Network Centrality as Statistical Inference in Large Networks

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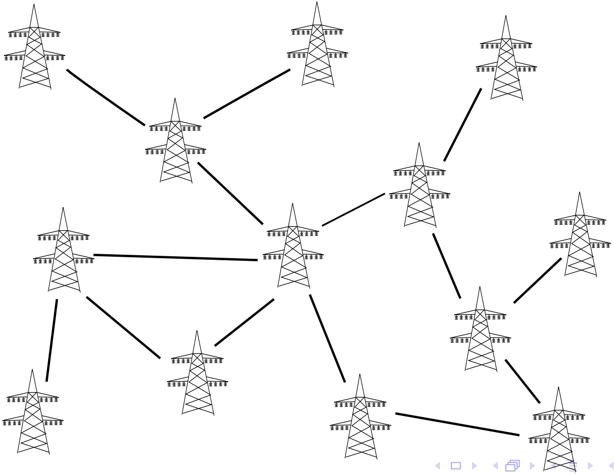
Problem Statement

Averting Cascading Failures in Networked Infrastructures: Poset-constrained Graph Algorithms, IEEE Journal of Selected Topics in Signal Processing, 2018 [1]



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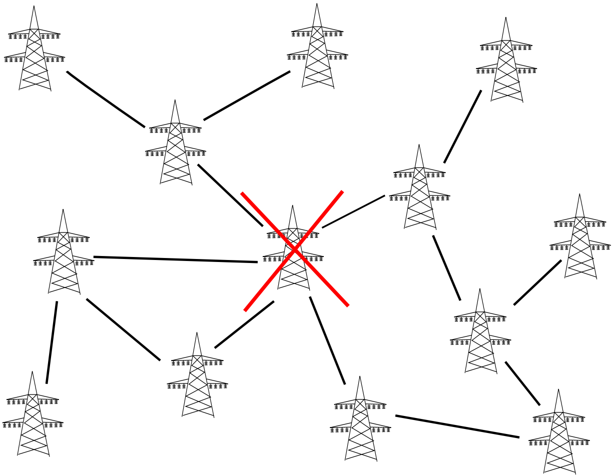

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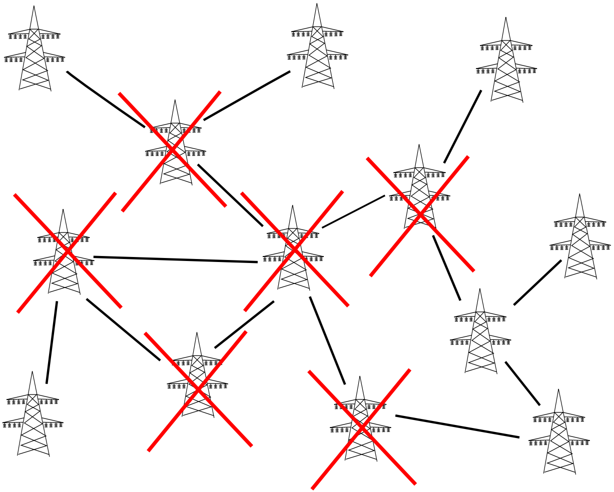

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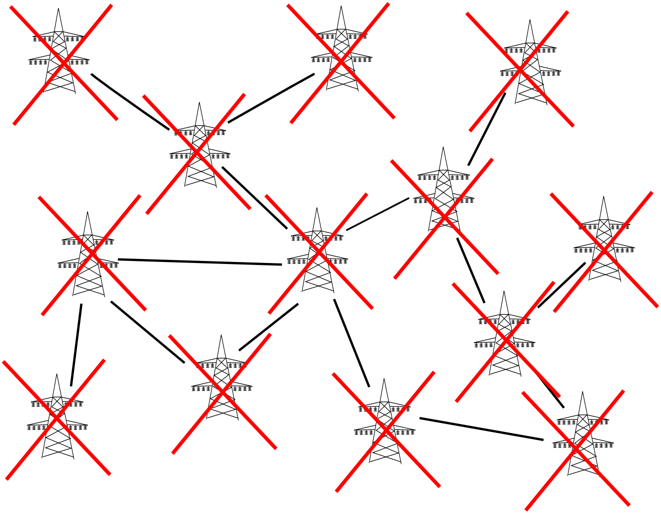

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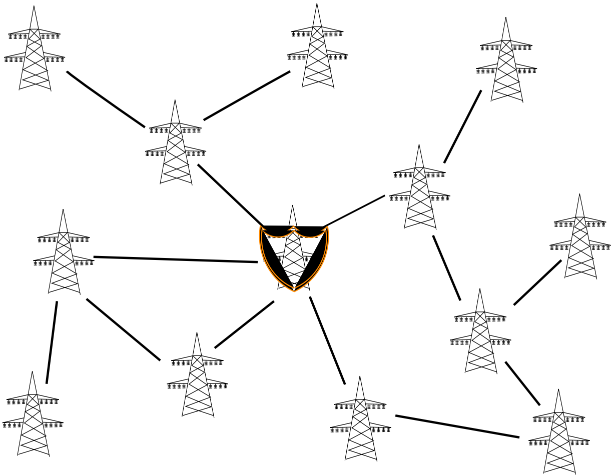

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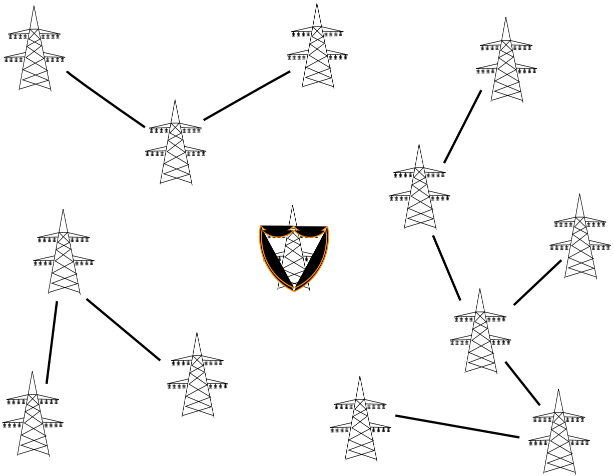

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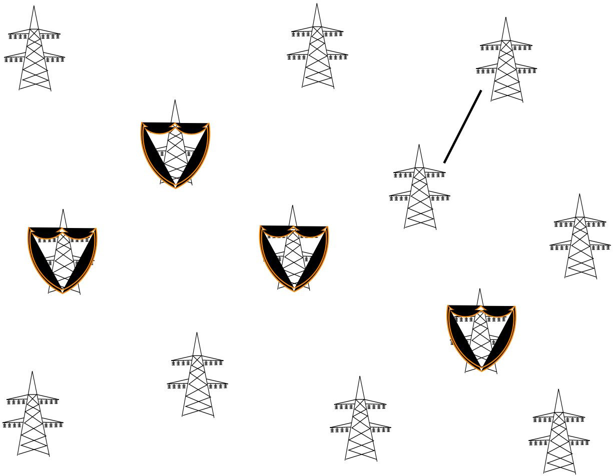
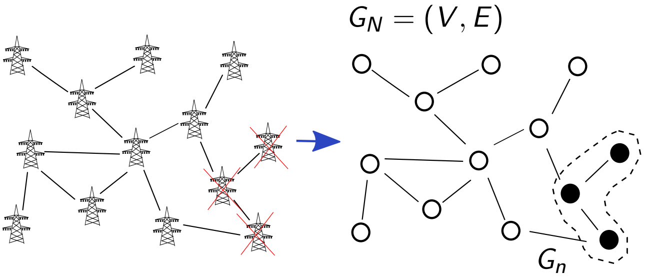


The Model and Assumptions



The Model and Assumptions

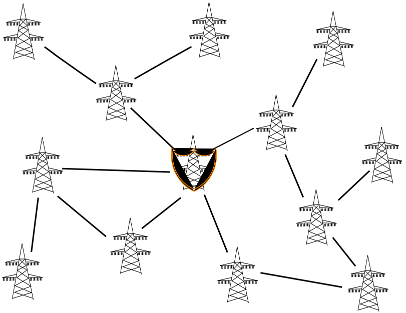
In the extended SI model, we have three types of nodes described as following:

- Susceptible node: Nodes that are susceptible to failure.
- Infected node: Nodes that are under the effect of failure.
- Protected node: Nodes that are protected and can not spread the failure further.
- Every vertex is equally like to be the source
- Assume that in each time period, one vertex is uniformly chosen from the neighbors of those infected vertices to be infected.

The Protection Node Placement Problem

$$
\begin{aligned}
\underset{v \in G_n}{\text{minimize}} \quad & \mathbf{E}(|G_n|) \\
\text{subject to} \quad & |V_P| = k,
\end{aligned} \tag{1}
$$

Example: $|V_P| = 1$



$$\mathbf{E}(|G_n|) = \frac{1}{13} \cdot [(3+3+3) + (3+3+3) + (5+5+5+5+5)]$$
$$= \frac{1}{13} \cdot [3^2 + 3^2 + 5^2]$$

The Protection Node Placement Problem

$$
\begin{array}{ll}
\underset{V_P \subseteq V(G_n)}{\text{minimize}} & (C_1^{\{V_P\}})^2 + (C_2^{\{V_P\}})^2 + \ldots + (C_m^{\{V_P\}})^2 \\
\text{subject to} & |V_P| = k,
\end{array}
\qquad (2)
$$

where $C_1^{\{V_P\}}, C_2^{\{V_P\}}, \ldots,$ and $C_m^{\{V_P\}}$ are the connected components after removing vertices in V_P from G_N.

Posets and Linear Extensions

Definition

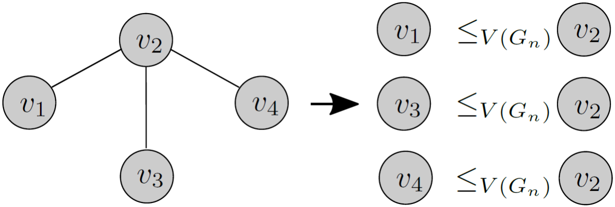
A non-strict partial order is a relation $\leq_S$ over a set S satisfying the following rules, for all $v_1, v_2, v_3 \in S$:

- $v_1 \leq_S v_1$ (reflexivity)
- if $v_1 \leq_S v_2$ and $v_2 \leq_S v_1$, then $v_1 = v_2$ (antisymmetry)
- if $v_1 \leq_S v_2$ and $v_2 \leq_S v_3$, then $v_1 \leq_S v_3$ (transitivity)

A **total order** has one more rule that every two elements in the set must be assigned a relation.

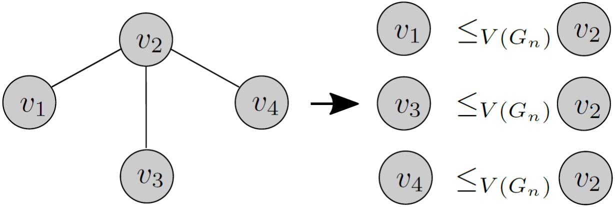
A **linear extension** $\leq_S^*$ of a partial order $\leq_S$ is a total order which preserve the relation in $\leq_S$, i.e., for all $v_1 \leq_S^* v_2$ whenever $v_1 \leq_S v_2$.

Posets and Rooted Trees



There is no relation between v_1, v_3 and v_4, hence this order is a **partial** order.

Linear Extensions and Cascading Failure



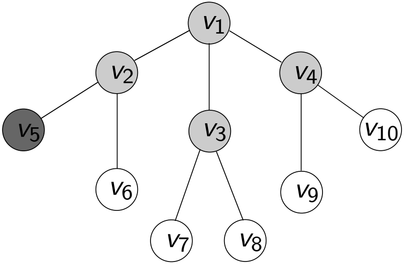
Consider a cascading failure on this graph with a specific order, for example $v_2 \rightarrow v_1 \rightarrow v_3 \rightarrow v_4$, then there is relation between any two vertices in this set, i.e., this specific order is a linear extensions on this posets (rooted tree). Intuitively, choosing the vertex with the maximum number of linear extensions to be protected is a good choice! [2]

Network Centrality to Determine Maximum Number of Linear Extensions of a Poset

Definition

Let G_n be a tree with n vertices, for any $u, v \in G_n$, let t_v^u be the subtree rooted at v by removing the edge (u, v) from G_n and slightly abusing the notation of the subtree size t_v^u as t_v^u.

For example, $t_{v_1}^{v_2} = 7$ and $t_{v_2}^{v_1} = 3$.

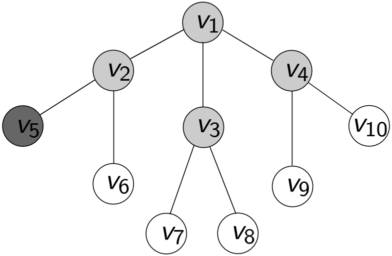


Definition

Define the branch weight of a vertex v in G_n by

$$\text{weight}(v) = \max_{c \in \text{child}(v)} t_c^v.$$

The vertex of G_n with the *minimum weight* is called the *centroid* of G_n [3]. For example, v_1 has the minimum weight, hence v_1 is the centroid.



Theorem

Let G_N be a general tree graph. Then, the rooted tree with the maximum number of linear extensions is rooted at v^* if and only if v^* is a centroid of G_N(proved in [4]).

Message Passing Algorithm to compute the Centroid of a Graph

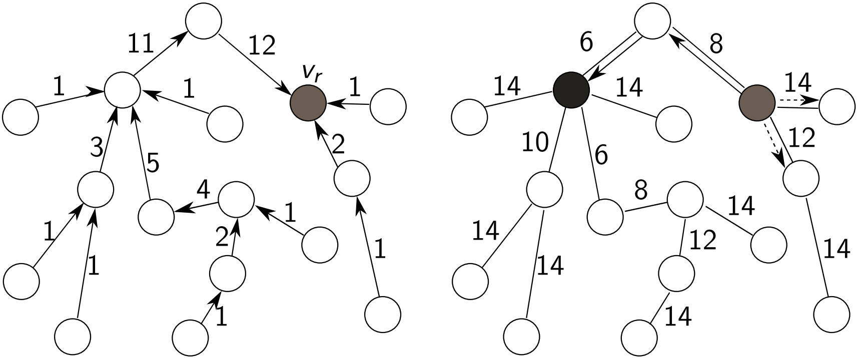
Let $M^{i \to j}$ denote the message from vertex i to vertex j. Let $\mathrm{Diff}(i,j)$ be defined by $\mathrm{Diff}(i,j) = |M^{i \to j} - M^{j \to i}|$.

Theorem

Given a tree G_n with n vertices.
$v_c \in G_n$ is the centroid if and only if $\forall v$ adjacent to v_c and $v_i, v_j \in V(G_n)$, $\min_{(v,v_c) \in E(G_n)} \{\mathrm{Diff}(v_c, v)\} \le \{\mathrm{Diff}(v_i, v_j)\}$. Moreover, for any $u \in G_n$, on the path from v_c to u say $(v_1, v_2, ..., v_D)$, where $v_1 = v_c$ and $v_D = u$. The sequence of $\mathrm{Diff}(v_i, v_{i+1})$ for $i = 1, 2...D$ is increasing.

Message Passing Algorithm to compute the Centroid of a Graph

$n = 12 + 1 + 2 + 1 = 16$



Assume G_N is a tree:

- When $|V_p| = 1$, we choose the centroid to be the solution.
- When $|V_p| > 1$, we use the centroid decomposition to select the protection set.

This may not be the optimal solution, but the performance can be bounded above.

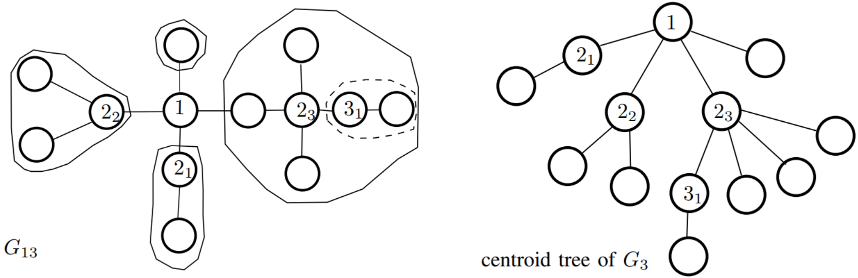
Theorem

Let $f(\{V_p\})$ denote the objective function in (2) and let V_p^* denote the optimal solution of (2). The centroid decomposition approach guarantees that

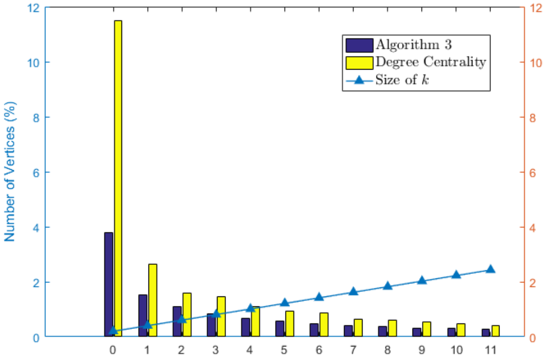
$$1 \leq \frac{f(\{V_p\})}{f(\{V_p^*\})} \leq c\frac{N}{k+1},$$

where k is the size of the protection set V_p and c is a small constant.

Centroid Decomposition

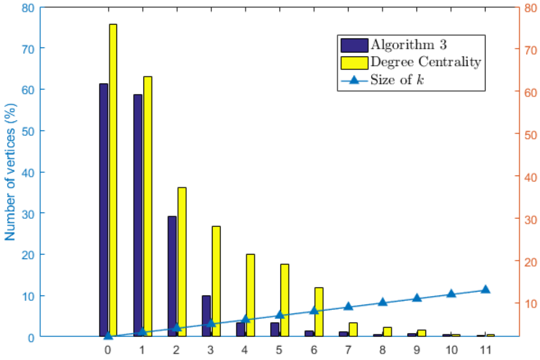


Experimental Results: $N = 4941$



A simulation result when G_N is a random tree. The y-axis represents the number of vertices in percentage and the x-axis represents each trial with different size of k.

Experimental Results: $N = 4941$



A simulation result when G_N is a real world network: Western United State Power Grid Network. The y-axis represents the number of vertices and the x-axis represents each trial with different size of k.

Network Centrality as Statistical Inference

In the reverse engineering perspective, we ask:

- Given a network centrality, what are the statistical inference optimization problems that it implicitly solves?
- Distance centrality and branch weight centrality solve the rumor source detection problem for degree-regular tree graphs.
- Betweenness centrality solves the protection node placement problem for a single node special case.
- Network centrality provides guiding principle on algorithm design and can compute exact or approximate solutions.

In the reverse engineering perspective, we ask:

- Given a stochastic optimization formulation over a network, how to transform it or to decompose it to one whose subproblems are graph-theoretic and can utilize network centrality, then solve or approximate the whole problem?
- Rumor source detection as a maximum-likelihood estimation problem solved by rumor centrality.
- Expected cascade size minimization problem solved by vaccine centrality.
- New algorithms can be designed based on message-passing (belief propagation) graph analysis
- Deep connections between network centrality on induced abstract data types with probability on trees and graphs.

Thank You!

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