DC Optimal Power Flow: Uniqueness and Algorithms

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Abstract—**The optimal power flow (OPF) problem minimizes the power loss in an electrical network by optimizing the voltage and power delivered at the network nodes, and is generally hard to solve. We study the direct current special case by leveraging recent developments on the zero duality gap of OPF. We study the uniqueness of the OPF solution using differential topology especially the Poincare–Hopf Index Theorem, and characterize its global uniqueness for simple network topologies, e.g., line and mesh networks. This serves as a starting point to design local algorithms with global behavior that have low complexity and are computationally fast for practical smart power grids.**

I. BACKGROUND

The Optimal Power Flow (OPF) problem is a classical nonlinear nonconvex optimization problem that minimizes the power generation costs and transmission loss in a power network subject to physical constraints governed by Kirchhoff's and Ohm's law [1]–[3]. There is a huge body of work on solving OPF since Carpentier's first formulation in 1962 [1]. To overcome the nonlinearity, the majority of these work uses linearization and approximation techniques to simplify the OPF, e.g., simplifying by the small angle approximation [4] or other relaxation methods [5]–[8]. In this paper, we use the bus injection model of OPF that focuses on the voltages and power injection at each node of the network. We study the Direct Current (DC) Optimal Power Flow (DC-OPF), i.e., for a purely resistive network. The DC-OPF can be practically important in a smart grid, where there are renewables, e.g., solar cells that produce DC power, and MW-oriented applications [9].

Recently, the authors in [10] showed that the Lagrange duality gap between the OPF problem and its convex dual can be zero in a radial network or under some mild conditions on the system model, and this was numerically verified to be true for a number of practical IEEE power networks. The implication is that the OPF can be optimally solved by a reformulation-relaxation technique using semidefinite programming (SDP) [11]. In the DC special case, this relaxed problem is in fact exact when there is load oversatisfaction [10], [12]. In this paper, we leverage this result to study the minimization of power transmission loss in DC-OPF and characterize the uniqueness of its solutions. This has implications on how algorithms can be designed to solve the OPF in a smart grid.

Different from prior work on algorithm design for solving OPF, e.g., in [5], [7], [8], we leverage the zero duality gap condition and dual decomposition to design decentralized algorithms to solve the OPF. Each bus (either the generator bus or the demand bus) exchanges local information with its one-hop neighbors. The uniqueness characterization provides an interesting perspective on the convergence proof of local algorithms to the global optimal solution.

Overall, the contributions of the paper are as follows:

1. We characterize the uniqueness of the DC-OPF solution using the Poincare-Hopf Index Theorem for simple network topologies.

2. We solve the DC-OPF problem using dual decomposition and iterative fixed-point analysis. Computationally fast convergent local algorithms with low complexity are proposed to compute the global optimal solution of the DC-OPF.

II. SYSTEM MODEL AND PROBLEM FORMULATION

Let us consider a DC power network with a set of buses $\mathcal{N} = \{1, 2, \ldots, N\}$ and a set of transmission lines $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$. We assume that each bus is either a generation bus or a demand bus. A demand bus i can model the aggregate of users (loads) in a distribution network. For each bus i, we use Ω_i to represent the set of buses connecting to bus i and $|\Omega_i| \geq 1$. Moreover, we assume that the line admittance satisfies $Y_{ij} = Y_{ji} \in \mathbb{R}_+$, if $(i, j) \in \mathcal{E}$; and $Y_{ij} = Y_{ji} = 0$, otherwise. We use $\mathbf{V}$ and $\mathbf{I}$ to denote the voltage vector $(V_i)_{i \in \mathcal{N}}$ and current vector $(I_i)_{i \in \mathcal{N}}$ respectively.

A. DC-OPF Problem Formulation

We consider nodal power and voltage constraints given by $V_i I_i \leq \bar{p}_i$ and $V_i \in [\underline{V}_i, \overline{V}_i]$, $\forall i \in \mathcal{N}$ respectively. If bus i is a generation bus, then $\bar{p}_i$ represents the generator capacity and $\bar{p}_i > 0$. If bus i is a demand bus, then $\bar{p}_j < 0$ and this constraint corresponds to the minimum demand that has to be satisfied at bus i. We assume that demand can be oversatisfied, which is a practical assumption when there are power storage devices available at the demand buses. Therefore, a demand bus i not only absorbs $|\bar{p}_i|$ amount of power, but it also

absorbs additional power to charge the power storage device attached to it. Moreover, for each line $(i, j) \in \mathcal{E}$, we impose the line capacity constraint $Y_{ij}(V_i - V_j)^2 \le c_{ij}$. The DC-OPF problem can be formulated as minimizing the power losses during transmission over the network subject to the bus and line constraints:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i \in \mathcal{N}} I_i V_i \\
\text{subject to} \quad & I_i V_i \le \bar{p}_i \quad \forall i \in \mathcal{N}, \\
& \underline{V}_i \le V_i \le \overline{V}_i \quad \forall i \in \mathcal{N}, \\
& Y_{ij}(V_i - V_j)^2 \le c_{ij} \quad \forall (i, j) \in \mathcal{E}, \\
\text{variables:} \quad & V_i, I_i, i \in \mathcal{N}.
\end{aligned}
\tag{1}
$$

By Ohm's Law and Kirchoff's current law, we have $\mathbf{I} = \mathbf{YV}$, i.e.,

$$
\begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{pmatrix} = \begin{pmatrix} \sum_{j \in \Omega_1} Y_{1j} & \cdots & -Y_{1N} \\ -Y_{21} & \cdots & -Y_{2N} \\ \vdots & \ddots & \vdots \\ -Y_{N1} & \cdots & \sum_{j \in \Omega_N} Y_{Nj} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix},
\tag{2}
$$

where $\mathbf{Y}$ is the system admittance matrix. Substituting the above relationship between $\mathbf{Y}$ and $\mathbf{I}$ into (1), the optimal power flow problem in (1) is equivalent to the following optimization problem:

$$
\begin{aligned}
\text{minimize} \quad & \mathbf{V}^T \mathbf{YV} \\
\text{subject to} \quad & \mathbf{V}^T \mathbf{Y}_i \mathbf{V} \le \overline{p}_i \quad \forall i \in \mathcal{N}, \\
& \underline{\mathbf{V}} \le \mathbf{V} \le \overline{\mathbf{V}}, \\
& \mathbf{V}^T \mathbf{Y}_{ij} \mathbf{V} \le c_{ij} \quad \forall (i, j) \in \mathcal{E}, \\
\text{variables:} \quad & \mathbf{V},
\end{aligned}
\tag{3}
$$

where $\mathbf{Y}_i = \frac{1}{2}(\mathbf{E}_i \mathbf{Y} + \mathbf{Y} \mathbf{E}_i)$ and $\mathbf{Y}_{ij} = Y_{ij}(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T$, where $\mathbf{e}_i$ is the standard basis vector in $\mathbb{R}^n$ and $\mathbf{E}_i = \mathbf{e}_i \mathbf{e}_i^T \in \mathbb{R}^{n \times n}$ [10].

III. UNIQUENESS CHARACTERIZATION

In this section, we study the uniqueness of the solution to (3). Our approach is to leverage the zero duality gap property in (3) and the Poincare-Hopf Index Theorem in [13], [14] together with the nonnegativity associated with the variables and problem parameters of (3). The uniqueness characterization of (3) has implications on how local algorithms with low complexity can be designed to solve (3) in Section IV.

We first use the Lagrange dual decomposition to relax (3). Define the following partial Lagrangian function:

$$
\begin{aligned}
L(\mathbf{V}, \{\boldsymbol{\lambda}, \boldsymbol{\mu}\}) \doteq & \sum_{i,j} \lambda_i Y_{ij} V_i (V_i - V_j) + \sum_{i,j} \mu_{ij} Y_{ij}(V_i - V_j)^2 \\
& - \sum_i \lambda_i \overline{p}_i - \sum_{i,j} \mu_{ij} c_{ij},
\end{aligned}
$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are the nonnegative dual variables associated

with the constraints $\mathbf{V}^T \mathbf{Y}_j \mathbf{V} \le \overline{p}_i \; \forall i \in \mathcal{N}$ and $\mathbf{V}^T \mathbf{Y}_{ij} \mathbf{V} \le c_{ij} \; \forall (i, j) \in \mathcal{E}$ respectively. For any given feasible λ_i at each node and μ_{ij} at each line, consider the following partial Lagrangian minimization problem:

$$
\begin{aligned}
\text{minimize} \quad & L(\mathbf{V}, \{\boldsymbol{\lambda}, \boldsymbol{\mu}\}) \\
\text{subject to} \quad & \underline{V}_i \le V_i \le \overline{V}_i \quad \forall i \in \mathcal{N}.
\end{aligned}
\tag{4}
$$

Note that (4) is nonconvex in $\mathbf{V}$. However, since the objective is smooth and the constraints satisfy the linearity constraint qualification, the optimal solution to (4) must satisfy the Karush-Kuhn Tucker (KKT) conditions. By rewriting the KKT conditions, it can be shown that the optimal solution satisfies:

$$
\mathbf{V} = \max \left\{ \underline{\mathbf{V}}, \min \left\{ \overline{\mathbf{V}}, \mathbf{B}(\boldsymbol{\lambda}, \boldsymbol{\mu}) \mathbf{V} \right\} \right\},
\tag{5}
$$

where the nonnegative matrix $\mathbf{B}$ has entries in terms of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$:

$$
B_{ij} = \begin{cases} \frac{2Y_{ij} + \lambda_i Y_{ij} + \lambda_j Y_{ij} + 2\mu_{ij} Y_{ij}}{2 \left((1 + \lambda_i) \sum_{j \in \Omega_i} Y_{ij} + \sum_{(i,j) \in \mathcal{E}} \mu_{ij} Y_{ij} \right)}, & \forall (i, j) \in \mathcal{E} \\ 0, & \text{otherwise.} \end{cases}
\tag{6}
$$

Suppose that the optimal dual variables $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ of (3) are given (recall that they exist due to zero duality gap), then the optimal voltage $\mathbf{V}^*$ must satisfy $\mathbf{V}^* = \max \left\{ \underline{\mathbf{V}}, \min \left\{ \overline{\mathbf{V}}, \mathbf{B}(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \mathbf{V}^* \right\} \right\}$.

Now, it is interesting to ask whether $\mathbf{V}^*$ is unique. There are several implications. First, from (5), it implies that the Lagrange dual function is smooth at the optimality of (3). Second, the smoothness of the Lagrange dual at optimality implies that the optimal dual variables are unique and stable, and thus can be suitably used as power and line prices in pricing schemes. Third, it enlarges the space of designing simple local algorithms with low complexity to solve (3), and we address this in Section IV.

To prove uniqueness, we leverage a result in [13], [14] to inspect the generalized critical points of $\nabla L(\mathbf{V}, \{\boldsymbol{\lambda}, \boldsymbol{\mu}\})$ (the first order derivative of $L(\mathbf{V}, \{\boldsymbol{\lambda}, \boldsymbol{\mu}\})$ with respect to $\mathbf{V}$) over the box constraint set in (4), which is given in the following.

Lemma 1 ([13], Corollary 1): Let M be a box-constrained region. Let U be an open set containing M, and $f : U \to \mathbb{R}$ be a twice differentiable function. Let $\mathrm{KKT}(f, M)$ denote the Karush-Kuhn-Tucker stationary points of f over the region M and assume that for every $x \in \mathrm{KKT}(f, M), H(x)|_{I^{\nabla f}(x)}$ is a P-matrix, where $H(x)|_{I^{\nabla f}(x)}$ denotes the principal submatrix of the Hessian matrix $H(x)$ containing precisely the entries H_{ij} where $i, j \in I^{\nabla f}(x)$ and $I^{\nabla f}(x) = \{i \in \{1, 2, \ldots, N\} | \nabla f_i = 0\}$. Then, $\mathrm{KKT}(f, M)$ has a unique element which is also the unique local (global) minimum of f over M.

Now, the derivative of $\nabla L(\mathbf{V}, \{\boldsymbol{\lambda}, \boldsymbol{\mu}\})$, i.e., the Hessian in (4),

is given by a square matrix $\mathbf{A}$ (in terms of $\{\boldsymbol{\lambda}, \boldsymbol{\mu}\}$). In general, the uniqueness of the solution in (3) depends on the network topology (as $\mathbf{A}$ contains $\mathbf{Y}$ and $\mathbf{Y}_{ij}$) and the dual variables (how they pair up with the optimal primal variable $\mathbf{V}^*$ to satisfy the KKT conditions). In the following, we study the application of Lemma 1 to $\mathbf{A}$ for simple network topologies.

Example 1: We study the uniqueness of the solution in (3) for a 2-bus line network as shown in Figure 1(a) with a generator at bus 1 and a load at bus 2. Also, $\overline{p}_1 > 0$ and $\overline{p}_2 < 0$. Then the Hessian $\mathbf{A}$ in (4) is given by $\mathbf{A} = (\mathbf{Y} + \lambda_1 \mathbf{Y}_1 + \lambda_2 \mathbf{Y}_2 + \mu_{12} \mathbf{Y}_{12})$. We first use the spectral property of $\mathbf{B}$ in (6) to show that at least one of the box constraints is binding. Now, the Perron-Frobenius eigenvalue $\rho(\mathbf{B})$ and its right eigenvector of $\mathbf{B}$ are given respectively as:

$$\rho(\mathbf{B}) = \frac{\lambda_1 + \lambda_2 + 2 + 2\mu_{12}}{2\sqrt{(\mu_{12} + 1 + \lambda_1)(\mu_{12} + 1 + \lambda_2)}}$$

and

$$\begin{pmatrix} \frac{\lambda_1 + \lambda_2 + 2 + 2\mu_{12}}{2\sqrt{(\mu_{12}+1+\lambda_1)(\mu_{12}+1+\lambda_2)}} \\ \frac{\lambda_1 + \lambda_2 + 2 + 2\mu_{12}}{2(\mu_{12}+1+\lambda_2)} \end{pmatrix}.$$

Observe that $\rho(\mathbf{B}) \geq 1$ in the above. In fact, $\rho(\mathbf{B}) = 1$ if and only if $\lambda_1 = \lambda_2$ and the right eigenvector is proportional to the all-one vector. Hence, if $\mathbf{V}^* = \mathbf{B}\mathbf{V}^*$, then $V_1 = V_2$, i.e., no power flows to bus 2, which is infeasible. Thus, $\lambda_1 \neq \lambda_2$ at optimality and $\mathbf{V}^*$ is not in the interior of the box constraint, i.e., at least one of the box constraints is binding.

Next, it is easy to see that $\mathbf{A}$ is nonsingular and it can be shown that the determinant of the principal submatrix (a scalar) of $\mathbf{A}$ is given by $(1 + \lambda_i + \mu_{12})Y_{12}$, $i = 1$ or 2, which is always positive. Using Lemma 1, (4) has a unique solution. To illustrate, Figure 1(b) plots the solution space over V_1 and V_2 in the box constraint $[1, 2]$ for fixed dual variables, and there is only one optimal solution, i.e., $V_1 = 2V$, $V_2 = 1.5V$.
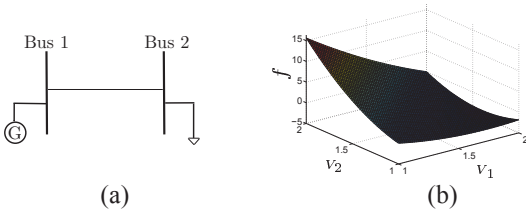

Fig. 1. (a) In this example, two connected buses (bus 1 and bus 2) are respectively attached by a generator and a load. (b) The value of f by varying V_1 and V_2 for fixed dual variables.

Moreover, we have the following relationships:

$$\frac{V_1^*}{V_2^*} = \frac{2 + \lambda_1 + \lambda_2 + 2\mu_{12}}{2 + 2\lambda_1 + 2\mu_{12}}, \quad \text{if the box constraint of } V_2^* \text{ is binding,}$$

$$\frac{V_2^*}{V_1^*} = \frac{2 + \lambda_1 + \lambda_2 + 2\mu_{12}}{2 + 2\lambda_2 + 2\mu_{12}}, \quad \text{if the box constraint of } V_1^* \text{ is binding.}$$

Note that λ_1 and λ_2 are inversely proportional to the nodal voltage, and furthermore $\lambda_2 > \lambda_1$. This has the natural interpretation that the price to be paid at bus 2 by the demand is strictly greater than the price to generate the power (since power is necessarily lost over transmission).

Example 2: We consider a general $(n+1)$-bus line network as shown in Figure 2(a). In this case, $\mathbf{A}$ is a symmetric tridiagonal matrix:

$$\mathbf{A} = \begin{pmatrix} A_{11} & -A_{12} & & & \\ -A_{12} & A_{22} & -A_{23} & & \\ & -A_{23} & \ddots & \ddots & \\ & & \ddots & \ddots & -A_{n(n+1)} \\ & & & -A_{n(n+1)} & A_{(n+1)(n+1)} \end{pmatrix},$$

where

$$A_{ij} = \begin{cases} (1 + \lambda_1 + \mu_{12})Y_{12}, & i = j = 1 \\ (1 + \lambda_{(n+1)} + \mu_{n(n+1)})Y_{n(n+1)}, & i = j = n+1 \\ (1 + \lambda_i + \mu_{(i-1)i})Y_{(i-1)i} & i = j = 2, \ldots, n \\ \quad + (1 + \lambda_i + \mu_{i(i+1)})Y_{i(i+1)}, & \\ (\frac{\lambda_i}{2} + \frac{\lambda_j}{2} + \mu_{ij} + 1)Y_{ij}, & j = i + 1 \text{ or } i - 1 \\ 0, & \text{Otherwise.} \end{cases}$$

When $n = 2$, it can be proved that $\rho(\mathbf{B}) \geq 1$. Thus, at least one of the nodal voltage constraints is binding. For example, when V_1^* has a binding box constraint, we delete the first row and the first column of $\mathbf{A}$ and use Lemma 1 to check whether the following reduced 2×2 matrix $\mathbf{A}_{2\times2}$ is a P-matrix:

$$\mathbf{A}_{2\times2} = \begin{pmatrix} A_{22} & -A_{23} \\ -A_{23} & A_{33} \end{pmatrix}.$$

By elementary row operations, we have the following upper triangular matrix

$$\mathbf{A}'_{2\times2} = \begin{pmatrix} A_{22} & -A_{23} \\ 0 & A_{33} - A_{23}^2/A_{22} \end{pmatrix}.$$

Next, from (4) and the relationship between $\mathbf{A}$ and $\mathbf{B}$, i.e., $B_{ij} = A_{ij}/A_{ii}$ if $(i, j) \in \mathcal{E}$ and $B_{ij} = 0$, otherwise, we obtain

$$A_{33} - A_{23}^2/A_{22} = A_{23}\frac{V_2^*}{V_3^*} + \frac{A_{12}A_{23}}{A_{22}}\frac{V_1^*}{V_3^*} - A_{23}\frac{V_2^*}{V_3^*} > 0.$$

Thus, the determinant of the upper triangular matrix $\mathbf{A}'_{2\times2}$ is positive, which implies that all principal minors of $\mathbf{A}_{2\times2}$ are positive. Therefore, using Lemma 1, we conclude that the optimal solution is unique when V_1^* is binding.

Next, if V_2^* has a binding box constraint, then the reduced matrix $\mathbf{A}_{2\times2}$ becomes

$$\mathbf{A}_{2\times2} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{33} \end{pmatrix},$$

which is a P-matrix, and thus the optimal solution is unique.

Since bus 1 and bus 3 are symmetric in the line network, we can also deduce that the optimal solution is unique when V_3^* has a binding box constraint. Lastly, if the voltage of any two buses have binding box constraints, then the reduced matrix only contains one positive element, which implies that the optimal solution is unique. To summarize, the optimal solution in the 3-bus line network is unique. A similar argument can be applied to the general $(n + 1)$-bus line network.
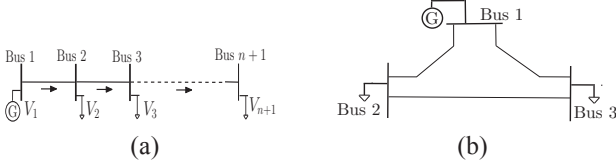

Fig. 2. Different topologies. (a) A line network. The power generated at bus 1 is transmitted to the consecutive n buses. (b) A mesh network that consists of three buses. Bus 1 is connected to a generator and the other two buses are attached by loads.

Example 3: We consider a mesh network with three buses as shown in Figure 2(b). For this mesh network, we can construct the matrix $\mathbf{A}$ as

$$\mathbf{A} = \begin{pmatrix} A_{11} & -A_{12} & -A_{13} \\ -A_{12} & A_{22} & -A_{23} \\ -A_{13} & -A_{23} & A_{33} \end{pmatrix},$$

where $A_{11} = (1 + \lambda_1 + \mu_{12})Y_{12} + (1 + \lambda_1 + \mu_{13})Y_{13}, A_{33} = (1 + \lambda_3 + \mu_{23})Y_{23} + (1 + \lambda_3 + \mu_{13})Y_{13}, A_{13} = (1 + \lambda_1/2 + \lambda_3/2 + \mu_{13})Y_{13}$ and all other problem parameters are the same as that in the 3-bus line network. In the mesh network, if V_1^* has a binding box constraint, the reduced matrix is the same as that in the 3-bus line network where V_1^* is binding. Checking the determinant of the submatrix, we have

$$A_{33} - A_{23}^2/A_{22} = A_{13}\frac{V_1^*}{V_3^*} + A_{23}\frac{V_2^*}{V_3^*} + \frac{A_{12}A_{23}}{A_{22}}\frac{V_1^*}{V_3^*} - A_{23}\frac{V_2^*}{V_3^*} > 0,$$

which implies that the submatrix is a P-matrix. Thus, we conclude that the optimal solution is unique when V_1^* is binding. The same argument can be applied to buses 2 and 3 to deduce that the optimal solution is unique when either V_2^* or V_3^* has a binding box constraint. Likewise, if the voltage of any two buses have binding box constraints, the optimal solution is unique. Therefore, we can conclude that the optimal solution for the mesh network is unique.

IV. LOCAL ALGORITHMS

In this section, we design simple local algorithms to solve (3). This is achieved by first solving (4) for given $\{\lambda, \mu\}$, and then using the projected gradient method in [15] to update the dual variables, which in turn are used as the input to solving (4) iteratively.

We propose the following fixed point algorithm that computes the fixed point $\mathbf{V}$ in (5) for a given set of feasible dual variables λ and μ.

Algorithm 1:

Compute voltage $\mathbf{V}$:

$$V_i(k + 1) = \max\left\{ \underline{V}_i, \min\left\{ \overline{V}_i, \sum_{j\in\Omega_i} B_{ij}V_j(k) \right\} \right\}, \quad (7)$$

for all $i \in \mathcal{N}$, where B_{ij} is given in (6) for all i, j.

Theorem 1: Suppose (4) has a unique optimal solution. Then, given any $\mathbf{V}(0)$ which satisfies $\underline{\mathbf{V}} \leq \mathbf{V}(0) \leq \overline{\mathbf{V}}$, $\mathbf{V}(k)$ in Algorithm 1 converges to the unique optimal solution of (4).

Proof: Let $\mathbf{f}(\mathbf{V}) = \max\left\{ \underline{\mathbf{V}}, \min\left\{ \overline{\mathbf{V}}, \mathbf{B}\mathbf{V} \right\} \right\}$. First, we show that if $\mathbf{V}(0) = \underline{\mathbf{V}}$, then $\mathbf{V}(k)$ is a monotonic increasing and bounded sequence. Clearly, we have

$$\mathbf{V}(1) \geq \mathbf{f}(\mathbf{V}(0)) \geq \underline{\mathbf{V}} = \mathbf{V}(0). \quad (8)$$

Now, observe that for any $\mathbf{V}(k)$ and $\mathbf{V}(k+1)$ satisfying $\mathbf{V}(k+1) \geq \mathbf{V}(k)$, we have

$$\mathbf{V}(k + 2) = \mathbf{f}(\mathbf{V}(k + 1)) \geq \mathbf{f}(\mathbf{V}(k)) = \mathbf{V}(k + 1). \quad (9)$$

Here, the inequality follows from the fact that the entries of $\mathbf{B}$ are nonnegative. By induction, it follows that $\mathbf{V}(k)$ is a monotonic increasing sequence. Clearly, $\mathbf{V}(k) = \mathbf{f}(\mathbf{V}(k - 1)) \leq \overline{\mathbf{V}}$ for all k so it is bounded.

By a similar argument, we have that if $\mathbf{V}(0) = \overline{\mathbf{V}}$, then $\mathbf{V}(k)$ is a monotonic decreasing and bounded sequence.

Now, given any $\mathbf{V}(0)$ satisfying $\underline{\mathbf{V}} \leq \mathbf{V}(0) \leq \overline{\mathbf{V}}$, we have

$$\lim_{k\to\infty} \mathbf{f}^k(\underline{\mathbf{V}}) \leq \lim_{k\to\infty} \mathbf{f}^k(\mathbf{V}(0)) \leq \lim_{k\to\infty} \mathbf{f}^k(\overline{\mathbf{V}}). \quad (10)$$

Since $\mathbf{f}^k(\underline{\mathbf{V}})$ is a monotonic increasing and bounded sequence, it must converge. Suppose that there is a unique solution to $\mathbf{V} = \mathbf{f}(\mathbf{V})$ (which is true for the network topologies considered in Section III and can be verified using Lemma 1 for general network topologies). Hence $\mathbf{f}^k(\underline{\mathbf{V}})$ must converge to the unique fixed point. Let us denote by $\mathbf{V}^*$ the unique fixed point. Using a similar argument, we conclude that $\mathbf{f}^k(\underline{\mathbf{V}})$ must also converge to $\mathbf{V}^*$. Hence, we have

$$\mathbf{V}^* \leq \lim_{k\to\infty} \mathbf{f}^k(\mathbf{V}(0)) \leq \mathbf{V}^* \quad (11)$$

which implies that $\lim_{k\to\infty} \mathbf{f}^k(\mathbf{V}(0)) = \mathbf{V}^*$. ∎

We next leverage Algorithm 1 together with a gradient projection method in [15] that updates the dual variables λ and μ to solve (4) in the following.

Algorithm 2:

---

1) Set the step sizes $\beta, \rho \in (0, 1)$.
2) Run Algorithm 1 with input $\lambda_i(t)$ and $\mu_{ij}(t)$ for the

---

entries of $\mathbf{B}$.

3) Compute:
$$\lambda_i(t+1) = \max\{0, \lambda_i(t) + \beta(\sum_{j \in \Omega_i} Y_{ij} V_i(k) V_j(k) - \overline{p}_i)\}, \quad (12)$$
for all $i \in \mathcal{N}$.
$$\mu_{ij}(t+1) = \max\{0, \mu_{ij}(t) + \rho(Y_{ij}(V_i(k) - V_j(k))^2 - c_{ij})\}, \quad (13)$$
for all $(i,j) \in \mathcal{E}$.

Update β and ρ according to Theorem 2.

Theorem 2: Assume that $V_i(k)$ and $\hat{V}_i(k)$ denote the (prematurely terminated) approximated and the exact solution (on convergence) at step 2 of Algorithm 2 for $\lambda_i(t)$ and $\mu_{ij}(t)$, respectively. Suppose the output of Algorithm 1 satisfies

$$\sum_{i \in \mathcal{N}} \limsup_k |\sum_{j \in \Omega_i} Y_{ij} \hat{V}_i(k)(\hat{V}_j(k) - \hat{V}_i(k)) - \overline{p}_i| \le \epsilon_1$$

and

$$\sum_{(i,j) \in \mathcal{E}} \limsup_k |Y_{ij}(\hat{V}_j(k) - \hat{V}_i(k))^2 - c_{ij}| \le \epsilon_2$$

for some sufficiently small positive ϵ_1 and ϵ_2. We choose the diminishing step sizes $\beta(t)$ and $\rho(t)$ to satisfy

$$\sum_{t=0}^{\infty} \beta(t) = \infty, \qquad \sum_{t=0}^{\infty} (\beta(t))^2 < \infty,$$

$$\sum_{t=0}^{\infty} \rho(t) = \infty, \qquad \sum_{t=0}^{\infty} (\rho(t))^2 < \infty.$$

Then $\boldsymbol{\lambda}(t+1)$ and $\boldsymbol{\mu}(t+1)$ converge to a closed neighborhood of $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ when $t \to \infty$, respectively.

Remark 1: Note that computing (7) is approximately optimal due to the finite iterations in the inner loop. These errors can accumulate and lead to an approximated (instead of the exact) gradient computed in (12). The proof of Theorem 2 is based on the approximated gradient projection method in [15]. Theorem 2 shows that the (sufficiently small) errors do not affect the asymptotic convergence so long as the step sizes are chosen appropriately. In Section V, we illustrate that the optimal solution obtained by Algorithm 2 is arbitrarily close to the solution obtained by centralized interior point methods.
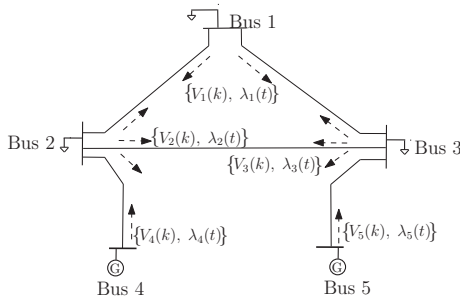

Fig. 3. Message passing in the IEEE 5-bus system.

A. Further Discussions

Algorithm 2 can be run in a distributed manner using message passing to transmit and receive iterates of the voltage and dual variables with neighboring one-hop nodes. This is illustrated in Figure 3 for a IEEE 5-bus system. Each node $i \in \mathcal{N}$ randomly chooses a feasible $V_i(0)$ and broadcast to its neighbors. Once a node $i \in \mathcal{N}$ receives all V_j where $j \in \Omega_i$, it calculates λ_i and μ_{ij} according to (12) and (13). Next, the updated λ_i is sent to its neighbors. Since μ_{ij} can be obtained at each pair of neighboring nodes, there is no need to exchange μ_{ij}. After collecting all λ_j ($j \in \Omega_i$), V_i can be calculated by (7) at bus i. This is repeated until convergence.

The dual variables $\{\boldsymbol{\lambda}, \boldsymbol{\mu}\}$ have a shadow price interpretation (cf. Chapter 5 in [11]) in the sense that $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ are the equilibrium prices for the resource availability of the nodal power and line capacity respectively. For example, at node i, we increase the price λ_i if $\mathbf{V}^T \mathbf{Y}_i \mathbf{V} - \overline{p}_i \ge 0$ and decrease it otherwise. As future work, it is interesting to understand how Algorithms 1 and 2 can be used in Locational Marginal Pricing (LMP) scheme.
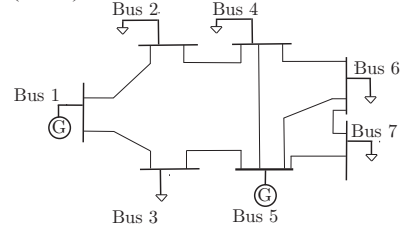

Fig. 4. An example with 7 buses and 9 lines.

V. NUMERICAL EXAMPLES

We evaluate the performance of Algorithm 2 using an example with 7 buses and 9 lines and an IEEE 5-bus system example. For the 7-bus system, as shown in Figure 4, there are two generators, i.e., bus 1 and bus 5, and five demand buses with loads. For the IEEE 5-bus system, bus 4 and bus 5 are the generators. The parameter settings are as follows. In the 7-bus system, the admittance for each line is $\mathbf{y} = [Y_{12} \ Y_{13} \ Y_{24} \ Y_{35} \ Y_{45} \ Y_{46} \ Y_{56} \ Y_{57} \ Y_{67}] = [5 \ 2 \ 5 \ 6 \ 4 \ 6 \ 4 \ 3 \ 6]^T S$. The power constraint $\overline{\mathbf{p}}$ is $[2.7306 \ -0.6466 \ -0.7046 \ -0.6151 \ 3.2626 \ -0.5831 \ -0.6571]^T W$. In the 5-bus system, the admittance for each line is $\mathbf{y} = [Y_{12} \ Y_{13} \ Y_{23} \ Y_{24} \ Y_{35}] = [4 \ 3 \ 4 \ 6 \ 7]^T S$. The power constraint $\overline{\mathbf{p}}$ is $[-0.2468 \ -3.0204 \ -0.32 \ 4.0203 \ 7.7234]^T W$. For comparison, a baseline solution can be obtained numerically by the SDP relaxation in [10] using CVX [16]. Numerically, we check that our examples satisfy the P-matrix requirement in Lemma 1, which implies that the optimal solution is unique.

1) **Example with high line capacity constraints.** We let each transmission line have a large enough capacity, e.g., we
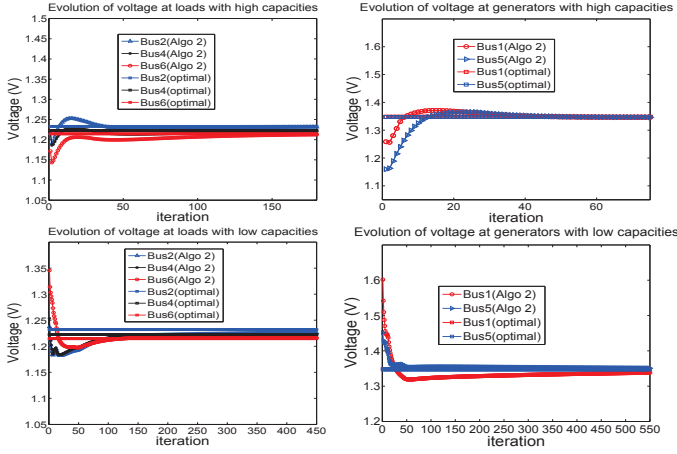
Fig. 5. Illustration of the convergence of Algorithm 2 in 7-bus system. The top two figures show the voltage convergence at loads (buses 2, 4 and 6) and generators in the high capacity lines scenario, respectively. The below two figures show the voltage convergence at loads (buses 2, 4 and 6) and generators in the low capacity lines scenario, respectively.
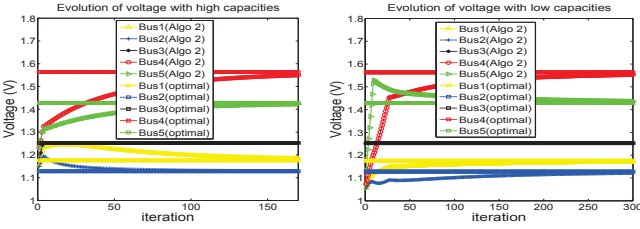

Fig. 6. Illustration of the convergence of Algorithm 2 in IEEE 5-bus system. The left figure shows the voltage convergence in the high capacity lines scenario. The right figure shows the voltage convergence in the low capacity lines scenario.

set the line capacity of each line as $3W$ in the 7-bus system and $1W$ in the 5-bus system, which leads to that none of the capacity constraint in (3) is tight. Thus, at optimality, $\mu_{ij}, \forall (i,j) \in \mathcal{E}$ are all zeros and only $\lambda_i, \forall i \in \mathcal{N}$ matter. We plot the iterations of Algorithm 2 for the 7-bus system at the top lefthand and righthand sides of Figure 5 for the loads and generators, respectively. The iterations for the 5-bus system are plotted on the lefthand side of Figure 6.

2) **Example with low line capacity constraints.** We set the line capacities as $\mathbf{c} = [0.0664 \ 1.0097 \ 1.0008 \ 1.0284 \ 2.0624 \ 2.0004 \ 1.0707 \ 0.0671 \ 2.0017]^T W$ and $\mathbf{c} = [0.1021 \ 0.1313 \ 0.1938 \ 0.8432 \ 0.3459]^T W$ in the 7-bus and 5-bus systems respectively. We plot iterations of Algorithm 2 at the bottom lefthand and righthand sides of Figure 5 for the loads and generators, respectively. We plot iterations of Algorithm 2 for the IEEE 5-bus system on the righthand side of Figure 6. From Figures 5 and 6, we observe that Algorithm 2 have fast convergence time. In the high line capacity scenario, as only $\lambda_i, \forall i \in \mathcal{N}$ need to be updated, convergence can be faster than the low capacity scenario.

VI. CONCLUSIONS

We studied the DC-OPF problem, which is a nonconvex quadratic constrained quadratic programming problem that optimizes the voltage and the power delivered at nodes in a power network. Leveraging recent results on the zero duality gap of the OPF problem, we characterized the uniqueness of its solution using differential topology especially the Poincare–Hopf Index Theorem, and characterized its global uniqueness for simple network topologies, e.g., a 2-bus line network, the general line network and a mesh network. Based on the uniqueness characterization, we proposed local algorithms with low complexity that converged to the unique global optimal solution of the DC-OPF problem. Our numerical evaluations showed that both of these two algorithms were computationally fast running on the IEEE 5-bus and the 7-bus systems.

VII. ACKNOWLEDGEMENT

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