Fast Algorithms and Performance Bounds for Sum Rate Maximization in Wireless Networks

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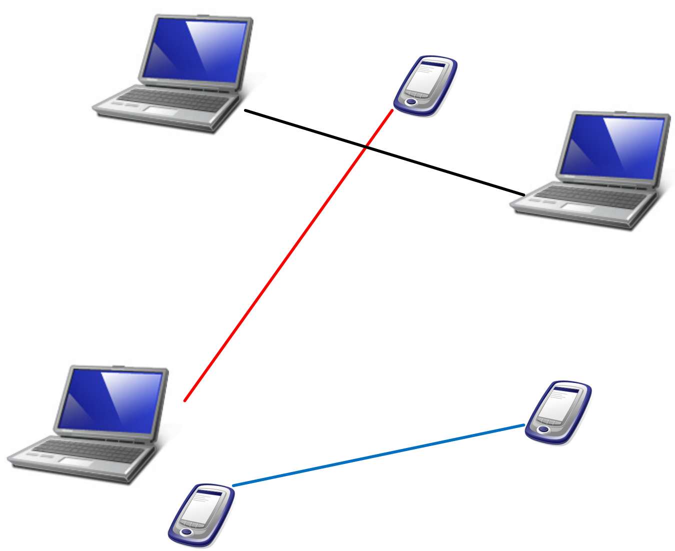
Outline

- Motivations
- System Model
- Nonconvex Sum Rate Maximization
- Power Control Algorithms with Performance Guarantees
- Geometry of Rate Region
- On-going Work

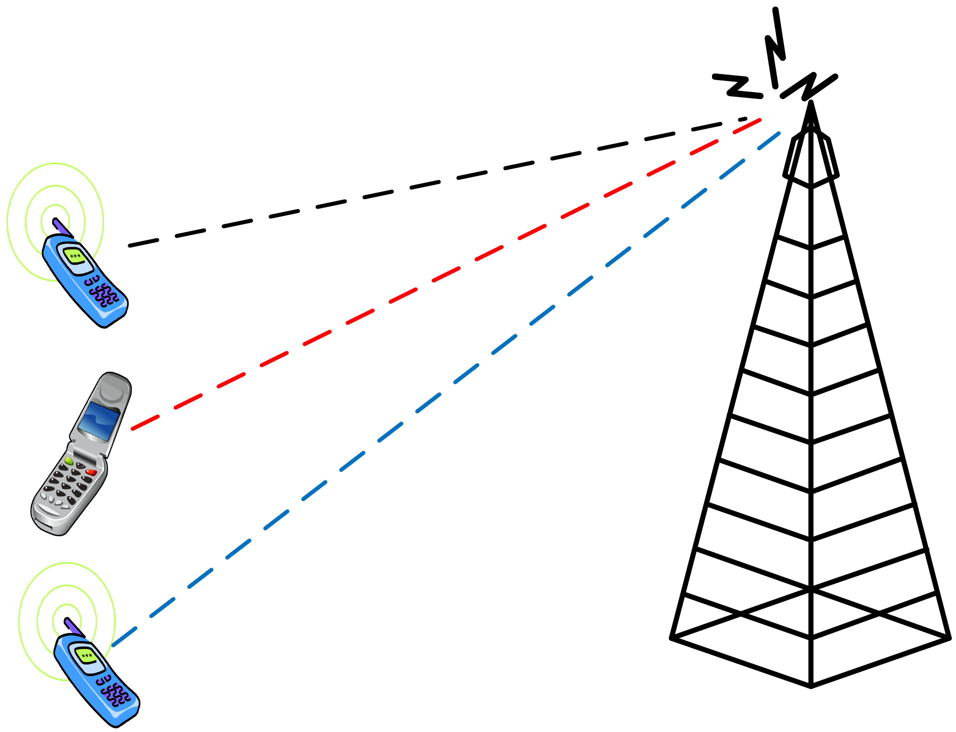
Motivation 1

- IEEE 802.11b ad hoc network cross-layer (TCP/IP/MAC)
- TCP/IP and application layers demand data rate
- MAC/Physical layers build variable capacity 'pipes' as supply



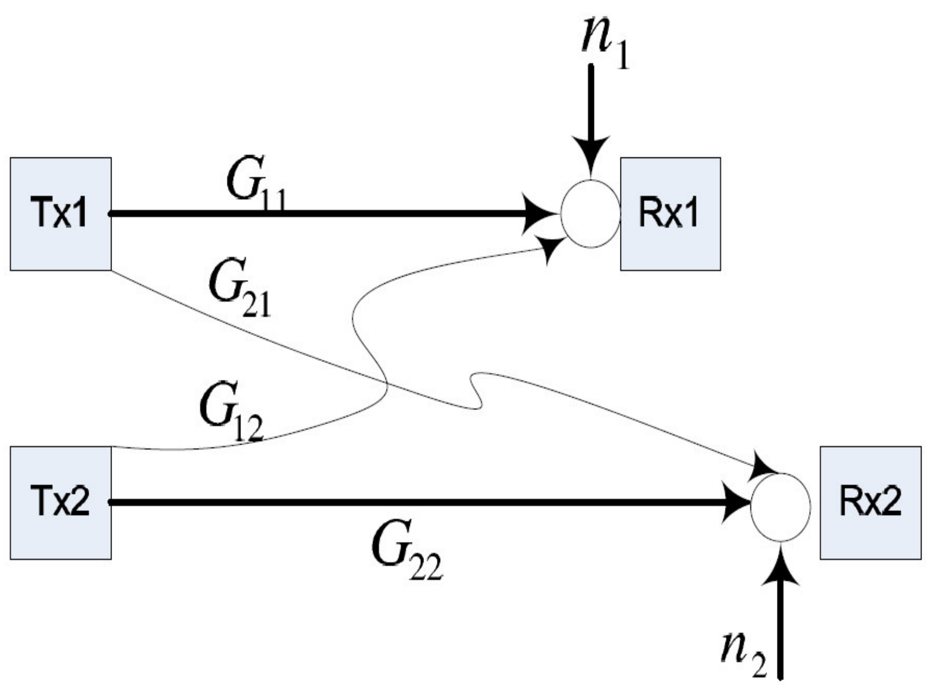
Motivation 2

- 3G CDMA2000 EV-DO cellular network link adaptation maximizes uplink/downlink rate using power control



System Model

- Interference channel with single-user decoding: Treat interference as additive Gaussian noise
- Control interference and meet objective using power control



Performance Metrics

- Signal-to-Interference Ratio:

$$\mathsf{SIR}_l(\mathbf{p}) = \frac{G_{ll}p_l}{\sum_{j \neq l} G_{lj}p_j + n_l}$$

with G_{lj} the channel gains from transmitter j to receiver l and n_l the additive white Gaussian noise (AWGN) power at receiver l

- Attainable data rate (nats per channel use) is a function of SIR, e.g., Shannon capacity formula $r_l = \log(1 + \mathsf{SIR}_l)$

- Mean Squared Error (MSE) of received signal, e.g., $(1 + \mathsf{SIR}_l)^{-1}$

- Power constraints $\mathbf{p} \leq \bar{\mathbf{p}}$

Interference Parameters

- Let $\mathbf{F}$ be a nonnegative matrix with entries:

$$F_{ij} = \begin{cases} 0, & \text{if } i = j \\ \frac{G_{ij}}{G_{ii}}, & \text{if } i \neq j \end{cases}$$

  and

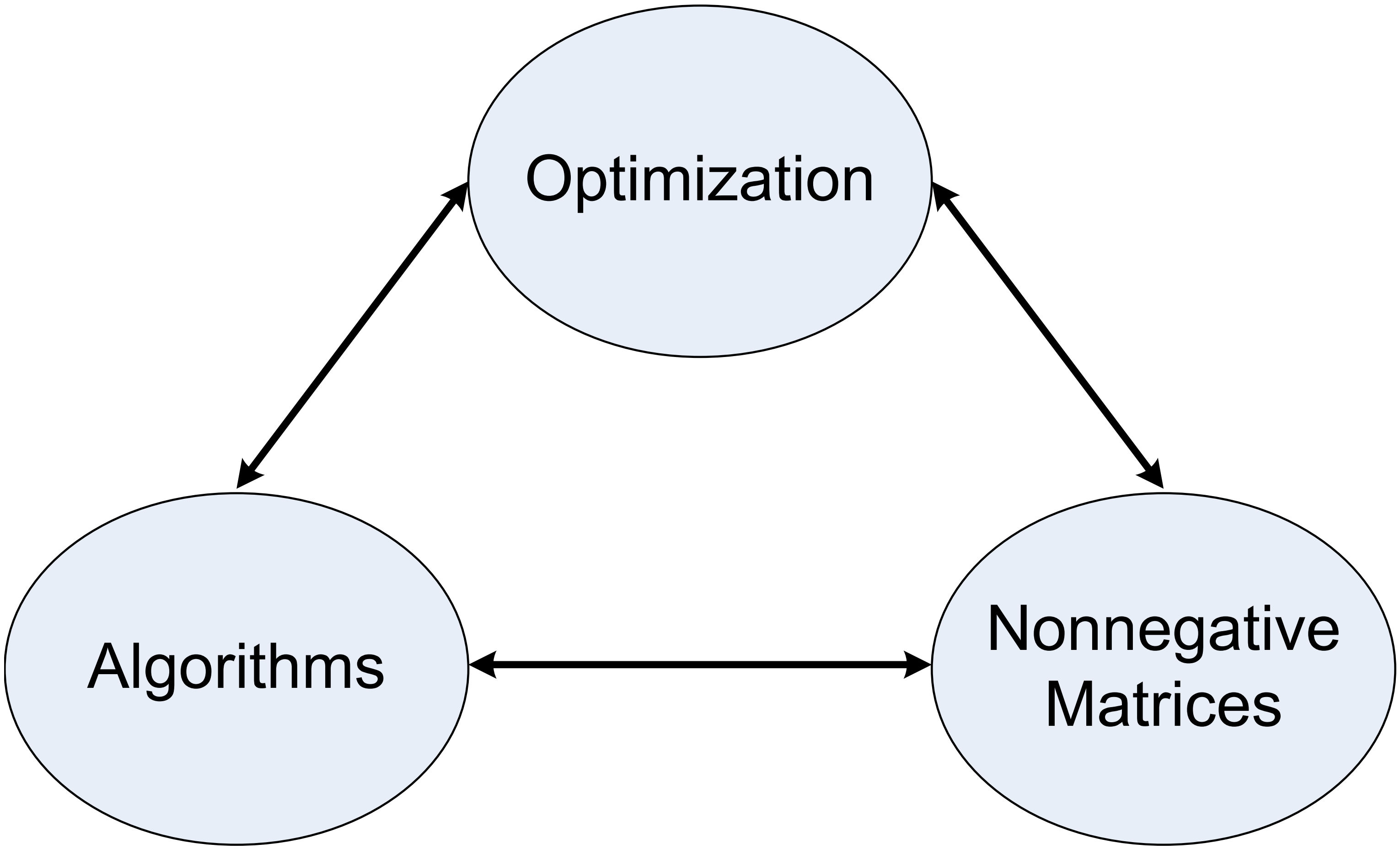
$$\mathbf{v} = \left(\frac{n_1}{G_{11}}, \frac{n_2}{G_{22}}, \ldots, \frac{n_L}{G_{LL}} \right)^\top .$$

- $\mathbf{F}$ is irreducible (each user has at least one interferer)

System Considerations

- How to solve optimally nonconvex power control problems?
- How many ways to characterize optimality?
- How to design distributed power control algorithms with fast convergence and good performance guarantees?
- How fast is fast?
- Can we leverage existing technology?
- What is the industry impact?

Interplay of Mathematical Tools



Problem: Maximize Sum Shannon Rates

- Find $\mathbf{p}^\star = \arg \max_{\mathbf{0} \le \mathbf{p} \le \bar{\mathbf{p}}} \sum_l w_l \log(1 + \mathsf{SIR}_l(\mathbf{p}))$ where $\mathbf{1}^\top \mathbf{w} = 1$

- Characterize the achievable rate region: $r_l = \log(1 + \mathsf{SIR}_l(\mathbf{p}))\ \forall\, l$

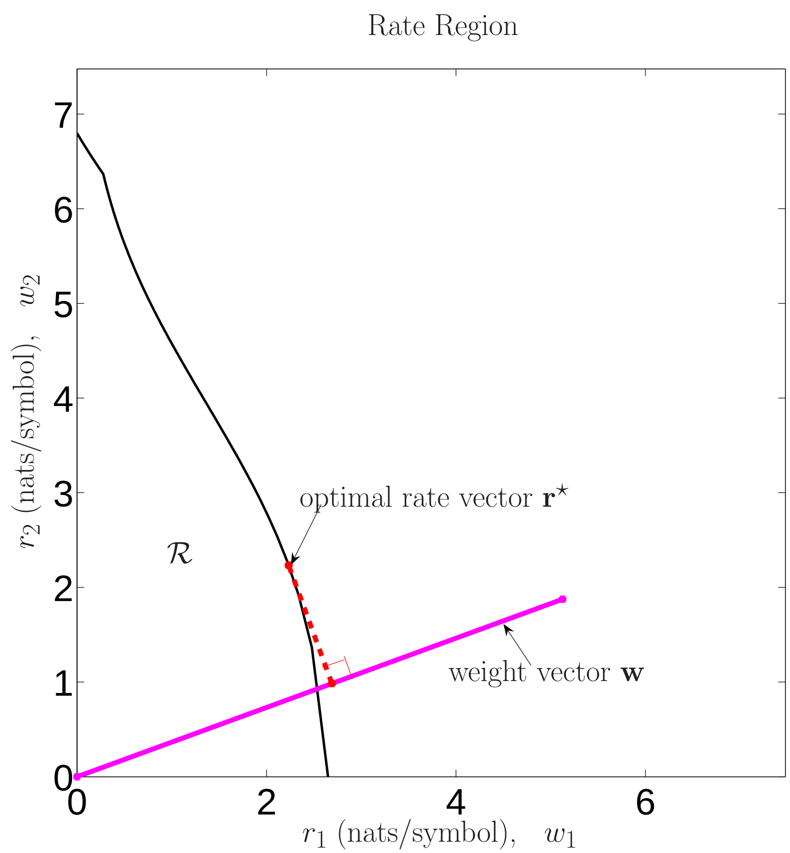
- Two-User case:

$$\max \quad w_1 \log\left(1 + \frac{G_{11} p_1}{G_{12} p_2 + n_1}\right) + w_2 \log\left(1 + \frac{G_{22} p_2}{G_{21} p_1 + n_2}\right)$$

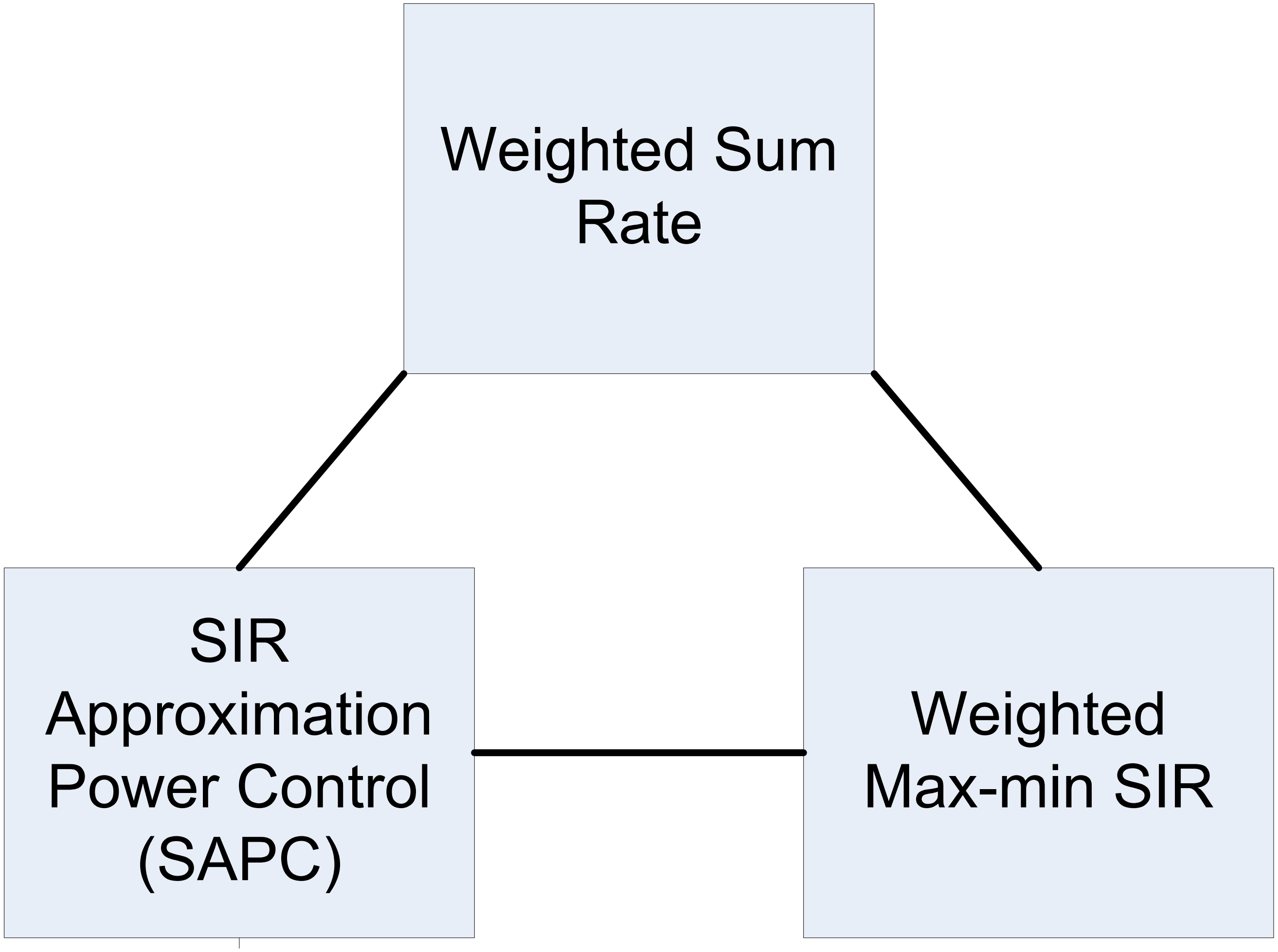
subject to: $\quad 0 \le p_1 \le \bar{p}_1,\ \ 0 \le p_2 \le \bar{p}_2$

Sum Rate Geometry Illustration

$$\begin{array}{rl} \text{maximize} & \sum_l w_l \log(1 + \mathsf{SIR}_l(\mathbf{p})) \;=\; \sum_l w_l\, r_l \\ \text{subject to} & 0 \le p_l \le \bar{p}_l \quad \forall\, l, \\ \text{variables:} & p_l \quad \forall\, l. \end{array}$$



Fast Algorithms with Performance Guarantees



Tan, Chiang and Srikant, *Fast Algorithms and Performance Bounds for Sum Rate Maximization in Wireless Networks*, IEEE INFOCOM, 2009

SAPC: New Perspective

- Drop the '1' approach:

$$\begin{array}{ll} \text{maximize} & \sum_l w_l \log \mathsf{SIR}_l(\mathbf{p}) \\ \text{subject to} & 0 \le p_l \le \bar{p}_l \quad \forall\, l, \\ \text{variables:} & p_l \quad \forall\, l. \end{array}$$

- Geometric programming ($\tilde{p}_l = \log p_l$)

  Chiang, Tan, Palomar, O'Neill, Julian, *Power Control by Geometric Programming*, IEEE Trans Wireless Comms, 2007

- 1) Connection with Weighted max-min SIR
  2) New algorithm with faster convergence

SAPC: Algorithm

- **Algorithm 1. [SAPC Algorithm]**

  1. Update $\mathbf{p}(k+1)$:

$$p_l(k+1) = \min\left\{ w_l / \left(\sum_{j \neq l} \frac{w_j F_{jl} \mathsf{SIR}_j(\mathbf{p}(k))}{p_j} \right), \bar{p}_l \right\}$$

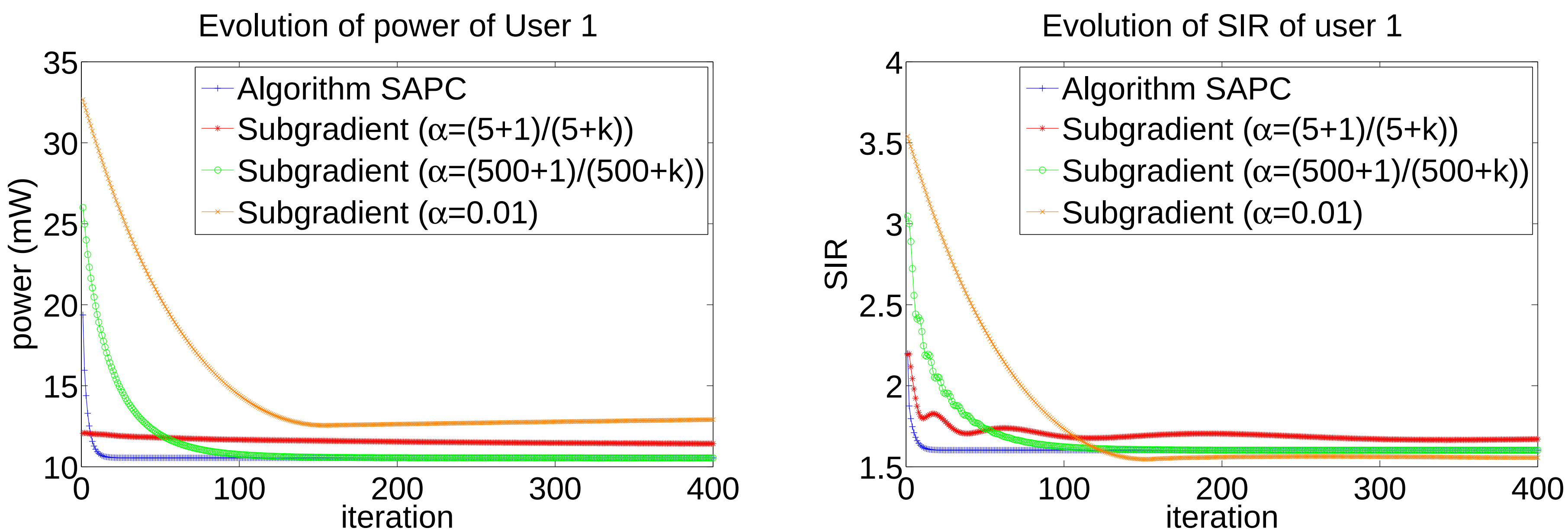
  for all l, where k indexes discrete time slots.

  Theorem 1. *Starting from any initial point $\mathbf{p}(0)$, $\mathbf{p}(k)$ in Algorithm 1 converges to $\mathbf{p}'$ asymptotically, the optimal solution to SAPC under synchronous and asynchronous updates.*

- Geometrically fast when the initial point is $\overline{\mathbf{p}}$.

SAPC: Examples

- Algorithm SAPC is faster than the gradient algorithm (stepsize α)



(a) (b)

Weighted Max-Min SIR

- Consider $\max_{\mathbf{p}\geq\mathbf{0}} \min_{l} \dfrac{\mathsf{SIR}_l(\mathbf{p})}{\beta_l}$ subject to $p_l \leq \bar{p}_l \quad \forall\, l$

- **Theorem 2.** *The optimal solution is such that the value* SIR_l/β_l *for all users are equal. The optimal weighted max-min* SIR *is given by*

$$\gamma^* = \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))},$$

where

$$i = \arg\min_{l} \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))}.$$

Further, all links i *transmit at peak power* $\bar{p}_i$ *and the rest do not. Further, the optimal* $\mathbf{p}$*, denoted by* $\mathbf{p}^*$*, is* $t\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))$ *for some constant* $t > 0$.

Connecting SAPC & Max-min SIR

- Let $\mathbf{x}$ and $\mathbf{y}$ be the Perron and left eigenvectors of $\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top$ respectively, where $i = \arg\min_l \dfrac{1}{\rho(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)}$

- Set $\mathbf{w} = \mathbf{x} \circ \mathbf{y}$ in SAPC:

$$\begin{array}{rl} \text{maximize} & \sum_l x_l y_l \log \mathsf{SIR}_l(\mathbf{p}) \\ \text{subject to} & 0 \leq p_l \leq \bar{p}_l \quad \forall\, l, \\ \text{variables:} & p_l \quad \forall\, l. \end{array}$$

$\mathbf{p}^* = \mathbf{x}$ (unique up to a scaling constant)

Max-min SIR: Primal-Dual Algorithm

- **Algorithm 2. [Weighted Max-min** SIR **Algorithm]**

  1. *Initialize an arbitrarily positive* $\mathbf{w}(t)$ *and small* $\epsilon, \alpha(1)$*.*
  2. *Set* $\mathbf{p}(0) = \bar{\mathbf{p}}$*. Repeat*

  $$p_l(k+1) = \min\left\{ w_l(t) \Big/ \left(\sum_{j \neq l} \frac{w_j(t) F_{jl} \mathsf{SIR}_j(\mathbf{p}(k))}{p_j(k))} \right), \bar{p}_l \right\}$$

  until $\|\mathbf{p}(k+1) - \mathbf{p}(k)\| \leq \epsilon$*.*

  3. *Compute*

  $$\begin{aligned} w_l(t+1) = \max\{w_l(t) \quad & +\alpha(t)(\textstyle\sum_j w_j(t) \log(\mathsf{SIR}_j(\mathbf{p}(k+1))/\beta_j) - \\ & \log(\mathsf{SIR}_l(\mathbf{p}(k+1))/\beta_l)),\ 0\} \end{aligned}$$

  for all l*, where* t *indexes discrete time slots much larger than* k*.*

  4. *Normalize* $\mathbf{w}(t+1)$ *so that* $\mathbf{1}^\top \mathbf{w}(t+1) = 1$*. Go to Step 2.*

Nonlinear Perron-Frobenius Theory

- Find $(\check{\lambda}, \check{\mathbf{s}})$ in

$$\lambda \mathbf{s} = \mathbf{A}\mathbf{s} + \mathbf{b}, \quad \lambda \in \mathbb{R}, \quad \mathbf{s} \geq \mathbf{0}, \quad \|\mathbf{s}\| = 1,$$

where $\mathbf{A}$ and $\mathbf{b}$ is a square irreducible nonnegative matrix and nonnegative vector, respectively and $\|\cdot\|$ a monotone vector norm.

- $(\check{\lambda}, \check{\mathbf{s}})$ is the Perron-Frobenius eigenvalue and vector pair of $\mathbf{A} + \mathbf{b}\mathbf{c}_*^\top$, where

$$\mathbf{c}_* = \arg \max_{\|\mathbf{c}\|_* = 1} \rho(\mathbf{A} + \mathbf{b}\mathbf{c}^\top),$$

where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$, and $\check{\mathbf{s}} = (\mathbf{A}\check{\mathbf{s}} + \mathbf{b}) / \|\mathbf{A}\check{\mathbf{s}} + \mathbf{b}\|$.

V. D. Blondel, L. Ninove and P. Van Dooren, *An affine eigenvalue problem on the nonnegative orthant*, Linear Algebra & its Applications, 2005

Nonlinear Perron-Frobenius Theory: Max-min SIR

- Individual power constraints ($\bar{p}_1 = \bar{p}_2 = \cdots = \bar{p}_L$):

$$\mathsf{SIR}_l(\mathbf{p}^*) = \tau^*\beta_l \;\Rightarrow\; \frac{(p_l^*/\bar{p}_i)}{\sum_{j\neq l} F_{lj}(p_l^*/\bar{p}_i) + (v_l/\bar{p}_i)} = \tau^*\beta_l$$

Let $\mathbf{s}^* = (1/\bar{p}_i)\mathbf{p}^*$:

$$(1/\tau^*)\mathbf{s}^* = \mathsf{diag}(\boldsymbol{\beta})\mathbf{F}\mathbf{s}^* + (1/\bar{p}_i)\mathsf{diag}(\boldsymbol{\beta})\mathbf{v}, \quad \|\mathbf{s}\|_\infty = 1$$

- ▪ $s_l = p_l/\bar{p}_l$, $\mathbf{A} = \mathrm{diag}(\boldsymbol{\beta})\mathbf{F}$, $\mathbf{b} = (1/\bar{p}_i)\,\mathrm{diag}(\boldsymbol{\beta})\mathbf{v}$ and $\lambda = 1/\tau^*$
  - ▪ $\|\cdot\| = \|\cdot\|_\infty \longleftrightarrow \|\cdot\|_* = \|\cdot\|_1$ & $\mathbf{c}_* = \mathbf{e}_i$
  - ▪ $(\check{\lambda}, \check{\mathbf{s}})$ is the Perron-Frobenius eigenvalue and vector pair of $\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$

A Faster Max-min SIR Algorithm

- **Algorithm 3. [Equal power constrained Max-min** SIR**]**

  1. Update power $\mathbf{p}(k+1)$*:*

$$p_l(k+1) = \frac{\beta_l}{\mathsf{SIR}_l(\mathbf{p}(k))} p_l(k) \;\; \forall\, l.$$

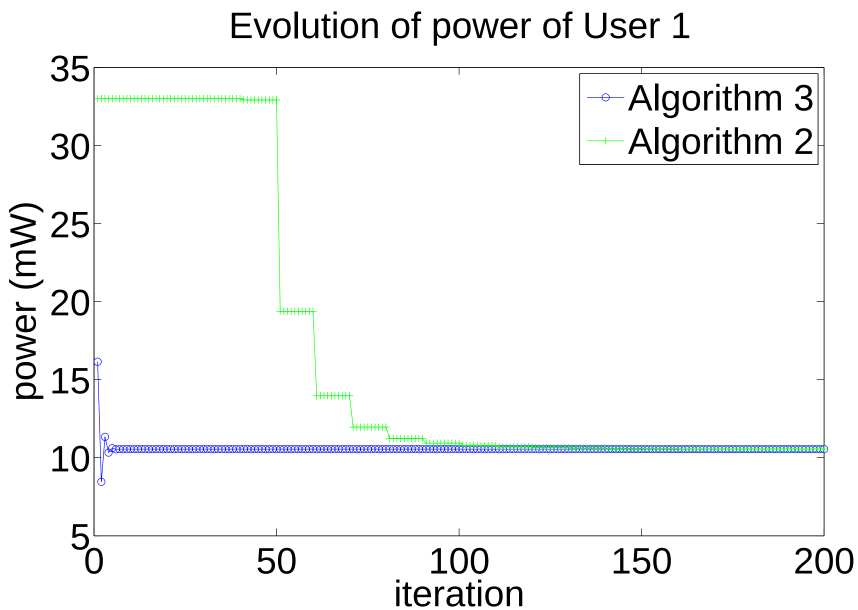
  2. Normalize $\mathbf{p}(k+1)$*:*

$$p_l(k+1) = p_l(k+1) / \max_j p_j(k+1) \cdot \bar{p}_i \;\; \forall\, l.$$

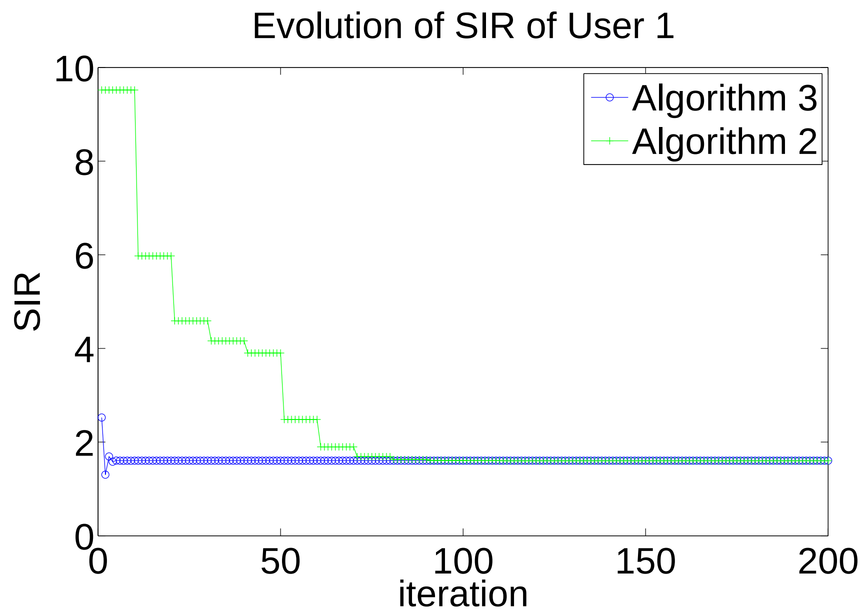
- **Theorem 3.** *Starting from any initial point* $\mathbf{p}(0)$*,* $\mathbf{p}(k)$ *in Algorithm 3 converges geometrically fast to* $\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))$ *(unique up to a scaling constant).*

Max-min SIR: Examples

- The nonlinear Perron-Frobenius theory based algorithm is much faster than the subgradient algorithm



(a)



(b)

Goodness of Suboptimality

- Max-min SIR, SAPC, All-$\bar{\mathbf{p}}$, One-On-Others-Off, ...

- Good suboptimal solutions may have attractive implementation quality
  - Simplicity, distributed protocol, fairness, backward compatibility

- Strongly NP-hard and Inapproximability **[LuoZhang07]**

  Z.-Q. Luo and S. Zhang, *Dynamic Spectrum Management: Complexity and Duality*, IEEE J. of Selected Topics in Signal Processing, 2007

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$$\text{Objective}(\mathbf{p}_{\text{approx}}) \leq \text{Objective}(\mathbf{p}^{\star}) \leq \eta \cdot \text{Objective}(\mathbf{p}_{\text{approx}})$$
  where $\eta \geq 1$

Quasi-Inverse of Nonnegative Matrices

- Definition **[Wong54]**: $\mathbf{B}$ is a quasi-inverse of $\tilde{\mathbf{B}} \geq \mathbf{0}$ if $\mathbf{B} - \tilde{\mathbf{B}} = \mathbf{B}\tilde{\mathbf{B}} = \tilde{\mathbf{B}}\mathbf{B} \geq \mathbf{0}$

- Instrumental property of Minkowski-Leontief (ML) matrices in mathematical economy **[Wong54]**

- $(\mathbf{I} + \mathbf{B})^{-1}\mathbf{B} = \mathbf{B}(\mathbf{I} + \mathbf{B})^{-1} \geq \mathbf{0}$

- $\rho(\tilde{\mathbf{B}}) = \frac{\rho(\mathbf{B})}{1+\rho(\mathbf{B})}$

- $\mathbf{x}(\tilde{\mathbf{B}}) = \mathbf{x}(\mathbf{B})$ & $\mathbf{y}(\tilde{\mathbf{B}}) = \mathbf{y}(\mathbf{B})$

Interference & SNR Regime

- Consider the matrix

$$\mathbf{B} = \mathbf{F} + \sum_{l} \frac{1}{\mathbf{1}^\top \overline{\mathbf{p}}} \mathbf{v} \mathbf{e}_l^\top$$

- (High SNR regime) $\tilde{\mathbf{B}}$ does not exist

  or any nonnegative matrix with a zero trace & positive off-diagonals

- (Low SNR regime) $\tilde{\mathbf{B}}$ always exists

  or any nonnegative matrix that is a dyad

- (Low interference/moderate SNR regime) $\tilde{\mathbf{B}}$ almost always exists

Tight Upper Bound: Key Theorem

- If $\tilde{\mathbf{B}} \geq \mathbf{0}$, then

$$\sum_l w_l \log(1 + \mathsf{SIR}_l(\mathbf{p}^\star)) \leq \|\mathbf{w}\|_\infty^{\mathbf{x}\circ\mathbf{y}} \log\left(1 + 1/\rho(\mathbf{B})\right),$$

where $\mathbf{x}, \mathbf{y}$ are the Perron and left eigenvectors of $\mathbf{B}$ respectively.

- Main ideas of proof:
  - Relaxation of nonconvexity
  - Quasi-invertibility of nonnegative matrix **[Wong54]**
  - Friedland-Karlin Inequalities **[FriedlandKarlin75]**

- Physical and operational meaning of upper bound

Physical Interpretation of Upper Bound (I)

- $\sum_l w_l \log(1 + \mathsf{SIR}_l(\mathbf{p}^\star)) \leq \|\mathbf{w}\|_\infty^{\mathbf{x}\circ\mathbf{y}} \log\left(1 + 1/\rho(\mathbf{B})\right).$

- ▪ $1 \leq \|\mathbf{w}\|_\infty^{\mathbf{x}\circ\mathbf{y}} \leq \frac{1}{\min_l\,(\mathbf{x}\circ\mathbf{y})_l}$ as an approximation ratio using

$$\begin{array}{ll} \text{maximize} & \min_l \mathsf{SIR}_l(\mathbf{p}) \\ \text{subject to} & \mathbf{1}^\top \mathbf{p} \leq \mathbf{1}^\top \bar{\mathbf{p}} \\ \text{variables:} & \mathbf{p}. \end{array}$$

▪ Closed-form solution (via Nonlinear Perron-Frobenius Theory):

$$\begin{array}{ll} \text{Optimal solution :} & 1/\rho(\mathbf{B}), \quad \mathbf{B} = \mathbf{F} + (1/\mathbf{1}^\top \bar{\mathbf{p}})\mathbf{v}\mathbf{1}^\top; \\ \text{Optimizer :} & \mathbf{x}(\mathbf{B}) \end{array}$$

General Bounds

- A subset of users $\mathcal{C} = \{l \,|\, l = 1, \ldots, L\}$ with $|\mathcal{C}| \leq L$. Users in $\mathcal{C}$ transmit with positive power. Users that belong to $\bar{\mathcal{C}}$ are removed (delete rows/columns of $\mathbf{B}$)

- L users $\Rightarrow \sum_{l=1}^{L-2} \binom{L}{l} + 2$ possible configurations

- General upper bound (subject to $\tilde{\mathbf{B}}_{\mathcal{C}} \geq \mathbf{0}$):

$$\sum_{l=1}^{L} w_l \log(1 + \mathsf{SIR}_l(\mathbf{p}^\star))$$
$$\leq \max_{l \in \mathcal{C}} \frac{w_l}{(\mathbf{x}(\mathbf{B}_{\mathcal{C}}) \circ \mathbf{y}(\mathbf{B}_{\mathcal{C}}))_l} \log\left(1 + \frac{1}{\rho(\mathbf{B}_{\mathcal{C}})}\right) + \sum_{l \in \bar{\mathcal{C}}} w_l \log(1 + \frac{G_{ll}\bar{p}_l}{n_l})$$

Performance Guarantee: Weighted Max-min SIR

- **Theorem 4.** *Suppose* $\tilde{\mathbf{B}} \geq \mathbf{0}$*. Let*

$$\eta = \frac{\sum_l w_l \log(1 + w_l/\rho(\mathit{diag}(\mathbf{w})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)))}{\|\mathbf{w}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})} \log(1 + 1/\rho(\mathbf{B}))},$$

where

$$i = \arg\min_l \frac{1}{\rho(\mathit{diag}(\mathbf{w})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))}.$$

Then, η *is an approximation ratio by solving the constrained max-min weighted SIR problem:*

$$\begin{array}{rl} \textit{maximize} & \min_l \dfrac{\mathsf{SIR}_l(\mathbf{p})}{w_l} \\ \textit{subject to} & \mathbf{p} \leq \bar{\mathbf{p}} \\ \textit{variables:} & \mathbf{p}. \end{array}$$

Quasi-invertibility in Wireless Network: Examples

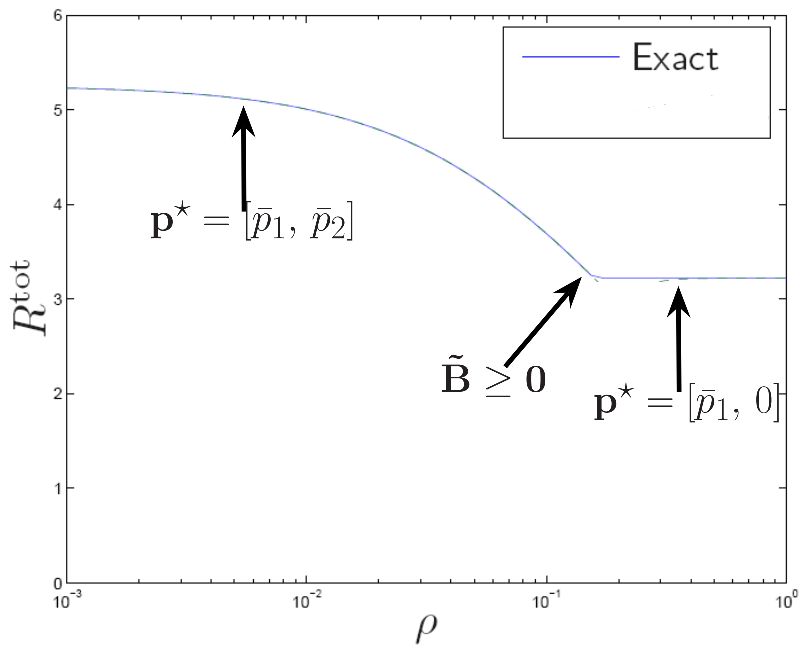
Parameter	Avg. % of $\tilde{\mathbf{B}} \geq \mathbf{0}$	SAPC (η)	Max-min SIR (η)	On-off sched. (η)
$\bar{p}_l = 33\text{mW}\,\forall l$ SNR $= 7$dB	99	0.97 (0.93)	0.99 (0.96)	0.89 (0.84)
$\bar{p}_l = 1\text{W}\,\forall l$ SNR $= 40$dB	65	0.87 (0.82)	0.91 (0.83)	0.87 (0.82)

Table 1: A typical numerical example in a ten-user network with two different maximum power constraint settings.

A Simple & Useful Engineering Indicator

$$\mathbf{F} = \begin{bmatrix} 0 & \rho/1.2 \\ 1.4\rho & 0 \end{bmatrix}, \qquad \mathbf{v} = [0.0417,\ 0.15]^T, \qquad \bar{\mathbf{p}} = [1,\ 1]^T$$

$$\boxed{\tilde{\mathrm{B}} \geq \mathbf{0} \qquad \Rightarrow \qquad F_{12}F_{21}(\bar{p}_1 + \bar{p}_2) + F_{12}v_2 + F_{21}v_1 \leq \min\{v_1, v_2\}}$$



As Eigenvalue Problem: SIR Domain

Theorem 5. *Consider the following maximization problem:*

$$\begin{array}{rl} \textit{maximize} & \sum_l w_l \log(1+\gamma_l) \\ \textit{subject to} & \rho(\textit{diag}(\boldsymbol{\gamma})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \leq 1 \quad \forall\, l, \\ \textit{variables:} & \gamma_l, \quad \forall\, l. \end{array}$$

The optimal SIR *vector* $\boldsymbol{\gamma}^\star$ *is related to the optimal power vector* $\mathbf{p}^\star$ *as follows:*

$$\mathbf{p}^\star = (\mathbf{I} - \textit{diag}(\boldsymbol{\gamma}^\star)\mathbf{F})^{-1}\,\textit{diag}(\boldsymbol{\gamma}^\star)\mathbf{v}.$$

Further, there exists a link i *such that*

$$\rho(\textit{diag}(\boldsymbol{\gamma}^\star)(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \leq \rho(\textit{diag}(\boldsymbol{\gamma}^\star)(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)) = 1$$

for all l. *Further,* $\mathbf{p}^\star$ *is the Perron eigenvector of* $\textit{diag}(\boldsymbol{\gamma}^\star)(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top)$ *for some* i *corresponding to Perron eigenvalue of 1.*

Power Controlled Achievable Rate Region (I)

- Rate region with power control only as nonnegative matrix eigenvalue constraint set:

$$\mathcal{R}$$

where $\mathcal{R} = r \in \{\rho(\mathrm{diag}(e^{\mathbf{r}} - \mathbf{1})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \leq 1 \quad \forall\, l\}$

- Rate region with power control and time sharing:

$$co\mathcal{R}$$

where $\mathcal{R} = r \in \{\rho(\mathrm{diag}(e^{\mathbf{r}} - \mathbf{1})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \leq 1 \quad \forall\, l\}$

- Low and High SNR regime

On-going Related Work

- Eigenvalue characterization enables efficient global optimization of sum rate maximization
  - *Maximizing Sum Rates in Multiuser Communication Systems: Theory and Algorithms*, 20th Meeting of the International Symposium for Mathematical Programming

- Realize $\tilde{\mathbf{B}} \geq \mathbf{0}$ by
  - Scheduling by clusters (IEEE 802.11b RTS-CTS)
  - Beamforming

- Connection with Information Theory

- Nonconvex nonnegative cone programming and nonnegative matrix theory

Thank You

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