A Unified Analysis of Max-Min Weighted SINR for MIMO Downlink System

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***Abstract*—This paper studies the max-min weighted signal-to-interference-plus-noise ratio (SINR) problem in the multiple-input-multiple-output (MIMO) downlink, where multiple users are weighted according to priority and are subject to a weighted-sum-power constraint. First, we study the multiple-input-single-output (MISO) and single-input-multiple-output (SIMO) problems using nonlinear Perron-Frobenius theory. As a by-product, we solve the open problem of convergence for a previously proposed MISO algorithm by Wiesel, Eldar, and Shamai in 2006. Furthermore, we unify our analysis with respect to the previous alternate optimization algorithm proposed by Tan, Chiang, and Srikant in 2009, by showing that our MISO result can, in fact, be derived from their algorithm. Next, we combine our MISO and SIMO results into an algorithm for the MIMO problem. We show that our proposed algorithm is optimal when the channels are rank-one, or when the network is operating in the low signal-to-noise ratio (SNR) region. Finally, we prove the parametric continuity of the MIMO problem in the power constraint, and we use this insight to propose a heuristic initialization strategy for improving the performance of our (generally) sub-optimal MIMO algorithm. The proposed initialization strategy exhibits improved performance over random initialization.**

***Index Terms*—Beamforming, uplink-downlink duality, multiple-input-multiple-output (MIMO).**

I. Introduction

In bandwidth-limited wireless networks, the spatial diversity of networks arising from the use of antenna arrays at both the transmitter and the receivers can be exploited to mitigate the interference between multiple users and increase the overall performance. Beamforming is a spatial diversity technique used at both the transmitter and the receiver to increase the signal-to-interference-plus-noise ratio (SINR) of each user when multiple users share a common bandwidth. We consider the joint optimization of transmit power, linear transmit beamformers at the base station, and linear receive beamformers at the mobile users in a wireless multiple-input-multiple-output (MIMO) downlink system.

Two important performance metrics in a wireless network are the minimum weighted SINR and the total transmit power. Clearly, the total transmit power is directly related to the resource cost of the network. The minimum weighted SINR, at first glance, appears to characterize only the worst performing user relative to a set of user priorities. However, maximizing the minimum weighted SINR actually amounts to equalizing the weighted SINR performance of all users (where the weights reflect the user priorities); and in the case when all users are assigned the same priority, amounts to equalizing the SINR performance. Hence, maximizing the minimum weighted SINR is a strategy for enforcing the desired level of fairness in the network.

Thus, two optimization problems of interest are the *max-min weighted SINR problem* subject to a given total power constraint and the *total power minimization problem* subject to given minimum SINR constraints. Whichever optimization goal is chosen depends on the priorities of the system designer. Moreover, both optimization goals are closely related. It was proved that for multiple-input-single-output (MISO) downlink systems, these two optimization problems are inverse problems [1]. This means that, suppose we know the optimal minimum weighted SINR in the max-min weighted SINR problem, then by plugging in this optimal minimum weighted SINR as the weighted SINR constraint in the total power minimization problem, the optimal total transmit power will be equal to the total transmit power constraint in the max-min weighted SINR problem.

Despite the close relationship between the two problems, there are significantly more results on the total power minimization problem. Two of the earliest and most significant results were the distributed power control (DPC) algorithm [2] and the standard interference function framework for analyzing iterative power control systems [3]. The DPC algorithm, together with uplink-downlink duality [4], was extensively applied in alternate optimization approaches to derive optimal algorithms for the MISO problem [5]–[7], and suboptimal algorithms for the MIMO problem [8], [9]. In a completely different approach, the authors in [1] showed that the standard interference function framework can also be used to solve the MISO problem. By reformulating the MISO problem as a second-order cone program (SOCP), the authors derived a fixed-point equation in the dual uplink power, and applied the standard interference function properties to prove the

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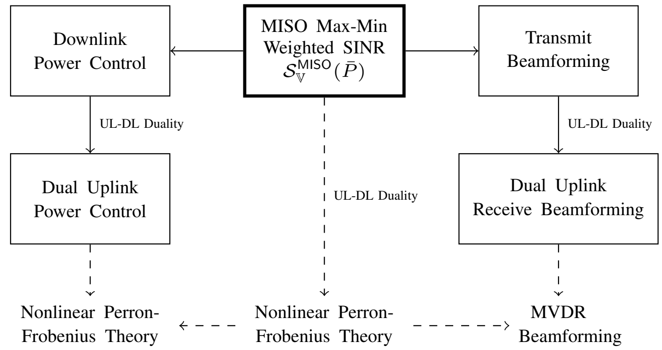
convergence of the fixed-point iteration. The authors also derived a similar fixed-point iteration for the MISO max-min weighted SINR problem without a convergence proof.

To our knowledge, there are no successful attempts to provide a complete analytical and algorithmic analysis to the max-min weighted SINR problem. The earliest strategy proposed to solve the max-min weighted SINR power control problem was the extended coupling matrix approach [10]. Unfortunately, this approach, when extended to the MISO/MIMO problems via alternate optimization [8], [11]–[15], required a centralized power update involving an eigenvector computation. Recently, the authors in [16], [17] derived a different algorithm for the max-min weighted SINR power control problem. The key insight was that the problem is equivalent to solving an affine eigenvalue equation, in which the optimal value is associated with the eigenvalue and the optimal power is given by the eigenvector. By applying nonlinear Perron-Frobenius theory, a DPC-like power control algorithm was derived which avoided the eigenvector computation and normalization necessary at each iterative step in the extended-coupling matrix approach.

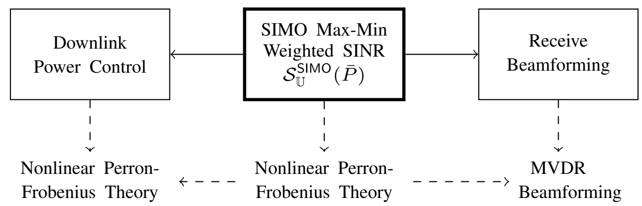
In this paper, we extend the recent results in [16] to the MIMO max-min weighted SINR problem with linear beamforming. First, we analyze the MISO and single-input-multiple-output (SIMO) problems using nonlinear Perron-Frobenius theory. Specifically, we derive a fixed-point equation for each problem and apply nonlinear Perron-Frobenius theory to prove the convergence of the fixed-point iterations. As a by-product, we solve the open problem of convergence for the previously proposed MISO algorithm in [1]. We also show that our fixed-point iteration can, in fact, be decoupled and interpreted as an alternate optimization of power, via nonlinear Perron-Frobenius theory [16], and beamformers, via minimum-variance distortionless response (MVDR) beamforming (cf. Fig. 1). Finally, we apply nonnegative matrix theory to gain further insight into the structure of the optimal solutions of both problems.

Next, we combine our MISO and SIMO results into an algorithm for the MIMO problem via an alternate optimization procedure. The nonconvexity of the MIMO problem imply that alternate optimization algorithms are in general suboptimal. We further our analysis by proving that our proposed algorithm is optimal when the channels are rank-one, or when the network is operating in the low signal-to-noise ratio (SNR) region. Based on the latter property, and by studying the continuity of the optimal solutions in the power constraint, we propose a heuristic strategy for initializing our alternate optimization algorithm. We call this the *parametric search initialization algorithm*. Numerical examples show that our proposed initialization strategy performs significantly better than random initialization. Finally, we briefly describe how a similar strategy can be employed for the MIMO total power minimization problem.

This paper is organized as follows. We begin in Section II by introducing the problem formulation. Next, in Section III, we analyze the MISO and SIMO problems using nonlinear Perron-Frobenius theory. In Section IV, we combine our MISO and SIMO results into an algorithm for the MIMO problem. We also prove the parametric continuity of the optimal solutions in the power constraint. Then, we exploit this property to propose the *parametric search initialization algorithm*, which improves the performance of the original MIMO algorithm. In Section V, we evaluate the performance of the proposed algorithms. Finally, Section VI concludes the paper.



(a) MISO Max-Min Weighted SINR



(b) SIMO Max-Min Weighted SINR

Fig. 1. These two diagrams illustrate how nonlinear Perron-Frobenius theory unifies key approaches for solving the MISO and SIMO max-min weighted SINR problems. For the MISO problem, uplink-downlink duality (UL-DL Duality) is used to convert the problems into corresponding problems in the dual uplink variables.

The following notations are used.

- Boldface upper-case letters denote matrices, boldface lowercase letters denote column vectors, and italics denote scalars.
- For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^L$, $\mathbf{u} \leq \mathbf{v}$ denotes $u_i \leq v_i$ for $1 \leq i \leq L$, $\mathbf{u} < \mathbf{v}$ denotes $u_i < v_i$ for $1 \leq i \leq L$, and $\mathbf{u} \lneq \mathbf{v}$ denotes $\mathbf{u} \leq \mathbf{v}$ and $\mathbf{u} \neq \mathbf{v}$.
- $\rho(\mathbf{F})$ denotes the Perron-Frobenius eigenvalue of a nonnegative matrix $\mathbf{F}$.
- $\mathbf{x}(\mathbf{F})$ and $\mathbf{y}(\mathbf{F})$ denote the Perron (right) and left eigenvectors of $\mathbf{F}$ associated with $\rho(\mathbf{F})$.
- $\mathrm{diag}(\mathbf{v})$ denotes the diagonal matrix formed by the components of the vector $\mathbf{v}$.
- $(\cdot)^{\mathsf{T}}$ and $(\cdot)^{\dagger}$ denote transpose and complex conjugate transpose respectively.
- $\mathbb{R}_+$ denotes the set of nonnegative real numbers, $\mathbb{R}_{++}$ denotes the set of positive real numbers, and $\mathbb{C}$ denotes the set of complex numbers.
- $\|\cdot\|$ denotes the ℓ_2 norm.

II. Problem Formulation

We consider a MIMO downlink system where L independent data streams are transmitted over a common frequency band. The transmitter is equipped with N antennas and the receiver of the lth stream, denoted by r_l, is equipped with N_l

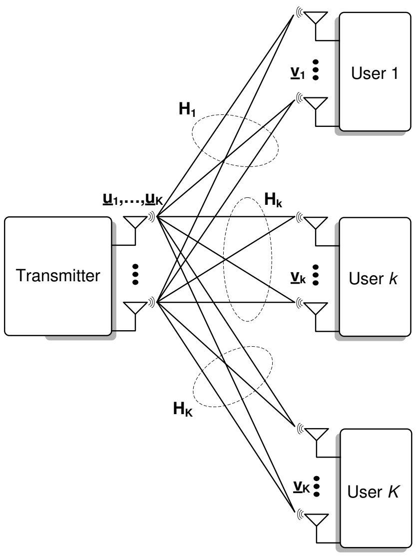


Fig. 2. Multiuser MIMO with one data stream per user.

antennas. The downlink channel can be modeled as a vector Gaussian broadcast channel given by

$$\mathbf{y}_l = \mathbf{H}_l \mathbf{x} + \mathbf{z}_l, \quad l = 1, \ldots, L$$

where $\mathbf{y}_l \in \mathbb{C}^{N_l \times 1}$ is the received signal vector at r_l, $\mathbf{H}_l \in \mathbb{C}^{N_l \times N}$ is the channel matrix between the transmitter and r_l, $\mathbf{x} \in \mathbb{C}^{N \times 1}$ is the transmitted signal vector, and $\mathbf{z}_l \in\sim \mathcal{CN}(0, n_l \mathbf{I})$ is the circularly symmetric Gaussian noise vector with covariance $n_l \mathbf{I}$ at r_l for some $n_l \in \mathbb{R}_{++}$.

We assume linear transmit and receive beamforming. The transmitted signal vector can be written as $\mathbf{x} = \sum_l \mathbf{x}_l = d_l \sqrt{p_l} \mathbf{u}_l$, where $\mathbf{x}_l \in \mathbb{C}^{N \times 1}$ is the signal intended for the lth stream, $\mathbf{u}_l \in \mathbb{C}^{N \times 1}$ is the normalized transmit beamformer, and d_l and $p_l \in \mathbb{R}_{++}$ are respectively the information signal and transmit power for that stream. The lth stream is decoded using a normalized receive beamformer $\mathbf{v}_l \in \mathbb{C}^{N_l \times 1}$. Define the power vector $\mathbf{p} = [p_1, \ldots, p_L]^\mathsf{T}$, the tuples of beamforming matrices $\mathbb{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_L)$ and $\mathbb{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_L)$, the noise vector $\mathbf{n} = [n_1, \ldots, n_L]^\mathsf{T}$, and the matrix $\mathbf{G} \in \mathbb{R}_{++}^{L \times L}$ such that the entry $G_{li} = |\mathbf{v}_l^\dagger \mathbf{H}_l \mathbf{u}_i|^2$. We assume that the receive beamformers are chosen such that all the effective links are coupled so $G_{li} > 0$ for all l and i. The SINR of the lth stream in the downlink can be expressed as

$$\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}, \mathbb{U}, \mathbf{v}_l) := \frac{p_l G_{ll}}{\sum_{i \neq l} p_i G_{li} + n_l}. \tag{1}$$

We are interested in the max-min weighted SINR problem in the downlink given by

$$\mathcal{S}(\bar{P}) := \begin{cases} \displaystyle\max_{\mathbf{p}, \mathbb{U}, \mathbb{V}} & \displaystyle\min_l \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}, \mathbb{U}, \mathbf{v}_l)}{\beta_l} \\ \text{subject to} & \mathbf{w}^\mathsf{T} \mathbf{p} \le \bar{P}, \\ & \mathbf{p} > \mathbf{0}, \\ & \|\mathbf{u}_l\| = 1, \quad l = 1, \ldots, L, \\ & \|\mathbf{v}_l\| = 1, \quad l = 1, \ldots, L \end{cases}$$

where $\bar{P} \in \mathbb{R}_{++}$ is the given power constraint, $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_L]^\mathsf{T}$ is the priority vector such that $\beta_l \in \mathbb{R}_{++}$ is the priority assigned by the network to the lth stream, and $\mathbf{w} = [w_1, \ldots, w_L]^\mathsf{T}$ is the weight vector such that $w_l \in \mathbb{R}_{++}$ is the weight associated with p_l in the power constraint. The restriction that the weights be positive is for technical purposes and is not a stringent limitation on the model. In cases where a certain stream must be excluded from a power constraint, the weight on that stream in the power constraint can simply be chosen to be extremely small. Since $G_{li} > 0$ for all l and i, we have that at the optimal solution to $\mathcal{S}(\bar{P})$, all the weighted SINR's are equal, that is, $\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}, \mathbb{U}, \mathbb{V})/\beta_l$ are equal for all l. Moreover, the power constraint must be active.

For ease of notation, we define the (cross channel interference) matrix $\mathbf{F} \in \mathbb{R}_+^{L \times L}$ and the normalized priority vector $\hat{\boldsymbol{\beta}} \in \mathbb{R}_{++}^{L \times 1}$ with the following entries:

$$F_{li} := \begin{cases} 0, & \text{if} \quad l = i \\ G_{li}, & \text{if} \quad l \neq i \end{cases}$$

$$\hat{\boldsymbol{\beta}} := \left(\frac{\beta_1}{G_{11}}, \ldots, \frac{\beta_L}{G_{LL}} \right)^\mathsf{T}.$$

Hence, $\mathcal{S}(\bar{P})$ can be rewritten as

$$\mathcal{S}(\bar{P}) = \begin{cases} \displaystyle\max_{\mathbf{p}, \mathbb{U}, \mathbb{V}} & \displaystyle\min_l \frac{p_l}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p} + \mathbf{n}))_l} \\ \text{subject to} & \mathbf{w}^\mathsf{T} \mathbf{p} \le \bar{P}, \\ & \mathbf{p} > \mathbf{0}, \\ & \|\mathbf{u}_l\| = 1, \quad l = 1, \ldots, L, \\ & \|\mathbf{v}_l\| = 1, \quad l = 1, \ldots, L. \end{cases}$$

III. Analysis by nonlinear Perron-Frobenius Theory

In this section, we apply nonlinear Perron-Frobenius theory to analyze the max-min weighted SINR problem in the MISO and SIMO scenarios. First, in Section III-A, we apply nonlinear Perron-Frobenius theory to propose an algorithm for solving the MISO problem. Then, in Section III-B, we show that nonlinear Perron-Frobenius theory can resolve the open problem of convergence of the MISO max-min SINR algorithm proposed in [1]. Next, in Section III-C, we show that our new MISO result can, in fact, be derived from the alternate optimization algorithm proposed in [16]. In Section III-D, we exploit the connection between Perron-Frobenius theory and the Friedland-Karlin inequalities in [18] to draw further insights on the optimal solutions. Finally, in Section III-E, we present a similar analysis of the SIMO problem.

A. MISO Optimization

In this section, we fix the receive beamformers $\mathbb{V}$ in $\mathcal{S}(\bar{P})$ and consider the MISO max-min weighted SINR problem given by

$$\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P}) := \begin{cases} \displaystyle\max_{\mathbf{p}, \mathbb{U}} & \displaystyle\min_l \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}, \mathbb{U})}{\beta_l} \\ \text{subject to} & \mathbf{w}^\mathsf{T} \mathbf{p} \le \bar{P}, \\ & \mathbf{p} > \mathbf{0}, \\ & \|\mathbf{u}_l\| = 1, \quad l = 1, \ldots, L. \end{cases}$$

For notational brevity, we omitted the dependence of $\mathsf{SINR}_l^{\mathsf{DL}}$ on $\mathbf{v}_l$. First, consider the following problem obtained by fixing the transmit beamformers $\mathbb{U}$ in $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$:

$$\begin{array}{cc} \max\limits_{\mathbf{p}} & \min\limits_{l} \frac{p_l}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}+\mathbf{n}))_l} \\ \text{subject to} & \mathbf{w}^{\mathsf{T}}\mathbf{p} \le \bar{P}, \quad \mathbf{p} > \mathbf{0}. \end{array} \tag{2}$$

It is well-known that (2) can be reformulated as a geometric program by introducing an auxiliary variable τ. Making the logarithmic change of variables $\tilde{\tau} = \log \tau$ and $\tilde{p}_l = \log p_l$, we arrive at the following equivalent convex problem:

$$\begin{array}{cl} \min\limits_{\tilde{\tau},\tilde{\mathbf{p}}} & -\tilde{\tau} \\ \text{subject to} & \log\left(\frac{e^{\tilde{\tau}}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}e^{\tilde{\mathbf{p}}}+\mathbf{n}))_l}{(e^{\tilde{\mathbf{p}}})_l}\right) \le 0, \quad l = 1,\ldots,L, \\ & \log\left(\frac{1}{\bar{P}}\mathbf{w}^{\mathsf{T}}e^{\tilde{\mathbf{p}}}\right) \le 0. \end{array} \tag{3}$$

The Lagrangian associated with (3) is

$$\mathcal{L}(\tilde{\tau},\tilde{\mathbf{p}},\boldsymbol{\lambda},\mu) = -\tilde{\tau} + \sum_l \lambda_l \log\left(\frac{e^{\tilde{\tau}}(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}e^{\tilde{\mathbf{p}}}+\mathbf{n}))_l}{(e^{\tilde{\mathbf{p}}})_l}\right) + \mu \log\left(\frac{1}{\bar{P}}\mathbf{w}^{\mathsf{T}}e^{\tilde{\mathbf{p}}}\right)$$

where λ_l and μ are the nonnegative Lagrange dual variables.

It is easy to check that the convex problem given by (3) satisfies Slater's condition. Hence, the Karush-Kuhn Tucker (KKT) conditions are necessary and sufficient conditions for optimality of (3). Recall that, at the optimal solution to (3), all the inequality constraints must be active. Hence, we can replace the inequality constraints (in the feasibility conditions) with equality, and drop the complementary slackness conditions. Making a change of variables back into τ and $\mathbf{p}$, we arrive at the following necessary and sufficient conditions for optimality of (2):

$$\tau_* = \frac{p_{*l}}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}\mathbf{p}_* + \mathbf{n}))_l}, \quad l = 1,\ldots,L \tag{4}$$

$$\mathbf{w}^{\mathsf{T}}\mathbf{p}_* = \bar{P} \tag{5}$$

$$\tau_* = \frac{q_{*l}}{(\mathrm{diag}(\hat{\boldsymbol{\beta}})(\mathbf{F}^{\mathsf{T}}\mathbf{q}_* + \mathbf{w}))_l}, \quad l = 1,\ldots,L \tag{6}$$

$$\mathbf{1}^{\mathsf{T}}\boldsymbol{\lambda}_* = 1 \tag{7}$$

$$\mathbf{q}_* = \frac{\tau_*\bar{P}}{\mu_*}\cdot\left(\frac{\lambda_{*1}\hat{\beta}_1}{p_{*1}},\ldots,\frac{\lambda_{*L}\hat{\beta}_L}{p_{*L}}\right)^{\mathsf{T}} \tag{8}$$

$$\lambda_{*l} > 0, \quad l = 1,\ldots,L \tag{9}$$

$$\mu_* > 0 \tag{10}$$

where τ_*, $\mathbf{p}_*$ and $\boldsymbol{\lambda}_*$, μ_* are the optimal primal and dual variables respectively. Here, (6) was obtained from $\frac{\partial \mathcal{L}}{\partial p_l}$ and (9)-(10) follows from the fact that λ_{*l} and μ_* must be strictly positive in order for $\tilde{p}_{*l}$ and $\mathcal{L}(\tilde{\tau}_*,\tilde{\mathbf{p}}_*,\boldsymbol{\lambda}_*,\mu_*)$ to be bounded from below. Note that the auxiliary variable $\mathbf{q}_*$ is strictly positive and finite.

From (6), we conclude that $\mathbf{q}_*$ is the optimal dual uplink power and $\mathbf{w}$ is the noise vector in the dual uplink network [4], [9], [19]. To obtain the equivalent power constraint in the dual uplink network, we first rewrite (6) in vector form as

$$\mathbf{F}^{\mathsf{T}}\mathbf{q}_* + \mathbf{w} = \frac{1}{\tau_*}\mathrm{diag}(\hat{\boldsymbol{\beta}})^{-1}\mathbf{q}_*.$$

Then, taking the inner product of this equation with $\mathbf{p}_*$ and substituting for $\mathbf{F}\mathbf{p}_*$ and $\mathbf{w}^{\mathsf{T}}\mathbf{p}_*$ from (4) and (5), we have

$$\mathbf{n}^{\mathsf{T}}\mathbf{q}_* = \bar{P}. \tag{11}$$

Note that (4)-(8) are also necessary conditions for the optimal solution of $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$. To obtain necessary conditions for the optimal transmit beamformers $\mathbf{u}_{*l}$, first apply the constraint $\|\mathbf{u}_{*l}\| = 1$ for all l and rewrite (6) as

$$q_{*l}\mathbf{u}_{*l}^{\dagger}\mathbf{H}_l^{\dagger}\mathbf{v}_l\mathbf{v}_l^{\dagger}\mathbf{H}_l\mathbf{u}_{*l} = \beta_l\tau_*\mathbf{u}_{*l}^{\dagger}\left(\sum_{i\neq l} q_{*i}\mathbf{H}_i^{\dagger}\mathbf{v}_i\mathbf{v}_i^{\dagger}\mathbf{H}_i + w_l\mathbf{I}\right)\mathbf{u}_{*l}, \quad l = 1,\ldots,L. \tag{12}$$

Let the mapping $\mathbf{M}_l^{\mathsf{UL}} : \mathbb{R}_+^{L\times 1} \to \mathbb{C}^{L\times L}$ be defined by:

$$\mathbf{M}_l^{\mathsf{UL}}(\mathbf{q}_*) := \sum_{i\neq l} q_{*i}\mathbf{H}_i^{\dagger}\mathbf{v}_i\mathbf{v}_i^{\dagger}\mathbf{H}_i + w_l\mathbf{I}, \quad l = 1,\ldots,L.$$

Since $w_l > 0$, we have that $\mathbf{M}_l^{\mathsf{UL}}(\mathbf{q}_*)$ is positive definite for any $\mathbf{q}_*$ and hence it is invertible. Define the normalized minimum variance distortionless response (MVDR) beamformers in the dual uplink network [20]:

$$\mathbf{u}_l^{\mathsf{MVDR}}(\mathbf{q}_*) = \frac{\mathbf{M}_l^{\mathsf{UL}}(\mathbf{q}_*)^{-1}\mathbf{H}_l^{\dagger}\mathbf{v}_l}{\|\mathbf{M}_l^{\mathsf{UL}}(\mathbf{q}_*)^{-1}\mathbf{H}_l^{\dagger}\mathbf{v}_l\|}, \quad l = 1,\ldots,L.$$

By applying the fact that the optimal transmit beamformers in the dual uplink network are the normalized MVDR beamformers, i.e., $\mathbf{u}_{*l} = \mathbf{u}_l^{\mathsf{MVDR}}(\mathbf{q}_*)$ for $l = 1,\ldots,L$, we eliminate $\mathbf{u}_{*l}$ from (12) and obtain the following fixed-point equation:

$$\frac{1}{\tau_*}\mathbf{q}_* = f^{\mathsf{UL}}(\mathbf{q}_*) \tag{13}$$

where the mapping $f : \mathbb{R}_+^{L\times 1} \to \mathbb{R}_+^{L\times 1}$ is defined by:

$$f_l^{\mathsf{UL}}(\mathbf{q}_*) := \frac{\beta_l}{\mathbf{v}^{\dagger}\mathbf{H}_l\mathbf{M}_l^{\mathsf{UL}}(\mathbf{q}_*)^{-1}\mathbf{H}_l^{\dagger}\mathbf{v}_l}, \quad l = 1,\ldots,L.$$

We now state the key result for solving $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$.

Theorem 1: Define the norm $\|\cdot\|_{\mathsf{UL}}$ on $\mathbb{R}^{L\times 1}$ as follows: $\|\mathbf{q}\|_{\mathsf{UL}} = (1/\bar{P})\sum_l n_l|q_l|$. The conditional eigenvalue problem given by $(1/\tau)q_l = f_l^{\mathsf{UL}}(\mathbf{q})$ and $\|\mathbf{q}\|_{\mathsf{UL}} = 1$ has a unique solution $\mathbf{q} = \mathbf{q}_*$, $\tau = \tau_*$, and $\mathbf{q}_* > \mathbf{0}$, $\tau_* > 0$. Moreover, the normalized fixed-point iteration $\tilde{f}^{\mathsf{UL}}(\mathbf{q}^{(n+1)}) = (1/\|f^{\mathsf{UL}}(\mathbf{q}^{(n)})\|_{\mathsf{UL}})f^{\mathsf{UL}}(\mathbf{q}^{(n)})$ converges to $\mathbf{q}_*$ geometrically fast.

Proof: Refer to Appendix A.

Theorem 1 implies that the optimal dual uplink power $\mathbf{q}_*$ for $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$ is unique. With the optimal dual uplink power, $\mathbb{U}_*$ can be computed using the normalized MVDR beamformers, and $\mathbf{p}_*$ can be computed using the max-min weighted SINR power control algorithm [16], [19]. In fact, since the fixed-point iteration for $\mathbf{q}$ does not depend on $\mathbf{p}$ and $\mathbb{U}$ (cf. steps 1 and 2 of Algorithm 1 below), the latter two quantities can be updated in parallel with $\mathbf{q}$. The following algorithm converges to the optimal solution of $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$. It can also be shown that the algorithm converges with monotonically increasing minimum weighted SINR.

Algorithm 1: MISO max-min weighted SINR

- Initialize: arbitrary $\mathbf{q}^{(0)} \in \mathbb{R}_{++}^{L\times 1}$, $\mathbf{p}^{(0)} \in \mathbb{R}_{++}^{L\times 1}$, and $\mathbf{u}_l \in \mathbb{C}^{N\times 1}$ for $l = 1, \ldots, L$, such that $\mathbf{w}^\mathsf{T}\mathbf{p}^{(0)} \leq \bar{P}$, $\mathbf{n}^\mathsf{T}\mathbf{q}^{(0)} \leq \bar{P}$, and $\|\mathbf{u}_l^{(0)}\| = 1$ for all l.

1) Compute dual auxiliary variables:
$$\tilde{q}_l = f_l^{\mathsf{UL}}(\mathbf{q}^{(n)}), \quad l = 1, \ldots, L.$$
2) Update dual uplink powers:
$$\mathbf{q}^{(n+1)} = \frac{\bar{P}}{\mathbf{n}^\mathsf{T}\tilde{\mathbf{q}}} \cdot \tilde{\mathbf{q}}.$$
3) Compute auxiliary variables:
$$\tilde{p}_l = \left(\frac{\beta_l}{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}^{(n)}, \mathbb{U}^{(n)})}\right) p_l^{(n)}, \quad l = 1, \ldots, L.$$
4) Update downlink powers:
$$\mathbf{p}^{(n+1)} = \frac{\bar{P}}{\mathbf{w}^\mathsf{T}\tilde{\mathbf{p}}} \cdot \tilde{\mathbf{p}}.$$
5) Compute normalized MVDR transmit beamformers:
$$\mathbf{u}_l^{(n+1)} = \mathbf{u}_l^{\mathsf{MVDR}}(\mathbf{q}^{(n+1)}), \quad l = 1, \ldots, L.$$

B. Resolving a Previously Open Problem on Convergence

Next, we briefly address an open problem regarding the convergence of an algorithm proposed for $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$ under the special case when all data streams have equal priority ($\boldsymbol{\beta} = \mathbf{1}$), the network has a total power constraint ($\mathbf{w} = \mathbf{1}$), and all users suffer from the same noise covariance ($\mathbf{n} = \mathbf{1}$). Under this conditions, $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$ can be rewritten as:

$$\begin{array}{ll} \max\limits_{\mathbf{p},\mathbb{U}} & \min\limits_{l} \frac{p_l|\mathbf{v}_l^\dagger \mathbf{H}_l \mathbf{u}_l|^2}{\sum_{i\neq l} p_i |\mathbf{v}_l^\dagger \mathbf{H}_l \mathbf{u}_i|^2 + 1} \\ \text{subject to} & \mathbf{1}^\mathsf{T}\mathbf{p} \leq \bar{P}, \\ & \mathbf{p} > \mathbf{0}, \\ & \|\mathbf{u}_l\| = 1, \quad l = 1, \ldots, L. \end{array} \tag{14}$$

In [1], it was proved that if $\mathbf{q}_* > \mathbf{0}$ and $\tau_* > 0$ satisfy
$$\frac{1+\tau_*}{\tau_*} q_{*l} = \frac{1}{\mathbf{v}_l^\dagger \mathbf{H}_l \mathbf{M}^{\mathsf{UL}}(\mathbf{q}_*)^{-1}\mathbf{H}_l^\dagger \mathbf{v}_l}, \quad l = 1, \ldots, L \tag{15}$$
where the mapping $\mathbf{M}^{\mathsf{UL}} : \mathbb{R}_+^{L\times 1} \to \mathbb{C}^{L\times L}$ is defined by:
$$\mathbf{M}^{\mathsf{UL}}(\mathbf{q}_*) := \sum_i q_{*i} \mathbf{H}_i^\dagger \mathbf{v}_i \mathbf{v}_i^\dagger \mathbf{H}_i + \mathbf{I}$$
then $\mathbf{q}_*$ is the optimal dual uplink power and τ_* is the optimal value of (14). Moreover, the authors proposed the following fixed-point iteration to solve for the optimal dual uplink powers:
$$\tilde{q}_l = \frac{1}{\mathbf{v}_l^\dagger \mathbf{H}_l \mathbf{M}^{\mathsf{UL}}(\mathbf{q}^{(n)})^{-1}\mathbf{H}_l^\dagger \mathbf{v}_l}, \quad l = 1, \ldots, L \tag{16}$$
$$\mathbf{q}^{(n+1)} = \frac{\bar{P}}{\mathbf{1}^\mathsf{T}\tilde{\mathbf{q}}} \cdot \tilde{\mathbf{q}}. \tag{17}$$

By comparing (15) with (13), it is easy to modify the proof of Theorem 1 to prove the geometric convergence of the fixed-point iteration (16)-(17). The complete algorithm for solving (14) is obtained by replacing steps 1-2 of Algorithm 1 with (16)-(17). This algorithm has the advantage that the denominator of (16) can be computed by estimating the covariance matrix at the base station in the uplink direction. However, this algorithm is limited to the special case when all data streams have the same priority.

C. Connection with Alternate Optimization

In this section, we provide more insight into Theorem 1 by analyzing the relationship between the fixed-point iteration and the alternate optimization algorithm proposed in [16] for $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$ for the special case when $\mathbf{w} = \mathbf{n} = \mathbf{1}$. There are two differences between Algorithm 1 and the algorithm in [16]. First, the algorithm in [16] only treated the specific case of a total power constraint while Algorithm 1 handles a general weighted-sum power constraint. Second, the algorithm in [16] was derived by applying nonlinear Perron-Frobenius theory to the downlink power control problem and then exploiting uplink-downlink duality to update the transmit beamformers. On the other hand, Algorithm 1 was derived by applying nonlinear Perron-Frobenius theory to the dual uplink fixed-point equation (13) and then exploiting uplink-downlink duality to update the downlink power and transmit beamformers. Nevertheless, we show that for the case when $\mathbf{w} = \mathbf{n} = \mathbf{1}$, Theorem 1 can, in fact, be derived from the results in [16].

Given the dual uplink power $\mathbf{q}^{(n)}$ at some time step, consider using MVDR beamforming to optimize $\mathbb{U}^{(n)}$. The uplink SINR of the lth stream after MVDR beamforming is given by
$$\begin{aligned} &\mathsf{SINR}_l^{\mathsf{UL}}(\mathbf{q}^{(n)}, \mathbf{u}_l^{\mathsf{MVDR}}(\mathbf{q}^{(n)}), \mathbb{V}) \\ &= \max_{\|\mathbf{u}_l\|=1} \mathsf{SINR}_l^{\mathsf{UL}}(\mathbf{q}^{(n)}, \mathbf{u}_l, \mathbb{V}) \\ &= q_l^{(n)} \mathbf{v}_l^\dagger \mathbf{H}_l \mathbf{M}_l^{\mathsf{UL}}(\mathbf{q}^{(n)})^{-1} \mathbf{H}_l^\dagger \mathbf{v}_l \end{aligned} \tag{18}$$
where
$$\mathsf{SINR}_l^{\mathsf{UL}}(\mathbf{q}, \mathbf{u}_l, \mathbb{V}) := \frac{q_l G_{ll}}{\sum_{i\neq l} q_i G_{il} + w_l}$$
is the SINR of the lth stream in the dual uplink network. Using (18), we can decouple the fixed-point iteration in Theorem 1 into the following steps:
$$\mathbf{u}_l^{(n+1)} = \mathbf{u}_l^{\mathsf{MVDR}}(\mathbf{q}^{(n)}), \quad l = 1, \ldots, L \tag{19}$$
$$\tilde{q}_l = \left(\frac{\beta_l}{\mathsf{SINR}_l^{\mathsf{UL}}(\mathbf{q}^{(n)}, \mathbf{u}_l^{(n+1)})}\right) q_l^{(n)}, \quad l = 1, \ldots, L \tag{20}$$
$$\mathbf{q}^{(n+1)} = \frac{\bar{P}}{\mathbf{n}^\mathsf{T}\tilde{\mathbf{q}}} \cdot \tilde{\mathbf{q}}. \tag{21}$$

For notational brevity, we omitted the dependence of $\mathsf{SINR}_l^{\mathsf{UL}}$ on $\mathbb{V}$.

Hence, the fixed-point iteration can be viewed as an alternate optimization of $\mathbf{q}$ and $\mathbb{U}$. Observe that when $\mathbf{w} = \mathbf{n} = \mathbf{1}$, (20)-(21) is exactly the max-min weighted SINR power control algorithm (applied in the uplink) from [16], which was derived by applying nonlinear Perron-Frobenius theory to the power control problem. Furthermore, since (19)-(21) do not depend on $\mathbf{p}$, the power control algorithm from [16] can also be used to update $\mathbf{p}$ in parallel with (19)-(21). The complete algorithm is exactly the algorithm proposed in [16] for the MISO problem. Hence, the fixed-point iteration in Theorem 1 can also be derived from the algorithm proposed in [16] by combining the updates for $\mathbf{q}$ and $\mathbb{U}$ in the algorithm in [16].

We illustrate the above analysis in Fig. 1a. Each block in the left and right portions represent a subproblem of $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$. The left portion shows nonlinear Perron-Frobenius theory being applied to the downlink power control problem via the dual uplink power control problem. The right portion shows MVDR beamforming being used to solve the transmit beamforming problem by using uplink-downlink duality to convert into a receive beamforming problem. The center portion shows nonlinear Perron-Frobenius theory being applied to the MISO problem directly via (13). This direct approach gives the same algorithm as that obtained by studying the two subproblems of $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$ separately and then alternately optimizing $\mathbf{p}$ and $\mathbb{U}$.

D. Characterization via Friedland-Karlin Spectral Radius Minimax Theorem

In this section, we apply nonnegative matrix theory to gain further insight into the solution of $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$. Define the nonnegative vector $\mathbf{b}^{\mathsf{UL}} \in \mathbb{R}_+^{L\times 1}$ as:

$$b_l^{\mathsf{UL}}(\mathbf{q}) := \frac{1}{\mathbf{v}_l^\dagger \mathbf{H}_l \mathbf{M}_l^{\mathsf{UL}}(\mathbf{q})^{-1}\mathbf{H}_l^\dagger \mathbf{v}_l}, \quad l = 1, \ldots, L.$$

Suppose $\mathbf{q}_*$ is the unique optimal dual uplink power of $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$ and τ_* is the optimal value. Since $\mathbf{q}_*$ and τ_* satisfy (11) and (13), it follows that

$$\frac{1}{\tau_*}\mathbf{q}_* = \left(\mathrm{diag}(\boldsymbol{\beta})\mathbf{b}^{\mathsf{UL}}(\mathbf{q}_*)\cdot \frac{1}{\bar{P}}\mathbf{n}^\top\right)\mathbf{q}_*.$$

Hence, at optimality, $\mathbf{q}_* = \mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})\mathbf{b}^{\mathsf{UL}}(\mathbf{q}_*)\cdot\frac{1}{\bar{P}}\mathbf{n}^\top)$ (normalized with respect to $\|\cdot\|_{\mathsf{UL}}$) and $\frac{1}{\tau_*} = \rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{b}^{\mathsf{UL}}(\mathbf{q}_*)\cdot\frac{1}{\bar{P}}\mathbf{n}^\top)$. Applying the Friedland-Karlin spectral radius minimax characterization (Lemma 2 of [21]) to $\rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{b}^{\mathsf{UL}}(\mathbf{q}_*)\cdot\frac{1}{\bar{P}}\mathbf{n}^\top)$, it can be observed that $(\mathbf{q}_*, \frac{1}{\bar{P}}\mathbf{n})$ are dual pairs with respect to $\|\cdot\|_{\mathsf{UL}}$. Moreover, from (18), it can be seen that $q_{*l}\mathbf{v}_l^\dagger\mathbf{H}_l\mathbf{M}_l^{\mathsf{UL}}(\mathbf{q}_*)^{-1}\mathbf{H}_l^\dagger\mathbf{v}_l$ is the optimal SINR of the lth stream. Hence, it can be deduced that at optimality, the SINR allocation in $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$ is a weighted geometric mean of the individual SINRs, where the weights are the dual uplink powers normalized with respect to $\|\cdot\|_{\mathsf{UL}}$, that is,

$$\prod_l \left(\frac{\mathsf{SINR}_l^{\mathsf{UL}}(\mathbf{q}_*, \mathbf{u}_l)}{\beta_l}\right)^{\frac{n_l q_{*l}}{\bar{P}}} = \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{b}^{\mathsf{UL}}(\mathbf{q}_*)\cdot\frac{1}{\bar{P}}\mathbf{n}^\top)}. \quad (22)$$

E. Analysis of SIMO Max-Min Weighted SINR

It is easy to see that all the analysis in Sections III-A-III-D can be modified for the SIMO max-min weighted SINR problem given by

$$\mathcal{S}_{\mathbb{U}}^{\mathsf{SIMO}}(\bar{P}) := \begin{cases} \max\limits_{\mathbf{p},\mathbb{V}} & \min\limits_{l} \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p},\mathbf{v}_l)}{\beta_l} \\ \text{subject to} & \mathbf{w}^\top\mathbf{p} \le \bar{P}, \\ & \mathbf{p} > \mathbf{0}, \\ & \|\mathbf{v}_l\| = 1, \quad l = 1, \ldots, L \end{cases}$$

which is obtained by fixing the transmit beamformers $\mathbb{U}$ in $\mathcal{S}(\bar{P})$. For notational brevity, we omitted the dependence of $\mathsf{SINR}_l^{\mathsf{DL}}$ on $\mathbb{U}$. We will briefly state the results for $\mathcal{S}_{\mathbb{U}}^{\mathsf{SIMO}}(\bar{P})$. These results, in particular the fixed-point iteration, are also relevant toward understanding Algorithm 2 later in Section IV-A.

Let $\mathbf{p}_*$ and $\mathbb{V}_*$ be the optimal downlink power and receive beamformers of $\mathcal{S}_{\mathbb{U}}^{\mathsf{SIMO}}(\bar{P})$, and let τ_* be the optimal value. Since the receive beamformers are already decoupled in the downlink network, we have that $\mathbf{p}_*$ and τ_* satisfy the fixed-point equation given by

$$\frac{1}{\tau}\mathbf{p}_* = f^{\mathsf{DL}}(\mathbf{p}_*) \quad (23)$$

where the mappings $f^{\mathsf{DL}} : \mathbb{R}_+^{L\times 1} \to \mathbb{R}_+^{L\times 1}$ and $\mathbf{M}_l^{\mathsf{DL}} : \mathbb{R}_+^{L\times 1} \to \mathbb{C}^{L\times L}$ are defined by:

$$f_l^{\mathsf{DL}}(\mathbf{p}_*) := \frac{\beta_l}{\mathbf{u}_l^\dagger\mathbf{H}_l^\dagger\mathbf{M}_l^{\mathsf{DL}}(\mathbf{p}_*)^{-1}\mathbf{H}_l\mathbf{u}_l}$$
$$\mathbf{M}_l^{\mathsf{DL}}(\mathbf{p}_*) := \sum_{i\neq l} p_{*i}\mathbf{H}_l\mathbf{u}_i\mathbf{u}_i^\dagger\mathbf{H}_l^\dagger + n_l\mathbf{I}$$

for $l = 1, \ldots, L$, and $\|\mathbf{p}_*\|_{\mathsf{DL}} = \frac{1}{\bar{P}}\mathbf{w}^\top\mathbf{p}_* = 1$. From (23), we can obtain a fixed-point iteration (independent of $\mathbb{V}$) which converges to $\mathbf{p}_*$. This fixed-point iteration can be decoupled into the following steps:

$$\mathbf{v}_l^{(n+1)} = \mathbf{v}_l^{\mathsf{MVDR}}(\mathbf{p}^{(n)}), \quad l = 1, \ldots, L \quad (24)$$
$$\tilde{p}_l = \left(\frac{\beta_l}{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}^{(n)}, \mathbf{v}_l^{(n+1)})}\right)p_l^{(n)}, \quad l = 1, \ldots, L \quad (25)$$
$$\mathbf{p}^{(n+1)} = \frac{\bar{P}}{\mathbf{w}^\top\tilde{\mathbf{p}}}\cdot\tilde{\mathbf{p}} \quad (26)$$

where

$$\mathbf{v}_l^{\mathsf{MVDR}}(\mathbf{p}) := \frac{\mathbf{M}_l^{\mathsf{DL}}(\mathbf{p})^{-1}\mathbf{H}_l\mathbf{u}_l}{\|\mathbf{M}_l^{\mathsf{DL}}(\mathbf{p})^{-1}\mathbf{H}_l\mathbf{u}_l\|}, \quad l = 1, \ldots, L \quad (27)$$

are defined as the normalized MVDR beamformers in the downlink network. We illustrate in Fig. 1b the correspondence between the algorithms obtained by applying nonlinear Perron-Frobenius theory to the power control problem and that obtained by applying it to the SIMO problem directly.

Define the nonnegative vector $\mathbf{b}^{\mathsf{DL}} \in \mathbb{R}_+^{L\times 1}$ as follows:

$$b_l^{\mathsf{DL}}(\mathbf{p}) := \frac{1}{\mathbf{u}_l^\dagger\mathbf{H}_l^\dagger\mathbf{M}_l^{\mathsf{DL}}(\mathbf{p})^{-1}\mathbf{H}_l\mathbf{u}_l}, \quad l = 1, \ldots, L.$$

Then, we have that

$$\frac{1}{\tau_*}\mathbf{p}_* = \left(\mathrm{diag}(\boldsymbol{\beta})\mathbf{b}^{\mathsf{DL}}(\mathbf{p}_*)\cdot\frac{1}{\bar{P}}\mathbf{w}^\top\right)\mathbf{p}_*$$

and $(\mathbf{p}_*, \frac{1}{\bar{P}}\mathbf{w})$ are dual pairs with respect to $\|\cdot\|_{\mathsf{DL}}$. Moreover, the optimal SINR allocation satisfies

$$\prod_l \left(\frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}_*, \mathbf{v}_{*l})}{\beta_l}\right)^{\frac{w_l p_{*l}}{\bar{P}}} = \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{b}^{\mathsf{DL}}(\mathbf{p}_*)\cdot\frac{1}{\bar{P}}\mathbf{w}^\top)}.$$

We summarize the connections between the fixed-point equations, the Perron-Frobenius theorem, and the Friedland-Karlin spectral radius minimax characterization in Table I. For

TABLE I
THE FIXED POINT EQUATIONS CHARACTERIZED THROUGH THE PERRON-FROBENIUS THEOREM AND THE FRIEDLAND-KARLIN SPECTRAL RADIUS MINIMAX THEOREM. EXCEPT FOR THE FIXED QUANTITIES, ALL OTHER QUANTITIES DENOTE OPTIMAL VALUES OF $\mathcal{S}(\bar{P})$.

Fixed	Nonnegative Matrix	Perron Eigenvalue	Right Eigenvector	Left Eigenvector	Geometric Mean Interpretation	Ref.
$\mathbb{U}$, $\mathbb{V}$	$\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + \frac{1}{\bar{P}}\mathbf{n} \cdot \mathbf{w}^{\mathsf{T}})$	$1/\tau_*$	$\mathbf{p}_*$	$\mathrm{diag}(\boldsymbol{\beta})^{-1}\mathbf{q}_*$	$\tau_* = \prod_l \left(\frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}_*)}{\beta_l}\right)^{\frac{p_{*l}q_{*l}/\beta_l}{\sum_j p_{*j}q_{*j}/\beta_j}}$	Generalized from [18]
$\mathbb{V}$	$\mathrm{diag}(\boldsymbol{\beta})\mathbf{b}^{\mathsf{UL}}(\mathbf{q}_*) \cdot \frac{1}{\bar{P}}\mathbf{n}^{\mathsf{T}}$	$1/\tau_*$	$\mathbf{q}_*$	$\frac{1}{\bar{P}}\mathbf{n}$	$\tau_* = \prod_l \left(\frac{\mathsf{SINR}_l^{\mathsf{UL}}(\mathbf{q}_*, \mathbf{u}_{*l})}{\beta_l}\right)^{\frac{n_l q_{*l}}{\bar{P}}}$	Herein
$\mathbb{U}$	$\mathrm{diag}(\boldsymbol{\beta})\mathbf{b}^{\mathsf{DL}}(\mathbf{p}_*) \cdot \frac{1}{\bar{P}}\mathbf{w}^{\mathsf{T}}$	$1/\tau_*$	$\mathbf{p}_*$	$\frac{1}{\bar{P}}\mathbf{w}$	$\tau_* = \prod_l \left(\frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}_*, \mathbf{v}_{*l})}{\beta_l}\right)^{\frac{w_l p_{*l}}{\bar{P}}}$	

completeness, we also list the Friedland-Karlin spectral radius minimax characterizations for the power control problem, i.e., with $\mathbb{U}$ and $\mathbb{V}$ fixed. Notice that while $(\mathbf{p}_*, \mathrm{diag}(\boldsymbol{\beta})^{-1}\mathbf{q}_*)$ form a dual pair in the power control problem, $(\mathbf{q}_*, \frac{1}{\bar{P}}\mathbf{n})$ and $(\mathbf{p}_*, \frac{1}{\bar{P}}\mathbf{w})$ are the dual pairs in the MISO and SIMO problems respectively.

IV. MIMO OPTIMIZATION

In this section, we study the MIMO max-min weighted SINR problem $\mathcal{S}(\bar{P})$. In general, global optimization of $\mathcal{S}(\bar{P})$ is an open problem because the nonconvexity and the mutual coupling of the transmit and receive beamformers make it difficult to jointly optimize the beamformers. We begin in Section IV-A by proposing a suboptimal algorithm for $\mathcal{S}(\bar{P})$ based on alternately implementing our MISO and SIMO algorithms. In addition, we show that our proposed algorithm is optimal when the channel matrices are rank-one, or when the network is operating in the low SNR region. Next, in Section IV-B, we study the continuity of the optimal solutions to $\mathcal{S}(\bar{P})$ in the parameter $\bar{P}$, and we propose a heuristic approach for tackling the sub-optimality of Algorithm 2. Finally, in Section IV-C, we briefly describe how the results in Sections IV-A and IV-B can be applied to the total power minimization problem.

A. Alternate Optimization Algorithm

In this section, we propose an alternate optimization algorithm for $\mathcal{S}(\bar{P})$ based on our fixed-point iterations for the MISO and SIMO problems in Section III. The main idea is to alternate between fixing $\mathbb{V}$ and $\mathbb{U}$. Recall that, when $\mathbb{U}$ is fixed, $\mathcal{S}(\bar{P})$ becomes $\mathcal{S}_{\mathbb{U}}^{\mathsf{SIMO}}(\bar{P})$. Hence, (24)-(26) can be used to update $\mathbf{p}$ and $\mathbb{V}$. Similarly, when $\mathbb{V}$ is fixed, $\mathcal{S}(\bar{P})$ becomes $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$, so (19)-(21) can be used to update $\mathbf{p}$ and $\mathbb{U}$. Since the fixed-point iteration for $\mathcal{S}_{\mathbb{V}}^{\mathsf{MISO}}(\bar{P})$ is in the dual uplink power $\mathbf{q}$, prior to updating $\mathbf{q}$, uplink-downlink duality must be used to compute $\mathbf{q}$ corresponding to the current $\mathbf{p}$, $\mathbb{U}$, and $\mathbb{V}$. Define the diagonal matrices $\mathbf{D}^{\mathsf{DL}} \in \mathbb{R}_+^{L\times L}$ and $\mathbf{D}^{\mathsf{UL}} \in \mathbb{R}_+^{L\times L}$ of normalized downlink and uplink SINR values respectively, i.e., $D_{ll}^{\mathsf{DL}} := \mathsf{SINR}_l^{DL}(\mathbf{p}, \mathbb{U}, \mathbf{v}_l)/G_{ll}$ and $D_{ll}^{\mathsf{UL}} := \mathsf{SINR}_l^{UL}(\mathbf{p}, \mathbf{u}_l, \mathbb{V})/G_{ll}$. Uplink-downlink duality theory states that if the downlink and dual uplink networks have the same SINR values, i.e., $\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}, \mathbb{U}, \mathbf{v}_l) = \mathsf{SINR}_l^{\mathsf{UL}}(\mathbf{q}, \mathbf{u}_l, \mathbb{V})$ for all l, and $\mathbf{w}^{\mathsf{T}}\mathbf{p} = \mathbf{n}^{\mathsf{T}}\mathbf{q}$, then the following relations hold [4], [9]:

$$\mathbf{p} = \left(\mathbf{I} - \mathbf{D}^{\mathsf{UL}}\mathbf{F}\right)^{-1}\mathbf{D}^{\mathsf{UL}}\mathbf{n} \tag{28}$$

$$\mathbf{q} = \left(\mathbf{I} - \mathbf{D}^{\mathsf{DL}}\mathbf{F}^{\mathsf{T}}\right)^{-1}\mathbf{D}^{\mathsf{DL}}\mathbf{w}. \tag{29}$$

Since the fixed-point iterations converge with monotonically increasing minimum-weighted SINR, we can alternate between one pass of (24)-(26) in the downlink domain, and one pass of (19)-(21) in the dual uplink domain, using (28) and (29) to transition between the downlink and dual uplink updates. We refer to the resultant algorithm as Algorithm 2.

Algorithm 2: MIMO max-min weighted SINR

- Initialize: arbitrary $\mathbf{p}^{(0)} \in \mathbb{R}_{++}^{L\times 1}$, and $\mathbf{u}_l^{(0)} \in \mathbb{C}^{N\times 1}$, $\mathbf{v}_l^{(0)} \in \mathbb{C}^{N_l\times 1}$ for $l = 1, \ldots, L$, such that $\mathbf{w}^{\mathsf{T}}\mathbf{p}^{(0)} \leq \bar{P}$, and $\|\mathbf{u}_l^{(0)}\| = 1$, $\|\mathbf{v}_l^{(0)}\| = 1$ for all l.

1) Update receive beamformers:
$$\mathbf{v}_l^{(n+1)} = \mathbf{v}_l^{\mathsf{MVDR}}(\mathbf{p}^{(n)}), \quad l = 1, \ldots, L.$$
2) Compute auxiliary variables:
$$\tilde{p}_l = \left(\frac{\beta_l}{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}^{(n)}, \mathbb{U}^{(n)}, \mathbf{v}_l^{(n+1)})}\right) p_l^{(n)}$$
for $l = 1, \ldots, L$.
3) Update downlink powers:
$$\mathbf{p}^{(n+1)} = \frac{\bar{P}}{\mathbf{w}^{\mathsf{T}}\tilde{\mathbf{p}}} \cdot \tilde{\mathbf{p}}.$$
4) Compute dual uplink powers:
$$\mathbf{q}^{(n)} = \left(\mathbf{I} - \mathbf{D}^{\mathsf{DL}}\mathbf{F}^{\mathsf{T}}\right)^{-1}\mathbf{D}^{\mathsf{DL}}\mathbf{w}.$$
5) Update transmit beamformers:
$$\mathbf{u}_l^{(n+1)} = \mathbf{u}_l^{\mathsf{MVDR}}(\mathbf{q}^{(n)}), \quad l = 1, \ldots, L.$$
6) Compute dual auxiliary variables:
$$\tilde{q}_l = \left(\frac{\beta_l}{\mathsf{SINR}_l^{\mathsf{UL}}(\mathbf{q}^{(n)}, \mathbb{V}^{(n+1)}, \mathbf{u}_l^{(n+1)})}\right) q_l^{(n)}$$
for $l = 1, \ldots, L$.
7) Compute dual uplink powers:
$$\mathbf{q}^{(n+1)} = \frac{\bar{P}}{\mathbf{n}^{\mathsf{T}}\tilde{\mathbf{q}}} \cdot \tilde{\mathbf{q}}.$$
8) Compute downlink powers:
$$\mathbf{p}^{(n+1)} = \left(\mathbf{I} - \mathbf{D}^{\mathsf{UL}}\mathbf{F}\right)^{-1}\mathbf{D}^{\mathsf{UL}}\mathbf{n}.$$

Remark 1: It is also possible to combine the MISO and SIMO iterations into an algorithm which does not require uplink-downlink transitions. Recall that the fixed-point iteration in Theorem 1 converges to the optimal solution of the MISO problem regardless of the initial value of the dual uplink power $\mathbf{q}$. Suppose steps 5-7 of Algorithm 2 are looped till convergence.[1] Then, regardless of the value of $\mathbf{q}^{(n)}$ passed into the loop by step 4, steps 5-7 will converge to the optimal $\mathbf{q}$ (for fixed receive beamformers $\mathbb{V}^{(n)}$). Hence, the downlink-to-uplink conversion in step 4 can be omitted. Similarly, by looping steps 1-3 till convergence, the uplink-to-downlink conversion in step 8 can be omitted. A slightly different modification which also enables us to drop the uplink-downlink transitions is to loop steps 2-3 and 6-7 till convergence. This is because steps 2-3 and 6-7 ensure that $\mathbf{p}^{(n)}$ and $\mathbf{q}^{(n)}$ always converge to the optimal $\mathbf{p}$ and $\mathbf{q}$ respectively for any fixed beamformers $(\mathbb{U}^{(n)}, \mathbb{V}^{(n)})$ [16].

Convergence of Algorithm 2 is guaranteed because each step in the Algorithm increases the minimum weighted SINR. The power constraint and the unity norm constraint on the beamformers imply the existence of a limiting value for the SINR so the sequence must converge monotonically. In addition, it is easily seen that Algorithm 2 is *coordinate-wise* optimal in $(\mathbf{p}, \mathbb{U})$ and $(\mathbf{p}, \mathbb{V})$. This means that, if Algorithm 2 converges to $(\mathbf{p}_*, \mathbb{U}_*, \mathbb{V}_*)$, then

$$\min_l \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}_*, \mathbb{U}_*, \mathbf{v}_{*l})}{\beta_l} = \mathcal{S}_{\mathbb{U}_*}^{\mathsf{SIMO}}(\bar{P})$$
$$\min_l \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p}_*, \mathbb{U}_*, \mathbf{v}_{*l})}{\beta_l} = \mathcal{S}_{\mathbb{V}_*}^{\mathsf{MISO}}(\bar{P}).$$

Unfortunately, the global optimality of Algorithm 2 is not guaranteed and depends on the initial values $(\mathbf{p}^{(0)}, \mathbb{U}^{(0)}, \mathbb{V}^{(0)})$. But we can identify two special cases under which Algorithm 2 is optimal. We give these results in Proposition 1 and Observation 2. In particular, Observation 2 is significant as it will form part of the intuition behind a heuristic strategy for initializing Algorithm 2, which we propose in the next section.

First, consider the case when $\text{rank}(\mathbf{H}_l) = 1$ for all l, and suppose $\mathbf{H}_l = a_l \mathbf{h}_l \tilde{\mathbf{h}}_l^\dagger$ for some $a_l \in \mathbb{R}_{++}$, $\mathbf{h}_l \in \mathbb{C}^{N_l \times 1}$, and $\tilde{\mathbf{h}}_l \in \mathbb{C}^{N \times 1}$, where $\|\mathbf{h}_l\| = 1$ and $\|\tilde{\mathbf{h}}_l\| = 1$. It is well-known that the optimal receive beamformer for the lth stream is the normalized matched filter (scaled by arbitrary complex phases) corresponding to $\mathbf{h}_l$. Notice that by substituting the decomposition for $\mathbf{H}_l$ into (27) and applying the Sherman-Morrison-Woodbury formula, (27) can be rewritten as

$$\mathbf{v}_l^{\mathsf{MVDR}} = \left(\frac{a_l \tilde{\mathbf{h}}_l^\dagger \mathbf{u}_l}{n_l + a_l^2 \|\mathbf{h}_l\|^2 \sum_{i \neq l} |\tilde{\mathbf{h}}_l^\dagger \mathbf{u}_i|^2} \right) \mathbf{h}_l, \quad l = 1, \ldots, L$$

up to a scaling factor, and hence the MVDR beamformers are always optimal. It follows that Algorithm 2 is optimal when all the channels are rank-one. We state this result formally in the following proposition.[2]

Proposition 1: If $\text{rank}(\mathbf{H}_l) = 1$ for all l, then Algorithm 2 converges to the optimal solution of $\mathcal{S}(\bar{P})$.

[1]In practice, this can be implemented in a two time-scale algorithm by running steps 5-7 on a significantly faster time-scale such that convergence can be assumed.

[2]Notice that although Algorithm 2 updates $\mathbf{v}_l$ repeatedly, every update gives a global optimal $\mathbf{v}_l$ so, in fact, $\mathbf{v}_l$ need only be updated once for a single channel realization.

Second, observe that as $\bar{P} \to 0$, the interference between different users can be neglected. Hence, the optimization over $\mathbb{V}$ and $\mathbb{U}$ can be decoupled into L independent single-user MIMO beamforming problems. It follows that for the lth stream, the optimal $\mathbf{v}_l \to \mathbf{v}_{l1}$ and the optimal $\mathbf{u}_l \to \mathbf{u}_{l1}$, where $\mathbf{v}_{l1}$ and $\mathbf{u}_{l1}$ are the left and right singular vectors associated with the largest singular value of $\mathbf{H}_l$. Next, notice that for small $\bar{P}$, steps 1 and 5 of Algorithm 2 can be approximated as

$$\mathbf{v}_l^{(n+1)} \approx \frac{\mathbf{H}_l \mathbf{H}_l^\dagger \mathbf{v}_l^{(n)}}{\|\mathbf{H}_l \mathbf{H}_l^\dagger \mathbf{v}_l^{(n)}\|}, \quad l = 1, \ldots, L$$
$$\mathbf{u}_l^{(n+1)} \approx \frac{\mathbf{H}_l^\dagger \mathbf{H}_l \mathbf{u}_l^{(n)}}{\|\mathbf{H}_l^\dagger \mathbf{H}_l \mathbf{u}_l^{(n)}\|}, \quad l = 1, \ldots, L.$$

Hence, if $\mathbf{v}_l^{(0)}$ and $\mathbf{u}_l^{(0)}$ are not orthogonal to $\mathbf{v}_{l1}$ and $\mathbf{u}_{l1}$ respectively, then as $\bar{P} \to 0$, steps 1 and 5 of Algorithm 2 will converge to the optimal receive and transmit beamformers respectively of $\mathcal{S}(\bar{P})$ [22, Ch. 8]. This insight is significant as it forms part of the basis for the heuristic algorithm which we will propose in the following section. Hence, we summarize this analysis in the following remark.

Remark 2: Let $\mathbf{v}_{l1}$ and $\mathbf{u}_{l1}$ be the left and right singular vectors associated with the largest singular value of $\mathbf{H}_l$. If $\mathbf{v}_{l1}^\dagger \mathbf{v}_l^{(0)} \neq 0$ and $\mathbf{u}_{l1}^\dagger \mathbf{u}_l^{(0)} \neq 0$ for all l, then as $\bar{P} \to 0$, Algorithm 2 converges to the optimal solution of $\mathcal{S}(\bar{P})$. In other words, Algorithm 2 is optimal in the asymptotically low SNR region.

B. Parametric Search Initialization Algorithm

In this section, we propose a method for tackling the sub-optimality of Algorithm 2, which we call *parametric search initialization*. Due to the existence of multiple coordinate-wise optimal points, the performance of Algorithm 2 depends on the initial values $(\mathbf{p}^{(0)}, \mathbb{U}^{(0)}, \mathbb{V}^{(0)})$. Our proposed strategy runs multiple instances of Algorithm 2 for different values of $\bar{P}$. It is based on the following two ideas:

1) Optimality of Algorithm 2 in the asymptotically low SNR region (cf. Remark 2).
2) Parametric continuity of the optimal solutions of $\mathcal{S}(\bar{P})$ with respect to $\bar{P}$ (cf. Theorem 2 below).

Theorem 2: Let $\tau^*(\bar{P})$ denote the optimal value of $\mathcal{S}(\bar{P})$, and let $(\mathbf{p}^*(\bar{P}), \mathbb{U}^*(\bar{P}), \mathbb{V}^*(\bar{P}))$ denote the optimal solution. Then $\tau^*(\bar{P})$ is continuous over $\bar{P}$ and $(\mathbf{p}^*(\bar{P}), \mathbb{U}^*(\bar{P}), \mathbb{V}^*(\bar{P}))$ is upper semicontinuous[3] over $\bar{P}$.

Proof: Refer to Appendix B.

From Theorem 2, it is reasonable to make the following hypothesis: *If $\bar{P}_1$ and $\bar{P}_2$ are sufficiently close to each other, then the global optimal solution for $\mathcal{S}(\bar{P}_1)$ is a good choice of initial power and beamformers for Algorithm 2 to use in solving $\mathcal{S}(\bar{P}_2)$*. We can exploit this insight repeatedly to obtain an intelligent strategy for initializing Algorithm 2. Suppose we have the optimal solution of $\mathcal{S}(\bar{P}_0)$ for some $\bar{P}_0$, then we can obtain good solutions of $\mathcal{S}(\bar{P})$ for any $\bar{P} > \bar{P}_0$

[3]Semicontinuity reduces to the usual notion of continuity if $\mathcal{S}(\bar{P})$ has a unique optimal solution.

by using Algorithm 2 to solve $\mathcal{S}(\bar{P})$ over fine increments of $\bar{P}$. At each value for $\bar{P}$, Algorithm 2 is initialized with the solution it returned at the previous power constraint. Clearly, this strategy requires an optimal solution to begin with. Fortunately, Algorithm 2 is optimal in the low SNR region. We summarize our proposed approach, which we refer to as parametric search initialization, as Algorithm 3.

Algorithm 3: Parametric Search Initialization Algorithm
Input:

- Desired power constraint $\bar{P}$.
- Fixed step-size $\bar{P}_\delta$.
- Fixed tolerance ϵ.
- A total power constraint $\bar{P}^{(1)}$ such that the network is operating in the low SNR region.
- Arbitrary $\mathbf{p}^{(0)} \in \mathbb{R}_{++}^{L\times 1}$, and $\mathbf{u}_l^{(0)} \in \mathbb{C}^{N\times 1}$, $\mathbf{v}_l^{(0)} \in \mathbb{C}^{N_l\times 1}$ for $l = 1, \dots, L$, such that $\mathbf{w}^\mathsf{T}\mathbf{p}^{(0)} \le \bar{P}^{(1)}$, and $\|\mathbf{u}_l^{(0)}\| = 1$, $\|\mathbf{v}_l^{(0)}\| = 1$ for all l.
- Iteration index $n = 1$.

Iterate:

1. Initialize Algorithm 2 with $(\mathbf{p}^{(n-1)}, \mathbb{U}^{(n-1)}, \mathbb{V}^{(n-1)})$. Use a total power constraint of $\bar{P}^{(n)}$ and run Algorithm 2 till convergence to within a tolerance ϵ. Store the solution in $(\mathbf{p}^{(n)}, \mathbb{U}^{(n)}, \mathbb{V}^{(n)})$.
2. If $\bar{P} = \bar{P}^{(n)}$, exit the iteration. Otherwise, update the power constraint as follows:
$$\bar{P}^{(n+1)} = \min\left\{\bar{P}^{(n)} + \bar{P}_\delta, \bar{P}\right\}.$$
3. Update $n \leftarrow n + 1$. Repeat from step 1.

Output: $(\mathbf{p}^{(n)}, \mathbb{U}^{(n)}, \mathbb{V}^{(n)})$.

At first glance, the above linear search strategy appears to incur significant computational cost. Theoretically, $\bar{P}_\delta$ must be infinitesimally small in order for the proposed strategy to successfully exploit the parametric continuity of the solutions. However, empirical tests reveal that $\bar{P}_\delta$ does not necessarily have to be small in order to obtain performance gains over random initialization.

C. Application to Total Power Minimization

In this section, we consider the weighted-sum-power minimization problem subject to given SINR constraints:

$$\mathcal{P}(\bar{\tau}) := \begin{cases} \min\limits_{\mathbf{p},\mathbb{U},\mathbb{V}} & \mathbf{w}^\mathsf{T}\mathbf{p} \\ \text{subject to:} & \dfrac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p},\mathbb{U},\mathbf{v}_l)}{\beta_l} \ge \bar{\tau}, \quad l = 1,\dots,L, \\ & \mathbf{p} > \mathbf{0}, \\ & \|\mathbf{u}_l\| = 1, \quad l = 1,\dots,L, \\ & \|\mathbf{v}_l\| = 1, \quad l = 1,\dots,L. \end{cases}$$

We will briefly explain how our results on $\mathcal{S}(\bar{P})$ can be extended to $\mathcal{P}(\bar{\tau})$, focusing only on an algorithm for $\mathcal{P}(\bar{\tau})$ proposed in [9] which we refer to as Algorithm E.

Observe that Algorithm E is largely similarly to Algorithm 2, the only difference being that Algorithm 2 has additional steps to normalize the power (steps 3 and 7). It is easy to see that the analysis leading to Proposition 1 and Remark 2 can be modified to show that Algorithm E converges to the optimal solution of $\mathcal{P}(\bar{\tau})$ when the channels are rank-one, and also when the network is operating in the asymptotically low SNR region, as long as $\mathbf{v}_{l1}^\dagger\mathbf{v}_l^{(0)} \neq 0$ and $\mathbf{u}_{l1}^\dagger\mathbf{u}_l^{(0)} \neq 0$ for all l. It is also possible to prove a similar parametric continuity result for $\mathcal{P}(\bar{\tau})$ in the parameter $\tilde{\tau}$, analogous to Theorem 2. Therefore, a similar parametric search strategy can be employed for $\mathcal{P}(\bar{\tau})$.

Recall that $\mathcal{S}(\bar{P})$ and $\mathcal{P}(\bar{\tau})$ are inverse problems so a solution for $\mathcal{S}(\bar{P})$ is also a solution for $\mathcal{P}(\bar{\tau})$. Hence, the parametric search initialization algorithms can be used to solve $\mathcal{S}(\bar{P})$ using $\mathcal{P}(\bar{\tau})$ and vice versa.[4] For instance, Algorithm 3 can be terminated when the minimum weighted SINR exceeds $\bar{\tau}$ and the power constraint at termination will be an approximation of the optimal value of $\mathcal{P}(\bar{\tau})$. A similar analysis can be made the other way round.

Furthermore, the inverse relationship between $\mathcal{S}(\bar{P})$ and $\mathcal{P}(\bar{\tau})$ also implies that a common framework can be used to compare the performance of Algorithm 2 and Algorithm E. Suppose Algorithm 2 achieves a minimum weighted SINR τ (which could be suboptimal) for $\mathcal{S}(\bar{P})$. Then we can compare $\bar{P}$ against the weighted-sum power achieved by Algorithm E for $\mathcal{P}(\bar{\tau})$. The better algorithm is the one that yields a smaller weighted sum power. This performance is relevant because of the sub-optimality of both algorithms. We will apply this framework in Section V to compare the performance of Algorithm 2 and Algorithm E, where we plot their performance on the same graph.

V. Numerical Examples

In this section, we evaluate and compare the performance of the various algorithms in a single-cell network with four users at $d = 0.3, 0.5, 0.7, 0.9$ km from the base station, which transmits one independent stream to each user. The performance is evaluated in terms of the minimum weighted SINR and the total power cost. It will be assumed for simplicity that every data stream has the same priority ($\boldsymbol{\beta} = \mathbf{1}$), all users suffer from the same noise variance ($\mathbf{n} = \mathbf{1}$), and the network is subject to a total power constraint ($\mathbf{w} = \mathbf{1}$). In all simulations, the base station is equipped with $N = 4$ antennas, and each user is equipped with $N_l = 2$ antennas. The noise power spectral density is set to -162 dBm/Hz. Each user communicates with the base station over independent MIMO rayleigh fading channels, with a path loss of $L = 128.1 + 37.6\log_{10}(d)$ dB, where d is the distance in kilometers. We assume an antenna gain of 15dB.

A. Performance

We plot the performance of Algorithm 2 and Algorithm E (with random initialization and parametric search initialization) on the same graph in Fig. 3. For the max-min weighted SINR problem, the horizontal axis represents the sum power constraint $\bar{P}$ while the vertical axis represents the performance

[4] A bisection search method was proposed to use an algorithm for $\mathcal{P}(\bar{\tau})$ to solve $\mathcal{S}(\bar{P})$. However, this method requires the algorithm to return the optimal solution, and hence it is only applicable to the MISO and SIMO scenarios.

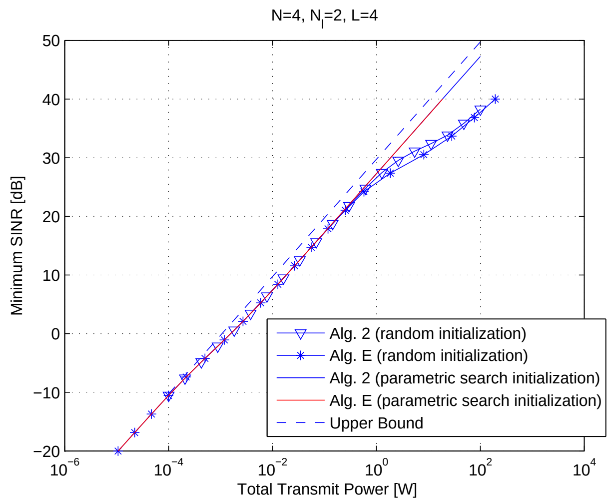


Fig. 3. Performance comparison of max-min weighted SINR and min power algorithms.

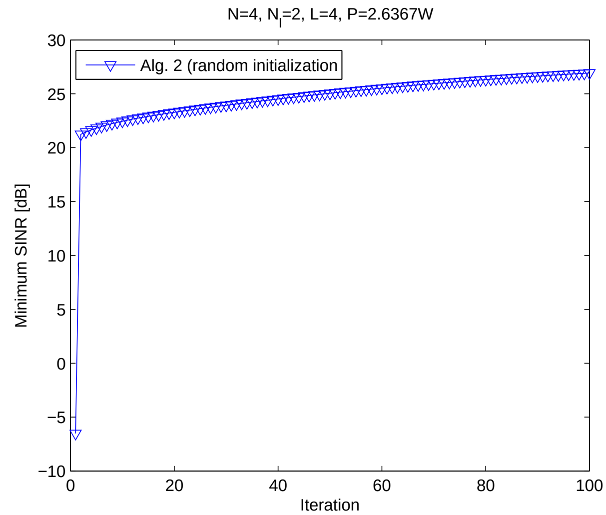


Fig. 4. Convergence plot of max-min weighted SINR algorithm for $\bar{P} = 2.6367$ W.

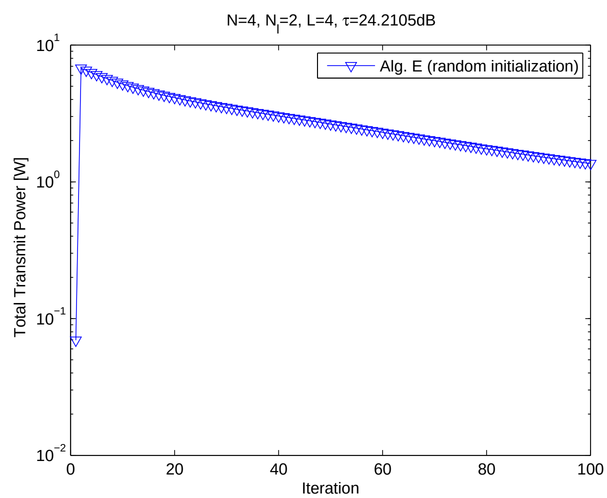


Fig. 5. Convergence plot of total power minimization algorithm for $\bar{\tau} = 24.2105$ dB.

in terms of minimum SINR. For the total power minimization problem, the vertical axis represents the SINR threshold $\bar{\tau}$ while the horizontal axis represents the performance in terms of total transmit power. In the parametric search initialization algorithm, the steps are chosen by dividing the range of $\bar{P}$ and $\bar{\tau}$ into 20 equally spaced values on the dB scale.

As stated in Remark 2, both Algorithm 2 and Algorithm E attain optimal performance for small $\bar{P}$ or small $\bar{\tau}$. As $\bar{P}$ or $\bar{\tau}$ increases, the performance of the algorithms deteriorate, with Algorithm 2 performing slightly better than Algorithm E. Hence, optimizing the cellular network according to a fixed desired total power provides approximately the same performance as optimizing according to a fixed desired SINR threshold. In the low to medium range of $\bar{P}$ or $\bar{\tau}$, both algorithms with random initialization or parametric search initialization achieve the same performance. However, as $\bar{P}$ or $\bar{\tau}$ increases further, parametric search initialization performs significantly better than random initialization. Moreover, the performance achieved by the parametric initialization is only about $2-3$ dB away from the interference-free bound (which is obtained by ignoring the interference between users while computing the SINR), demonstrating that parametric search initialization is effective at searching for good initial power and beamformers for Algorithm 2 and Algorithm E. There is no visible difference in performance between the parametric search over $\bar{P}$ (using Algorithm 2) or the parametric search over $\bar{\tau}$ (using Algorithm E). Similar results are obtained for a network in which the base station serves two users and transmits two independent streams to each user.

B. Convergence

We assume the same network configuration as in Fig. 3 and study the convergence of Algorithm 2 and Algorithm E with random initialization. Fig. 4 shows the average minimum SINR of Algorithm 2 as a function of iteration index n for $\bar{P} = 2.6367$ W while Fig. 5 shows the average total transmit power of Algorithm E as a function of iteration index n for an SINR threshold of $\bar{\tau} = 24.2105$ dB. Both algorithms converge fairly quickly. However, Algorithm E has a dominant spike in the total transmit power during the initial iterations. Further empirical tests attribute the spike to poor initial beamformers, which forces Algorithm E to increase the transmit power significantly in order to satisfy the SINR thresholds. Subsequently, when the SINR thresholds are satisfied, the total transmit power decreases monotonically (as proven in [9]). In contrast, Algorithm 2 maintains the power constraint at every iteration and converges with a monotonically increasing minimum weighted SINR.

C. Varying Step Size on Parametric Search Initialization

We assume the same network configuration as in Fig. 3 and investigate the effects of varying the step size on the parametric search initialization. For Algorithm 2, we divide the range of $\bar{P}$ between 10^{-5} and 10^{2} W into 2, 4, or 6 equally-spaced values (on the dB scale). For Algorithm E, we divide the range

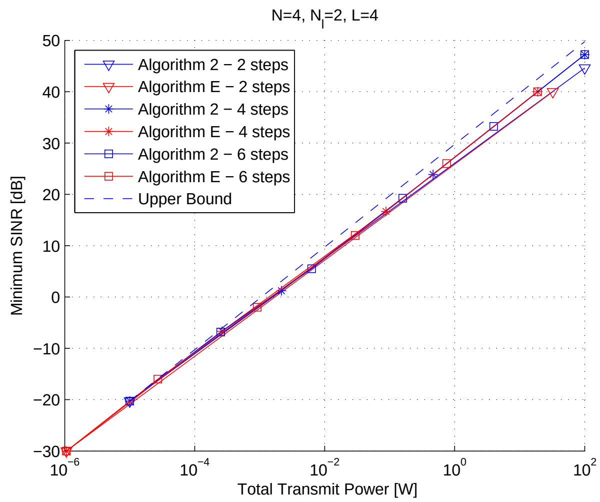


Fig. 6. Performance of parametric search initialization algorithm for various step sizes.

of $\bar{\tau}$ between -30 and 40 dB into 2, 4, or 6 equally-spaced values (on the dB scale). The results are given in Fig. 6. The performance of Algorithm 2 and Algorithm E are similar and there is no visible increase in performance beyond using 4 steps. The 2-step procedure deserves further comment. Notice that in the 2-step procedure, the initial beamformers in the second step are the optimal beamformers in the first step, which are approximately the singular vectors corresponding to the largest singular values of the channels. Thus, although singular-vector-decomposition (SVD) initialization does not perform better than a parametric search over ≥ 4 steps, it still performs significantly better than random initialization (cf. Fig. 3). Hence, although parametric search initialization, in theory, requires an extremely small step-size to search for good initializations, Fig. 6 shows that this might not be necessary in practice. It is possible to achieve significant gains in performance even with a large step-size.

VI. Conclusion

We began by analyzing the MISO and SIMO max-min weighted SINR problems using nonlinear Perron-Frobenius theory. Nonlinear Perron-Frobenius theory was key in the derivation of the DPC-like max-min weighted SINR power control algorithm in [16]. We extended the application of nonlinear Perron-Frobenius theory to the MISO and SIMO problems, and we showed that our result can be interpreted as an alternate optimization of power and beamformers. As a by-product, we resolved the open problem of convergence for the previously proposed MISO algorithm in [1]. In summary, we demonstrated that nonlinear Perron-Frobenius provides an analytical framework for the max-min weighted SINR problem, analogous to the relationship of the standard interference function framework with the total power minimization problem.

The nonconvexity of the MIMO max-min weighted SINR problem and the total power minimization problem, and the mutual coupling between the transmit and receive beamformers, imply that both problems are, in general, difficult to solve. Hence, most previous work are based on the approach of alternate optimization. In general, these approaches are suboptimal as they fail to tackle the nonconvexity of the problem. Nevertheless, we believe that an in-depth understanding of the optimality characteristics of the problems and their algorithms can provide insights into possible strategies for obtaining better solutions to the nonconvex problems.

Hence, in our study of the MIMO max-min weighted SINR problem, we conducted an in-depth analysis of the optimality characteristics. Specifically, we identified two scenarios under which this nonconvex problem can be optimally solved: (i) when the channels are rank-one, and (ii) when the network is operating in the low SNR region. We also proposed an algorithm which converges to the optimal solution under both scenarios. To improve the performance of our (generally) suboptimal algorithm, we proposed a method for initializing our algorithm, which is motivated by the parametric continuity of the problem in the power constraint. The proposed parametric search initialization algorithm exhibits better performance than random initialization, and can be easily extended to the MIMO total power minimization problem.

Appendix

A. Proof of Theorem 1

Proof: First, we show that f^{UL} is a concave mapping. It was shown in [1] that for a positive semidefinite matrix $\mathbf{A}$ and a vector $\mathbf{c}$ in the range of $\mathbf{A}$, we have

$$\frac{1}{\mathbf{c}^\dagger \mathbf{A}^{-1}\mathbf{c}} = \min_{\mathbf{z}:\mathbf{c}^\dagger\mathbf{z}=1} \mathbf{z}^\dagger \mathbf{A}\mathbf{z}$$

where the matrix inverse denotes the Moore Penrose pseudoinverse. Recall that $\mathbf{M}_l^{\mathsf{UL}}(\mathbf{q})$ is invertible so it has full rank. Hence, $\mathbf{H}_l^\dagger \mathbf{v}_l$ is in the range of $\mathbf{M}_l^{\mathsf{UL}}(\mathbf{q})$. Choosing $\mathbf{A} = \mathbf{M}_l^{\mathsf{UL}}(\mathbf{q})$ and $\mathbf{c} = \mathbf{H}_l^\dagger \mathbf{v}_l$, we have that

$$f_l^{\mathsf{UL}}(\mathbf{q}) = \min_{\mathbf{z}:\mathbf{v}_l^\dagger \mathbf{H}_l \mathbf{z}=1} \beta_l \mathbf{z}^\dagger \mathbf{M}_l^{\mathsf{UL}}(\mathbf{q})\mathbf{z}, \quad l = 1, \ldots, L.$$

Since $\mathbf{M}_l^{\mathsf{UL}}(\mathbf{q})$ is an affine mapping of $\mathbf{q}$, and the point-wise minimum of affine mappings is concave, we have that $f_l^{\mathsf{UL}}(\mathbf{q})$ is a concave mapping of $\mathbf{q}$. Since $\mathbf{H}_l^\dagger \mathbf{v}_l \neq \mathbf{0}$ (because $G_{li} > 0$) and $w_l > 0$, we have that $f_l^{\mathsf{UL}}(\mathbf{q}) > 0$ for all $\mathbf{q} \gneqq \mathbf{0}$. The rest of the proof follows from applying Theorem 1 in [23]. That the fixed-point iteration converges geometrically fast follows from the remark after Theorem 1 in [23]. □

B. Proof of Theorem 2

Proof: First, we need the following result from [24]:

Theorem 3: Let $f : \Gamma \times \bar{\mathcal{P}} \to \mathbb{R}$ be a continuous function and $\mathcal{D} : \bar{\mathcal{P}} \to P(\Gamma)$ be a compact-valued, continuous correspondence, where $P(\Gamma)$ denotes the power set of Γ. Let $f^* : \bar{\mathcal{P}} \to \mathbb{R}$ and $\mathcal{D}^* : \bar{\mathcal{P}} \to P(\Gamma)$ be defined by

$$f^*(\bar{P}) = \max_{x \in \mathcal{D}(\bar{P})} f(x, \bar{P}) \tag{30}$$

$$\mathcal{D}^*(\bar{P}) = \arg \max_{x \in \mathcal{D}(\bar{P})} f(x, \bar{P}). \tag{31}$$

Then f^* is a continuous function on $\bar{\mathcal{P}}$, and $\mathcal{D}^*$ is a compact-valued, upper-semi-continuous correspondence on $\bar{\mathcal{P}}$.

We apply Theorem 3 to prove our desired result. Let $\Gamma' = \mathbb{C}^{N\times 1}\times\ldots\times\mathbb{C}^{N\times 1}\times\mathbb{C}^{N_1\times 1}\times\ldots\times\mathbb{C}^{N_L\times 1}$, and $\Gamma = \mathbb{R}^{L\times 1}\times\Gamma'$, i.e., Γ is the cartesian product of $\mathbb{R}^{L\times 1}$ with L complex spaces of dimension $N_1,\ldots,N_L$ and L complex spaces of dimension N. Define $f:\Gamma\times\mathbb{R}_{++}\to\mathbb{R}$ and $\mathcal{D}:\mathbb{R}_{++}\to P(\Gamma)$ as follows:

$$f(\mathbf{p},\mathbb{U},\mathbb{V},\bar{P}) = \min_l \frac{\mathsf{SINR}_l^{\mathsf{DL}}(\mathbf{p},\mathbb{U},\mathbf{v}_l)}{\beta_l}$$

$$\mathcal{D}(\bar{P}) = \left\{ (\mathbf{p},\mathbb{U},\mathbb{V})\in\Gamma \,\middle|\, \begin{array}{l} \mathbf{p}\geq\mathbf{0}, \mathbf{w}^{\mathsf{T}}\mathbf{p}\leq\bar{P}, \\ \|\mathbf{u}_l\|=1, \quad l=1,\ldots,L, \\ \|\mathbf{v}_l\|=1, \quad l=1,\ldots,L. \end{array} \right\}.$$

Replacing $\bar{\mathcal{P}}$ in Theorem 3 with $\mathbb{R}_{++}$, we see that (30) and (31) give respectively the optimal value and solution of $\mathcal{S}(\bar{P})$. To satisfy the requirement of $\mathcal{D}(\bar{P})$ being compact, we relaxed the original constraint $\mathbf{p}>\mathbf{0}$ into $\mathbf{p}\geq\mathbf{0}$. This relaxation is valid because we always have $\mathbf{p}>\mathbf{0}$ at optimality of $\mathcal{S}(\bar{P})$ (otherwise some stream will have zero SINR).

It is easy to see that $f(\mathbf{p},\mathbb{U},\mathbb{V},\bar{P})$ is continuous. Hence, to complete the proof, it suffices to show that $\mathcal{D}$ is continuous. A correspondence $\mathcal{D}:\mathbb{R}_{++}\to P(\Gamma)$ is said to be continuous at $\bar{P}\in\mathbb{R}_{++}$ if $\mathcal{D}$ is both *upper-semicontinuous* (u.s.c) and *lower-semicontinuous* (l.s.c) at $\bar{P}$ [24]. First, we prove that $\mathcal{D}$ is u.s.c. at any $\bar{P}\in\mathbb{R}_{++}$. Define the correspondences $\mathcal{D}_1:\mathbb{R}_{++}\to\mathbb{R}^{L\times 1}$ and $\mathcal{D}_2:\mathbb{R}_{++}\to\Gamma'$ as follows:

$$\mathcal{D}_1(\bar{P}) = \left\{\mathbf{p}\in\mathbb{R}^{L\times 1} | \mathbf{p}\geq\mathbf{0}, \mathbf{w}^{\mathsf{T}}\mathbf{p}\leq\bar{P}\right\}$$

$$\mathcal{D}_2(\bar{P}) = \left\{ (\mathbb{U},\mathbb{V})\in\Gamma' \,\middle|\, \begin{array}{l} \|\mathbf{u}_l\|=1, \quad l=1,\ldots,L, \\ \|\mathbf{v}_l\|=1, \quad l=1,\ldots,L. \end{array} \right\}.$$

Hence, $\mathcal{D}$ can be rewritten as the cartesian product $\mathcal{D}_1\times\mathcal{D}_2$. Notice that $\mathcal{D}_2$ is a constant mapping so $\mathcal{D}_2$ is u.s.c. Since the cartesian product of a finite number of u.s.c. mappings is also u.s.c., it remains to to show that $\mathcal{D}_1$ is u.s.c. [25, Ch. 6]. Suppose A is an open set such that $\mathcal{D}_1(\bar{P})\subset A$. Since $\mathcal{D}_1(\bar{P})$ is compact, there exist some $\epsilon>0$ such that $\{\mathbf{p}\in\mathbb{R}^L : \mathbf{p}\geq\mathbf{0}, \mathbf{w}^{\mathsf{T}}\mathbf{p}<\bar{P}+\epsilon\}\subset A$. Choose δ such that $0<\delta<\epsilon$. Then for all $\bar{P}'\in(\bar{P}-\delta,\bar{P}+\delta)$, we have that $\mathcal{D}_1(\bar{P}')\subset A$. Hence, $\mathcal{D}_1$ is u.s.c.

Next, we prove that $\mathcal{D}$ is l.s.c. at any $\bar{P}\in\mathbb{R}_{++}$. The cartesian product of a finite number of l.s.c. mappings is also l.s.c. so it remains to show that $\mathcal{D}_1$ is l.s.c. Suppose A is an open set such that $A\cap\mathcal{D}_1\neq\emptyset$. Then, A contains some open subset of $\mathcal{D}_1$ so there exist some ϵ, where $0<\epsilon<\bar{P}$, such that $A\cap\{\mathbf{p}\in\mathbb{R}^{L\times 1} : \mathbf{p}\geq\mathbf{0}, \mathbf{w}^{\mathsf{T}}\mathbf{p}<\bar{P}-\epsilon\}\neq\emptyset$. Choose δ such that $0<\delta<\epsilon$. Then for all $\bar{P}'\in(\bar{P}-\delta,\bar{P}+\delta)$, we have that $\mathcal{D}_1(\bar{P}')\supset\{\mathbf{p}\in\mathbb{R}^{L\times 1}|\mathbf{p}\geq\mathbf{0}, \mathbf{w}^{\mathsf{T}}\mathbf{p}<\bar{P}-\epsilon\}$ so $A\cap\mathcal{D}_1(\bar{P}')\neq\emptyset$. Hence, $\mathcal{D}_1$ is l.s.c. $\square$

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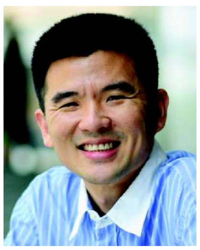

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