A Unified Framework for Wireless Max-Min Utility Optimization with General Monotonic Constraints

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***Abstract*—This paper presents a unifying and systematic framework to solve wireless max-min utility fairness optimization problems in multiuser wireless networks with generalized monotonic constraints. These problems are often challenging to solve due to their nonlinearity and non-convexity. Our framework leverages a general result in nonlinear Perron-Frobenius theory to characterize the global optimal solution of these problems analytically, and to design scalable and fast-convergent algorithms for the computation of the optimal solution. This work advances the state-of-the-art in handling wireless utility optimization problems with nonlinear monotonic constraints, which existing methodologies cannot handle, and also unifies previous works in this area. Several representative applications are considered to illustrate the effectiveness of the proposed framework, including max-min quality of service subject to robust interference temperature constraints in cognitive radio networks, min-max outage subject to outage constraints in heterogeneous networks, and min-max weighted MSE subject to SINR constraints in multiuser downlink system.**

I. Introduction

The demand for broadband mobile data services has grown significantly and rapidly in wireless networks. Interference sources, such as unlicensed spectrum usage in cognitive radio networks and spectrum sharing in small cell or heterogeneous networks, have begun to pose serious challenges to wireless network performance. Joint interference and wireless resource control is thus especially important to ensure that sharing is done fairly in a collaborative and efficient manner. However, maintaining a balanced fair operation becomes problematic since interference rises rapidly with the density of ad hoc wireless network deployment. Without appropriate resource coordination, the network can become unstable or may operate in a highly inefficient and unfair manner.

Rigorous studies of fairness in wireless networks can be modeled by nonlinear utility functions of wireless link metrics, e.g., signal-to-interference-and-noise ratio (SINR), mean-square error (MSE) and the outage probability [1]–[4]. The performance of wireless networks is often affected by channel conditions, interference, and the associated wireless quality-of-service constraints, e.g., the interference temperature constraints in cognitive radio networks (i.e., constraints on the received interference at the primary users) or outage probability constraints in heterogeneous networks. The total network fairness utility is then maximized over the joint solution space of all possible wireless link metrics, e.g., powers and interference. The main challenges of solving these wireless utility maximization problems come from the nonlinear dependency of link metrics on channel conditions and powers, as well as possible interference among the users. These are nonconvex problems that are notoriously difficult to solve, and designing scalable algorithms with low complexity to solve them optimally is even harder.

The authors in [2], [5], [6] applied geometric programming (GP) to solve a certain class of nonconvex wireless utility maximization problems that can be transformed into convex ones. The authors in [4] studied the use of Gibbs sampling techniques to solve nonconvex utility maximization, but the optimality of the solution cannot be guaranteed. The authors in [3], [7], [8] tackled deterministic wireless utility maximization involving only rates and powers, and cannot handle stochastic constraints. In [9]–[16], the authors studied the max-min utility fairness problem in wireless networks using the nonlinear Perron-Frobenius theory [17], [18]. They demonstrated that the solution to some widely-studied problems, e.g., max-min SINR and max-min rate problems, can be characterized analytically and also solved by fast fixed-point algorithms. These work however rely critically on simple power constraints, e.g., norm or affine constraints, and cannot handle other nonlinear and nontrivial constraints typically encountered in wireless networks, e.g., stochastic interference temperature constraints or outage probability constraints.

This paper proposes a general framework that not only unifies previous works, but also enables rigorous treatment of nonlinear monotonic constraints that are widely applicable in wireless networks. In particular, we leverage a generalized nonlinear Perron-Frobenius theory to show that a large class of utility fairness resource allocation problems with realistic nonlinear power and interference constraints, channel fading constraints, stochastic outage constraints can be transformed into nonlinear fixed-point problems with monotonic constraints. This enables a unifying framework to design algorithms (no

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configuration needed) to solve these nonconvex problems in a jointly optimal and scalable manner.

Overall, the contributions of the paper are as follows:

1) By leveraging a generalized result in nonlinear Perron-Frobenius theory, we propose a unified framework to optimally solve max-min utility fairness optimization problems with general monotonic constraints. These monotonic constraints encompass, e.g., nonlinear power constraints, interference temperature constraints, and outage probability constraints, that manifest in different kinds of wireless networks.
2) A low-complexity fast-convergent algorithm is proposed to efficiently compute the optimal solution in the general case and is further improved, in terms of convergence speed, in certain special cases by intelligent use of the standard interference function framework in [19].
3) Three representative examples are given in the fields of cognitive radio networks, heterogeneous cellular networks, and multiuser downlink system. Each example incorporates nonlinear monotonic constraints on the transmit powers, e.g., interference temperature constraints, outage probability constraints, and nonlinear power constraints, and illustrates how our framework can be applied to yield scalable algorithms systematically.

The structure of this paper is as follows. In Section II, we introduce the unifying framework for solving the wireless max-min utility fairness optimization problem. In Sections III-V, we apply our framework to solve the max-min SINR problem with interference temperature constraints in cognitive radio networks, the min-max weighted MSE subject to SINR constraints in multiuser downlink system, and the min-max outage subject to outage probability constraints in heterogeneous networks. We conclude the paper in Section VI.

The following notations are used in this paper. Let $\mathcal{R}^m$ be the m-dimensional Euclidean space. For $\mathbf{x} = [x_1, \ldots, x_m]^\top$ and $\mathbf{y} = [y_1, \ldots, y_m]^\top \in \mathcal{R}^m$, we say that $\mathbf{x} < \mathbf{y}$ if $x_i < y_i$, for all i; $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$, for all i, but $\mathbf{x} \neq \mathbf{y}$; and $\mathbf{x} \leqq \mathbf{y}$ if $x_i \leq y_i$, for all i. Moreover, let $\mathcal{R}^m_+$ be the set of vectors $\mathbf{x} \in \mathcal{R}^m$ with $\mathbf{x} \geqq \mathbf{0}$.

II. General Max-Min Utility Optimization with Monotonic Constraints

Let us consider a general wireless network with L users and let p_1, …, p_L be the transmit powers of the L users. In particular, p_i is the transmit power of the i-th user. We assume that the power vector $\mathbf{p} = [p_1, \ldots, p_L]^\top$ can be adjusted to optimize the overall network utility. Based on generalizations of nonlinear Perron-Frobenius theory [20], we develop in the following a unified treatment of a general class of max-min utility optimization problems under monotonic system constraints and requirements. The constraints and requirements can potentially be nonlinear and nonconvex.

A. Problem Formulation

Specifically, let $u_i : \mathcal{R}^L_+ \to \mathcal{R}_+$, for $i \in \{1, \ldots, L\}$, be a continuous function of $\mathbf{p}$ that specifies the utility of user i and let $g_k : \mathcal{R}^L_+ \to \mathcal{R}_+$, for $k \in \{1, \ldots, K\}$, be a continuous function of $\mathbf{p}$ that is used to specify the k-th system constraint. The class of problems that we consider in this work can be formulated as follows:

$$\begin{aligned} &\text{maximize} && \min_{i=1,\ldots,L} \; u_i(\mathbf{p}) && \text{(1a)}\\ &\text{subject to} && \mathbf{g}(\mathbf{p}) \leqq \bar{\mathbf{g}} && \text{(1b)}\\ &\text{variables}: && \mathbf{p} && \text{(1c)} \end{aligned}$$

where $\mathbf{g}(\mathbf{p}) = [g_1(\mathbf{p}), \ldots, g_K(\mathbf{p})]^\top$ is the vector of constraint functions and $\bar{\mathbf{g}} = [\bar{g}_1, \ldots, \bar{g}_K]^\top$ is the vector of constraint values. Due to the monotonicity of the functions $\{g_k\}_{k=1}^K$, we shall refer to (1b) as the set of monotonic constraints.

Specifically, we consider a class of utility functions that satisfy the following assumptions.

Assumption 1 (Competitive Utility Functions):

- *Positivity*: For all i, $u_i(\mathbf{p}) > 0$ if $\mathbf{p} > \mathbf{0}$ and, in addition, $u_i(\mathbf{p}) = 0$ if and only if $p_i = 0$.
- *Competitiveness*: For all i, u_i is strictly increasing with respect to p_i and is strictly decreasing with respect to p_j, for $j \neq i$, when $p_i > 0$.
- *Directional Monotonicity*: For $\lambda > 1$ and $p_i > 0$, $u_i(\lambda \mathbf{p}) > u_i(\mathbf{p})$, for all i.

Moreover, we consider a class of monotonic constraints that satisfy the following assumptions.

Assumption 2 (Monotonic Constraints):

- *Strict Monotonicity*: For all k, $g_k(\mathbf{p}_1) > g_k(\mathbf{p}_2)$ if $\mathbf{p}_1 > \mathbf{p}_2$, and $g_k(\mathbf{p}_1) \geqq g_k(\mathbf{p}_2)$ if $\mathbf{p}_1 \geqq \mathbf{p}_2$.
- *Feasibility*: The set $\{\mathbf{p} > \mathbf{0} : \mathbf{g}(\mathbf{p}) \leqq \bar{\mathbf{g}}\}$ is non-empty.
- *Validity*: For any $\mathbf{p} > \mathbf{0}$, there exists $\lambda > 0$ such that $g_k(\lambda \mathbf{p}) \geqq \bar{g}_k$, for some k.

For the utility functions, the competitiveness assumption models the interaction between users in a wireless network and the directional monotonicity captures the increase in utility as the total power consumption increases. For the constraints, the strict monotonicity captures the increase in cost or resource consumption as $\mathbf{p}$ increases, the feasibility ensures that there exists a positive power vector in the feasible set, and the validity ensures that the set of constraints is meaningful. If the validity condition does not hold, the corresponding constraint can be simply removed without loss of generality.

The class of utility functions satisfying Assumption 1 is fairly general and includes, as special cases, many standard performance measures in wireless networks, e.g.,

- the SINR under frequency-flat fading, i.e.,

$$u_i(\mathbf{p}) = \mathsf{SINR}_i(\mathbf{p}) \triangleq \frac{G_{ii} p_i}{\sum_{j \neq i} G_{ji} p_j + \eta_i}, \tag{2}$$

where G_{ji} is the channel fading gains between transmitter j and receiver i, and η_i is the receiver noise variance;

- the link capacity, i.e.,

$$u_i(\mathbf{p}) = \log\left(1 + \mathsf{SINR}_i(\mathbf{p})\right), \quad (3)$$

where the SINR is given by (2);
- the reliability function (which is the complement of the outage probability [6], [11]), i.e.,

$$\begin{aligned} u_i(\mathbf{p}) &= \Pr\left(\mathsf{SINR}_i^R(\mathbf{p}) \geq \gamma_i\right) \\ &\triangleq e^{\frac{-\gamma_i \eta_i}{G_{ii} p_i}} \prod_{j \neq i} \left(1 + \frac{\gamma_i G_{ji} p_j}{G_{ii} p_i}\right)^{-1}, \end{aligned} \quad (4)$$

where γ_i is the SINR threshold of the ith user and $\mathsf{SINR}_i^R(\mathbf{p})$ is the SINR received at the ith receiver under Rayleigh fading, defined as

$$\mathsf{SINR}_i^R(\mathbf{p}) \triangleq \frac{G_{ii} R_{ii} p_i}{\sum_{j \neq i} G_{ji} R_{ji} p_j + \eta_i}, \quad (5)$$

where $\{R_{ji}\}$ are independent and exponentially distributed with unit mean. Note that SINR under Rayleigh fading in (5) is a random variable in terms of $\mathbf{p}$.

The class of monotonic constraints satisfying Assumption 2 also includes a large class of constraints that are representative in wireless networks, such as

- linear power constraints, i.e.,

$$\mathbf{g}(\mathbf{p}) = \mathbf{A}\mathbf{p} \leqq \bar{\mathbf{p}}, \quad (6)$$

where $\bar{\mathbf{p}}$ is a $K \times 1$ positive vector of power constraint values and $\mathbf{A}$ is a $K \times L$ nonnegative weight matrix (e.g., $K = L$ and $\mathbf{A} = \mathbf{I}$ in the case of individual power constraints, and $K = 1$ and $\mathbf{A} = \mathbf{1}^\top$, i.e., an all 1 row vector, in the case of sum power constraints);
- interference temperature constraints, i.e.,

$$g_k(\mathbf{p}) = \Psi_k(\mathbf{p}) \triangleq \sum_{j \neq k} \alpha_{jk} p_j \leqq \bar{\Psi}_k, \quad (7)$$

where $\Psi_k(\mathbf{p})$ and $\bar{\Psi}_k$ are respectively the interference received at the kth receiver and the upper bound for the interference temperature, and $\alpha_{jk}, \forall\, j, k$ are positive parameters that model the extent of influence by the powers of other users on the kth user [21], [22]. In the case that α_{jk} are random variables, then $\Psi_k(\mathbf{p})$ is a random variable and $g_k(\mathbf{p})$ is a stochastic interference temperature constraint.
- nonlinear power constraints [23], i.e.,

$$\mathbf{g}(\mathbf{p}) = \mathbf{J}(\mathbf{p}) \leqq \bar{\mathbf{p}}, \quad (8)$$

where $\mathbf{J} : \mathcal{R}_+^L \rightarrow \mathcal{R}_+^K$ is a nonlinear monotonic function of the power vector which can be used to model nonlinearities in the circuit, e.g., the power amplifier [24];

In many cases, u_i and g_k are nonlinear and nonconvex functions in $\mathbf{p}$, making the problem difficult to solve using standard convex optimization approaches [25]. In some special cases, GP (or related approximation algorithms) [5] can be used to obtain efficient solutions, but it cannot be used to address the general case optimally. Leveraging results from nonlinear Perron-Frobenius theory, we give an efficient iterative algorithm to solve a more general class of such problems, namely, problems satisfying Assumptions 1 and 2.

B. General Solution

By introducing an auxiliary variable τ, the problem in (1) can be reformulated as:

$$\begin{aligned} &\text{maximize} && \tau && \text{(9a)} \\ &\text{subject to} && u_i(\mathbf{p}) \geq \tau, \quad \text{for } i = 1, \ldots, L, && \text{(9b)} \\ &&& g_k(\mathbf{p}) \leq \bar{g}_k, \quad \text{for } k = 1, \ldots, K, && \text{(9c)} \\ &\text{variables:} && \mathbf{p},\ \tau. && \text{(9d)} \end{aligned}$$

We shall refer to the constraints in (9b) as the objective constraints and those in (9c) as the system constraints.

Lemma 1: For $\{u_i\}_{i=1}^L$, $\{g_k\}_{k=1}^K$, and $\{\bar{g}_k\}_{k=1}^K$ that satisfy Assumptions 1 and 2, the optimal solution $(\tau^*, \mathbf{p}^*)$ is positive, i.e., $\tau^* > 0$ and $\mathbf{p}^* > \mathbf{0}$, and, at optimality, all objective constraints are tight and at least one system constraint is active. That is, $u_i(\mathbf{p}^*) = \tau^*$, for all i and $g_k(\mathbf{p}^*) = \bar{g}_k$, for some k.

The proof is given in the Appendix A. By Lemma 1, it follows that

$$\frac{1}{\tau^*} p_i^* = \frac{1}{u_i(\mathbf{p}^*)} p_i^* \triangleq T_i(\mathbf{p}^*). \quad (10)$$

This means that the optimal power vector $\mathbf{p}^*$ is a solution to the fixed point equation

$$\frac{1}{\tau^*} \mathbf{p}^* = [T_1(\mathbf{p}^*), \ldots, T_L(\mathbf{p}^*)]^\top \triangleq \mathbf{T}(\mathbf{p}^*). \quad (11)$$

Definition 1: The function $\beta : \mathcal{R}_+^L \rightarrow \mathcal{R}_+$ of $\mathbf{p}$ (called the *scale* of $\mathbf{p}$) is defined as

$$\beta(\mathbf{p}) \triangleq \min\{\beta' \geqq 0 : g_k(\mathbf{p}/\beta') \leqq \bar{g}_k, \forall k\}. \quad (12)$$

Lemma 2: The scale $\beta : \mathcal{R}_+^L \rightarrow \mathcal{R}_+$ defined in Definition 1 satisfies the following properties:

1) β is not identically zero and, in fact, $\beta(\mathbf{p}) > 0$, for all $\mathbf{p} > \mathbf{0}$;
2) $\beta(\lambda \mathbf{p}) = \lambda \beta(\mathbf{p})$ for $\mathbf{p} \geqq \mathbf{0}$ and $\lambda \geqq 0$ (i.e., positively homogeneous);
3) $\mathbf{0} \leqq \mathbf{p} \leqq \mathbf{q}$ implies $\beta(\mathbf{p}) \leqq \beta(\mathbf{q})$ (i.e., monotonic).

Notice that, when $\mathbf{p} > 0$, the scale $\beta(\mathbf{p}) > 0$ (c.f. Lemma 2) is chosen such that, after normalization, the power vector $\tilde{\mathbf{p}} = \mathbf{p}/\beta(\mathbf{p})$ yields the largest feasible solution in the direction of $\mathbf{p}$. In this case, at least one of the system constraints must be satisfied with equality, i.e., $g_k(\mathbf{p}/\beta(\mathbf{p})) = \bar{g}_k$ for some k. Therefore, the set of feasible solutions of $\mathbf{p}$ must include, as a subset, the set of $\mathbf{p}$ that has scale equal to 1, i.e., $\{\mathbf{p} \geqq \mathbf{0} : g_k(\mathbf{p}) \leqq \bar{g}_k, \forall k\} \supseteq \{\mathbf{p} \geqq \mathbf{0} : \beta(\mathbf{p}) = 1\} \triangleq \mathcal{U}$.

Lemma 3: The optimal solution of (1) is included in $\mathcal{U}$.

The proof is given in Appendix B. By Lemma 3, it follows that the solution of the optimization problem in (9) (and, thus, (1)) is a solution of the conditional eigenvalue problem [20], where the objective is to find $(\tau^*, \mathbf{p}^*)$ such that

$$\frac{1}{\tau^*}\mathbf{p}^* = \mathbf{T}(\mathbf{p}^*),\ \tau^* \in \mathcal{R},\ \mathbf{p}^* \in \mathcal{U}. \tag{13}$$

Hence, by exploiting the connection with nonlinear Perron-Frobenius theory [20], we propose in Algorithm 1 an efficient iterative procedure for the computation of $\mathbf{p}^*$. Here, we use $\mathbf{p}(t)$ to represent the power vector obtained in the t-th iteration of the algorithm.

Algorithm 1 Max-Min Utility Optimization under Monotonic Constraints

1) Initialize power vector $\mathbf{p}(0) > \mathbf{0}$.
2) Update power vector $\mathbf{p}(t+1)$:
$$p_i(t+1) = \frac{p_i(t)}{u_i(\mathbf{p}(t))}\left(\triangleq T_i(\mathbf{p}(t))\right),\ \forall i. \tag{14}$$
3) Scale power vector $\mathbf{p}(t+1)$:
$$\mathbf{p}(t+1) \leftarrow \frac{\mathbf{p}(t+1)}{\beta(\mathbf{p}(t+1))}. \tag{15}$$
4) Repeat Steps 2 and 3 until convergence.

The following theorem establishes the existence and the uniqueness of the solution of the conditional eigenvalue problem in (13) as well as the convergence of Algorithm 1.

Theorem 1: Suppose that $\mathbf{T}: \mathcal{R}_+^L \to \mathcal{R}_+^L$, defined as $\mathbf{T}(\mathbf{p}) = [T_1(\mathbf{p}), \ldots, T_L(\mathbf{p})]^\top$, where $T_i(\mathbf{p}) = p_i/u_i(\mathbf{p})$, satisfies the following conditions: (i) there exist numbers $a > 0$, $b > 0$, and a vector $\mathbf{e} > \mathbf{0}$ such that $a\mathbf{e} \leqq \mathbf{T}(\mathbf{p}) \leqq b\mathbf{e}$, for all $\mathbf{p} \in \mathcal{U}$; (ii) for any $\mathbf{p}$, $\mathbf{q} \in \mathcal{U}$ and $0 \leqq \lambda \leqq 1$: If $\lambda\mathbf{p} \leqq \mathbf{q}$, then $\lambda\mathbf{T}(\mathbf{p}) \leqq \mathbf{T}(\mathbf{q})$; and, if $\lambda\mathbf{p} \leq \mathbf{q}$ with $\lambda < 1$, then $\lambda\mathbf{T}(\mathbf{p}) < \mathbf{T}(\mathbf{q})$. Then, the following properties hold:

(a) The conditional eigenvalue problem in (13) has a unique solution $\mathbf{p}^* \in \mathcal{U}$ and $\tau^* > 0$.
(b) The power vector $\mathbf{p}(t)$ in Algorithm 1 converges to $\mathbf{p}^*$ (i.e., the solution of (13) and, thus, (1)) for any initial point $\mathbf{p}(0) \geqq \mathbf{0}$ with $\beta(\mathbf{T}(\mathbf{p}(0))) > 0$.

The proof of the theorem and the following corollary is a nontrivial extension of a generalized result in nonlinear Perron-Frobenius theory presented in [20].

Corollary 1: For $\beta(\mathbf{p})$ that is zero only when $\mathbf{p} = \mathbf{0}$, the properties in Theorem 1 hold for $\mathbf{T}$ that is positive and concave, i.e., $\mathbf{T}(\mathbf{p}) > \mathbf{0}$ for $\mathbf{p} \geq \mathbf{0}$ and

$$\mathbf{T}(\lambda\mathbf{p} + (1-\lambda)\mathbf{q}) \geqq \lambda\mathbf{T}(\mathbf{p}) + (1-\lambda)\mathbf{T}(\mathbf{q}), \tag{16}$$

for all $\mathbf{p} \geqq \mathbf{0}$, $\mathbf{q} \geqq \mathbf{0}$, and $0 \leqq \lambda \leqq 1$.

Even though conditions (i) and (ii) in Theorem 1 are more general, the conditions in Corollary 1 are easier to verify and are, in fact, sufficient for most applications. The requirement that $\beta(\mathbf{p})$ is zero only when $\mathbf{p} = \mathbf{0}$ is satisfied when the power of all users are constrained, i.e., no user's transmit power can go to infinity, which is the case in most practical applications.

C. Efficient Computation of $\beta(\mathbf{p})$

In Algorithm 1, it is necessary to compute the scale $\beta(\mathbf{p}(t))$ for $\mathbf{p}(t)$ obtained in each iteration t. In particular, for given $\mathbf{p} > \mathbf{0}$ (which is the case in each iteration when Algorithm 1 is initialized with a positive power vector, i.e., $\mathbf{p}(0) > \mathbf{0}$), the scale $\beta(\mathbf{p})$ is chosen such that the normalized power vector $\tilde{\mathbf{p}} \triangleq \mathbf{p}/\beta(\mathbf{p})$ guarantees that at least one system constraint in (1) is tight. In general, this scale can be found via a bisection search as described in Algorithm 2 below in which the upper bound of $\beta(\mathbf{p})$ is first established in the initialization phase and the actual value of $\beta(\mathbf{p})$ is then obtained by decreasing the interval between the upper and lower bounds geometrically.

Algorithm 2 Computation of β via Bisection Search

Initialization:

i) Set $i \leftarrow 0$, $L \leftarrow 0$, and $U \leftarrow 2^i$.
ii) If there exists k such that $g_k(\mathbf{p}/U) > \bar{g}_k$, then increment $i \leftarrow i+1$ and set $U \leftarrow 2^i$.
iii) Repeat (ii) until $g_k(\mathbf{p}/U) \leq \bar{g}_k$ for all k.

Bisection Search

1) Set $\beta \leftarrow (U+L)/2$. If $g_k(\mathbf{p}/\beta) > \bar{g}_k$ for some k, then set $L \leftarrow \beta$. Otherwise, set $U \leftarrow \beta$.
2) Repeat until $|U - L| < \epsilon$.

Alternatively, notice that, for $\mathbf{p} > \mathbf{0}$, finding $\beta(\mathbf{p})$ by (12) is equivalent to choosing the minimum value of $\beta' > 0$ such that

$$I_k(\beta') \triangleq \beta' g_k(\mathbf{p}/\beta')/\bar{g}_k \leqq \beta', \tag{17}$$

for $k = 1, \ldots, K$. Based on this observation, an efficient iterative algorithm can be proposed for the computation of $\beta(\mathbf{p})$ by generalizing the results from [19]. Following [19], let us define the following class of standard functions.

Definition 2 ([19]): A function $I: \mathcal{R}_+ \to \mathcal{R}_+$ is *standard* if the following conditions[1] are satisfied for all $\beta > 0$:

1) Monotonicity: If $\beta^{(a)} \geqq \beta^{(b)}$, then $I(\beta^{(a)}) \geqq I(\beta^{(b)})$.
2) Scalability: For $\lambda > 1$, $\lambda I(\beta) > I(\lambda\beta)$.

For $\{I_k\}_{k=1}^K$ that are standard, we propose the following algorithm for the computation of $\beta(\mathbf{p})$. With a slight abuse of notation, we denote $\beta(t)$ in the following as the value of β in the t-th iteration.

Algorithm 3 Computation of β via Fixed-Point Iteration

1) Set initial value $\beta(0) > 0$.
2) Set $\beta(t+1) \leftarrow \max_k I_k(\beta(t))$.
3) Repeat Step 2 until convergence.

The following theorem shows the convergence and the validity of the above algorithm.

[1] Notice that, even though $\beta \geq 0$ was required in [19], the results hold equally for just $\beta > 0$. Moreover, notice that positivity (i.e., $I(\beta) > 0$) was required explicitly in [19], but can actually be implied by monotonicity and scalability since the latter two yield $tI(\beta) > I(t\beta) \geq I(\beta)$, for $t > 1$.

Theorem 2: For given $\mathbf{p} > 0$ and for $\{g_k\}_{k=1}^K$ such that $I_k(\beta) \triangleq \beta g_k(\mathbf{p}/\beta)$ is standard, for all k, the following properties hold:

1) Algorithm 3 converges to a unique fixed point β^*.
2) The fixed point β^* is equal to $\beta(\mathbf{p})$ defined in (12).

The above theorem is proved following similar procedures as in [19] and is omitted for brevity. However, it is worthwhile to note that, in [19], the properties in Definition 2 are required to hold for $\beta = 0$ as well. However, by initializing Algorithm 3 with $\beta(0) > 0$, the value of $\beta(t)$ will always be positive for all t and, thus, the convergence proof presented in [19] holds in our case as well. In the following, we present a sufficient condition on g_k for efficient evaluation of whether I_k is standard.

Corollary 2: The properties in Theorem 2 hold for $\mathbf{p} > \mathbf{0}$ and for $\{g_k\}_{k=1}^K$ that are concave and satisfy Assumption 2.

The proof relies on showing that I_k is standard if g_k is concave and monotone.

In the following sections, we utilize the mathematical tool developed above to solve representative problems in cognitive radio networks, heterogeneous cellular networks, and multiuser downlink networks.

III. Case Study I: Cognitive Radio Networks with Stochastic Interference Temperature Constraints

Let us consider a cognitive radio network with M primary receivers and L secondary transmitter-receiver pairs. Here, we consider a representative example where the power vector $\mathbf{p} = [p_1, \dots, p_L]^\top$ associated with the L secondary transmitters is adjusted to maximize the minimum SINR among secondary receivers subject to interference temperature constraints at primary receivers and individual power constraints at secondary transmitters. This problem was previously investigated in [14], [21] for the case where the interference temperature constraints are deterministic. Here, using the tool developed in Section II-B, we show that the problem with stochastic interference temperature constraints can also be solved exactly.

Specifically, let us consider the case where the instantaneous channels between all secondary transmitters and secondary receivers and those between all secondary transmitters and primary receivers $1, \dots, M$ are perfectly known, i.e., the frequency-flat fading case. The SINR at secondary receiver i is given by (2) and the interference temperature at primary receiver m is given by (7). The representative problem that we consider in this section can thus be formulated as follows:

$$\text{maximize} \quad \min_{i=1,\dots,L} \text{SINR}_i(\mathbf{p}) \tag{18a}$$

$$\text{subject to} \quad \Pr\left(\Psi_m(\mathbf{p}) \geq \bar{\Psi}_m\right) \leq \epsilon, \quad m = 1, \dots, M, \tag{18b}$$

$$p_i \leq \bar{p}_i, \quad i = 1, \dots, L, \tag{18c}$$

$$\text{variables}: \quad \mathbf{p}, \tag{18d}$$

where (18b) represents the stochastic interference temperature constraints with ϵ being a small given constant, and (18c) represents the individual power constraints. It is easy to verify that all the constraints mentioned above satisfy Assumption 2.

Moreover, to justify the use of Algorithm 1, it is necessary to verify that the utility functions satisfy Assumption 1 and the conditions in Theorem 1 (or those in Corollary 1). The former is straightforward to show and, thus, we focus only on the latter. To do so, let us first define the $L \times L$ nonnegative (cross-channel interference) matrix $\mathbf{F}$ with entries

$$F_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \dfrac{G_{ji}}{G_{ii}}, & \text{if } i \neq j, \end{cases} \tag{19}$$

and the vector

$$\mathbf{v} = \left(\frac{\eta_1}{G_{11}}, \dots, \frac{\eta_L}{G_{LL}}\right)^\top, \tag{20}$$

and express the objective functions as

$$u_i(\mathbf{p}) = \mathsf{SINR}_i(\mathbf{p}) = \frac{p_i}{(\mathbf{F}\mathbf{p} + \mathbf{v})_i}, \tag{21}$$

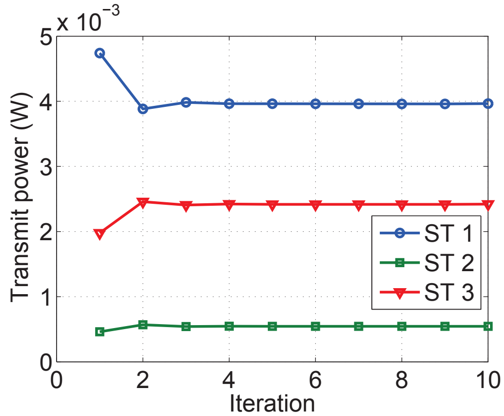
for $i = 1, \dots, L$. Notice that $\mathbf{T}(\mathbf{p}) = [T_1(\mathbf{p}), \dots, T_L(\mathbf{p})]^\top = \mathbf{F}\mathbf{p} + \mathbf{v}$ is affine and, thus, is concave. Moreover, since $\mathbf{F}$ is a nonnegative matrix and $\mathbf{v}$ has elements that are strictly positive, $\mathbf{T}$ is a positive mapping, i.e., $\mathbf{T}(\mathbf{p}) > \mathbf{0}$, for all $\mathbf{p} \geq \mathbf{0}$. Hence, the conditions in Corollary 1 are satisfied and, thus, the optimal solution can be obtained iteratively with guaranteed convergence using Algorithm 1. The value of $\beta(\mathbf{p}(t))$ in each iteration t is computed using the bisection search described in Algorithm 2.

Example I: Assume that there are $M = 2$ primary receivers and $L = 3$ secondary transmitter-receiver pairs in the cognitive radio network. Following [26], we model the $\alpha_{ji}, \forall j, i$ as $\alpha_{ji} = d_{ji}^{-\nu} s_{ji}$, where d_{ji} and s_{ji} are the distance and the shadowing effect between secondary transmitter j and secondary receiver i, respectively, and $\nu = 3.5$ is the path-loss exponent. s_{ji} is defined as $10^{X/10}$ with X being a Gaussian random variable with mean 0 and standard deviation 10 (in dB). Assume that all the primary receivers have instantaneous knowledge of the channels between itself and secondary transmitters. The instantaneous channels between secondary transmitters and secondary receivers are assumed to be known and are given by

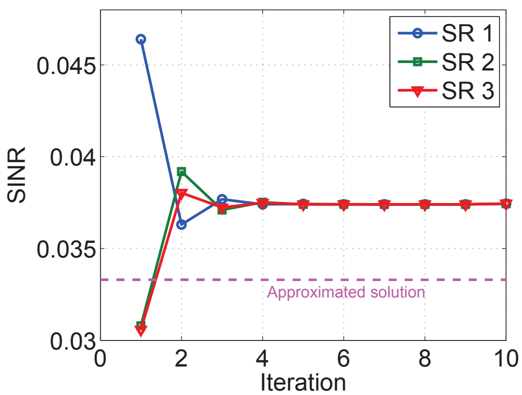
$$\mathbf{G} = \begin{bmatrix} 71^{-3.5} \times 0.60 & 61^{-3.5} \times 0.74 & 253^{-3.5} \times 3.13 \\ 83^{-3.5} \times 44.98 & 65^{-3.5} \times 2.37 & 199^{-3.5} \times 0.22 \\ 222^{-3.5} \times 0.32 & 204^{-3.5} \times 0.39 & 84^{-3.5} \times 1.96 \end{bmatrix}.$$

The interference thresholds for both primary receivers are set to -80 dBW (i.e., $\bar{\Psi}_1 = \bar{\Psi}_2 = 10^{-8}$) [26]. The maximum transmit power for all secondary transmitters is 5 W (i.e., $\bar{p}_1 = \bar{p}_2 = \bar{p}_3 = 5$).

In Fig. 1, we can see that both the secondary transmitter transmit powers and the SINR at the secondary receivers converge rapidly within 5 iterations using our proposed algorithm. The interference temperature outage probability is evaluated



(a) Evolution of transmit power.



(b) Evolution of SINR.

Fig. 1. In Example I, the transmit powers of the secondary transmitters converge rapidly and the SINR converges to a common value using Algorithm 1.

in each iteration of the bisection search for $\beta(\mathbf{p})$ using Monte Carlo experiments (averaged over 10^7 realizations). The terminating parameter in Algorithm 2 is set as $\epsilon = 5 \times 10^{-5}$. The "approximated solution" indicated in Fig. 1(b) represents the converged SINR value obtained in [26], which performs suboptimally compared to our result.

IV. Case Study II: Reliable Multiuser Downlink System with SINR requirement constraints

In this section, we consider a single cell multiuser downlink system, where the base stations (the transmitter) is equipped with an antenna array and each user (the receiver) has a single receive antenna, transmitting simultaneously on a shared spectrum. Assume that L users operate over a common frequency-flat channel. Channel State Information (CSI) is available at both the receiver and transmitter sides. Under this system setting, the antenna array forms a multiple-input-single-output (MISO) channel to obtain transmit diversity, which provides the extra degree of freedom. The accuracy and computational complexity of decoding depends on the mean-square error (MSE), thereby optimization of MSE is required for the reliable transmission. In addition, SINR requirements are imposed on all the users. Thus, we seek to minimize the maximum weighted MSE between the transmitted and estimated symbols subject to SINR constraints.

Let the vector $\mathbf{p} = [p_1, \ldots, p_L]^\top$ denote the transmit power vector. The SINR of the ith user is given by (2), and we use the nonnegative matrix $\mathbf{F}$ and the vector $\mathbf{v}$ defined in (19) and (20) respectively. Assuming that all the receivers use the linear minimum mean-square error (LMMSE) filter for estimating the received symbols of all the users, we can express the MSE of the ith user as

$$\mathsf{MSE}_i(\mathbf{p}) = \frac{1}{1 + \mathsf{SINR}_i(\mathbf{p})}. \tag{22}$$

Our objective is to minimize the maximum weighted MSE, i.e., $\min_{\mathbf{p}} \max_{i=1,\ldots,L} w_i \mathsf{MSE}_i(\mathbf{p})$, where $\mathbf{w}$ is a positive vector with the entry w_i assigned to the ith link to reflect some priority. A larger w_i denotes a higher priority. Notice that solving $\min_{\mathbf{p}} \max_{i=1,\ldots,L} w_i \mathsf{MSE}_i(\mathbf{p})$ is equivalent to solving $\max_{\mathbf{p}} \min_{i=1,\ldots,L} u_i(\mathbf{p})$, where

$$u_i(\mathbf{p}) = \frac{1 + \mathsf{SINR}_i(\mathbf{p})}{w_i} = \frac{(\mathbf{F}\mathbf{p} + \mathbf{v})_i + p_i}{w_i(\mathbf{F}\mathbf{p} + \mathbf{v})_i}. \tag{23}$$

Hence, the problem can be formulated as

$$\begin{aligned} &\text{maximize} \quad \min_{i=1,\ldots,L} \; u_i(\mathbf{p}) && \text{(24a)}\\ &\text{subject to} \quad \mathsf{SINR}_i(\mathbf{p}) \geq \gamma_i, \quad i = 1, \ldots, L && \text{(24b)}\\ &\text{variables :} \quad \mathbf{p}, && \text{(24c)} \end{aligned}$$

where $\boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_L]^\top$ is a given SINR threshold vector that represents the SINR requirement, i.e., the received SINR of all the users must be higher than $\boldsymbol{\gamma}$. This constraint set satisfies Assumption 2 if we rewrite it in an equivalent matrix form as $(\mathbf{I} - \mathrm{diag}(\boldsymbol{\gamma})\mathbf{F})\mathbf{p} \geq \mathrm{diag}(\boldsymbol{\gamma})\mathbf{v}$. Notably, $g_i(\mathbf{p}) = (\mathrm{diag}(\boldsymbol{\gamma})\mathbf{F} - \mathbf{I})\mathbf{p}$ is affine, and thus, is concave. Therefore, Corollary 2 holds and Algorithm 3 can be adopted to compute $\beta(\mathbf{p})$.

Notice that $\{u_i\}_{i=1}^L$ satisfies Assumption 1. We can observe that the function in (23) is positive and monotonically increases with respect to p_i. Furthermore, the first order derivative of $u_i(\mathbf{p})$ at p_j, for $j \neq i$, is given by $\partial u_i(\mathbf{p})/\partial p_j = -w_i^{-1} p_i F_{ij} (\mathbf{F}\mathbf{p} + \mathbf{v})^{-2} < 0$, and thus the competitiveness of (23) follows. For $\lambda > 1$, we have

$$u_i(\lambda\mathbf{p}) = \frac{1}{w_i}\left(1 + \frac{p_i}{(\mathbf{F}\mathbf{p} + \frac{1}{\lambda}\mathbf{v})_i}\right) \geqq u_i(\mathbf{p}),$$

for all i, which implies the directional monotonicity.

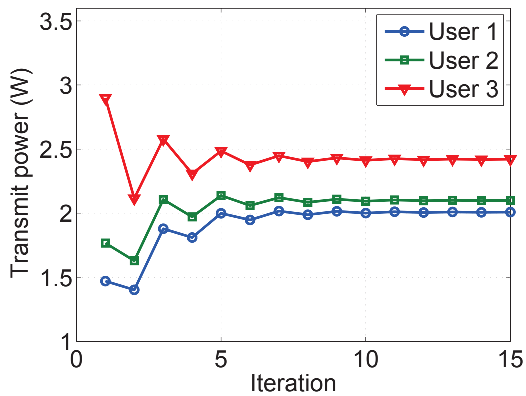
Next, we verify that $\{u_i\}_{i=1}^L$ satisfies the conditions in Theorem 1 (or equivalently, Corollary 1). Let us define $\mathbf{T}(\mathbf{p}) = [T_1(\mathbf{p}), \ldots, T_L(\mathbf{p})]^\top$, where

$$T_i(\mathbf{p}) = \frac{w_i p_i}{1 + \mathsf{SINR}_i(\mathbf{p})} = w_i(\mathbf{F}\mathbf{p} + \mathbf{v})_i \frac{\mathsf{SINR}_i(\mathbf{p})}{1 + \mathsf{SINR}_i(\mathbf{p})}. \tag{25}$$

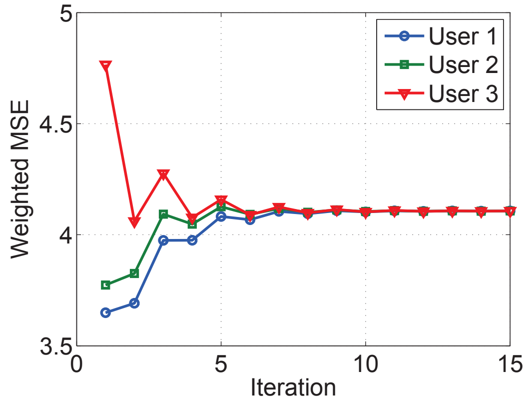
It is easy to see that $\mathbf{T}$ is positive. Moreover, to show that $\mathbf{T}$ is concave in $\mathbf{p}$, let us invoke the following lemma from [13].

Lemma 4 ([13]): If $h : \mathcal{R}^m \to \mathcal{R}$ is concave, and $\mathbf{A} \in \mathcal{R}^{m \times n}$, $\mathbf{b} \in \mathcal{R}^m$, $\mathbf{c} \in \mathcal{R}^n$, and $d \in \mathcal{R}$, then

$$g(\mathbf{x}) = (\mathbf{c}^\top\mathbf{x} + d)h((\mathbf{A}\mathbf{x} + \mathbf{b})/(\mathbf{c}^\top\mathbf{x} + d)) \tag{26}$$



(a) Evolution of transmit power.



(b) Evolution of weighted MSE.

Fig. 2. In Example II, the transmit powers and weighted MSE converge rapidly (within 15 iterations) using Algorithm 1.

is concave on $\{\mathbf{x}|\mathbf{c}^\top\mathbf{x}+d>0,(\mathbf{A}\mathbf{x}+\mathbf{b})/(\mathbf{c}^\top\mathbf{x}+d)\in \mathbf{dom}(h)\}$.

Since $h(s)=\frac{s}{1+s}$ is concave and by letting $m=1$, $\mathbf{A}=\mathbf{e}_i^\top$, $\mathbf{b}=0$, $\mathbf{c}^\top$ the i-th row of $\mathbf{F}$, $\mathbf{x}=\mathbf{p}$, $d=\eta_i/G_{ii}$ in Lemma 4, we have that T_i (and, thus, $\mathbf{T}$) is concave. Hence, it follows from Corollary 1 that the optimal solution to this problem can be obtained iteratively with guaranteed convergence using Algorithm 1.

Example II: Assume that there are $L=3$ users in a single cell multiuser downlink system, and only one base station serves all the users in the cell, i.e., User 1, User 2 and User 3 (whose downlink transmit powers p_1, p_2 and p_3 are measured in Watt (W)). In the channel gain matrix $\mathbf{G}=[G_{ij}]_{i,j=1}^L>0_{L\times L}$, the diagonal entries G_{ii}, for all i, represents the channel gain from the base station to the ith user, and G_{ji}, for $j\neq i$, is the channel gain from the transmission of the jth user to the transmission of the ith user, which leads to cross channel interference. We use the following channel gain matrix $\mathbf{G}$:

$$\mathbf{G}=\begin{bmatrix}0.65 & 0.22 & 0.14\\ 0.15 & 0.78 & 0.26\\ 0.19 & 0.13 & 0.72\end{bmatrix}, \tag{27}$$

and the following weight for the weighted MSE objective: $\mathbf{w}=[0.42\ 0.45\ 0.5]^\top$. The noise power of each user is 1 W, and we set the SINR threshold $\boldsymbol{\gamma}=[0.52\ 0.77\ 0.65]^\top$.

Fig. 2 plots the evolution of the base station transmit powers to all the users and the weighted MSE in the multiuser downlink system. We set the initial power vector to a positive random value, and run Algorithm 1 for 15 iterations before it terminates. In this numerical example, $\beta(\mathbf{p}(t))$ is evaluated using Algorithm 3. Fig. 2 illustrates that both the transmit power and the MSE converge very fast within 9 iterations to the optimal solution (verifying Theorem 1).

V. Case Study III: Heterogeneous Wireless Cellular Networks with Outage Constraints

In this section, we consider the downlink case in a heterogeneous cellular network with $L+M$ base stations transmitting to their corresponding users under Rayleigh fading. Among these base stations, we assume that L base stations can adjust transmission power to optimize the overall network utility whereas M base stations maintain a nonadjustable and fixed transmission power over time. Specifically, we seek to minimize the worst outage probability among the L base stations whose powers are adjustable subject to constraints on the outage probability of the M base stations whose powers are nonadjustable (e.g., high priority base stations such as macrocell base stations).

Let $p_1,\ldots,p_L$ be the transmit powers of the L base stations with adjustable power and let $p_{L+1},\ldots,p_{L+M}$ be the transmit powers of base stations with nonadjustable power. The transmissions are subject to cochannel interference and, thus, the signals received at the user served by base station i can be defined as (5). Note that, in this case, we assume a same noise variance η_n for all the users. Thus, the outage probability for the user served by base station i is can be written in closed-from as [6], [11]:

$$O_i(\mathbf{p})=\Pr(\mathsf{SINR}_i(\mathbf{p})<\gamma_i) \tag{28}$$

$$=1-e^{\frac{-\gamma_i\eta_n}{G_{ii}p_i}}\prod_{j\neq i}\left(1+\frac{\gamma_i G_{ji}p_j}{G_{ii}p_i}\right)^{-1}, \tag{29}$$

where γ_i is the SINR threshold of the user served by base station i.

Our goal is to find $\mathbf{p}$ to minimize the worst outage probability, i.e., $\min_{\mathbf{p}}\max_{i=1,\ldots,L}O_i(\mathbf{p})$. Notice that $\min_{\mathbf{p}}\max_i O_i(\mathbf{p})=\max_{\mathbf{p}}\min_i u_i(\mathbf{p})$, where

$$u_i(\mathbf{p})=[-\log(1-O_i(\mathbf{p}))]^{-1} \tag{30}$$

$$=\left[\frac{\gamma_i\eta_n}{G_{ii}p_i}+\sum_{j\neq i}\log\left(1+\frac{\gamma_i G_{ji}p_j}{G_{ii}p_i}\right)\right]^{-1}. \tag{31}$$

Hence, the problem can be formulated as

$$\begin{aligned}
&\text{maximize} && \min_{i=1,\ldots,L}\ u_i(\mathbf{p}) && (32a)\\
&\text{subject to} && O_m(\mathbf{p})\leq\bar{O}_m,\ \text{for } m>L, && (32b)\\
& && p_i\leq\bar{p}_i,\ \text{for } m\leq L, && (32c)\\
&\text{variables}: && \mathbf{p}=[p_1,\ldots,p_L]^\top. && (32d)
\end{aligned}$$

Again, it is straightforward to show that the above constraints satisfy Assumption 2. In particular, for the constraints in (32b), we can see that increasing $\mathbf{p}$ raises the interference at the user served by base station m and, thus, increases its outage probability, i.e., $O_m(\mathbf{p})$.

Next, we show that the utility functions $\{u_i\}_{i=1}^L$ considered here satisfy Assumption 1 and the conditions in Theorem 1 (or, more practically, those in Corollary 1). First, notice that positivity and competitiveness can be seen directly from the definition in (30). In particular, we can see that $u_i(\mathbf{p})$ is equal to 0 when $p_i = 0$ and that it increases monotonically with p_i but decreases monotonically with p_j, for all $j \neq i$, when $p_i > 0$. Moreover, for $\lambda \geqq 1$, we have

$$u_i(\lambda\mathbf{p}) = \left[\frac{\gamma_i\eta_n}{G_{ii}\lambda p_i} + \sum_{j\neq i}\log\left(1+\frac{\gamma_i G_{ji}p_j}{G_{ii}p_i}\right)\right]^{-1} \geqq u_i(\mathbf{p}),$$

for all i, and, thus, directional monotonicity follows.

Next, let $\mathbf{T}(\mathbf{p}) = [T_1(\mathbf{p}),\ldots,T_L(\mathbf{p})]^\top$, where

$$T_i(\mathbf{p}) = \frac{p_i}{u_i(\mathbf{p})} = \frac{\gamma_i\eta_n}{G_{ii}} + \sum_{j\neq i} p_i\log\left(1+\frac{\gamma_i G_{ji}p_j}{G_{ii}p_i}\right). \quad (33)$$

It is straightforward to see that $\mathbf{T}$ is positive. To show that $\mathbf{T}$ is concave, let us recall the fact that $t\log(1+x/t)$ is strictly concave in (x,t) for strictly positive t since it is the perspective function of the strictly concave function $\log(1+x)$ [25]. Hence, $T_i(\mathbf{p})$ can be viewed as the sum of strictly concave perspective functions and, therefore, $\mathbf{T}$ is strictly concave in $\mathbf{p}$. Hence, by Corollary 1, the optimal solution can be obtained efficiently with guaranteed convergence using Algorithm 1.

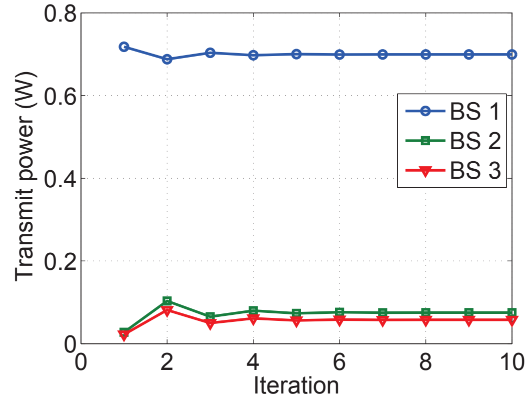
It is interesting to remark that, in computing $\beta(\mathbf{p})$, the constraints in (32b) can be reformulated to allow for the adoption of Algorithm 3. In particular, the constraints in (32b) can be reformulated as

$$\begin{aligned} g_m(\mathbf{p}) &\triangleq -\log\left[1-O_m(\mathbf{p})\right] && (34)\\ &= \frac{\gamma_m\eta_n}{G_{mm}p_m} + \sum_{j\neq m}\log\left(1+\frac{\gamma_m G_{jm}p_j}{G_{mm}p_m}\right) && (35)\\ &\leq -\log\left[1-\bar{O}_m\right] \triangleq \bar{g}_m, && (36)\end{aligned}$$

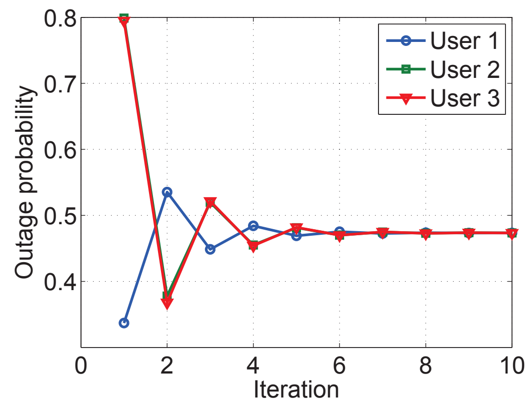
where p_m and p_j, for $j > L$, are constant. In this case, g_m is a concave and monotone function of $\mathbf{p}$. Hence, Corollary 2 holds and Algorithm 3 is applicable.

Example III: We consider three base stations (base station 1, base station 2, and base station 3) whose transmit powers are adjustable and one base stations (base stations 4) whose transmit power is nonadjustable. Assume that one base station only serves one user, i.e., User 1, User 2, User 3, and User 4 are served by base stations 1, base station 2, base station 3, and base station 4, respectively. For base station 1, base station 2, and base station 3 whose transmit powers are adjustable, the maximum transmit power is 1 W (i.e., $\bar{p}_1 = \bar{p}_2 = \bar{p}_3 = 1$) and the SINR threshold is $\gamma_1 = \gamma_2 = \gamma_3 = 1$. For base station 4 whose transmit power is nonadjustable, we set the transmit power to 2 W (i.e., $p_4 = 2$). The SINR threshold of User 4 is set to 10 (i.e., $\gamma_4 = 10$), and the outage constraint $\bar{O}_4 = 0.1$. The noise variance η_n is assumed to be 1 pW for all users in the network. By considering only the path-loss, we use the following path gain matrix $\mathbf{G}$:

$$\mathbf{G} = \begin{bmatrix} 93^{-4} & 124^{-4} & 152^{-4} & 127^{-4}\\ 67^{-4} & 55^{-4} & 215^{-4} & 191^{-4}\\ 119^{-4} & 213^{-4} & 58^{-4} & 188^{-4}\\ 169^{-4} & 164^{-4} & 179^{-4} & 53^{-4}\end{bmatrix}. \quad (37)$$



(a) Evolution of transmit power.



(b) Evolution of outage probability.

Fig. 3. In Example III, the transmit powers converge fast and the outage probabilities of the users converge to a unique value using Algorithm 1.

Fig. 3 plots the evolution of the base station transmit powers and the outage probabilities for all the users in the heterogeneous network. The fast convergence (within 6 iterations) for both transmit power and outage probability are illustrated. Here, $\beta(\mathbf{p}(t))$ is evaluated using Algorithm 3.

VI. Conclusion

In this paper, we proposed a unifying framework based on a generalized nonlinear Perron-Frobenius theory to solve max-min utility fairness optimization problems with nonlinear monotonic constraints. The proposed techniques were shown to be widely applicable to many problems in wireless networks. Our framework broadly enlarges the class of utility fairness resource allocation problems with realistic nonlinear power constraints, interference constraints, and stochastic outage constraints etc. In particular, algorithms with low complexity (no configuration needed) were designed to solve these nonconvex problems in a jointly optimal and scalable manner. Three representative case studies in cognitive radio networks, multiuser downlink system and heterogeneous networks were presented to illustrate the applicability of our framework.

APPENDIX

A. *Proof of Lemma 1*

Suppose that $(\tau^*, \mathbf{p}^*)$ is an optimal solution. To show that $(\tau^*, \mathbf{p}^*)$ is positive, we first suppose that there exists i such that $p_i^* = 0$. Then, $u_i(\mathbf{p}^*) = 0$ and $\tau^* = \min_i u_i(\mathbf{p}^*) = 0$. However, by the feasibility assumption (c.f. Assumption 2), there must exist a vector $\mathbf{p}' > \mathbf{0}$ that is feasible and, by the positivity assumption (c.f. Assumption 1), we can find $\tau \triangleq \min_i u_i(\mathbf{p}') > 0 = \tau^*$, which contradicts the assumption that $(\tau^*, \mathbf{p}^*)$ is optimal. Hence, $(\tau^*, \mathbf{p}^*)$ is positive.

Next, we show that all utility constraints are tight at optimality. Suppose that there exists i such that $u_i(\mathbf{p}^*) > \tau^*$. Then, by the fact that $\mathbf{p}^* > \mathbf{0}$, as shown above, we can choose $\hat{\mathbf{p}}$ such that $0 < \hat{p}_i < p_i^*$ and $\hat{p}_j = p_j^*$, for all $j \neq i$, and such that $\tau^* < u_i(\hat{\mathbf{p}}) < u_i(\mathbf{p}^*)$, due to the competitiveness of the utility functions (c.f. Assumption 1). However, this yields $u_j(\hat{\mathbf{p}}) > u_j(\mathbf{p}^*)$, for all $j \neq i$. In this case, we can choose τ such that $\tau \triangleq \min_i u_i(\hat{\mathbf{p}}) > \min_i u_i(\mathbf{p}^*) = \tau^*$, which contradicts the assumption that τ^* is optimal. Hence, all the objective constraints must be tight at optimality.

Finally, we show that at least one system constraint is tight at optimality. Suppose that $g_k(\mathbf{p}^*) < \bar{g}_k$, for all k. Since $\mathbf{p}^* > \mathbf{0}$, there exists $\lambda > 1$ and $\mathbf{p}' \triangleq \lambda \mathbf{p}^*$ such that $g_k(\mathbf{p}^*) < g_k(\mathbf{p}') \leq \bar{g}_k$, for all k. Then, by the fact that $\mathbf{p}^* > \mathbf{0}$ and the directional monotonicity of the utility functions (c.f. Assumption 1), it follows that $u_i(\mathbf{p}') > u_i(\mathbf{p}^*)$, for all i. This contradicts the assumption that $\mathbf{p}^*$ is optimal. Hence, at least one of the system constraints must be tight at optimality.

B. *Proof of Lemma 3*

Let $\mathbf{p}^*$ be an optimal solution of (1). We first show that $\beta(\mathbf{p}^*) \leqq 1$. Suppose that $\beta(\mathbf{p}^*) > 1$ and recall, from Lemma 1, that $\mathbf{p}^*$ must be nonzero in all components. It then follows by the strict monotonicity and validity of the constraints that $g_k(\mathbf{p}^*) > g_k(\mathbf{p}^*/\beta(\mathbf{p}^*)) = \bar{g}_k$ for some k. Therefore, $\mathbf{p}^*$ is infeasible and contradicts the assumption that $\mathbf{p}^*$ is optimal.

Moreover, by Lemma 1, we know that $g_k(\mathbf{p}^*) \leqq \bar{g}_k$, for all k, and there exists k' such that $g_{k'}(\mathbf{p}^*) = \bar{g}_{k'}$. Suppose that $\beta(\mathbf{p}^*) < 1$, then the power vector $\tilde{\mathbf{p}} = \mathbf{p}^*/\beta(\mathbf{p}^*)$ is also feasible. By the fact that $\mathbf{p}^* > \mathbf{0}$ (c.f. Lemma 1) and by the directional monotonicity of the utility functions, it follows that $u_i(\tilde{\mathbf{p}}) > u_i(\mathbf{p}^*)$, for all i. This implies that $\min_i u_i(\tilde{\mathbf{p}}) > \min_i u_i(\mathbf{p}^*)$, which contradicts the assumption that $\mathbf{p}^*$ is optimal. Hence, $\beta(\mathbf{p}^*) = 1$ and, thus, $\mathbf{p}^* \in \mathcal{U}$.

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