Convex Optimization of Cognitive Radio Networks

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Acknowledgement

- Mung Chiang (Princeton)
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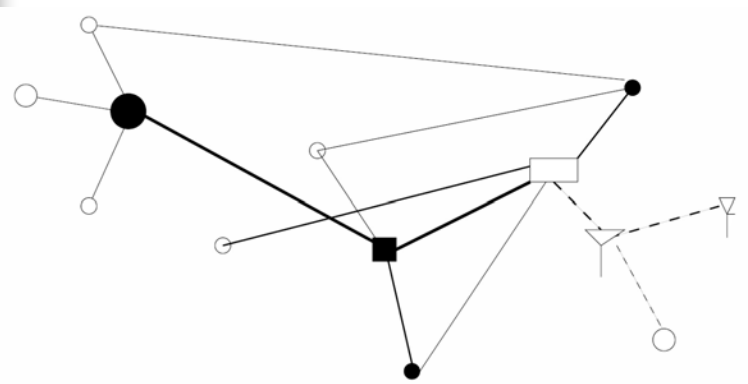
Reference: Convex Optimization by Stephen Boyd & Lieven Vandenberghe, Cambridge University Press, 2004

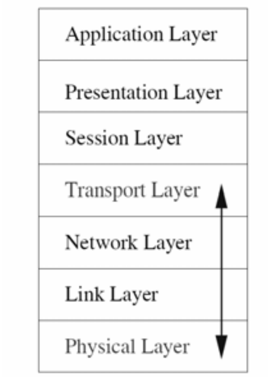
Convex Optimization:
Theory & Algorithms

Communication Networks



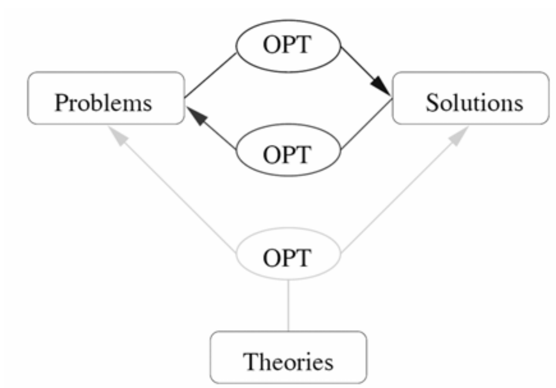
- Functionalities, Dynamics, Protocols, Connectivity, Competition etc

Communication Network Protocols



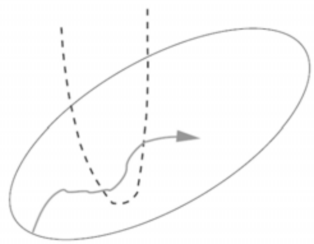
- Layers, Decentralization, Decomposability, Control

Optimization of Communication Networks



- Speed, Robustness, Complexity, Adaptability, Optimality

Optimization



minimize $f(x)$
subject to $x \in C$

Optimization variables: x. Constant parameters describe objective function f and constraint set C

Questions

- How to describe the constraint set?
- How many optimal solution?
- Can the problem be solved efficiently?
- What fundamental properties? How does it relate to another optimization problem?
- Can we find the problem to a given existing solution

Perspective

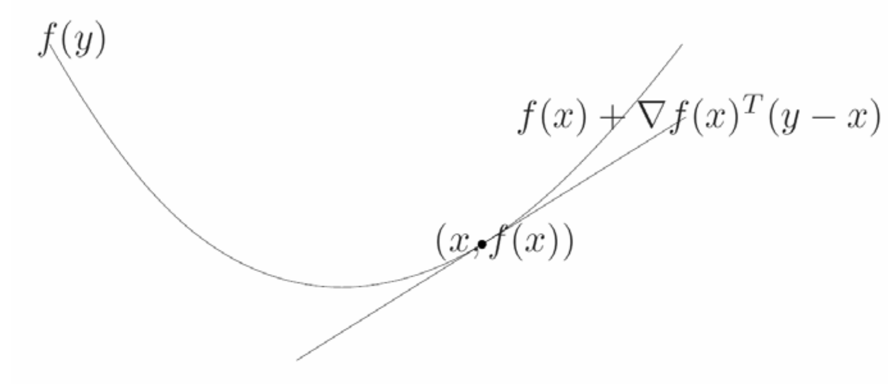
Widely known: Linear programming is powerful and easy to solve

Modified view: Watershed between easy and hard problems is not linearity, but convexity

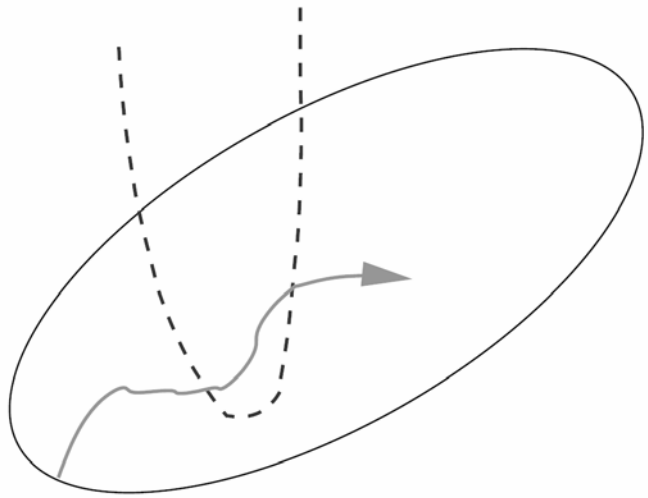
- Local optimality is also global optimality
- Lagrange duality well developed
- Efficiently compute the solutions numerically

Know how to recognize and formulate convex optimization problems

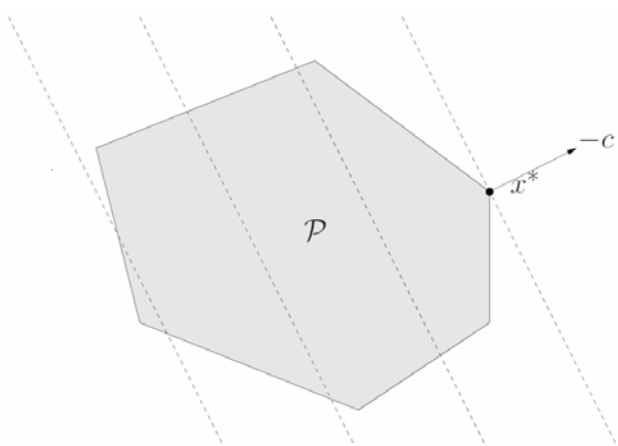
Convex Sets and Convex Functions



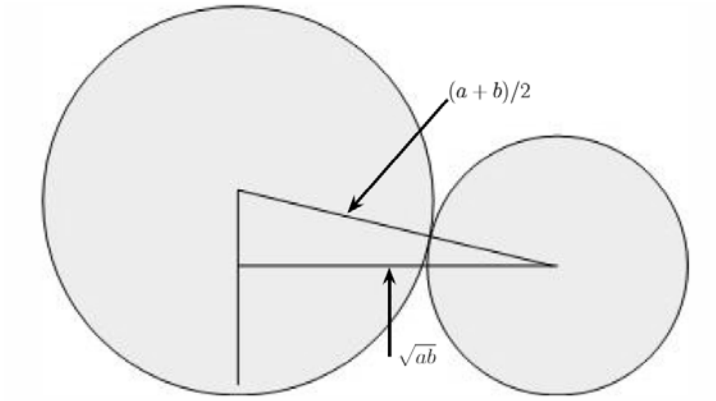
Convex Optimization and Lagrange Duality



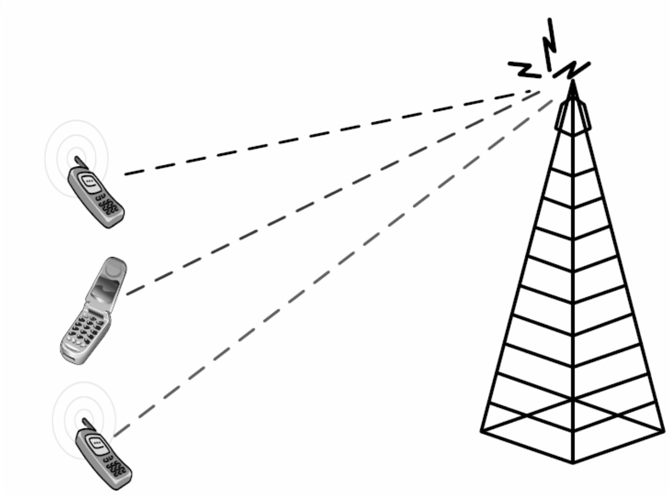
Linear Programming and Quadratic Programming



Geometric Programming



Power Control in Wireless



Outline

- Convex set and examples
- Separating and supporting hyperplanes
- Convex function and examples
- Conjugate functions
- Convex optimization
- Lagrange Duality

Convex Set

Set C is a convex set if the line segment between any two points in C lies in C, ie, if for any $x_1, x_2 \in C$ and any $\theta \in [0,1]$, we have

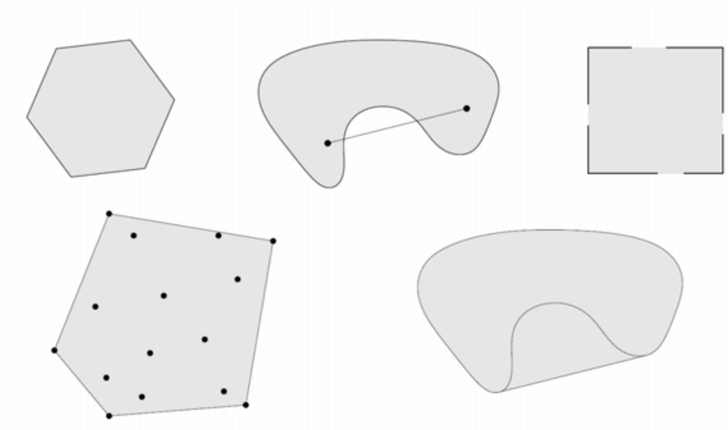
$$\theta x_1 + (1-\theta)x_2 \in C$$

Convex hull of C is the set of all convex combinations of points in C:

$$\left\{ \sum_{i=1}^{k} \theta_i x_i | x_i \in C, \theta_i \geq 0, i = 1, 2, \ldots, k, \sum_{i=1}^{k} \theta_i = 1 \right\}$$

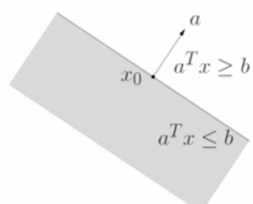
Can generalize to infinite sums and integrals

Examples



Examples of Convex Set

- Hyperplane in $\mathbf{R}^n$ is a set: $\{x|a^Tx = b\}$ where $a \in \mathbf{R}^n, a \neq 0, b \in \mathbf{R}$
  Divides $\mathbf{R}^n$ into two halfspaces: eg, $\{x|a^Tx \leq b\}$ and $\{x|a^Tx > b\}$



- Polyhedron is the solution set of a finite number of linear equalities and inequalities (intersection of finite number of halfspaces and hyperplanes)

Examples of Convex Set

- Euclidean ball in $\mathbf{R}^n$ with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

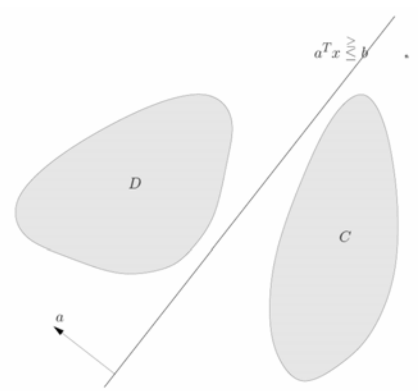
Verify its convexity by triangle inequality

- Generalize to ellipsoids:

$$\mathcal{E}(x_c, P) = \{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$$

P: symmetric and positive definite. Lengths of semi-exes of $\mathcal{E}$ are $\sqrt{\lambda_i}$ where λ_i are eigenvalues of P

Separating Hyperplane Theorem



- C and D: non-intersecting convex sets, *i.e.*, $C \bigcap D = \phi$. Then there exist $a \neq 0$ and b such that $a^T x \le b$ for all $x \in C$ and $a^T x \ge b$ for all $x \in D$.

Separating Hyperplane Theorem: Application

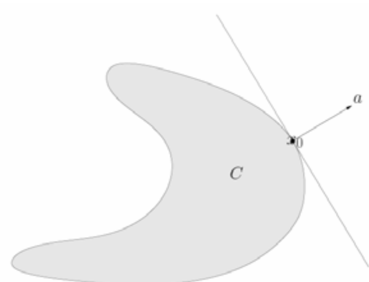
- Theorem of alternatives for strict linear inequalities:

$$Ax \prec b$$

are infeasible if and only if there exists $\lambda \in \mathbf{R}^m$ such that

$$\lambda \neq 0, \quad \lambda \succeq 0, \quad A^T\lambda = 0, \quad \lambda^T b \leq 0.$$

Supporting Hyperplane Theorem



- Given a set $C \in \mathbf{R}^n$ and a point x_0 on its boundary, if $a \neq 0$ satisfies $a^T x \leq a^T x_0$ for all $x \in C$, then $\{x | a^T x = a^T x_0\}$ is called a supporting hyperplane to C at x_0

Convex Functions

$f : \mathbf{R}^n \to \mathbf{R}$ is a convex function if $\mathbf{dom}\, f$ is a convex set and for all $x, y \in \mathbf{dom}\, f$ and $\theta \in [0, 1]$, we have

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

f is strictly convex if strict inequality above for all $x \neq y$ and $0 < \theta < 1$

f is concave if $-f$ is convex

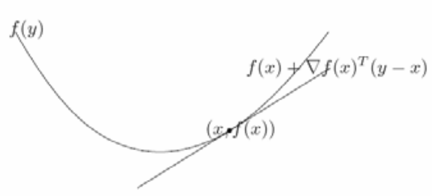
Affine functions are convex and concave

Conditions of Convex Functions

1. For differentiable functions, f is convex iff

$$f(y) - f(x) \geq \nabla f(x)^T (y - x)$$

for all $x, y \in \mathbf{dom}\, f$, and $\mathbf{dom}\, f$ is convex



Conditions of Convex Functions

2. For twice differentiable functions, f is convex iff

$$\nabla^2 f(x) \succeq 0$$

for all $x \in \mathbf{dom}\, f$ (upward slope) and $\mathbf{dom}\, f$ is convex

3. f is convex iff for all $x \in \mathbf{dom}\, f$ and all v,

$$g(t) = f(x + tv)$$

is convex on its domain $\{t \in \mathbf{R} | x + tv \in \mathbf{dom}\, f\}$

Examples of Convex or Concave Functions

- e^{ax} is convex on $\mathbf{R}$, for any $a \in \mathbf{R}$
- x^a is convex on $\mathbf{R}_{++}$ when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$
- $|x|^p$ is convex on $\mathbf{R}$ for $p \geq 1$
- $\log x$ is concave on $\mathbf{R}_{++}$
- $x \log x$ is strictly convex on $\mathbf{R}_{++}$
- Every norm on $\mathbf{R}^n$ is convex
- $f(x) = \max\{x_1, \ldots, x_n\}$ is convex on $\mathbf{R}^n$

Conjugate Functions

Given $f : \mathbf{R}^n \to \mathbf{R}$, conjugate function $f^* : \mathbf{R}^n \to \mathbf{R}$ defined as:

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$

with domain consisting of $y \in \mathbf{R}^n$ for which the supremum if finite

- $f^*(y)$ always convex: it is the pointwise supremum of a family of affine functions of y
- Fenchel's inequality: $f(x) + f^*(y) \geq x^T y$ for all x, y (by definition)
- $f^{**} = f$ if f is convex and closed
- Useful for Lagrange duality theory

Examples of Conjugate Functions

- $f(x) = ax + b,\ f^*(a) = -b$
- $f(x) = -\log x,\ f^*(y) = -\log(-y) - 1$ for $y < 0$
- $f(x) = e^x,\ f^*(y) = y \log y - y$
- $f(x) = x \log x,\ f^*(y) = e^{y-1}$
- $f(x) = \frac{1}{2} x^T Q x,\ f^*(y) = \frac{1}{2} y^T Q^{-1} y$ (Q is positive definite)
- $f(x) = \log \sum_{i=1}^n e^{x_i},\ f^*(y) = \sum_{i=1}^n y_i \log y_i$ if $y \succeq 0$ and $\sum_{i=1}^n y_i = 1$ ($f^*(y) = \infty$ otherwise)

Convex Optimization

A convex optimization problem with variables x:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, 2, \ldots, p \end{array}$$

where $f_0, f_1, \ldots, f_m$ are convex functions.

- Minimize convex objective function (or maximize concave objective function)
- Upper bound inequality constraints on convex functions ($\Rightarrow$ Constraint set is convex)
- Equality constraints must be affine

Convex Optimization

- Epigraph form:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, 2, \ldots, p \end{array}$$

- Not in convex optimization form:

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & \frac{x_1}{1+x_2^2} \leq 0 \\ & (x_1 + x_2)^2 = 0 \end{array}$$

Convex Optimization

Now transformed into a convex optimization problem:

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 + x_2 = 0 \end{array}$$

Optimality Condition for Differentiable Objective

x is optimal for a convex optimization problem iff x is feasible and for all feasible y:

$$\nabla f_0(x)^T(y - x) \geq 0$$

$-\nabla f_0(x)$ is supporting hyperplane to feasible set

Unconstrained convex optimization: condition reduces to:

$$\nabla f_0(x) = 0$$

Unconstrained Quadratic Optimization

Minimize $f_0(x) = \frac{1}{2}x^T P x + q^T x + r$

P is positive semidefinite. So it's a convex optimization problem

x minimizes f_0 iff (P, q) satisfy this linear equality:

$$\nabla f_0(x) = Px + q = 0$$

- If $q \notin \mathcal{R}(P)$, no solution. f_0 unbounded below
- If $q \in \mathcal{R}(P)$ and $P \succ 0$, there is a unique minimizer $x^* = -P^{-1}q$
- If $q \in \mathcal{R}(P)$ and P is singular, set of optimal x: $-P^{\dagger}q + \mathcal{N}(P)$

Duality Mentality

Bound or solve an optimization problem via a different optimization problem!

We'll develop the basic Lagrange duality theory for a general optimization problem, then specialize for convex optimization

Lagrange Dual Function

An optimization problem in standard form:

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \le 0, \quad i = 1, 2, \ldots, m \\
& h_i(x) = 0, \quad i = 1, 2, \ldots, p
\end{array}
$$

Variables: $x \in \mathbf{R}^n$. Assume nonempty feasible set

Optimal value: p^*. Optimizer: x^*

Idea: augment objective with a weighted sum of constraints

Lagrangian $L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x)$

Lagrange multipliers (dual variables): $\lambda \succeq 0, \nu$

Lagrange dual function: $g(\lambda, \nu) = \inf_x L(x, \lambda, \nu)$

Lower Bound on Optimal Value

Claim: $g(\lambda, \nu) \le p^*, \quad \forall \lambda \succeq 0, \nu$

Proof: Consider feasible $\tilde{x}$:

$$L(\tilde{x}, \lambda, \nu) = f_0(\tilde{x}) + \sum_{i=1}^m \lambda_i f_i(\tilde{x}) + \sum_{i=1}^p \nu_i h_i(\tilde{x}) \le f_0(\tilde{x})$$

since $f_i(\tilde{x}) \le 0$ and $\lambda_i \ge 0$

Hence, $g(\lambda, \nu) \le L(\tilde{x}, \lambda, \nu) \le f_0(\tilde{x})$ for all feasible $\tilde{x}$

Therefore, $g(\lambda, \nu) \le p^*$

Lagrange Dual and Conjugate Functions

- Lagrange dual function $g(\lambda, \nu)$
- Conjugate function: $f^*(y) = \sup_{x \in \textbf{dom}\, f}(y^T x - f(x))$

Consider linearly constrained optimization:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Ax \preceq b \\ & Cx = d \end{array}$$

$$\begin{aligned} g(\lambda, \nu) &= \inf_x \left(f_0(x) + \lambda^T(Ax - b) + \nu^T(Cx - d)\right) \\ &= -b^T\lambda - d^T\nu + \inf_x \left(f_0(x) + (A^T\lambda + C^T\nu)^T x\right) \\ &= -b^T\lambda - d^T\nu - f_0^*(-A^T\lambda - C^T\nu) \end{aligned}$$

Example

We'll use the simplest version of entropy maximization as our example for the rest of this lecture on duality. Entropy maximization is an important basic problem in information theory:

$$\begin{array}{ll} \text{minimize} & f_0(x) = \sum_{i=1}^n x_i \log x_i \\ \text{subject to} & Ax \preceq b \\ & \mathbf{1}^T x = 1 \end{array}$$

Since the conjugate function of $u \log u$ is e^{y-1}, by independence of the sum, we have

$$f_0^*(y) = \sum_{i=1}^n e^{y_i - 1}$$

Example

Therefore, dual function of entropy maximization is

$$g(\lambda, \nu) = -b^T\lambda - \nu - e^{-\nu-1}\sum_{i=1}^{n} e^{-a_i^T\lambda}$$

where a^i are columns of A

Lagrange Dual Problem

Lower bound from Lagrange dual function depends on (λ, ν). What's the best lower bound that can be obtained from Lagrange dual function?

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

This is the Lagrange dual problem with dual variables (λ, ν)

Always a convex optimization! (Dual objective function always a concave function since it's the infimum of a family of affine functions in (λ, ν))

Denote the optimal value of Lagrange dual problem by d^*

Weak Duality

What's the relationship between d^* and p^*?

Weak duality always hold (even if primal problem is not convex):

$$d^* \leq p^*$$

Optimal duality gap:

$$p^* - d^*$$

Efficient generation of lower bounds through (convex) dual problem

Strong Duality

Strong duality (zero optimal duality gap):

$$d^* = p^*$$

If strong duality holds, solving dual is 'equivalent' to solving primal.
But strong duality does not always hold

Convexity and constraint qualifications $\Rightarrow$ Strong duality

A simple constraint qualification: Slater's condition (there exists strictly feasible primal variables $f_i(x) < 0$ for non-affine f_i)

Another reason why convex optimization is 'easy'

Saddle Point Interpretation

Assume no equality constraints. We can express primal optimal value as

$$p^* = \inf_x \sup_{\lambda \succeq 0} L(x, \lambda)$$

By definition of dual optimal value:

$$d^* = \sup_{\lambda \succeq 0} \inf_x L(x, \lambda)$$

Weak duality (max min inequality):

$$\sup_{\lambda \succeq 0} \inf_x L(x, \lambda) \leq \inf_x \sup_{\lambda \succeq 0} L(x, \lambda)$$

Saddle Point Interpretation

Strong duality (saddle point property):

$$\sup_{\lambda \succeq 0} \inf_x L(x, \lambda) = \inf_x \sup_{\lambda \succeq 0} L(x, \lambda)$$

Complementary Slackness

Assume strong duality holds:

$$\begin{aligned} f_0(x^*) &= g(\lambda^*, \nu^*) \\ &= \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x) \right) \\ &\leq f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* h_i(x^*) \\ &\leq f_0(x^*) \end{aligned}$$

So the two inequalities must hold with equality. This implies:

$$\lambda_i^* f_i(x^*) = 0, \quad i = 1, 2, \ldots, m$$

Complementary Slackness

Complementary Slackness Property:

$$\begin{aligned} \lambda_i^* > 0 &\;\Rightarrow\; f_i(x^*) = 0 \\ f_i(x^*) < 0 &\;\Rightarrow\; \lambda_i^* = 0 \end{aligned}$$

Karush Kuhn Tucker (KKT) Conditions

Since x^* minimizes $L(x, \lambda^*, \nu^*)$ over x, we have

$$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0$$

Karush-Kuhn-Tucker optimality conditions:

$$f_i(x^*) \leq 0, \quad h_i(x^*) = 0, \quad \lambda_i^* \succeq 0$$

$$\lambda_i^* f_i(x^*) = 0$$

$$\nabla f_0(x^*) + \textstyle\sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0$$

- Any optimization (with differentiable objective and constraint functions) with strong duality, KKT condition is necessary condition

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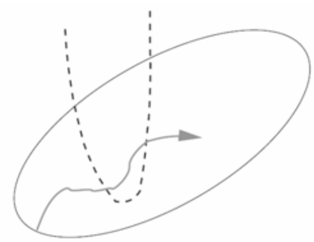
- Any optimization (with differentiable objective and constraint functions) with strong duality, KKT condition is necessary condition

Optimization



minimize $f(x)$
subject to $x \in C$

Optimization variables: x. Constant parameters describe objective function f and constraint set C

Linear Programming

Minimize linear function over linear inequality and equality constraints:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

Variables: $x \in \mathbf{R}^n$.

Standard form LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

Most well-known, widely-used and efficiently-solvable optimization

Transformation to Standard Form

Introduce slack variables s_i for inequality constraints:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx + s = h \\ & Ax = b \\ & s \succeq 0 \end{array}$$

Express x as difference between two nonnegative variables $x^+, x^- \succeq 0$: $x = x^+ - x^-$

$$\begin{array}{ll} \text{minimize} & c^T x^+ - c^T x^- \\ \text{subject to} & Gx^+ - Gx^- + s = h \\ & Ax^+ - Ax^- = b \\ & x^+, x^-, s \succeq 0 \end{array}$$

Norm Minimization Problem

- l_1 norm: $\|x\|_1 = \sum_{i=1}^n |x_i|$

  Minimize $\|Ax - b\|_1$ is equivalent to this LP in $x \in \mathbf{R}^n, s \in \mathbf{R}^n$:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T s \\ \text{subject to} & Ax - b \preceq s \\ & Ax - b \succeq -s \end{array}$$

- l_∞ norm: $\|x\|_\infty = \max_i\{|x_i|\}$

  Minimize $\|Ax - b\|_\infty$ is equivalent to this LP in $x \in \mathbf{R}^n, t \in \mathbf{R}$:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \preceq t\mathbf{1} \\ & Ax - b \succeq -t\mathbf{1} \end{array}$$

Norm Minimization Problem

- l_1 norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

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Algorithms

- Simplex Method
- Interior-point Method
- Ellipsoid Method

  item Cutting-plane Method

Simplex method is very efficient in practice but specialized for LP: move from one vertex to another without enumerating all the vertices

Interior point algorithms are fierce competitors of Simplex since 1984

Convex QCQP

- (Convex) QP (with linear constraints) in x:

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

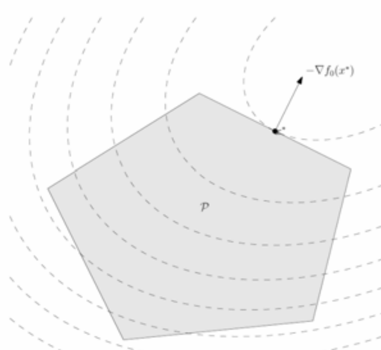
where $P \in \mathbf{S}^n_+, G \in \mathbf{R}^{m\times n}, A \in \mathbf{R}^{p\times n}$

- (Convex) QCQP in x:

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, 2, \ldots, m \\ & Ax = b \end{array}$$

Convex QCQP

where $P \in \mathbf{S}^n_+, \quad i = 0, \ldots, m$



Least Squares

- Minimize $\|Ax - b\|_2^2 = x^T A^T A x - 2b^T A x + b^T b$ over x. Unconstrained QP, Regression analysis, Least-squares approximation

  Analytic solution: $x^* = A^\dagger b$ where, for $A \in \mathbf{R}^{m\times n}$, $A^\dagger = (A^T A)^{-1} A^T$ if rank of A is n, and $A^\dagger = A^T(AA^T)^{-1}$ if rank of A is m. If not full rank, then by singular value decomposition.

- Constrained least-squares (no general analytic solution). For example:

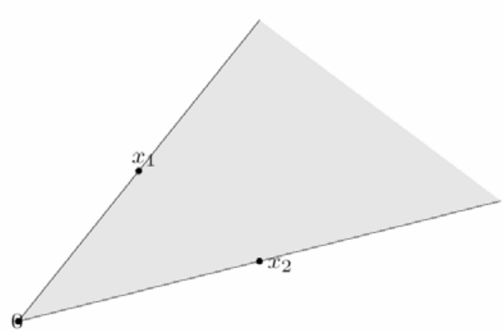
$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & l_i \le x_i \le u_i, \quad i = 1, \ldots, n \end{array}$$

Cones and Convex Cones

C is a cone if for every $x \in C$ and $\theta \ge 0$, we have $\theta x \in C$

C is a convex cone if it is convex and a cone: for any $x_1, x_2 \in C$ and $\theta_1, \theta_2 \ge 0$

$$\theta_1 x_1 + \theta_2 x_2 \in C$$



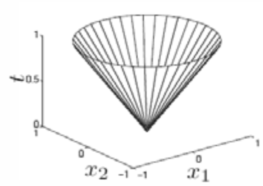
Norm Cones

- Given a norm, norm cone is a convex cone:

$$C = \{(x,t) \in \mathbf{R}^{n+1} | \|x\| \le t\}$$

Example: second order cone:

$$\begin{aligned} C &= \{(x,t) \in \mathbf{R}^{n+1} | \|x\|_2 \le t\} \\ &= \left\{ \begin{bmatrix} x \\ t \end{bmatrix} | \begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \le 0, t \ge 0 \right\} \end{aligned}$$



Second Order Cone Programming (SOCP)

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \ldots, m \\ & Fx = g \end{array}$$

Variables: $x \in \mathbf{R}^n$. And $A_i \in \mathbf{R}^{n_i \times n}, F \in \mathbf{R}^{p \times n}$

If $c_i = 0, \ \forall i$, SOCP is equivalent to QCQP If $A_i = 0, \ \forall i$, SOCP is equivalent to LP

Robust Linear Program

Consider inequality constrained LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

Parameters a_i are not accurate. They are only known to lie in given ellipsoids described by $\bar{a}_i$ and $P_i \in \mathbf{R}^{n \times n}$:

$$a_i \in \mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}$$

Since $\sup\{a_i^T x | a_i \in \mathcal{E}\} = \bar{a}_i^T x + \|P_i^T x\|_2$,

Robust LP (satisfy constraints for all possible a_i) formulated as

Robust Linear Program

SOCP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

Positive Semidefinite Cone

Matrix $A \in \mathbf{R}^{n \times n}$ is positive semidefinite $A \succeq 0$ if for all $x \in \mathbf{R}^n$,

$$x^T A x \geq 0$$

Matrix $A \in \mathbf{R}^{n \times n}$ is positive definite $A \succ 0$ if for all $x \in \mathbf{R}^n$,

$$x^T A x > 0$$

Set of symmetric positive semidefinite matrices:

$$\mathbf{S}_+^n = \{X \in \mathbf{R}^{n \times n} | X = X^T, X \succeq 0\}$$

$\mathbf{S}_+^n$ is a convex cone: if $\theta_1, \theta_2 \geq 0$ and $A, B \in \mathbf{S}_+^n$, then $\theta_1 A + \theta_2 B \in \mathbf{S}_+^n$, since for all $x \in \mathbf{R}^n$:

$$x^T(\theta_1 A + \theta_2 B)x = \theta_1 x^T A x + \theta_2 x^T B x \geq 0$$

Semidefinite Program

SDP: Minimize linear objective over linear equalities and LMI on variables $x \in \mathbf{R}^n$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + \ldots + x_n F_n + G \preceq 0 \\ & Ax = b \end{array}$$

SDP in standard form: Minimize a matrix inner product over equality constraints on inner products on variables $X \in \mathbf{S}^n$

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(CX) \\ \text{subject to} & \mathbf{tr}(A_i X) = b_i, \quad i = 1, 2, \ldots, p \\ & X \succeq 0 \end{array}$$

LP & SOCP as SDP

LP as SDP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{diag}(Gx - h) \preceq 0 \\ & Ax = b \end{array}$$

SOCP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, N \\ & Fx = g \end{array}$$

SOCP as SDP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \left[\begin{array}{cc} (c_i x + d_i)I & A_i x + b_i \\ (A_i x + b_i)^T & (c_i x + d_i)I \end{array}\right] \succeq 0, \quad i = 1, \ldots, N \\ & Fx = g \end{array}$$

KKT Conditions for QP

Primal (convex) QP with linear equality constraints:

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Ax = b \end{array}$$

KKT conditions:

$$Ax^* = b, \quad Px^* + q + A^T \nu^* = 0$$

which can be written in matrix form:

$$\left[\begin{array}{cc} P & A^T \\ A & 0 \end{array}\right] \left[\begin{array}{c} x^* \\ \nu^* \end{array}\right] = \left[\begin{array}{c} -q \\ b \end{array}\right]$$

Solving a system of linear equations is equivalent to solving equality constrained convex quadratic minimization

Monomials and Posynomials

Monomial as a function $f : \mathbf{R}^n_+ \to \mathbf{R}$:

$$f(x) = d x_1^{a^{(1)}} x_2^{a^{(2)}} \dots x_n^{a^{(n)}}$$

where the multiplicative constant $d \geq 0$ and the exponential constants $a^{(j)} \in \mathbf{R}, j = 1, 2, \dots, n$

Sum of monomials is called a posynomial:

$$f(x) = \sum_{k=1}^{K} d_k x_1^{a_k^{(1)}} x_2^{a_k^{(2)}} \dots x_n^{a_k^{(n)}}.$$

where $d_k \geq 0,\ k = 1, 2, \dots, K$, and $a_k^{(j)} \in \mathbf{R},\ j = 1, 2, \dots, n, k = 1, 2, \dots, K$

Example

Example: $\sqrt{2} x^{-0.5} y^{\pi} z$ is a monomial, $x - y$ is not a posynomial

GP Standard Form

- GP standard form with variables x:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, 2, \ldots, m, \\ & h_l(x) = 1, \quad l = 1, 2, \ldots, M \end{array}$$

where $f_i, i = 0, 1, \ldots, m$ are posynomials and $h_l, l = 1, 2, \ldots, M$ are monomials

GP Convex Form

Log transformation: $y_j = \log x_j, b_{ik} = \log d_{ik}, b_l = \log d_l$

- GP convex form with variables y:

$$\begin{array}{ll} \text{minimize} & p_0(y) = \log \sum_{k=1}^{K_0} \exp(a_{0k}^T y + b_{0k}) \\ \text{subject to} & p_i(y) = \log \sum_{k=1}^{K_i} \exp(a_{ik}^T y + b_{ik}) \leq 0, \quad i = 1, 2, \ldots, m, \\ & q_l(y) = a_l^T y + b_l = 0, \quad l = 1, 2, \ldots, M \end{array}$$

In convex form, GP with only monomials reduces to LP

Example

- The following problem can be turned into an equivalent GP:

$$\begin{array}{ll} \text{maximize} & x/y \\ \text{subject to} & 2 \le x \le 3 \\ & x^2 + 3y/z \le \sqrt{y} \\ & x/y = z^2 \end{array}$$

$$\begin{array}{ll} \text{minimize} & x^{-1}y \\ \text{subject to} & 2x^{-1} \le 1, \quad (1/3)x \le 1 \\ & x^2 y^{-1/2} + 3y^{1/2}z^{-1} \le 1 \\ & xy^{-1}z^{-2} = 1 \end{array}$$

Example

- Let p, q be posynomials and r monomial

$$\begin{array}{ll} \text{minimize} & p(x)/(r(x) - q(x)) \\ \text{subject to} & r(x) > q(x) \end{array}$$

which is equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & p(x) \le t(r(x) - q(x)) \\ & (q(x)/r(x)) < 1 \end{array}$$

which is in turn equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & (p(x)/t + q(x))/r(x) \le 1 \\ & (q(x)/r(x)) < 1 \end{array}$$

GP Solver

Generalized GP: minimize generalized posynomials over upper bound inequality constraints on other generalized posynomials Generalized GP can be turned into equivalent GP

- Freely available software: Stanford CVX & GPLAB, MOSEK
  - http://stanford.edu/~boyd/ggplab

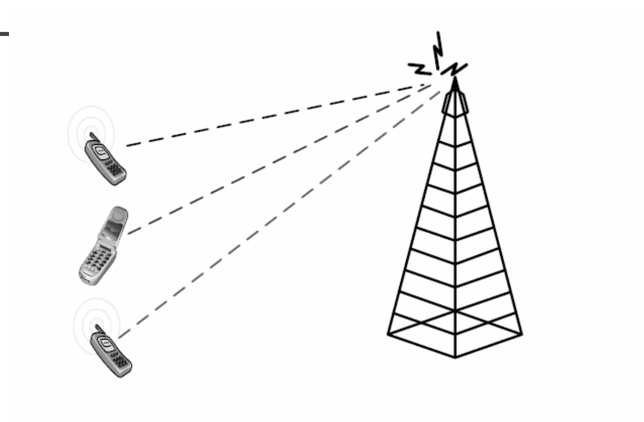
Interior Point Method Solver

- Stanford CVX
  - Matlab software for disciplined convex programming developed by Michael Grant and Stephen Boyd
  - Solve LP, QP, SOCP, SDP, GP etc.
- Freely available from Stanford
- http://www.stanford.edu/~boyd/cvx/

Outline

- Wireless Power Control
- Foschini-Miljanic Power Minimization
- Max-min Weighted SIR
- Extensions to Multiple Antennae
- Sum Rate Maximization

Wireless Cognitive Radio Network

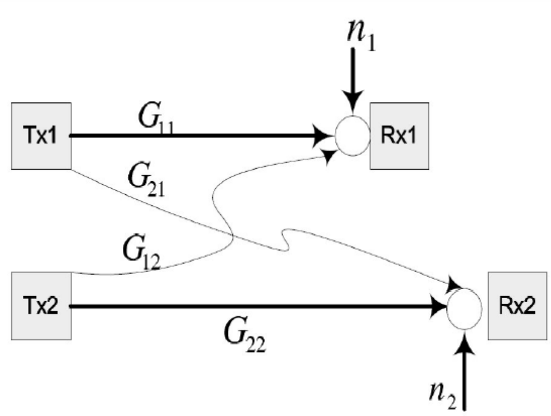


- Link adaptation, uplink-downlink, power control

System Considerations

- How to solve optimally nonconvex power control problems?
- How many ways to characterize optimality?
- How to design distributed power control algorithms with fast convergence and good performance guarantees?
- How fast is fast?
- Can we leverage existing technology?

System Models



- Interference channel model

Performance Metrics

- Signal-to-Interference Ratio:

$$\mathsf{SIR}_l(\mathbf{p}) = \frac{G_{ll}p_l}{\sum_{j \neq l} G_{lj}p_j + n_l}$$

  with G_{lj} the channel gains from transmitter j to receiver l and n_l the additive white Gaussian noise (AWGN) power at receiver l

- Attainable data rate (nats per channel use) is a function of SIR, e.g., Shannon capacity formula $r_l = \log(1 + \mathsf{SIR}_l)$

- Power constraints $\mathbf{1}^\top \mathbf{p} \leq \bar{\mathbf{P}}$

Interference Parameters

- Let $\mathbf{F}$ be a nonnegative matrix with entries:

$$F_{ij} = \begin{cases} 0, & \text{if } i = j \\ \frac{G_{ij}}{G_{ii}}, & \text{if } i \neq j \end{cases}$$

  and

$$\mathbf{v} = \left(\frac{n_1}{G_{11}}, \frac{n_2}{G_{22}}, \ldots, \frac{n_L}{G_{LL}} \right)^\top .$$

- Let $\mathbf{B}$ be:

$$\mathbf{B} = \mathbf{F} + (1/\bar{P})\mathbf{v}\mathbf{1}^\top .$$

Foschini-Miljanic's 1993 Work

-

$$\begin{array}{ll} \text{minimize} & \sum_l p_l \\ \text{subject to} & \text{SIR}_l(\mathbf{p}) = \frac{G_{ll}p_l}{\sum_{j\neq l} G_{lj}p_j + n_l} \geq \gamma_l \;\; \forall l. \end{array}$$

- Matrix notation:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^\top \mathbf{p} \\ \text{subject to} & (\mathbf{I} - \text{diag}(\gamma)\mathbf{F})\mathbf{p} \geq \text{diag}(\gamma)\mathbf{v}. \end{array}$$

G. J. Foschini & Z. Miljanic, *A Simple Distributed Autonomous Power Control Algorithm and its Convergence*, IEEE Trans. Vehicular Technology, 1993

- IS-95 CDMA Systems, Qualcomm 3G Systems

Foschini-Miljanic's 1993 Work

- The optimal power vector $\mathbf{p}^\star$ is:

$$\mathbf{p}^\star = (\mathbf{I} - \text{diag}(\gamma)\mathbf{F})^{-1}\,\text{diag}(\gamma)\mathbf{v}.$$

- Fixed-point algorithm: $\mathbf{p}(k+1) = \text{diag}(\boldsymbol{\gamma})\mathbf{F}\mathbf{p}(k) + \text{diag}(\boldsymbol{\gamma})\mathbf{v}$
- Distributed Power Control (DPC) algorithm (more illuminating form):

$$p_l(k+1) = \frac{\gamma_l}{\text{SIR}_l(\mathbf{p}(k))} p_l(k) \;\; \forall l.$$

- Geometric convergence to $(\mathbf{I} - \text{diag}(\boldsymbol{\gamma})\mathbf{F})^{-1}\text{diag}(\boldsymbol{\gamma})\mathbf{v}$ if $\rho(\text{diag}(\boldsymbol{\gamma})\mathbf{F}) < 1$

Power Minimization with Beamforming

- Base station has M transmit antennas, each mobile user has one receive antenna (Single-cell multiuser MISO channel)
- Let h_l denote the M-dimensional channel response from base station to lth user, u_l denote the M-dimensional transmit beamforming vector, instantaneous transmitted signal denote

$$\text{xmt} = \sum_{l=1}^{L} b_l u_l$$

where b_l is the transmitted data signal for the lth user

- Received signal at the lth user:

$$\text{rcv}_l = b_l u_l^\dagger h_l + \sum_{j \neq l} b_j u_j^\dagger h_l + n_l.$$

Power Minimization with Beamforming

- Downlink SIR dependent on power $\mathbf{p}$ and transmit beamformers $\mathbf{U} = [\mathbf{u}_1 \dots \mathbf{u}_L]$

$$\text{SIR}_l(\mathbf{p}, \mathbf{U}) = \frac{|\boldsymbol{h}_l^\dagger \mathbf{u}_l|^2}{\sum_{j \neq l} |\boldsymbol{h}_l^\dagger \mathbf{u}_j|^2 + v_l}.$$

Power Minimization with Beamforming

- Total power minimization:

$$\begin{array}{ll} \text{minimize} & \sum_l \|\mathbf{u}_l\|^2 \\ \text{subject to} & \gamma_l(\sum_{j\neq l} |\boldsymbol{h}_l^\dagger \mathbf{u}_j|^2 + v_l) - |\boldsymbol{h}_l^\dagger \mathbf{u}_l|^2 \leq 0 \,\forall\, l \\ \text{variables:} & \mathbf{u}_l, \quad \forall\, l. \end{array}$$

- Change-of-variable technique: $\tilde{\mathbf{U}}_l = \mathbf{u}_l \mathbf{u}_l^\dagger$ $\qquad \tilde{\mathbf{H}}_l = \mathbf{h}_l \mathbf{h}_l^\dagger$

- Equivalent problem in new variables:

$$\begin{array}{ll} \text{minimize} & \sum_l \mathrm{Tr}(\tilde{\mathbf{U}}_l) \\ \text{subject to} & \gamma_l(\mathrm{Tr}(\tilde{\mathbf{H}}_l \tilde{\mathbf{U}}_j) + v_l) - \mathrm{Tr}(\tilde{\mathbf{H}}_l \tilde{\mathbf{U}}_l) \leq 0 \,\forall\, l, \\ & \tilde{\mathbf{U}}_l \succeq \mathbf{0}, \;\; \mathrm{Rank}(\tilde{\mathbf{U}}_l) = 1 \,\forall\, l \\ \text{variables:} & \tilde{\mathbf{U}}_l, \quad \forall\, l. \end{array}$$

- Relax the problem by removing the rank-one constraint leads to semidefinite program (SDP)

Power Minimization with Beamforming

- There exists a rank-one solution for $\tilde{\mathbf{U}}_l$ for all l of the SDP

- Many other extensions including second order cone program (SOCP) preprocessing

- Stanford CVX, SeDuMi toolbox for SDP, MOSEK

- State of the art of SDP & SOCP solvers for large (but not huge) problems is quite satisfactory

1. E. Visotsky & U. Madhow, *Optimum beamforming using transmit antenna arrays*, Proc. of IEEE VTC, 1999
2. M. Chiang, P. Hande, T. Lan & T, *Power control in wireless cellular networks*, NOW Foundations & Trends in Networking, 2008

Max-min Weighted SIR

- Consider $\max_{\mathbf{p}\geq 0} \min_{l} \frac{\mathsf{SIR}_l(\mathbf{p})}{\beta_l}$ subject to $\sum_l p_l \leq \overline{P} \quad \forall\, l$
- **Theorem 1.** *The optimal solution is such that the value* SIR_l/β_l *for all users are equal. The optimal weighted max-min* SIR *is given by*

$$\gamma^* = \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B})},$$

where

$$\mathbf{B} = \mathbf{F} + (1/\bar{P})\mathbf{v}\mathbf{1}^{\top}$$

Further, the optimal $\mathbf{p}$*, denoted by* $\mathbf{p}^*$*, is* $t\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B})$ *for some constant* $t > 0$.

T, M. Chiang & R. Srikant, , *Maximizing Sum Rate and Minimizing MSE on Multiuser Downlink: Optimality, Fast Algorithms, and Equivalence via Max-min SIR*, IEEE ISIT, 2009

Max-min Weighted SIR: Proof Outline

- Convexify & Dual decomposition
  - Primal Convex problem:

$$\begin{array}{rl} \text{maximize} & \tilde{\tau} \\ \text{subject to} & \tilde{\tau} \leq \log(\mathsf{SIR}_l(\tilde{\mathbf{p}})/\beta_l) \;\; \forall\, l, \\ & \sum_l e^{\tilde{p}_l} \leq \bar{P} \\ \text{variables:} & \tilde{\tau},\; \tilde{\mathbf{p}}. \end{array}$$

  - Partial Lagrangian:

$$L(\{\tilde{\tau}, \tilde{\mathbf{p}}\}, \{\boldsymbol{\lambda}\}) = \tilde{\tau}(1 - \sum_l \lambda_l) + \sum_l \lambda_l \log(\mathsf{SIR}_l(\tilde{\mathbf{p}})/\beta_l).$$

Max-min Weighted SIR: Proof Outline

- Dual Convex Problem:

$$\begin{array}{ll} \text{minimize} & \max_{\tilde{\tau}, \sum_l e^{\tilde{p}_l} \leq \bar{P} \ \forall l} L(\{\tilde{\tau}, \tilde{\mathbf{p}}\}, \{\boldsymbol{\lambda}\}) \\ \text{subject to} & \mathbf{1}^\top \boldsymbol{\lambda} = 1, \ \boldsymbol{\lambda} \geq \mathbf{0}, \text{variables:} \quad \boldsymbol{\lambda}. \end{array}$$

- Primal-dual inequality:

$$\tilde{\tau} \leq \tilde{\tau}^\star \leq \sum_l \lambda_l \log(\mathsf{SIR}_l(\mathbf{p}(\boldsymbol{\lambda}))/\beta_l).$$

- Closed-formed dual solution:

Lemma 1. *For any power vector* $\mathbf{p}$*,*

$$\sum_l x_l y_l \log(\mathsf{SIR}_l(\mathbf{p})/\beta_l) \leq -\log \rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}).$$

Equality is achieved in (16) if and only if $\mathbf{p} = \mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B})$ *(unique up to a scaling constant).*

Nonlinear Perron-Frobenius Theorem

- Find $(\check{\lambda}, \check{\mathbf{s}})$ in

$$\lambda \mathbf{s} = \mathbf{A}\mathbf{s} + \mathbf{b}, \quad \lambda \in \mathbb{R}, \quad \mathbf{s} \geq \mathbf{0}, \quad \|\mathbf{s}\| = 1,$$

where $\mathbf{A}$ and $\mathbf{b}$ is a square irreducible nonnegative matrix and nonnegative vector, respectively and $\|\cdot\|$ a monotone vector norm.

- $(\check{\lambda}, \check{\mathbf{s}})$ is Perron-Frobenius eigenvalue-vector pair of $\mathbf{A} + \mathbf{b}\mathbf{c}_*{}^\top$, where

$$\mathbf{c}_* = \arg \max_{\|\mathbf{c}\|_* = 1} \rho(\mathbf{A} + \mathbf{b}\mathbf{c}^\top),$$

where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$, and $\check{\mathbf{s}} = (\mathbf{A}\check{\mathbf{s}} + \mathbf{b})/\|\mathbf{A}\check{\mathbf{s}} + \mathbf{b}\|$.

V. D. Blondel, L. Ninove and P. Van Dooren, *An affine eigenvalue problem on the nonnegative orthant*, Linear Algebra & its Applications, 2005

Fast Max-min SIR Algorithm

- Based on Nonlinear Perron-Frobenius Theory
- **Algorithm 1.**

  1. *Update power* $\mathbf{p}(k+1)$*:*

  $$p_l(k+1) = \frac{\beta_l}{\mathsf{SIR}_l(\mathbf{p}(k))} p_l(k) \quad \forall\, l.$$

  2. *Normalize* $\mathbf{p}(k+1)$*:*

  $$p_l(k+1) \leftarrow p_l(k+1) / \sum_j p_j(k+1) \cdot \bar{P} \quad \forall\, l.$$

- **Theorem 2.** *Starting from any initial point* $\mathbf{p}(0)$, $\mathbf{p}(k)$ *in Algorithm 1 converges geometrically fast to* $\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B})$ *(unique up to a scaling constant).*

Uplink-downlink Duality

- Downlink SIR dependent on power $\mathbf{p}$ and transmit beamformers $\mathbf{U} = [\mathbf{u}_1 \ldots \mathbf{u}_L]$

$$\mathsf{SIR}_l(\mathbf{p}, \mathbf{U}) = \frac{p_l |\boldsymbol{h}_l^\dagger \mathbf{u}_l|^2}{\sum_{j \neq l} p_j |\boldsymbol{h}_l^\dagger \mathbf{u}_j|^2 + v_l}.$$

- Uplink-downlink duality: Reverse the role of transmitters and receivers
  P. Viswanath & D. Tse, *Sum Capacity of the Multiple Antenna Gaussian Broadcast Channel and Uplink-Downlink Duality*, IEEE Transactions on Information Theory, 2003
- For a fixed total power, the optimal transmit beamformer in the downlink is the same as the optimal receive beamformer in the uplink, and any SIR achievable in the uplink can be achieved in the downlink
- Simple proof based on nonnegative matrix theory:

$$\rho(\mathsf{diag}(\gamma)(\mathbf{F} + (1/\bar{P})\mathbf{1}\mathbf{1}^\top)) = \rho(\mathsf{diag}(\gamma)(\mathbf{F}^\top + (1/\bar{P})\mathbf{1}\mathbf{1}^\top))$$

Joint Power Control and Beamforming

- Downlink SIR dependent on power $\mathbf{p}$ and transmit beamformers $\mathbf{U} = [\mathbf{u}_1 \dots \mathbf{u}_L]$

$$\mathsf{SIR}_l(\mathbf{p}, \mathbf{U}) = \frac{p_l |\boldsymbol{h}_l^\dagger \mathbf{u}_l|^2}{\sum_{j \neq l} p_j |\boldsymbol{h}_l^\dagger \mathbf{u}_j|^2 + n_l}.$$

- Uplink-downlink duality + Max-min weighted SIR algorithm = Distributed geometrically fast algorithm

- Introduce virtual uplink $\hat{\mathsf{SIR}}_l$:

$$\hat{\mathsf{SIR}}_l(\mathbf{p}, \mathbf{U}) = \frac{q_l |\boldsymbol{h}_l^\dagger \mathbf{u}_l|^2}{\sum_{j \neq l} q_j |\boldsymbol{h}_j^\dagger \mathbf{u}_l|^2 + 1}$$

Joint Power Control and Beamforming: Algorithm

Algorithm 2. [Max-min Weighted SIR–Joint Power Control & Beamforming]

1. *Update (virtual) uplink power* $\mathbf{q}(k+1)$*:*

$$q_l(k+1) = \left(\frac{\beta_l}{\hat{\mathsf{SIR}}_l(\mathbf{q}(k), \mathbf{U}(k))} \right) q_l(k) \quad \forall\, l.$$

2. *Normalize* $\mathbf{q}(k+1)$*:*

$$\mathbf{q}(k+1) \leftarrow \mathbf{q}(k+1) \cdot \bar{P} / \mathbf{1}^\top \mathbf{q}(k+1).$$

3. *Update transmit beamforming matrix* $\mathbf{U}(k) = [\mathbf{u}_1(k) \dots \mathbf{u}_L(k)]$*:*

$$\mathbf{u}_l(k) = \left(\sum_{j \neq l} q_j(k) \mathbf{h}_j \mathbf{h}_j^\dagger + \mathbf{I} \right)^+ \mathbf{h}_l \quad \forall\, l.$$

Joint Power Control and Beamforming: Algorithm

4. *Update downlink power* $\mathbf{p}(k+1)$*:*

$$p_l(k+1) = \left(\frac{\beta_l}{\mathsf{SIR}_l(\mathbf{p}(k), \mathbf{U}(k))}\right) p_l(k) \quad \forall\, l.$$

5. *Normalize* $\mathbf{p}(k+1)$*:*

$$\mathbf{p}(k+1) \leftarrow \mathbf{p}(k+1) \cdot \bar{P}/\mathbf{1}^\top \mathbf{p}(k+1).$$

Fast & Simple Algorithms

Sum Rate Maximization

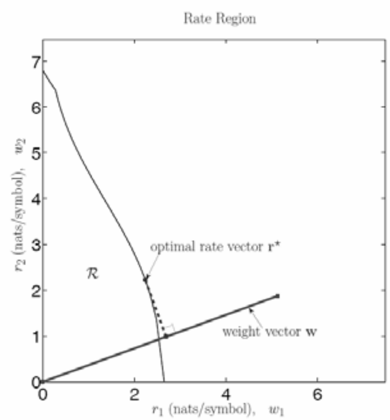
- Find $\mathbf{p}^\star = \arg \max_{\mathbf{1}^\top \mathrm{p}=\bar{P}} \sum_l w_l \log(1+\mathsf{SIR}_l(\mathrm{p}))$ where $\mathbf{1}^\top \mathbf{w} = 1$
- Characterize the achievable rate region: $r_l = \log(1+\mathsf{SIR}_l(\mathbf{p}))\ \forall\, l$
- Two-User case:

$$\max \quad w_1 \log\left(1+\frac{G_{11}p_1}{G_{12}p_2+n_1}\right) + w_2 \log\left(1+\frac{G_{22}p_2}{G_{21}p_1+n_2}\right)$$

subject to: $\quad p_1 + p_2 \leq \bar{P}$

Sum Rate Maximization

maximize $\sum_l w_l \log(1+\mathsf{SIR}_l(\mathbf{p})) = \sum_l w_l r_l$
subject to $\sum_l p_l \le \bar{P}$
variables: $p_l \quad \forall\, l.$



Sum Rate Maximization

Theorem 3. *Under sufficiently low signal-to-noise ratio or interference,*

$$\sum_{l=1}^{L} w_l \log(1+\mathsf{SIR}_l(\mathbf{p})) \le \|\mathbf{w}\|_\infty^{\mathbf{x}(\mathrm{B})\circ\mathbf{y}(\mathrm{B})} \log\left(1+1/\rho(\mathrm{B})\right)$$

for all feasible $\mathbf{p}$.

Equality is achieved if and only if $\mathbf{w}=\mathbf{x}(\mathrm{B})\circ\mathbf{y}(\mathrm{B})$, *and* $\mathsf{SIR}_l(\mathbf{p}^\star) = (1/\rho(\mathrm{B}))\mathbf{1}\ \forall\, l$. *In this case,* $\mathbf{p}^\star = \mathbf{x}(\mathrm{B})$.

- Max-min SIR ($\boldsymbol{\beta}=\mathbf{1}$) as an approximation algorithm with guarantees

Sum Rate Maximization: Proof Sketch

- Sum-of-ratios fractional program (positive $f(\mathbf{p})$, $g(\mathbf{p})$) **[FrenkSchaible05]**:

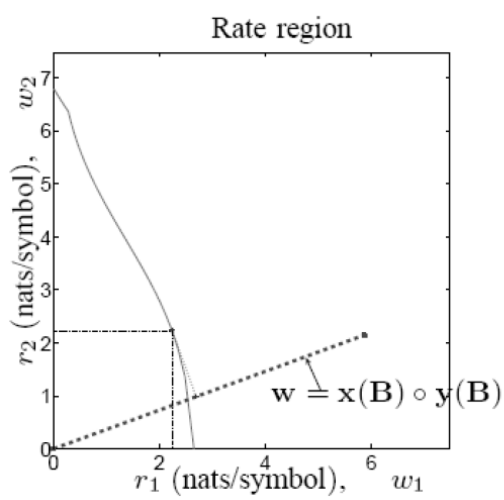
$$\min_{\mathbf{p}} \sum_l \frac{f_l(\mathbf{p})}{g_l(\mathbf{p})}.$$

- Domain transformation: $\mathbf{z} = g(\mathbf{p})$
- Transform optimization over $\mathbf{p}$ to $\mathbf{z}$ (when $g^{-1}(\mathbf{z}) \geq \mathbf{0}$):

$$\min_{\mathbf{z}} \sum_l \frac{(f(g^{-1}(\mathbf{z})))_l}{z_l}$$

- Link to minimax theorems in Nonlinear Perron-Frobenius theory when $f(g^{-1}(\mathbf{z}))$ is concave and monotone

Link Sum Rate Maximization to Max-min SIR



Global Optimization

- Equivalent Sum-Rate Problem convex maximization ('dB' domain):

$$\begin{array}{ll} \text{maximize} & \sum_l w_l \log(1+\exp(\tilde{\gamma}_l)) \\ \text{subject to} & \log \rho(\mathsf{diag}(\exp(\tilde{\boldsymbol{\gamma}}))\mathbf{B}) \leq 0 \\ \text{variables:} & \tilde{\gamma}_l, \quad \forall\, l. \end{array}$$

- Nonlinear map between power $\mathbf{p}$ and SIR $\boldsymbol{\gamma} = \exp(\tilde{\boldsymbol{\gamma}})$:

$$\mathbf{p}^\star = (\mathbf{I} - \operatorname{diag}(\exp(\tilde{\gamma}^\star))\mathbf{F})^{-1} \operatorname{diag}(\exp(\tilde{\gamma}^\star))\mathbf{v}$$

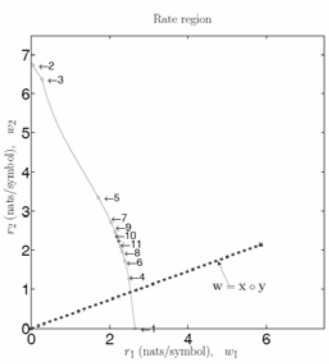
- Relaxation of the constraint set by the Friedland-Karlin Inequalities:

$$\prod_l \gamma_l^{x_l(\mathbf{B})y_l(\mathbf{B})} \rho(\mathbf{B}) \leq \rho(\mathsf{diag}(\boldsymbol{\gamma})\mathbf{B})$$

$$\sum_l x_l(\mathbf{B})y_l(\mathbf{B})\tilde{\gamma}_l + \log \rho(\mathbf{B}) \leq \log \rho(\mathsf{diag}(\exp(\tilde{\boldsymbol{\gamma}}))\mathbf{B}) \quad (\text{'dB' domain}).$$

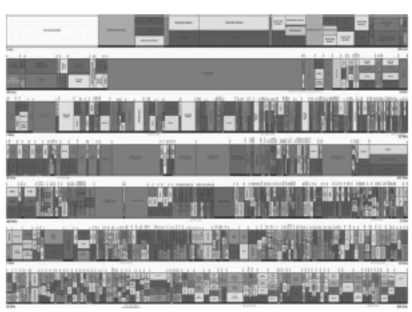
Example

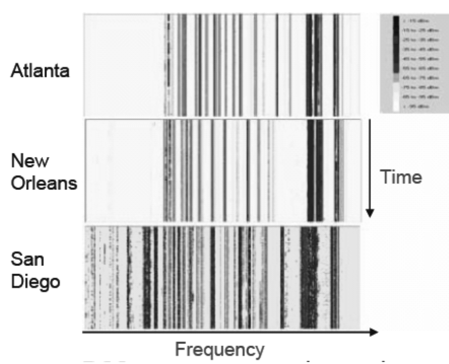
- $\lim_{k\to\infty} \left(\mathbf{I} - \operatorname{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))\mathbf{F}\right)^{-1} \operatorname{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))\mathbf{v} = \mathbf{p}^\star$



Spectrum Sharing Models and Interference Management in Cognitive Radio Networks

Motivation





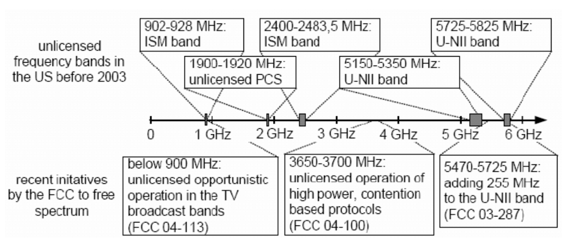
- Available spectrum looks scarce, but measurements show the allocated spectrum is vastly underutilized in both space and time.

Everyone Seems to Agree That:

- Spectrum use is inefficient
- “White space” is abundant
  - A bandwidth is considered white space if it is wider than 1 MHz and remains unoccupied for at least 10 minutes
  - Measurements conducted by the Shared Spectrum Company in 2003 showed that 62% of the spectrum was white space in the most crowded area near downtown Washington, DC (consisting of government and commercial spectrum use).
- Dynamic spectrum access (i.e. cognitive radio) makes sense
- Resource allocation is important

Cognitive Radio for Dynamic Spectrum Access

- Improved spectral utilization!
- Allow secondary network to run in background without interrupting communication of primary network.
- Advances in software-defined radios will enable smart and agile (secondary) users.
- FCC’s push for policy reform:



Taxonomies of Spectrum Access Models

- Taxonomy: a science or technique of classification
  - Specify existence of licensed vs. unlicensed users
  - Specify freedom (or lack-there-of) for DSA
- Works of M. Buddhikot and J. Peha
- Different taxonomies of spectrum access models [M. Buddhikot] :
  - Command and control
  - Exclusive use
  - Shared use
  - Spectrum commons

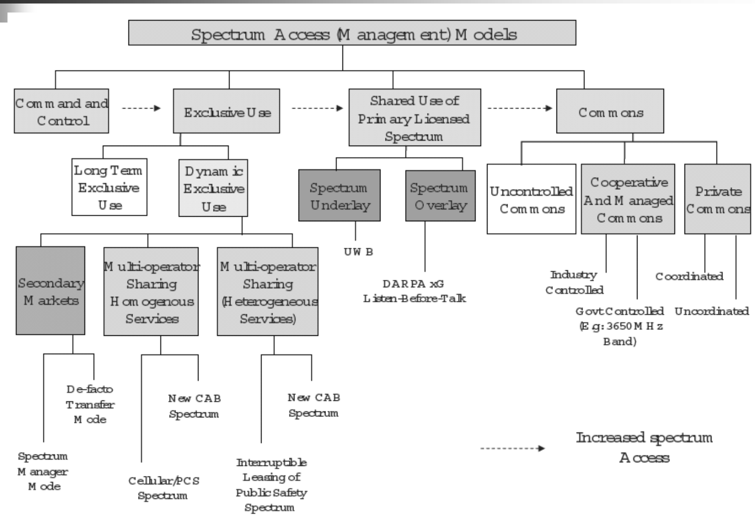
Taxonomies

- Command and Control
  - Ideal for military, government, aeronautical operations
  - ...no DSA allowed !!!
  - Static and out-dated for commercial applications
- Exclusive Use
  - License allows owner exclusive rights
  - 2 variants:
    - Long-term exclusive use
    - Dynamic exclusive use: allows for “secondary markets” and multi-operator sharing

Taxonomies

- Shared Use of Primary Licensed Spectrum:
  - Consists of licensee (primary user) and non-license holder (secondary user)
  - SUs are expected to have minimal impact on PUs
  - Stresses peaceful coexistence
- Spectrum Commons:
  - Spectrum is public resource accessible to all
  - Mitigates spectrum under-utilization

A More-Detailed View [Buddhikot 2007]



Radio Resource Management and Spectrum Taxonomies

- Different taxonomy => different RRM strategy
  - Command and Control: no DSA allowed… (RRM for unlicensed users is a moot discussion)
  - Exclusive Use: too general to fixate a specific RRM strategy. Consider on case-by-case basis.
  - Shared Use: a premium on RRM to allow peaceful coexistence among Pus and SUs
  - Spectrum Commons: all users are equal, thus no differentiation among RRM for PUs and SUs
- Big dependency of RRM on the legacy system

Some Caveats of Radio Resource Management for Cognitive Radio Networks

Taxonomies, Primary Networks, and Interference Models

- Multitude of primary networks (i.e. legacy systems):
  - Commercial cellular network
  - Ad hoc network
  - TV broadcast service
  - Emergency service
- Considered taxonomies:
  - Shared Use
  - Dynamic Exclusive Use
- Interference model:
  - A CDMA-like MAC
  - Coupling among all active users
  - Users mutually interfere

Radio Resource Management and Cognitive Radio

- Primary and secondary users are different !
- RRM decisions should take a user's license into account
- Classical RRM analysis+results may not hold when considering coexistence of users with different spectrum rights
- Question: How robust are RRM techniques to etiquette?
- Question: RRM re-considered?

A Note on Etiquette

- Webster's definition:

  "the conduct or procedure required by good breeding or prescribed by authority to be observed in social or official life."

- Haykin's analogy [Haykin 2005]:

  Etiquette and *protocol*. Such provisions may be likened to the use of traffic lights, stop signs, and speed limits, which are intended for motorists (using a highly dense transportation system of roads and highways) for their individual safety and benefits.

- Our interpretation:
  - Co-existence of primary and secondary users with a caveat.

A Power Control Optimization Problem + Solution in Cognitive Radio Networks

Overview

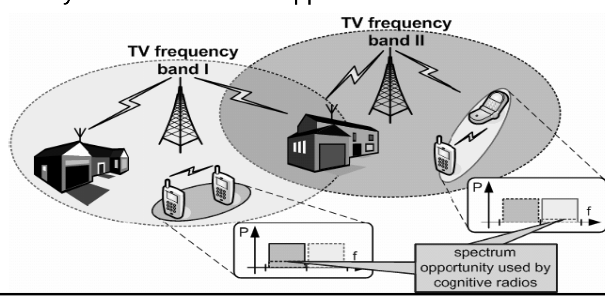
- Power control for cognitive radio: some attributes...
- Dual Priority Class Power Control (DPCPC) policies and etiquette
- Autonomous Interference-Aware Power Control (AIPC)
- Simulation results and discussion

Power Control Basics...

- Non-orthogonal users => Interference
- Dynamic power adaptation by a Tx following the reception of feedback
- Users try to satisfy QoS criterion:
  - Signal-to-interference plus noise ratio (SIR)
- A single priority class of users per network
- Autonomous power control is more difficult than centralized power control

Power Control for Cognitive Radio Networks

- Not much work done…
- Primary users and secondary users are different!!!
- Flexibility of policy:
  - Primary users satisfy “hard” QoS constraints
  - Secondary users should have opportunistic communication



Essential Attributes

- QoS Protection:
  - Primary users maintain QoS irrespective of secondary user operation.
- Opportunism:
  - Secondary users’ QoS improves upon dormancy of a primary user.
- Admissibility:
  - Policy allows for control over the admission of a user into the network.
- Autonomous Operation:
  - Tx’s only have access to local info via feedback from intended Rx’s.
  - No global info, message passing, etc.

Desirable Attributes

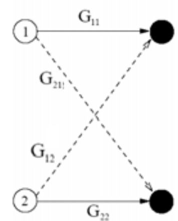
- Licensing:
  - Policy allows for the assigning of priorities to the applications of various users… different priorities = different licenses
- Versatility:
  - Policy is flexible enough to be deployed by all network users whether primary or secondary.
- …others???

Some Cautionary Notes…

- Different taxonomies of spectrum access models [M. Buddhikot] :
  - Command and control
  - Exclusive use
  - Shared use
  - Spectrum commons
- Different primary networks (i.e. legacy systems):
  - Emergency service
  - TV broadcast service
  - Ad hoc network
  - Commercial cellular network
- Power control will affect several layers (e.g. physical, MAC, network).

System Model

- Distributed power control problem
  - Only local information available to each user



- Active users of transmit power: $\{P_i(k)\}$
  - No power constraint: $P_i^{min} \leq P_i(k)$
- Propagation effects modeled by link gains: $\{G_{ij}(k)\}$
  - Assume slow fading: $\{G_{ij}(k)\} \rightarrow \{G_{ij}\}$
  - Not critical for formulation
- Users have application-specific QoS requirements: $\{\mathrm{SIR}_i^{tar}\}$

User Parameters

- Signal-to-interference plus noise ratio (SIR):
  - SIR of i th user:

$$\mathrm{SIR}_i(k) = \frac{P_i(k)G_{ii}}{\sum_{j \neq i} P_j(k)G_{ij} + \eta_i} = \frac{P_i(k)G_{ii}}{I_{-i}(k)}$$

- Interference:
  - Perceived interference:

$$I_{-i}(k) = \sum_{j \neq i} P_j(k)G_{ij} + \eta_i$$

  - Aggregate interference:

$$I_i(k) = \sum_j P_j(k)G_{ij} + \eta_i = I_{-i}(k) + P_i(k)G_{ii}$$

3 Axioms (that define a DPCPC policy)

Axiom 1: The policy should provide a power allocation such that $\partial P_i/\partial I_{-i} > 0 : i \in \mathsf{P}$ and $\partial P_i/\partial I_{-i} < 0 : i \in \mathsf{S}$.

- Ensures QoS Protection

Axiom 2: The target QoS of a user should be dependent upon the user's channel state via $\partial \mathrm{SIR}_i^{tar}/\partial I_{-i} < 0 : i \in \mathsf{S}$ and $\partial \mathrm{SIR}_i^{tar}/\partial I_{-i} = 0 : i \in \mathsf{P}$.

- Ensures Opportunism

Axiom 3: The entrance of a secondary user should not cause the outage of a primary user via $\mathrm{SIR}_i < \mathrm{SIR}_i^{tar} : i \in \mathsf{P}$.

- Ensures Admissibility

DPCPC and Etiquette

A more concrete definition of etiquette:

QoS Protection

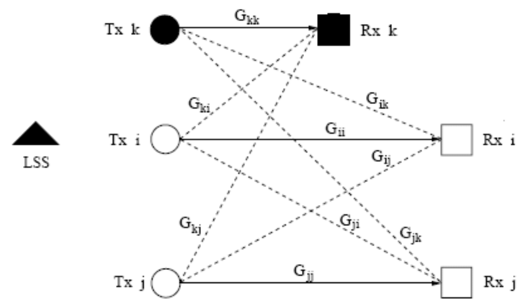
Opportunism

Admissibility

dictate etiquette among primary + secondary users in a cognitive radio network.

Cognitive Radio Network Model

- Primary network, secondary network, Local Spectrum Server (LSS)



- LSS:
  - Can: Mediate spectrum among secondary users.
  - Can: Interact with backbone of primary network.
  - Can't: Coordinate actions of users.

QoS Protection

- Definitions:
  - Primary users: $\mathrm{P} = \{1, 2, \ldots, N\}$
  - Secondary users: $\mathrm{S} = \{N+1, N+2, \ldots, N+M\}$
- Primary QoS vs. secondary QoS:
  - Primary user: hard QoS constraint $\mathrm{SIR}_i^{tar} : i \in \mathrm{P}$
  - Secondary user: soft QoS requirement $\widetilde{\mathrm{SIR}}_i^{tar} : i \in \mathrm{S}$
- Upon convergence:

$$\begin{aligned}
&\text{minimize} && \sum_{i=1}^{N+M} P_i \\
&\text{subject to} && \mathrm{SIR}_i^{tar} \le \frac{P_i G_{ii}}{\sum_{j \ne i} P_j G_{ij} + \eta_i} && i \in \mathrm{P}, \quad N = |\mathrm{P}| \\
& && \widetilde{\mathrm{SIR}}_i^{tar} \le \frac{P_i G_{ii}}{\sum_{j \ne i} P_j G_{ij} + \eta_i} && i \in \mathrm{S}, \quad M = |\mathrm{S}|
\end{aligned}$$

- ...some math...
  In matrix form: $(\mathbf{I}-\mathbf{A})\mathbf{p} \geq \mathbf{b}$

  with: $$A(i,j) = \begin{cases} 0 & \text{if} \quad i = j \\ \frac{\mathrm{SIR}_i^{tar} G_{ij}}{G_{ii}} & \text{if} \quad i \leq N,\ i \neq j \\ \frac{\widetilde{\mathrm{SIR}}_i^{tar} G_{ij}}{G_{ii}} & \text{if} \quad i > N,\ i \neq j \end{cases}$$

  $$\mathbf{b} = \left[\frac{\mathrm{SIR}_1^{tar}\eta_1}{G_{11}}, \ldots, \frac{\mathrm{SIR}_N^{tar}\eta_N}{G_{NN}}, \frac{\widetilde{\mathrm{SIR}}_{N+1}^{tar}\eta_{N+1}}{G_{N+1\,N+1}}, \ldots, \frac{\widetilde{\mathrm{SIR}}_{N+M}^{tar}\eta_{N+M}}{G_{N+M\,N+M}}\right]^T$$

  Pareto optimal solution: $\mathbf{p}^* = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}$

  feasibility condition: $\rho(\mathbf{A}) < 1$

  ...or equivalently:

  $$\max_{\mathbf{x}\geq 0, \mathbf{x}\neq 0} \min_{1\leq i\leq N+M, x_i\neq 0} \left\{ \frac{1}{x_i}\sum_{j=1}^{N} A(i,j)x_j + \frac{1}{x_i G_{ii}} \sum_{j=N+1}^{N+M} \widetilde{\mathrm{SIR}}_i^{tar} G_{ij} x_j \right\} < 1$$

Theorem 1: Consider a feasible network consisting of N primary users. With DPCPC, a network of N primary users and M secondary users will be feasible.

Key to Proof: Axioms 1 and 2

Theorem 2: With DPCPC policy the entrance of secondary users will only adversely affect the QoS of other secondary users. The resulting system will remain feasible with the QoS of the primary users undeterred.

Key to Proof: Axiom 1, and $\underline{\mathrm{SIR}}_i^{tar} < \widetilde{\mathrm{SIR}}_i^{tar}$ for previously existing secondary users.

A Cautionary Note: With entrance of secondary user, each primary user will transmit with higher power!!!

Opportunism

- Opportunistic spectrum access by a secondary user due to
  - Dormancy of primary user
  - Favorable change in the channel

Theorem 3: With DPCPC the dormancy of primary users will lead to a power allocation such that:

- The remaining primary users maintain their QoS while conserving power.
- The secondary users witness improved QoS.

Key to Proof: Axiom 2, and $\partial P_i / \partial I_{-i} > 0 : i \in \mathrm{P}$

Corollary 1: With DPCPC an improvement in the channel between a primary user and it's intended Rx, or a degradation in the channel between a primary user and the intended Rx of any other primary user will result in:

- The primary users maintain their QoS while conserving power.
- The secondary users witness improved QoS.

Admissibility

- Regulation of power is natural means of dictating admission of a user into a network.
- Type I vs. Type II admission control errors.
- Many works have considered integration of power and admission control…for single QoS class of users.
- Fundamental aspects of admission control policies be reconsidered within primary/secondary framework?

- Reconsideration of admission errors:
  - Secondary-primary type I error: a new secondary user is erroneously admitted causing the outage of a primary user.
  - Primary-secondary type I error: a new primary user is erroneously admitted causing the outage of a secondary user.
  - Primary (Secondary) type I error: a new primary (secondary) user is erroneously admitted causing the outage of a primary (secondary) user.
  - Primary (Secondary) type II error: a new primary (secondary) user is erroneously denied admission while it could have been supported.

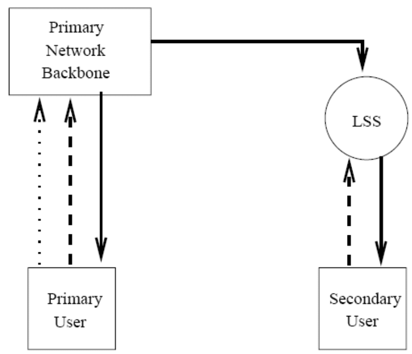
Theorem 4: A DPCPC policy is:

- secondary-primary type I error free (i.e. Axiom 3).
- secondary type II error free.
- prone to secondary type I errors.

Key to Proof:

- DPCPC allows for:
  - Voluntary drop-out (VDO)
    - [Bambos, Chen, Pottie 2000]
  - Distributed interactive admission control (DIAC)
    - [Andersin, Rosberg, Zander 1997]

- An architecture for admission control in power-controlled cognitive radio network.



Autonomous Interference-Aware Power Control (AIPC)

- Specify cost function for *i* th user:

$$J_i(k) = \rho_{i1} E_i^2(k+1) + \rho_{i2} I_i^2(k+1)$$

  - SIR deviation: $E_i(k) = \mathrm{SIR}_i^{tar} - \mathrm{SIR}_i(k)$
  - Satisfy QoS requirement subject to interference constraint

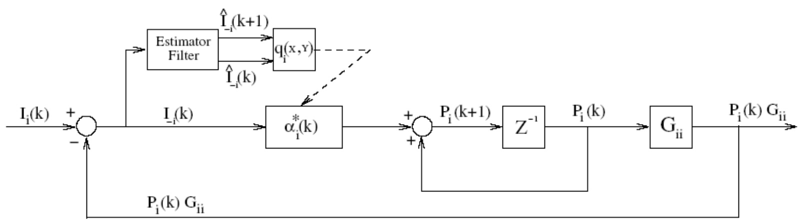
- Power update of *i* th user: $P_i(k+1) = P_i(k) + \alpha_i(k) I_{-i}(k)$

- Optimal gain derived via $\alpha_i^*(k) = \arg\min_{\alpha_i(k)}\{\ J_i(k)\ \}$:

$$\alpha_i^*(k) = \frac{\rho_i I_{-i}^2(k+1)\left(-P_i(k)G_{ii} - I_{-i}(k+1)\right) + \mathrm{SIR}_i^{tar} I_{-i}(k+1) - \mathrm{SIR}_i(k) I_{-i}(k)}{I_{-i}(k) G_{ii}(1+\rho_i I_{-i}^2(k+1))}$$

and $\rho_i \triangleq \rho_{i2}/\rho_{i1}$

- Closed-loop implementation of AIPC algorithm:



- Estimated and predicted perceived interference: $\hat{I}_{-i}(k)$, $\hat{I}_{-i}(k+1)$

Interpretation of AIPC

- Priority of *i* th user's QoS is controlled in distributed fashion via ρ_i .
  - $\rho_i = 0$:
    - User has hard QoS constraint.
    - *i* th user may cause unregulated amount of interference to satisfy QoS.
    - Nearly equivalent to (greedy) DPC algorithm of [Foschini, Miljanic 1993].
  - $\rho_i = \infty$:
    - *i* th user will transmit with minimum allowable power (P_i^{min}) regardless of its channel state.
  - $0 < \rho_i < \infty$:
    - *i* th user has soft QoS constraint.

- $P_i^*(k) = P_i^{min}$ (for $0 < \rho_i < \infty$) :
  - Indicates that the interference of *i* th user is too high/costly to the network
  - *i* th user has decided to "opt-out" and transmit with minimum allowable power

$$I_{-i}(k) > \max\left\{\frac{P_i^{min} - P_i(k)}{\alpha_i^*(k)},\ \frac{\mathrm{SIR}_i^{tar} I_{-i}(k+1)\ -\ \rho_i I_{-i}^2(k+1)\,(P_i(k)\,G_{ii} + I_{-i}(k+1))}{\mathrm{SIR}_i(k)}\right\}$$

- Upon convergence of AIPC, *i* th user with:

$$I_{-i} \geq \sqrt{\frac{\mathrm{SIR}_i^{tar}}{\rho_i}}$$

would have $P_i^* = P_i^{min}$

- At convergence of AIPC, *i* th user attains:

$$\widetilde{\mathrm{SIR}}_i^{tar} \triangleq \beta_i \mathrm{SIR}_i^{tar}$$

with $\beta_i \in [0,1]$ via:

$$\beta_i = \max\left\{0,\ \frac{\mathrm{SIR}_i^{tar} - \rho_i I_{-i}^2(k^{ss})}{\mathrm{SIR}_i^{tar}(1 + \rho_i I_{-i}^2(k^{ss}))}\right\}$$

note: $\beta_i = 1$ for $\rho_i = 0$.

Is AIPC a DPCPC Policy?

Theorem 5: The AIPC algorithm is a DPCPC policy so long as: $I_{-i} > \sqrt{\mathrm{SIR}_i^{tar}/\rho_i}/(\sqrt{\mathrm{SIR}_i^{tar}+\rho_i I_{-i}^2+3})$ for $\rho_i > 0$.

Key to Proof: verify Axiom 1, 2, 3, and note that:

$$\begin{aligned} \partial P_i^*/\partial I_{-i} &\gtreqless 0 \quad \text{for} \quad \sqrt{\mathrm{SIR}_i^{tar}/\rho_i} \gtreqless I_{-i}\sqrt{\mathrm{SIR}_i^{tar}+\rho_i I_{-i}^2+3} \\ P_i^* &= P_i^{min} \quad \text{for} \quad I_{-i} \geq \sqrt{\mathrm{SIR}_i^{tar}/\rho_i}. \end{aligned}$$

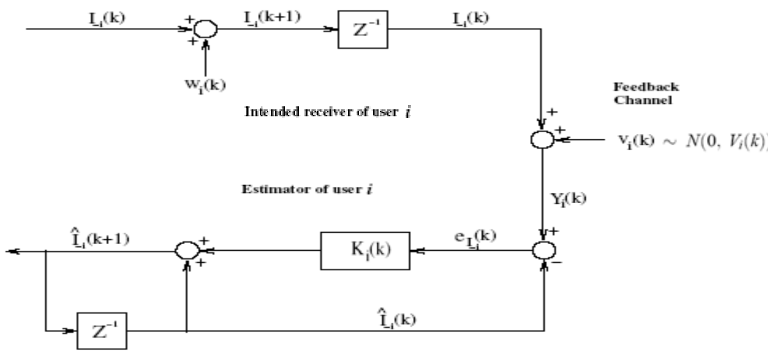
Corollary 2: With AIPC the dormancy of primary users, an improvement in the channel between a primary user and it's intended Rx, or a degradation in the channel between a primary user and the intended Rx of any other primary user will result in:

- The primary users maintain their QoS while conserving power.
- The secondary users witness improved QoS.
- The *i* th secondary user conserves power if $I_{-i} < \sqrt{\mathrm{SIR}_i^{tar}/\rho_i}/(\sqrt{\mathrm{SIR}_i^{tar}+\rho_i I_{-i}^2+3})$

AIPC and Autonomous Operation

- To perform an AIPC power update, *i* th user needs local info:
  - $\hat{I}_{-i}(k)$
  - $\hat{I}_{-i}(k+1)$
  - G_{ii}
- Feedback from intended Rx is necessary.
- Autonomous interference estimation/prediction is possible.
- Dynamic system model of perceived interference:

$$\begin{aligned} I_{-i}(k+1) &= I_{-i}(k) + w_i(k) \\ Y_i(k) &= I_{-i}(k) + v_i(k) \end{aligned}$$



- Estimator/predictor: $\hat{I}_{-i}(k+1) = \hat{I}_{-i}(k) + K_i(k)(Y_i(k) - \hat{I}_{-i}(k))$
- Consider a Kalman Filter: $K_i(k) = B_i(k)/(B_i(k) + V_i(k))$

  with algebraic Riccati equation: $B_i(k+1) = B_i(k) - B_i^2(k)(B_i(k) + V_i(k))^{-1} + W_i(k)$
- Gaussian assumption on driving disturbance: $w_i(k) \sim N(b_i(k), W_i(k))$
- Local computation of disturbance statistics ($\approx$ ML estimates):

  $\hat{b}_i(k) = \frac{1}{K} \sum_{n=k-K+1}^{k} \hat{I}_{-i}(n+1) - \hat{I}_{-i}(n)$ $\quad$ $\hat{W}_i(k) = \frac{1}{K} \sum_{n=k-K+1}^{k} [\hat{I}_{-i}(n+1) - \hat{I}_{-i}(n)]^2 - \hat{b}_i^2(k)$

AIPC and Licensing

- AIPC allows for exclusive access for primary users:
  - Greedy via $\rho_i = 0 : i \in \mathrm{P}$
- AIPC allows for lower priority for secondary users:
  - Interference-conscious via $\rho_i > 0 : i \in \mathrm{S}$
- Network performs Licensing via assignment of $\{\rho_i\}$
- “Degree” of *i* th user’s license dictated by assigned ρ_i value
- Differentiation among priority = Differentiation among license

AIPC and Versatility

- Deployment of a single power control policy (i.e. AIPC) allows for definitive statements on:
  - Convergence and feasibility analysis
  - The union of:
    - Qos protection
    - opportunism
    - admissibility
    - licensing

- AIPC incorporates:
  - Multi-criterion objective function

- AIPC allows for:
  - Differing dynamics for two classes of users

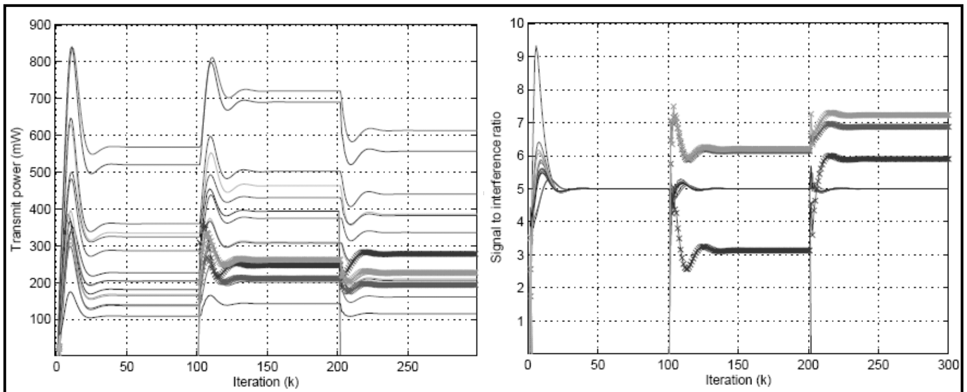
Simulation Results

- Uplink of a single-cell CDMA system
  - PNB is the base station
  - Primary network: voice users with $\mathrm{SIR}_i^{tar} = 5 \,:\, i \in \mathrm{P}$
  - Secondary network: delay-insensitive data users with $\mathrm{SIR}_i^{tar} = 10 \,:\, i \in \mathrm{S}$
  - Matched filter receiver used at base station:

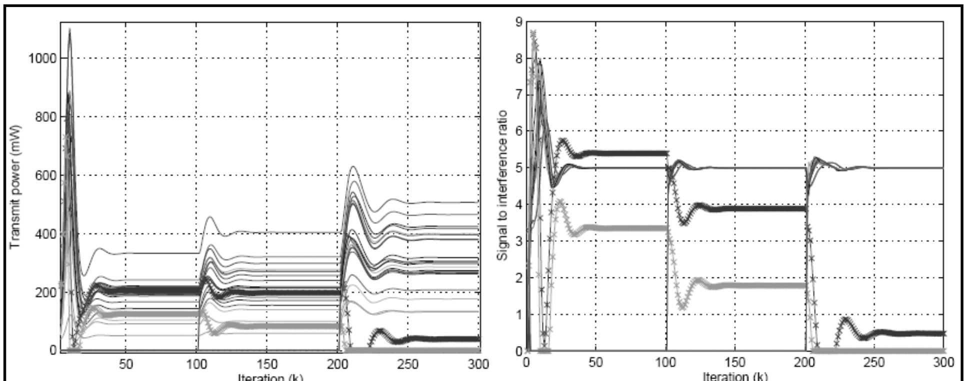
$$\mathrm{SIR}_i(k) = \frac{P_i(k)h_i\left[\mathbf{c}_i^T(k)\mathbf{s}_i(k)\right]^2}{\sum\limits_{j\neq i} P_j(k)h_j\left[\mathbf{c}_i^T(k)\mathbf{s}_j(k)\right]^2 + \mathbf{c}_i^T(k)\mathbf{c}_i(k)\eta_i} = \frac{P_i(k)h_i}{\sum\limits_{j\neq i} P_j(k)h_j\kappa_{ij}^2 + \eta_i}$$

with $\kappa_{ij} = \mathbf{s}_i^T\mathbf{s}_j$, $\mathbf{s}_i = \frac{1}{\sqrt{L}}[s_{i1},\ s_{i2},\ \ldots,\ s_{iL}]^T$, $s_{ij} \in \{-1, 1\}$

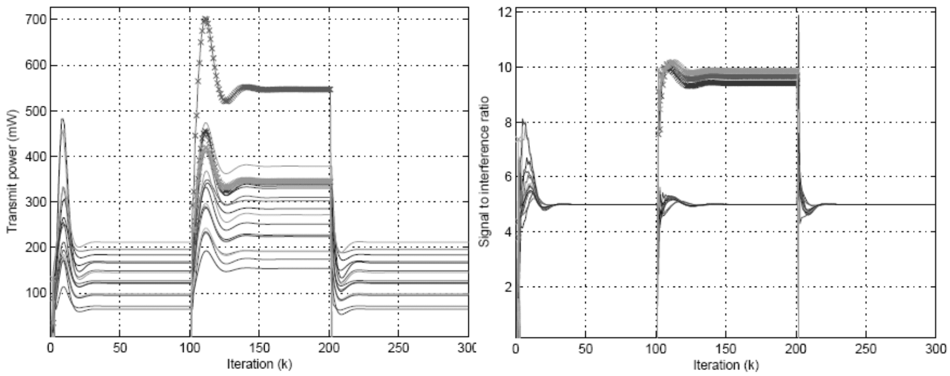
  - Path loss model: $h_i = P_R\left(\frac{d_R}{d_i}\right)^n = \frac{A}{d_i^n}$ with $d_i \in (0, r]$ $\quad r = 1\ \mathrm{km}$, $n = 4$, $A = 10^{-4}$
  - processing gain of L=128 per user
  - $\eta_i = 10^{-3}\ \mathrm{mW}$, $P_i^{min} = 0.0\ \mathrm{mW}$



- $0 < k \leq 100$: N=15 primary users $\rho_i = 0 : i \in P = \{1, 2, \ldots, 15\}$
  M=0 secondary users $S = \emptyset$
- $100 < k \leq 200$: N=15 primary users $\rho_i = 0 : i \in P = \{1, 2, \ldots, 15\}$
  M=3 secondary users $\rho_i = 200 : i \in S = \{16, 17, 18\}$
  ...QoS Protection
- $200 < k \leq 300$: N=14 primary users $\rho_i = 0 : i \in P' = \{1, 2, \ldots, 14\}$
  M=3 secondary users $\rho_i = 200 : i \in S = \{16, 17, 18\}$
  ...Opportunism



- $0 < k \leq 100$: N=15 primary users $\rho_i = 0 : i \in P = \{1, 2, \ldots, 15\}$
  M=2 secondary users $\rho_i = 200 : i \in S = \{16, 17\}$
- $100 < k \leq 200$: N=17 primary users $\rho_i = 0 : i \in P = \{1, 2, \ldots, 15, 18, 19\}$
  M=2 secondary users $\rho_i = 200 : i \in S = \{16, 17\}$
  ...QoS Protection
- $200 < k \leq 300$: N=21 primary users $\rho_i = 0 : i \in P = \{1, 2, \ldots, 15, 18, \ldots, 23\}$
  M=2 secondary users $\rho_i = 200 : i \in S = \{16, 17\}$
  ...QoS Protection and Admissibility



- $0 < k \leq 100$: N=15 primary users $\rho_i = 0 : i \in P = \{1,2,\ldots,15\}$
  M=0 secondary users $S = \oslash$
- $100 < k \leq 200$: N=15 primary users $\rho_i = 0 : i \in P = \{1,2,\ldots,15\}$
  M=3 secondary users $\rho_i = 20 : i \in S = \{16,17,18\}$
  ...Licensing
- $200 < k \leq 300$: N=15 primary users $\rho_i = 0 : i \in P = \{1,2,\ldots,15\}$
  M=0 secondary users $\rho_i = 10^4 : i \in S = \{16,17,18\}$
  ...QoS Protection and Admissibility

Convex Optimization
Resource Allocation
Problems + Solutions
in
Cognitive Radio Networks

Secondary User Capacity Maximization with Multiple Antennas [Zhang, Liang 2008]

Problem statement:

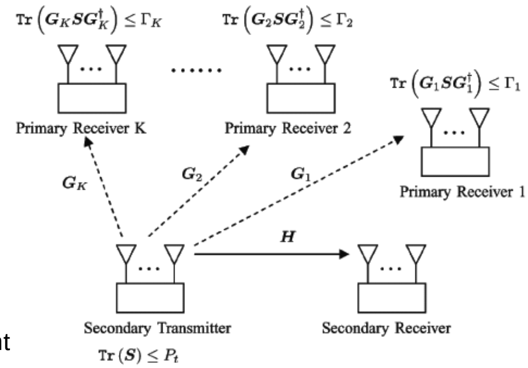
- Maximize SU's capacity subject to PUs interference constraints

Rx'd signal of SU:

- $\boldsymbol{y}(n) = \boldsymbol{H}\boldsymbol{x}(n) + \boldsymbol{z}(n)$

Definitions:

- 1 SU, K PUs
- $\boldsymbol{S} = \mathbb{E}[\boldsymbol{x}(n)\boldsymbol{x}^{\dagger}(n)]$
- $\boldsymbol{H} \in \mathbb{C}^{M_{r,s} \times M_{t,s}}$
- $\boldsymbol{G}_k \in \mathbb{C}^{M_k \times M_{t,s}}$
- $\boldsymbol{z}(n) \sim \mathcal{CN}(0, \boldsymbol{I})$
- P_t : SU's power constraint
- Γ_k: *k* th PU's inter. constraint



- Optimization problem:

$$
\begin{aligned}
\texttt{Maximize} \quad & \log_2 \left| \boldsymbol{I} + \boldsymbol{H}\boldsymbol{S}\boldsymbol{H}^{\dagger} \right| \\
\texttt{Subject to} \quad & \texttt{Tr}(\boldsymbol{S}) \le P_t, \\
& \texttt{Tr}\left(\boldsymbol{G}_k\boldsymbol{S}\boldsymbol{G}_k^{\dagger}\right) \le \Gamma_k, \quad k = 1, \ldots, K \\
& \boldsymbol{S} \succeq 0.
\end{aligned}
$$

  - The obje
  - The constraints satisfy a convex set of
  - Semidefinite constraint on

$\boldsymbol{S}$

$\boldsymbol{S}$

$\boldsymbol{S}$

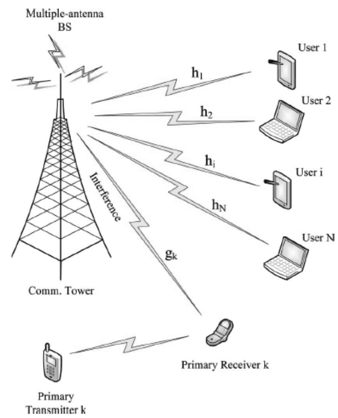
- Solution:
  - Interior-point method
  - Two SVD-based solutions...
  - See "*Dynamic Resource Allocation in Cognitive Radio Networks: A Convex Optimization Perspective*", R. Zhang, Y.C. Liang and S. Cui, IEEE Signal Processing Magazine 2010

Beamforming for QoS-Aware Secondary Multicast [Phan, et al. 2009]

Problem statement:
- Satisfy SUs QoS subject to interference constraint to PUs

Rx'd SNR at SU i: $\mathrm{SNR}_i = \frac{|\mathbf{w}^H \mathbf{h}_i|^2}{\sigma_i^2}$

Rx'd interference at PU k: $|\mathbf{w}^H \mathbf{g}_k|^2$



Definitions:
- N SUs, K PUs
- M antennas at BS
- $\mathbf{h}_i, \mathbf{g}_k$: Mx1 complex gains to SUs, PUs
- $\mathbf{w}$: beamforming weight vector
- σ_i^2: noise variance at user receiver
- η_0 : allowable interference threshold

- Optimization problem:

$$\begin{aligned} \text{minimize}_{\mathbf{w}} \ & \|\mathbf{w}\|_2^2 \\ \text{subject to} \ & \frac{|\mathbf{w}^H \mathbf{h}_i|^2}{\sigma_i^2} \geq \mathrm{SNR}_i^{\min},\ i = 1, \ldots, N \\ & |\mathbf{w}^H \mathbf{g}_k|^2 \leq \eta_0,\ k = 1, \ldots, K \end{aligned}$$

  - Quadratically constrained quadratic programming (QCQP) problem
  - NP-hard

- Solution:
  - Relax to a convex optimization problem via a SDP Relaxation

- Use the fact that: $\mathbf{h}_i^H \mathbf{w}\mathbf{w}^H \mathbf{h}_i = \text{trace}(\mathbf{w}\mathbf{w}^H \mathbf{h}_i \mathbf{h}_i^H)$
- Introduce the designations: $\mathbf{H}_i \triangleq \mathbf{h}_i \mathbf{h}_i^H / \sigma_i^2$ and $\mathbf{G}_k \triangleq \mathbf{g}_k \mathbf{g}_k^H$
- Introduce PSD matrix: $\mathbf{X} \triangleq \mathbf{w}\mathbf{w}^H$
- Now, recast optimization problem as:

$$\begin{aligned} \text{minimize}_{\mathbf{X}} \quad & \text{trace}\ (\mathbf{X}) \\ \text{subject to} \quad & \text{trace}(\mathbf{X}\mathbf{H}_i) \geq \text{SNR}_i^{\min},\ i=1,\ldots,N \\ & \text{trace}\ (\mathbf{X}\mathbf{G}_k) \leq \eta_0,\ k=1,\ldots,K \\ & \mathbf{X} \succcurlyeq \mathbf{0},\ \ \text{rank}(\mathbf{X}) = 1 \end{aligned}$$

  - objective function and trace constraints are linear in **X**
  - set of symmetric PSD matrices is convex
  - rank constraint is nonconvex!

- Dropping the rank constraint (the so-called SDP relaxation) gives the SDP problem:

$$\begin{aligned} \text{minimize}_{\mathbf{X}} \quad & \text{trace}\ (\mathbf{X}) \\ \text{subject to} \quad & \text{trace}(\mathbf{X}\mathbf{H}_i) \geq \text{SNR}_i^{\min},\ i=1,\ldots,N \\ & \text{trace}\ (\mathbf{X}\mathbf{G}_k) \leq \eta_0,\ k=1,\ldots,K \\ & \mathbf{X} \succcurlyeq \mathbf{0} \end{aligned}$$

- The SDP problem is convex (solved via an interior point method)

- Drawback: this does not solve the original problem in the truest sense. …however, it's been shown that rank-relaxation of a general QCCP problem provides the Lagrange bi-dual problem (the closest convex problem to the original NP-hard problem).

Max-Min Fairness Based Beamforming [Phan, et al. 2009]

Problem statement:

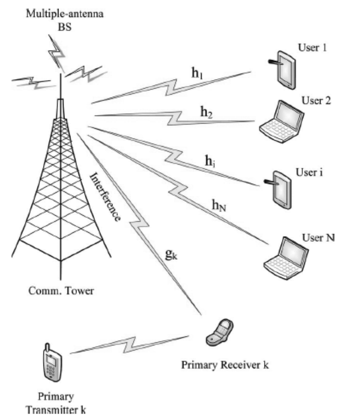
- Give a “meaningful” level of QoS to each SU

Rx’d SNR at SU i: $\mathrm{SNR}_i = \dfrac{|\mathbf{w}^H \mathbf{h}_i|^2}{\sigma_i^2}$

Rx’d interference at PU k: $|\mathbf{w}^H \mathbf{g}_k|^2$

Definitions:

- N SUs, K Pus, M antennas at BS
- P : total available power at BS
- $\mathbf{h}_i, \mathbf{g}_k$: Mx1 complex gains to SUs, PUs
- $\mathbf{w}$: beamforming weight vector
- σ_i^2: noise variance at user receiver
- η_0 : allowable interference threshold



- Optimization problem:

$$\begin{aligned} &\text{maximize}_{\mathbf{w}} \min_{i=1,\ldots,N} \left\{ \frac{|\mathbf{w}^H \mathbf{h}_i|^2}{\sigma_i^2} \right\} \\ &\text{subject to } \|\mathbf{w}\|_2^2 \le P \\ &\qquad |\mathbf{w}^H \mathbf{g}_k|^2 \le \eta_0, \; k = 1, \ldots, K. \end{aligned}$$

  - Nonconvex

- Solution:
  - Relax to a convex optimization problem via a SDP Relaxation

- Re-write the optimization problem in equivalent form:

$$
\begin{aligned}
\text{minimize}_{\mathbf{w},\, t} \quad & -t \\
\text{subject to} \quad & \frac{|\mathbf{w}^H \mathbf{h}_i|^2}{\sigma_i^2} \geq t, \; i = 1, \ldots, N \\
& \|\mathbf{w}\|_2^2 \leq P, \; t \geq 0 \\
& |\mathbf{w}^H \mathbf{g}_k|^2 \leq \eta_0, \; k = 1, \ldots, K.
\end{aligned}
$$

- Dropping the rank constraint (i.e. SDP relaxation) gives SDP problem:

$$
\begin{aligned}
\text{minimize}_{\mathbf{X},\, t} \quad & -t \\
\text{subject to} \quad & \text{trace}\,(\mathbf{X}\mathbf{H}_i) \geq t, \; i = 1, \ldots, N \\
& \text{trace}\,(\mathbf{X}) \leq P \\
& \text{trace}\,(\mathbf{X}\mathbf{G}_k) \leq \eta_0, \; k = 1, \ldots, K \\
& \mathbf{X} \succcurlyeq \mathbf{0}, \;\; t \geq 0.
\end{aligned}
$$

  - Convex on X and t (solved via an interior point method)

Scheduling of Variable Rate Links via a Spectrum Server [Yates, Raman, Mandayam 2006]

Problem statement:

- Optimal transmission schedule that maximizes avg sum rate subject to avg rate constraint for each link.

Mechanism:

- A spectrum server coordinates the activity of a set of links sharing a common spectrum to ensure efficient use of the spectrum.

Definitions:

- N nodes, L links
- On-off, fixed power users
- M possible Tx modes
- $\mathbf{C} = [c_{li}]$ is L x M matrix with $c_{li} = \log(1 + \gamma_{li})$ and $\gamma_{li} = \dfrac{t_{li} G_{ll} P_l}{\sum_{k \in \mathcal{E}, k \neq l} t_{ki} G_{lk} P_k + \sigma_l^2}$
- Binary mode activity vector $t_i = (t_{1i}, t_{2i}, \ldots, t_{Li})$ with $t_{li} = \begin{cases} 1, & \text{link } l \text{ is active under transmission mode } i, \\ 0, & \text{otherwise.} \end{cases}$

- r_l : avg data rate of link l (i.e $r_l = \sum_i c_{li} x_i$)
- $\mathbf{r} = \mathbf{C}\mathbf{x}$: rate vector for the L links with
- $r_l^{\min}$: min avg data rate requirement of link l
- x_i : fraction of time that Tx-mode i is active

- Optimization Problem:

$$\begin{array}{rl} \max & \mathbf{1}^T \mathbf{r} \\ \text{subject to} & \mathbf{r} = \mathbf{C}\mathbf{x}, \\ & \mathbf{r} \geq \mathbf{r}_{\min}, \\ & \mathbf{1}^T \mathbf{x} = 1, \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

  - Optimization variables are **r** and **x**

- Re-writing the optimization problem in terms of the variable **x** only:

$$\begin{array}{rcl} c_{\text{opt}}(\mathbf{r}_{\min}) = \max & & \mathbf{1}^T \mathbf{C}\mathbf{x} \\ \text{subject to} & & \mathbf{C}\mathbf{x} \geq \mathbf{r}_{\min}, \\ & & \mathbf{1}^T \mathbf{x} \leq 1, \\ & & \mathbf{x} \geq \mathbf{0}. \end{array}$$

- This is a LP

- Solution:
  - Case #1, no min rate constraint
  - Noted that: $\mathbf{1}^T \mathbf{C}\mathbf{x} = \sum_{l=1}^{L} \sum_{i=1}^{M} c_{li} x_i = \max_i \sum_{l=1}^{L} c_{li}$
  - Obtain:

$\mathbf{x}_{\text{opt}} = [0 \ 0 \ldots 1 \ldots 0 \ 0]^T$ where position of 1 in $\mathbf{x}_{\text{opt}}$ is $\hat{i} = \arg\max_i \sum_{l=1}^{L} c_{li}$

- Note that for Case #1 Efficiency >> Fairness

- <u>Case #2</u>, min rate constraint $\mathbf{r}_{\min}$ enforced
- The Lagrangian for the LP is:

$$L(\mathbf{x}, \mathbf{u}, v) = \mathbf{1}^T\mathbf{C}\mathbf{x} + \mathbf{u}^T(\mathbf{C}\mathbf{x} - \mathbf{r}_{\min}) + v(1 - \mathbf{1}^{\mathbf{T}}\mathbf{x})$$

  with $\mathbf{u} \in \mathcal{R}^L$ and $v \in \mathcal{R}$ as the dual variables
- The Lagrange dual is

$$\begin{aligned} g(\mathbf{u}, v) &= \sup_{\mathbf{x} \geq \mathbf{0}} L(\mathbf{x}, \mathbf{u}, v) \\ &= -\mathbf{u}^T\mathbf{r}_{\min} + v \\ &\quad + \sup_{\mathbf{x} \geq \mathbf{0}} (\mathbf{1}^{\mathbf{T}}\mathbf{C} + \mathbf{u}^T\mathbf{C} - v\mathbf{1}^T)\mathbf{x} \\ &= \begin{cases} -\mathbf{u}^T\mathbf{r}_{\min} + v, & \mathbf{1}^T\mathbf{C} + \mathbf{u}^T\mathbf{C} - v\mathbf{1}^T \leq \mathbf{0} \\ \infty, & \text{otherwise} \end{cases} \end{aligned}$$

- Thus the dual problem is:

$$\begin{array}{ll} \text{minimize} & -\mathbf{r}_{\min}^T\mathbf{u} + v \\ \text{subject to} & \mathbf{C}^T(\mathbf{1} + \mathbf{u}) \leq v\mathbf{1}, \\ & \mathbf{u} \geq \mathbf{0}, v \geq 0. \end{array}$$

- By strong duality principle, the optimal value of the dual problem is equal to $c_{\text{opt}}(\mathbf{r}_{\min})$
- Let $(\mathbf{u}^*, v^*)$ denote the solution of the dual problem. It's noted that the optimal value of the dual problem yields:

$$\begin{aligned} c_{\text{opt}}(\mathbf{r}_{\min}) &= -\mathbf{r}_{\min}^T\mathbf{u}^* + v^* \\ &\geq -\mathbf{r}_{\min}^T\mathbf{u}^* + c_{\text{opt}}(\mathbf{0}) \end{aligned}$$

- This indicates $c_{\text{opt}}(\mathbf{0}) - c_{\text{opt}}(\mathbf{r}_{\min}) \leq \mathbf{r}_{\min}^T\mathbf{u}^*$. In other words, the min fairness constraint causes a maximal efficiency loss of $\mathbf{r}_{\min}^T\mathbf{u}^*$

Thank You

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