\sum_j \widehat{\lambda}_j \log\left(\mathsf{SINR}_j\left(\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}})\right)\beta_j\right)\right. \\
&\qquad\qquad \left. - \log\left(\mathsf{SINR}_l\left(\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}})\right)\beta_l\right)\right) \\
&\overset{(d)}{=} \max_{\tilde{\tau},\tilde{p}_l\leq\log\bar{p}_l\ \forall l} L\left(\{\tilde{\tau},\tilde{\mathbf{p}}\},\{\widehat{\boldsymbol{\lambda}}\}\right) - \mathbf{g}^\top(\boldsymbol{\lambda} - \widehat{\boldsymbol{\lambda}}).
\end{aligned} \tag{41}$$

From Lemma 6, this implies $\mathsf{SINR}_l(a\mathbf{x})/\beta_l = \mathsf{SINR}_j(a\mathbf{x})/\beta_j\ \forall j \neq l$. This means that a feasible power allocation $\mathbf{p} = a\mathbf{x}$ for some positive a solves (8) optimally. Hence, we deduce that $\tau^\star = 1/\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$. This completes the proof of Theorem 2.

C. Proof of Theorem 3

Algorithm 2 is based on maximizing the partial Lagrangian given by (32) of (31) using Lagrange dual decomposition (cf. Proof of Theorem 1). Since the duality gap is zero, solving the Lagrange dual problem is equivalent to solving (31) [thereby solving (8)].

First, we compute a subgradient of $\max_{\tilde{\tau},\tilde{p}_l\leq\log\bar{p}_l\ \forall l} L(\{\tilde{\tau},\tilde{\mathbf{p}}\},\{\boldsymbol{\lambda}\})$ for any feasible $\boldsymbol{\lambda}$ in (33). Observe the following chain of inequalities in (41). In (41), (a) is due to choosing a feasible $\tilde{\tau}$ and

$$\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}}) = \arg\max_{\tilde{p}_l\leq\log\bar{p}_l\ \forall l} L\left(\{\tilde{\tau},\tilde{\mathbf{p}}\},\{\widehat{\boldsymbol{\lambda}}\}\right) \tag{42}$$

such that

$$\log\left(\mathsf{SINR}_l\left(\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}})\right)\beta_l\right) \geq \tilde{\tau} \tag{43}$$

for a feasible $\widehat{\boldsymbol{\lambda}}$, (b) is due to $\sum_l \widehat{\lambda}_l \log(\mathsf{SINR}_l(\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}}))/\beta_l) \geq \tilde{\tau}$, which is implied by (43), (c) is due to the definition of $\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}})$ in (42), and in (d) we have

$$(\mathbf{g})_l = \sum_j \widehat{\lambda}_j \log\left(\mathsf{SINR}_j\left(\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}})\right)/\beta_j\right) - \log\left(\mathsf{SINR}_l\left(\tilde{\mathbf{p}}(\widehat{\boldsymbol{\lambda}})\right)/\beta_l\right). \tag{44}$$

We thus deduce that $-\mathbf{g}$ is a subgradient of $\max_{\tilde{\tau},\tilde{p}_l\leq\log\bar{p}_l\ \forall l} L(\{\tilde{\tau},\tilde{\mathbf{p}}\},\{\boldsymbol{\lambda}\})$ at $\widehat{\boldsymbol{\lambda}}$. The subgradient method generates a sequence of dual feasible points according to the iteration

$$\lambda_l(t+1) = \max\left\{\lambda_l(t) + \alpha(t)\left(\mathbf{g}(t)\right)_l, 0\right\} \qquad \forall l \tag{45}$$

where $\mathbf{g}(t)$ is the subgradient at $\boldsymbol{\lambda}(t)$ and $\alpha(t)$ is a positive scalar step-size. For a given feasible $\boldsymbol{\lambda}(t)$, the Lagrange dual function is evaluated by $\tilde{\mathbf{p}}(\boldsymbol{\lambda}(t))$ or equivalently $\mathbf{p}(\boldsymbol{\lambda}(t))$, which is simply equivalent to solving (6) by letting the weight vector $\mathbf{w} = \boldsymbol{\lambda}(t)$.

There are many standard convergence results of the subgradient method, depending on the choice of step-size used [23]. For example, if $\alpha(t) = \alpha$, then (45) converges to within some range of the optimal value, i.e., $L(\{\tilde{\tau}(t),\tilde{\mathbf{p}}(t)\},\{\widehat{\boldsymbol{\lambda}}(t)\}) - \tilde{\tau}^\star < \epsilon_0$ for a small positive ϵ_0. If $\{\alpha(t)\}_{k=1}^{\infty}$ satisfies $\lim_{k\to\infty}\alpha(t) = 0$ and $\sum_{k=1}^{\infty}\alpha(t) = \infty$, $\inf_k L(\{\tilde{\tau}(t),\tilde{\mathbf{p}}(t)\},\{\widehat{\boldsymbol{\lambda}}(t)\}) = \tilde{\tau}^\star$ [23].

Lemma 7 ([23, Proposition 6.3.1]): If $\boldsymbol{\lambda}(t)$ is not optimal, then for every dual optimal solution $\boldsymbol{\lambda}^\star$, we have $\|\boldsymbol{\lambda}(k+1) - \boldsymbol{\lambda}^\star\| < \|\boldsymbol{\lambda}(t) - \boldsymbol{\lambda}^\star\|$, for all step-sizes $\alpha(t)$ such that

$$0 < \alpha(t) < \frac{2\left(L(\boldsymbol{\lambda}^\star) - L\left(\boldsymbol{\lambda}(t)\right)\right)}{\|\mathbf{g}(t)\|^2} \tag{46}$$

where $L(\boldsymbol{\lambda}(t))$ and $\mathbf{g}(t)$ are the Lagrange dual function and the subgradient at $\boldsymbol{\lambda}(t)$, respectively.

Now, since the optimal Lagrange dual function in (33) is $\tilde{\tau}^\star = -\log\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$, we apply Lemma 7 to our problem to deduce that if $\alpha(t)$ in Algorithm 2 is strictly less than (11) [let $\mathbf{w}(t) = \boldsymbol{\lambda}(t)$ in (11)] where $\mathbf{g}(t)$ is the subgradient function given by (44) at $\boldsymbol{\lambda}(t)$, then Algorithm 2 converges linearly. This proves the overall convergence of $\mathbf{p}(k)$ to the solution of (8).

D. Proof of Corollary 1

Using Lemma 5 in the proof of Theorem 2, we show that $\lambda_l^\star = x_l y_l$ for all l. From (34)

$$\sum_l \lambda_l^\star \log\mathsf{SINR}_l\left(\mathbf{p}(\boldsymbol{\lambda}^\star)\right) = \tilde{\tau}^\star = \log\frac{1}{\rho\left(\mathrm{diag}(\boldsymbol{\beta})\left(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top\right)\right)}. \tag{47}$$

Now, by the Perron–Frobenius theorem, $\mathbf{x}$ and $\mathbf{y}$ are unique up to a scaling constant. Also, $\boldsymbol{\lambda}^\star$ is unique by strict convexity of the constraint set in (31). Combining (47) and the equality case in (37), we have $\boldsymbol{\lambda}^\star = \mathbf{x} \circ \mathbf{y}$. Considering SAPC in (6) with $\mathbf{w} = \mathbf{x} \circ \mathbf{y}$ thus yields the solution $\mathbf{p} = \mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$ (unique up to a scaling constant).

E. Proof of Theorem 4

Our proof is based on the nonlinear Perron–Frobenius theory [31]. We let the optimal max-min weighted $\mathsf{SINR}(\mathbf{p}^*)$ in (8) be τ^*. A key observation is that all the SINR constraints are tight at optimality. This implies, at optimality of (8)

$$\frac{(p_l^*/\bar{p})}{\sum_{j\neq l} F_{lj}\,(p_l^*/\bar{p}) + (v_l\bar{p})} = \tau^*\beta_l \qquad (48)$$

for all l. Letting $\mathbf{s}^* = (1/\bar{p})\mathbf{p}^*$, (48) can be rewritten as

$$(1/\tau^*)\mathbf{s}^* = \mathrm{diag}(\boldsymbol{\beta})\mathbf{F}\mathbf{s}^* + (1/\bar{p})\mathrm{diag}(\boldsymbol{\beta})\mathbf{v}. \qquad (49)$$

We first state the following result.

Lemma 8 (Conditional Eigenvalue [31, Corollary 14]): Let $\mathbf{A}$ be a nonnegative matrix and $\mathbf{b}$ be a nonnegative vector. If $\rho(\mathbf{A} + \mathbf{b}\mathbf{e}_i^\top) > \rho(\mathbf{A})$, where $i = \arg\min_l 1/\rho(\mathbf{A} + \mathbf{b}\mathbf{e}_l^\top)$, then the conditional eigenvalue problem

$$\lambda\mathbf{s} = \mathbf{A}\mathbf{s} + \mathbf{b}, \qquad \lambda \in \mathbb{R}, \quad \mathbf{s} \geq \mathbf{0}, \quad \max_l s_l = 1$$

has a unique solution given by $\lambda = \rho(\mathbf{A} + \mathbf{b}\mathbf{e}_i^\top)$ and $\mathbf{s}$ being the unique normalized Perron eigenvector of $\mathbf{A} + \mathbf{b}\mathbf{e}_i^\top$.

Letting $\lambda = 1/\tau^*, \mathbf{A} = \mathrm{diag}(\boldsymbol{\beta})\mathbf{F}, \mathbf{b} = (1/\bar{p})\mathrm{diag}(\boldsymbol{\beta})\mathbf{v}$ in Lemma 8 and noting that $\max_l s_l^* = 1$ shows that $\mathbf{p}^* = (\bar{s}_i/\bar{p})\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})\mathbf{F} + (1/\bar{p})\mathrm{diag}(\boldsymbol{\beta})\mathbf{v})$ is a fixed point of (49) as it should be (cf. Theorem 2). Now, the fixed point in Lemma 8 is also a unique fixed point of the following iteration [31]:

$$\mathbf{s}(k+1) = \frac{\mathbf{A}\mathbf{s}(k) + \mathbf{b}}{\|\mathbf{A}\mathbf{s}(k) + \mathbf{b}\|_\infty}. \qquad (50)$$

Applying the power method in (50) to the system of equations in (49) and letting $\mathbf{p}(k+1) = \mathbf{s}(k+1)\bar{p}$ yield Algorithm 3. The geometric convergence of (50) follows from the more general concave Perron–Frobenius theory in [32]. This completes the proof of Theorem 4.

F. Proof of Lemmas 1 and 2

Since $\sum_l w_l \log(1 + \mathsf{SINR}_l(\mathbf{p}^\star))$ is upper-bounded (due to relaxation) by the optimal value of (12), its upper bound has been shown in [26] to be $\mathbf{w}\|_\infty^{\mathbf{x}\circ\mathbf{y}} \log(1 + 1/\rho(\mathbf{B}))$. Hence, $\sum_l w_l \log(1 + \mathsf{SINR}_l(\mathbf{p}^\star)) \leq \mathbf{w}\|_\infty^{\mathbf{x}\circ\mathbf{y}} \log(1 + 1/\rho(\mathbf{B}))$. This proves Lemma 1. Building on Lemma 1, we will prove the condition for equality in (14), thereby proving Lemma 2. We first prove the equality $\rho(\mathbf{B}) = \rho(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$, which is equivalent to $\mathbf{x}(\mathbf{B})$ being a feasible solution in (8).

Lemma 9: The vector $\mathbf{x}(\mathbf{B})$ is feasible in (8) with $\boldsymbol{\beta} = \mathbf{1}$ if and only if

$$\rho(\mathbf{B}) = \rho\left(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top\right). \qquad (51)$$

In this case, the optimal value of (8) is $1/\rho(\mathbf{B})$.

We first prove $\rho(\mathbf{B}) \geq \rho(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)$ for all l (implying $\rho(\mathbf{B}) \geq \rho(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$) by using the following result.

Lemma 10 ([31, Lemma 4]): Let $\mathbf{A} \geq \mathbf{0}$ and $\mathbf{b}, \mathbf{c}$ be nonnegative vectors. If $\rho(\mathbf{A} + \mathbf{b}\mathbf{c}^\top) > \rho(\mathbf{A})$, then for any nonnegative vector $\mathbf{d}$

$$\mathrm{sign}\left(\rho(\mathbf{A} + \mathbf{b}\mathbf{c}^\top) - \rho(\mathbf{A} + \mathbf{b}\mathbf{d}^\top)\right) = \mathrm{sign}(\mathbf{c}^\top\mathbf{x} - \mathbf{d}^\top\mathbf{x}) \qquad (52)$$

where $\mathbf{x}$ is the Perron right eigenvector of $\mathbf{A} + \mathbf{b}\mathbf{c}^\top$.

For any l, let $\mathbf{A} = \mathbf{F}$, $\mathbf{b} = \mathbf{v}$, $\mathbf{c} = (1/\sum_l \bar{p}_l)\mathbf{1}$ and $\mathbf{d} = (1/\bar{p}_l)\mathbf{e}_l$ in Lemma 10. Then, $\rho(\mathbf{B}) \geq \rho(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)$ is equivalent to $(1/\sum_l \bar{p}_l)\mathbf{1}^\top\mathbf{x}(\mathbf{B}) \geq (1/\bar{p}_l)\mathbf{e}_l^\top\mathbf{x}(\mathbf{B})$ or $\bar{p}_l \geq (\mathbf{x}(\mathbf{B}))_l$ since $\mathbf{1}^\top\mathbf{x}(\mathbf{B}) = \sum_l \bar{p}_l$. However, this is equivalent to $\rho(\mathbf{B}) \geq \rho(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$, where i is given in (10).

Next, to prove the inequality $\rho(\mathbf{B}) \leq \rho(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$, we note that (let $\mathbf{p} = \mathbf{x}(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$)

$$\mathbf{B}\mathbf{p} \leq \mathbf{F}\mathbf{p} + \mathbf{v} = \left(\mathbf{F} + \frac{1}{\bar{p}_i}\mathbf{v}\mathbf{e}_i^\top\right)\mathbf{p} = \rho\left(\mathbf{F} + \frac{1}{\bar{p}_i}\mathbf{v}\mathbf{e}_i^\top\right)\mathbf{p}.$$

By the subinvariance theorem ([33, p. 23, Theorem 1.6]), this implies $\rho(\mathbf{B}) \leq \rho(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$. Thus, $\rho(\mathbf{B}) = \rho(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$ is necessary and sufficient for $\mathbf{x}(\mathbf{B})$ to be an optimizer of (8) with $\boldsymbol{\beta} = \mathbf{1}$.

Hence, if $\mathbf{w} = \mathbf{x}(\mathbf{B}) \circ \mathbf{y}(\mathbf{B})$, we have $\sum_l w_l \log(1 + \mathsf{SINR}_l(\mathbf{p}^\star)) = \log(1 + 1/\rho(\mathbf{B}))$ (cf. Lemma 9). This is also when all the $\mathsf{SINR}(\mathbf{p}^\star)$ are equal. Therefore, the equality is achieved by the optimizer of (8) with $\boldsymbol{\beta} = \mathbf{1}$, i.e., $\mathbf{p}^\star = \mathbf{x}(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top) = \mathbf{x}(\mathbf{B})$ (unique up to a scaling constant).

G. Proof of Theorem 5

By solving (8) with $\boldsymbol{\beta} = \mathbf{w}$, the lth link achieves the SINR value: $w_l/\rho(\mathrm{diag}(\mathbf{w})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$, where i is given in (10). Substituting this quantity into the objective function of (1), the sum rate is thus given by $\sum_l w_l \log(1 + w_l/\rho(\mathrm{diag}(\mathbf{w})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)))$.

Now, suppose $\tilde{\mathbf{B}} \geq \mathbf{0}$. Then, by using the upper bound to the optimal objective in Lemma 1, the quantity

$$\eta = \frac{\sum_l w_l \log\left(1 + w_l/\rho\left(\mathrm{diag}(\mathbf{w})\left(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top\right)\right)\right)}{\|\mathbf{w}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})} \log\left(1 + 1/\rho(\mathbf{B})\right)} \qquad (53)$$

bounds the ratio of the optimal value of (1) and the value of the sum rates evaluated at the solution of (8) with $\boldsymbol{\beta} = \mathbf{w}$.

H. Proof of Theorem 6

Theorem 6 is proved similarly to Theorem 5, where the optimal solution $\mathbf{p}'$ to (6) is used instead. Hence, the quantity

$$\eta = \frac{\sum_l w_l \log\left(1 + \mathsf{SINR}(\mathbf{p}')\right)}{\|\mathbf{w}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})} \log\left(1 + 1/\rho(\mathbf{B})\right)} \qquad (54)$$

bounds the ratio of the optimal value of (1) and the value of the sum rates evaluated at the solution of (6).

I. Proof of Theorem 7

We will prove Theorem 7 using Theorem 5 in Section IV and [15, Proposition 3] (with extension to time-varying capacity found in [15, Section IV-B] and similar results are also found in [14]). Our proof approach follows the argument in [15, Proposition 3]. The key to proving [15, Proposition 3] is to define the Lyapunov function (omitting the time-slot index kT)

$$\mathcal{V}(\mathbf{n},\mathbf{q}) = V_n(\mathbf{n}) + V_q(\mathbf{q}),$$

where

$$V_n(\mathbf{n}) = \sum_{s=1}^{S} \frac{\phi_s n_s^2}{2\lambda_s} \quad V_q(\mathbf{q}) = \sum_{l=1}^{L} \frac{q_l^2}{2\alpha_l}$$

and show that $\mathcal{V}(\mathbf{n},\mathbf{q})$ has a negative drift for an appropriately chosen α_l for all l.

First, we note that $\mathcal{R}$ in (21) (and hence $\mathrm{Co}(\mathcal{R})$) is closed and bounded for all time-slots. Following [15], we can show that

$$\begin{aligned}\mathbf{E}&\left[\mathcal{V}\left(\mathbf{n}\left((k+1)T\right),\mathbf{q}\left((k+1)T\right)\right)\right.\\ &\left.-\mathcal{V}\left(\mathbf{n}(kT),\mathbf{q}(kT)\right)|\mathbf{n}(kT),\mathbf{q}(kT)\right]\\ &\leq T\sum_{l=1}^{L} q_l(kT)\left[\sum_{s=1}^{S} H_{ls}\varrho_s - r_l(kT)\right] + E_1(k) + E_2(k)\end{aligned} \quad (55)$$

where $E_1(k)$ and $E_2(k)$ are error terms roughly on the order of $|\varrho_s - n_s(kT)m_s(kT)|$ and $\left[\sum_{s=1}^{S} H_{ls}n_s(kT)m_s(kT) - r_l(kT)\right]^2$, respectively.

Now, suppose $\boldsymbol{\varrho}$ lies strictly inside $\eta(kT)\Lambda$, then there exists some $\epsilon > 0$ such that

$$(1+\epsilon)\sum_{s=1}^{S} H_{ls}\varrho_s \in \eta(kT)\mathrm{Co}(\mathcal{R}) \quad (56)$$

provided $\lim_{k\to\infty}\inf \eta(kT)$ is bounded away from zero when the maximum entry of $\mathbf{q}(kT)$ is very large. This is the case in [14] when $\lim_{k\to\infty}\inf \eta(kT)$ is a constant independent of k.

We now show that by solving a max-min weighted SINR problem (with $\mathbf{q}(kT)$ as weights) using Algorithm 3 to determine $\mathbf{r}(kT)$, we obtain $\mathbf{r}(kT) = \log(1 + q_l(kT)/\rho(\mathrm{diag}(\mathbf{q}(kT))(\mathbf{F} + (1/\bar{p}_{j^k})\mathbf{v}\mathbf{e}_{j^k}^\top)))$, where $j^k = \arg\max_l \rho(\mathrm{diag}(\mathbf{q}(kT))(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_l))$. Then, $\eta(kT)$ is given by (15) of Theorem 15 (replace $\mathbf{w}$ and i by $\mathbf{q}(kT)$ and j^k, respectively). Furthermore, this $\eta(kT)$ is bounded away from zero when $\|\mathbf{q}(kT)\|$ is large. More precisely, we shall find a lower bound to $\eta(kT)$ as $\max_j q_j(kT)$ (let $m = \arg\max_j q_j(kT)$) tends to a large value. Now, let $i = \arg\max_l \lim_{k\to\infty} \rho(\mathrm{diag}(\mathbf{q}(kT))(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_l))$. Observe the following chain of inequalities (omitting the time-slot index kT):

$$\begin{aligned}&\lim_{q_m\to\infty}\eta\\ &= \lim_{q_m\to\infty} \frac{\sum_j (q_j/\mathbf{1}^\top\mathbf{q})\log\left(1+\frac{q_j}{\rho\left(\mathrm{diag}(\mathbf{q})\left(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_{j^k}^\top\right)\right)}\right)}{\|\mathbf{q}/\mathbf{1}^\top\mathbf{q}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})}\log\left(1+1/\rho(\mathbf{B})\right)}\\ &\overset{(a)}{\geq} \lim_{q_m\to\infty} \frac{\sum_j (q_j/\mathbf{1}^\top\mathbf{q})\log\left(1+\frac{q_j}{\rho\left(\mathrm{diag}(\mathbf{q})\left(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i^\top\right)\right)}\right)}{\|\mathbf{q}/\mathbf{1}^\top\mathbf{q}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})}\log\left(1+1/\rho(\mathbf{B})\right)}\\ &= \left(\lim_{q_m\to\infty}\min_l\left(\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})\right)_l\frac{\mathbf{1}^\top\mathbf{q}}{q_l}\right)\frac{1}{\log\left(1+1/\rho(\mathbf{B})\right)}\\ &\quad\times\lim_{q_m\to\infty}\sum_j\frac{q_j}{\mathbf{1}^\top\mathbf{q}}\log\left(1+\frac{q_j}{\rho(\mathrm{diag}(\mathbf{q})(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i))}\right)\\ &\overset{(b)}{\geq}\left(\lim_{q_m\to\infty}\min_l\left(\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})\right)_l\frac{\mathbf{1}^\top\mathbf{q}}{q_l}\right)\frac{1}{\log\left(1+1/\rho(\mathbf{B})\right)}\\ &\quad\times\lim_{q_m\to\infty}\sum_j\frac{q_j}{\mathbf{1}^\top\mathbf{q}}\log\left(1+\frac{q_j}{q_m\rho(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i)}\right)\\ &\overset{(c)}{=}\frac{\min_l\left(\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})\right)_l}{\log\left(1+1/\rho(\mathbf{B})\right)}\log(1+1/\rho(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i))\end{aligned} \quad (57)$$

where (a) is due to the fact that the index j^k is the optimal index of the max-min SINR problem (since we consider only $q_l(kT) > 0$) at time-slot kT, (b) is due to the inequality $\rho(\mathrm{diag}(\mathbf{z})\mathbf{A}) \leq \max_l z_l\rho(\mathbf{A})$ for any nonnegative and irreducible square matrix $\mathbf{A}$ (e.g., see [30]), and (c) is due to the L'Hopital's rule.

Now, assume that ϱ lies strictly inside

$$\frac{\min_l\left(\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})\right)_l}{\log\left(1+1/\rho(\mathbf{B})\right)}\log\left(1+\frac{1}{\rho\left(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i\right)}\right)\Gamma \quad (58)$$

then there exists some $\epsilon > 0$ such that

$$\begin{aligned}&(1+\epsilon)\sum_{s=1}^{S} H_{ls}\varrho_s\\ &\in\frac{\min_l(\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B}))_l}{\log(1+1/\rho(\mathbf{B}))}\log\left(1+\frac{1}{\rho(\mathbf{F}+(1/\bar{p})\mathbf{v}\mathbf{e}_i)}\right)\mathrm{Co}(\mathcal{R}).\end{aligned} \quad (59)$$

Substituting (59) into (55), we have

$$\begin{aligned}\mathbf{E}&\left[\mathcal{V}\left(\mathbf{n}\left((k+1)T\right),\mathbf{q}\left((k+1)T\right)\right)\right.\\ &\left.-\mathcal{V}\left(\mathbf{n}(kT),\mathbf{q}(kT)\right)|\mathbf{n}(kT),\mathbf{q}(kT)\right]\\ &\leq -T\epsilon\sum_{l=1}^{L} q_l(kT)\sum_{s=1}^{S} H_{ls}\varrho_s + E_1(k) + E_2(k)\end{aligned} \quad (60)$$

thus showing that $\mathcal{V}(\mathbf{n}(kT),\mathbf{q}(kT))$ drifts to zero when queue sizes are large and when the error terms $E_1(k)$ and $E_2(k)$ are bounded [15]. This concludes the proof of Theorem 7.

Acknowledgment

The authors acknowledge helpful discussions with S. Low at the California Institute of Technology (Caltech), Pasadena, S. Friedland at the University of Illinois at Chicago (UIC), and K. Tang at Cornell University, Ithaca, NY.

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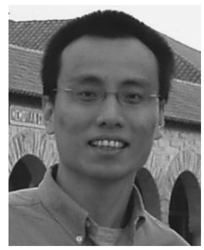

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