# Nonconvex Power Control in Ad Hoc Wireless Networks

Chee Wei Tan

City University Hong Kong

Lake Arrowhead IPAM Workshop
8th June 2010

# Acknowledgement

- Mung Chiang (Princeton)
- R. Srikant (Uni. of Illinois at Urbana-Champaign)
- Steven Low (Caltech)
- Shmuel Friedland (Uni. of Illinois at Chicago)

# What makes a problem easy or hard

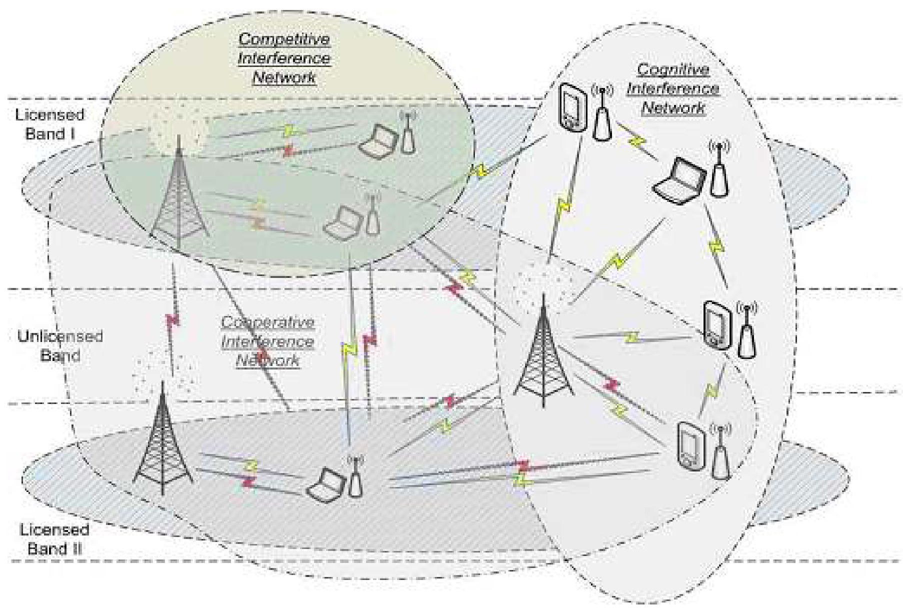
`... the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.`

– *SIAM Review 1993*, R. Rockafellar

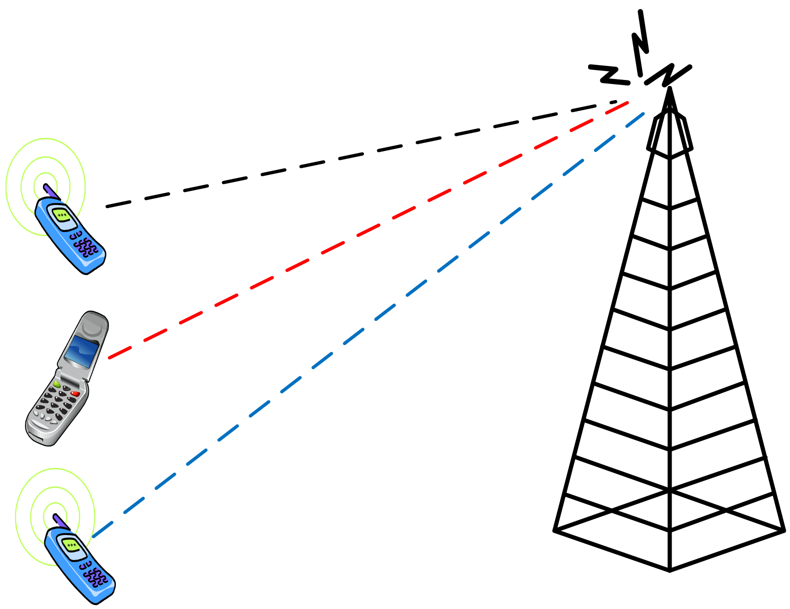
- Linear inequality theory & nonconvex integer programming (1947)
- Semidefinite matrix theory & nonconvex quadratic programming (1995)
- Nonnegative matrix theory & nonconvex cone programming (this talk)

# Motivation

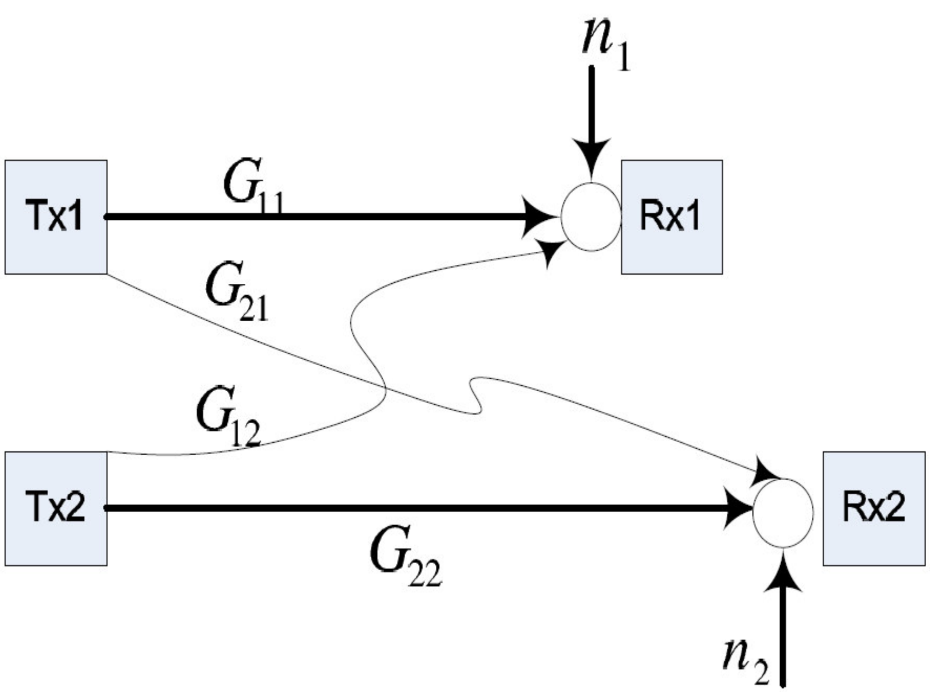
- Wireless Network Power Control in IS-95, CDMA 2000, 3G-4G
- Rayleigh Fading, Multiple Antennas etc



# System Model



(a) Cellular wireless network




(b) Interference channel

# Power Control & Performance Metrics

- Signal-to-Interference Ratio:

$$\mathsf{SIR}_l(\mathbf{p}) = \frac{G_{ll}p_l}{\sum_{j \neq l} G_{lj}p_j + n_l}$$

with $G_{lj}$ the channel gains from transmitter $j$ to receiver $l$ and $n_l$ the additive white Gaussian noise (AWGN) power at receiver $l$

- Power constraints $\mathbf{p} \in \mathcal{P}$, e.g., $\mathbf{p} \leq \bar{\mathbf{p}}$ (Uplink), $\mathbf{1}^\top \mathbf{p} \leq \bar{\mathbf{P}}$ (Downlink)

# Max-min Weighted SIR

- Let $\boldsymbol{\beta}$ be a priority weight vector

$$\begin{array}{rl} \text{maximize} & \min_{l} \dfrac{\mathsf{SIR}_l(\mathbf{p})}{\beta_l} \\ \text{subject to} & p_l \leq \bar{p}_l \quad \forall\, l \end{array}$$

- How to solve this nonconvex problem?

- Fast algorithm? Fast in what sense?

# Max-Min Weighted SIR: Analytical Solution

- **Theorem 1.** *The optimal solution is such that the value* $\mathsf{SIR}_l/\beta_l$ *for all users are equal. The optimal weighted max-min* SIR *is given by*

$$\gamma^* = \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))}, \tag{1}$$

*where*

$$i = \arg\min_l \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))}. \tag{2}$$

*Further, all links* $i$ *that achieve the minimum in (2) transmit at peak power* $\bar{p}_i$ *and the rest do not. Further, the optimal* $\mathbf{p}$*, denoted by* $\mathbf{p}^*$*, is* $t\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))$ *for some constant* $t > 0$*.*

# Conditional Affine Eigenvalue Problem

- Find $(\check{\lambda}, \check{\mathbf{s}})$ in

$$\lambda \mathbf{s} = \mathbf{A}\mathbf{s} + \mathbf{b}, \quad \lambda \in \mathbb{R}, \quad \mathbf{s} \geq \mathbf{0}, \quad \|\mathbf{s}\| = 1, \tag{3}$$

  where $\mathbf{A}$ and $\mathbf{b}$ is a square irreducible nonnegative matrix and nonnegative vector, respectively and $\|\cdot\|$ a monotone vector norm.

- $(\check{\lambda}, \check{\mathbf{s}})$ is the Perron-Frobenius eigenvalue and vector pair of $\mathbf{A} + \mathbf{b}\mathbf{c}_*^\top$, where

$$\mathbf{c}_* = \arg \max_{\|\mathbf{c}\|_* = 1} \rho(\mathbf{A} + \mathbf{b}\mathbf{c}^\top), \tag{4}$$

  where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$, and $\check{\mathbf{s}} = (\mathbf{A}\check{\mathbf{s}} + \mathbf{b}) / \|\mathbf{A}\check{\mathbf{s}} + \mathbf{b}\|$.

An affine eigenvalue problem on the nonnegative orthant, V. D. Blondel, L. Ninove and P. Van Dooren, Linear Algebra & its Applications, Vol. 404, 2005

# Max-min SIR & Conditional Affine Eigenvalue

- Individual power constraints ($\bar{p}_1 = \bar{p}_2 = \cdots = \bar{p}_L = \bar{p}$):

$$\mathsf{SIR}_l(\mathbf{p}^*) = \tau^* \beta_l \;\Rightarrow\; \frac{(p_l^*/\bar{p})}{\sum_{j \neq l} F_{lj}(p_l^*/\bar{p}) + (v_l/\bar{p})} = \tau^* \beta_l \qquad (5)$$

Let $\mathbf{s}^* = (1/\bar{p})\mathbf{p}^*$:

$$(1/\tau^*)\mathbf{s}^* = \mathsf{diag}(\boldsymbol{\beta})\mathbf{F}\mathbf{s}^* + (1/\bar{p})\mathsf{diag}(\boldsymbol{\beta})\mathbf{v}, \quad \|\mathbf{x}\|_\infty = 1 \qquad (6)$$

- Conditional eigenvalue problem:
  - $s_l = p_l/\bar{p}_l$, $\mathbf{A} = \mathrm{diag}(\boldsymbol{\beta})\mathbf{F}$, $\mathbf{b} = (1/\bar{p})\,\mathrm{diag}(\boldsymbol{\beta})\mathbf{v}$ and $\lambda = 1/\tau^*$
  - $\|\cdot\| = \|\cdot\|_\infty \longleftrightarrow \|\cdot\|_* = \|\cdot\|_1$ & $\mathbf{c}_* = \mathbf{e}_i$
  - $(\check{\lambda}, \check{\mathbf{s}})$ is the Perron-Frobenius eigenvalue and vector pair of $\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_i{}^\top)$

# A Max-min SIR Algorithm

- **Algorithm 1. [Equal power constrained Max-min** SIR**]**

  *1. Update power* $\mathbf{p}(k+1)$*:*

$$p_l(k+1) = \frac{\beta_l}{\mathsf{SIR}_l(\mathbf{p}(k))} p_l(k) \ \ \forall\, l.$$

  *2. Normalize* $\mathbf{p}(k+1)$*:*

$$p_l(k+1) = p_l(k+1) / \max_j p_j(k+1) \cdot \bar{p} \ \ \forall\, l.$$

- **Theorem 2.** *Starting from any initial point* $\mathbf{p}(0)$, $\mathbf{p}(k)$ *in Algorithm 1 converges geometrically fast to* $\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_i^\top))$ *(unique up to a scaling constant).*

# Interlude: Nonlinear Perron-Frobenius Theory

- A mapping $T : K \to K$ is concave if

  $$T(a\mathbf{x}+(1-a)\mathbf{y}) \geq aT\mathbf{x}+(1-a)T\mathbf{y} \text{ for all } \mathbf{x},\mathbf{y} \in K \text{ and } a \in [0,1],$$

  and montone if $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y}$ implies $\mathbf{0} \leq T\mathbf{x} \leq T\mathbf{y}$.

- **Theorem 3. [Krause01]** *Let $\|\cdot\|$ be a monotone norm on $\mathbb{R}^L$. For a concave mapping $f : \mathbb{R}_+^L \to \mathbb{R}_+^L$ with $f(\mathbf{z}) > 0$ for $\mathbf{z} \geq \mathbf{0}$, the following statements hold. The conditional eigenvalue problem $f(\mathbf{z}) = \lambda\mathbf{z}$, $\lambda \in \mathbb{R}$, $\mathbf{z} \geq \mathbf{0}$, $\|\mathbf{z}\| = 1$ has a unique solution $(\lambda^*, \mathbf{z}^*)$, where $\lambda^* > 0$, $\mathbf{z}^* > \mathbf{0}$. Furthermore, $\lim_{k\to\infty} \tilde{f}^k(\mathbf{z})$ converges geometrically fast to $\mathbf{z}^*$, where $\tilde{f}(\mathbf{z}) = f(\mathbf{z})/\|(\mathbf{z})\|$.*

# Interlude: Nonlinear Perron-Frobenius Theory

- $T\mathbf{p} = \mathbf{F}\mathbf{p}$, where $\mathbf{F}$ is a irreducible nonnegative matrix
  - Classical (Linear) Perron-Frobenius Theory
  - Power Method: $\mathbf{p}(k+1) \leftarrow \frac{\mathbf{F}\mathbf{p}(k)}{\|\mathbf{F}\mathbf{p}(k)\|}$, & $\lim_{k\rightarrow\infty} \mathbf{p}(k) = \mathbf{x}(\mathbf{F})$

- $T\mathbf{p} = \mathbf{F}\mathbf{p} + \mathbf{v}$, where $\mathbf{v}$ is a nonnegative vector
  - Conditional Affine Eigenvalue Problem **[BlondelNinoveVanDooren05]**

- Other interesting $T\mathbf{p}$?
  How to handle different constraints on $\mathbf{p}$?

# Probabilistic Max-min Weighted SIR

- Under Rayleigh fading, power received from the $j$th transmitter at $l$th receiver is given by $G_{lj}R_{lj}P_j$ and exponentially distributed with mean $E[G_{lj}R_{lj}p_j] = G_{lj}p_j$ ($R_{lj}$ models Rayleigh fading - unit mean exponential random variable).

$$
\begin{array}{rl}
\text{minimize} & \max_{l} \;\; P(\mathsf{SIR}_l(\mathbf{p}) < \beta_l) \\
\text{subject to} & \mathbf{p} \in \mathcal{P}, \\
\text{variables:} & \mathbf{p},
\end{array}
$$

- How to solve this nonconvex chance-constrained problem?

- Fast algorithm?

# An Equivalent Deterministic Problem

- Under Rayleigh Fading **[KandukuriBoyd02,Haenggi04]**:

$$P(\mathsf{SIR}_l(\mathbf{p}) < \beta_l) = 1 - e^{-v_l \beta_l / p_l} \prod_j \left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right)^{-1}$$

- Deterministic optimization to a chance-constrained problem:

$$\begin{array}{rl} \text{minimize} & \max_l \; 1 - e^{-v_l \beta_l / p_l} \prod_j \left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right)^{-1} \\ \text{subject to} & \mathbf{p} \in \mathcal{P}, \\ \text{variables:} & \mathbf{p}, \end{array}$$

- When $\mathbf{v} = \mathbf{0}$, use geometric programming **[KandukuriBoyd02]**

# Reformulation & Optimality

- A nonconvex problem with hidden convexity:

$$
\begin{array}{rl}
\text{minimize} & \alpha \\
\text{subject to} & v_l\beta_l/p_l + \sum_j \log\left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right) \le \alpha \ \ \forall\, l, \\
 & \mathbf{p} \in \mathcal{P}, \\
\text{variables:} & \mathbf{p},\ \alpha.
\end{array}
$$

- Use logarithmic change of variable and interior point method. But can we say more?

- At optimality:

$$
v_l\beta_l/p_l^\star + \sum_j \log\left(1 + \frac{\beta_l F_{lj} p_j^\star}{p_l^\star}\right) = \alpha^\star \ \ \text{for all } l
$$

# Connection to Nonlinear Perron-Frobenius Theory

- Denote a matrix $\mathbf{B}$ with the entries (that are functions of $\mathbf{p}$):

$$B_{lj} = \begin{cases} 0, & \text{if } l = j \\ \frac{p_l}{\beta_l p_j} \log\left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right), & \text{if } l \neq j. \end{cases}$$

- Examine optimality condition using the nonlinear Perron-Frobenius theory:

$$f_l(\mathbf{p}) = \beta_l((\mathbf{B}(\mathbf{p})\mathbf{p})_l + v_l) = \alpha_l p_l \text{ for all } l, \text{ with } \mathbf{p} \in \mathcal{P}.$$

- Prove Concavity by Perspective Function: If $f(x)$ is convex, then its perspective function $tf(x/t)$ is also convex.

# Comparison

| Concave Self-mapping $T\mathbf{p}$ | Perron eigenvalue |
|---|---|
| First part (deterministic max-min SIR) | |
| $(\mathbf{F}\mathbf{p} + (1/\bar{p})\mathbf{v})_l$,<br>$i = \arg\max_l \rho\left(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_l^\top\right)$ | $\rho\left(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_i^\top\right)$ |
| $(\mathbf{F}\mathbf{p} + (1/\bar{P})\mathbf{v})_l$ | $\rho\left(\mathbf{F} + (1/\bar{P})\mathbf{v}\mathbf{1}^\top\right)$ |

- The fixed point in each above system, i.e., optimal power, is the Perron-Frobenius eigenvector

# Comparison

| Concave Self-mapping $T\mathbf{p}$ | Perron eigenvalue |
|---|---|
| First part (deterministic max-min SIR) | |
| $(\mathbf{F}\mathbf{p} + (1/\bar{p})\mathbf{v})_l$,<br>$i = \arg\max_l \rho\left(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_l^\top\right)$ | $\rho\left(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_i^\top\right)$ |
| $(\mathbf{F}\mathbf{p} + (1/\bar{P})\mathbf{v})_l$ | $\rho\left(\mathbf{F} + (1/\bar{P})\mathbf{v}\mathbf{1}^\top\right)$ |
| Second part (probabilistic max-min SIR) | |
| $v_l\beta_l + \sum_j p_l \log\left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right)$<br>$i = \arg\max_l \rho\left(\mathbf{B}(\mathbf{p}^\star) + (1/\bar{p})\text{diag}(\boldsymbol{\beta})\mathbf{v}\mathbf{e}_l^\top\right)$ | $\rho\left(\mathbf{B}(\mathbf{p}^\star) + (1/\bar{p})\text{diag}(\boldsymbol{\beta})\mathbf{v}\mathbf{e}_i^\top\right)$ |
| $v_l\beta_l + \sum_j p_l \log\left(1 + \frac{\beta_l F_{lj} p_j}{p_l}\right)$ | $\rho\left(\mathbf{B}(\mathbf{p}^\star) + (1/\bar{P})\text{diag}(\boldsymbol{\beta})\mathbf{v}\mathbf{1}^\top\right)$ |

- The fixed point in each above system, i.e., optimal power, is the Perron-Frobenius eigenvector

# Solve Previous Open Problems

- No-noise ($\mathbf{v} = \mathbf{0}$) case **[KandukuriBoyd02]** 'heuristic':

$$\mathbf{p}(k+1) \leftarrow \text{diag}(\boldsymbol{\beta})\mathbf{B}(\mathbf{p}(k))\mathbf{p}(k)$$

  converges from *any* initial point

- Multiple antenna case **[WieselEldarShamai04]**:

$$(T\mathbf{p})_l = \frac{1}{\mathbf{u}_l^\dagger \mathbf{H}_l (\sum_{j=1}^{L} p_j \mathbf{H}_j \mathbf{H}_j^\dagger + \mathbf{I})^{-1} \mathbf{H}_l^\dagger \mathbf{v}_l}$$

  and convergence of 'heuristic'

- Concavity, Monotonicity & 'Power Method'

# Extensions & Applications

- Link nonconvex nonnegative cone programming and nonnegative matrix theory
- Fundamental spectral radius minimax theorem **[FriedlandKarlin75]**
- Convergence rate: How to quantify the 'nonlinear' second largest eigenvalue?
- Intriguing link to other applications:
  - network resource allocation (arbitrarily affine constraints)
  - network traffic estimation, network data mining (nonnegative matrix factorization) . . .
- Nonnegative nonconvex optimization is 'interesting'

# Thank You

http://www.cs.cityu.edu.hk/∼cheewtan

cheewtan@cityu.edu.hk