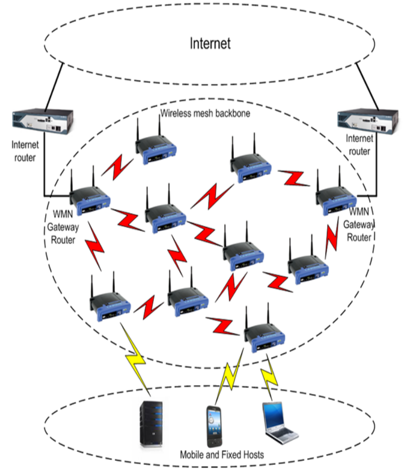
BackLog Based Random Access in Wireless Networks : (In)Stability Issues and Fluid Limits

Javad Ghaderi & Sem Borst & Phil Whiting

Macquarie Univ. & Bell Laboratories & Columbia
Hong Kong Talk
Hong Kong

January 2015

Background and Motivation



Wireless Mesh Networks - Applications and Issues:

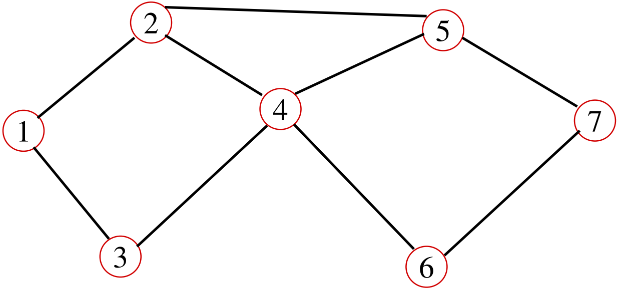
Wireless Mesh Networks - Applications and Issues:

- Applications
  - Electric Meters for Billing
  - One lap top per child program
  - Iridium satellite constellation

- Future Wireless Networks
  - Very Large Number of Users
  - No centralised control and no infrastructure
  - Autonomous radio nodes share resources
- Internet of things, sensor networks, ...

Challenge: Efficient use of resources in such networks

WirelessMesh Network: Interference Model



- Users cannot Transmit Simultaneously because of Interference
- Interference relationship represented by Conflict Graph
- Feasible schedules correspond to Independent Sets e.g $\{1, 5, 6\}$

Wireless Mesh Network: Traffic Model

- Packets arrivals Poisson
- Packet Transmission Times are unit exponential
- $U_i(t)$ activity r.v. for node i, time t
- $Q_i(t)$ backlog for node i, time t
- Interference Constraints Respected
- Performance: Throughput and Delay

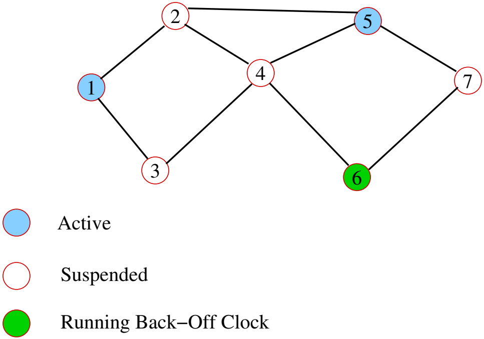
Medium Access Algorithms

- Capacity region: Convex Hull of Feasible Schedules
- Throughput-optimal algorithm: Stabilises the queues, for any traffic intensity inside the capacity region (positive recurrence)
- Centralised algorithms: High Complexity, throughput optimal
  - e.g Max. Weight - selects a schedule to maximise $\mathbf{Q}^T\mathbf{R}$
- Distributed Algorithms: (low complexity, not clear if optimal)
  - Random Access Algorithms: e.g. CSMA in 802.11
  - User listens to channel and transmits with some probability when idle

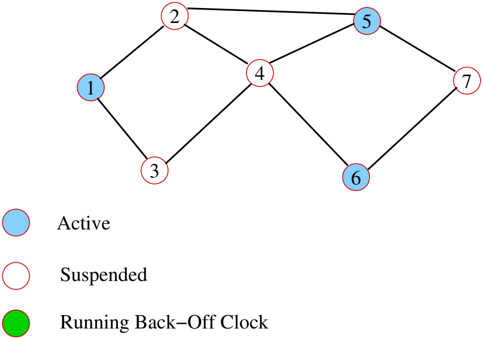
Random Access Algorithm: Description

- When node i ends activity at time t it then backs-off w.p. $g(Q_i(t))$
  - Back-off periods are unit exponential
- Back-off clock is suspended if any neighbour of i is active
  - Resumes when channel becomes idle
- At end of back-off period, node i begins transmission w.p. $f(Q_i(t))$

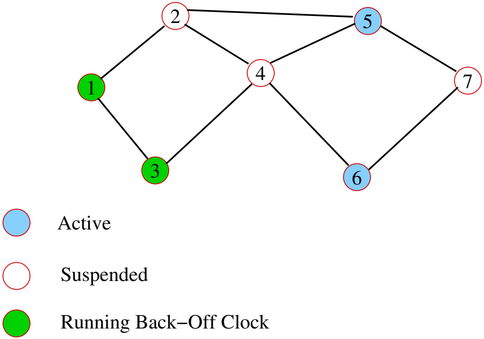
Random Access Algorithm: Example I



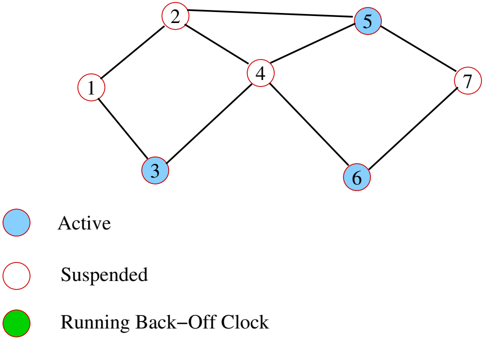
Random Access Algorithm: Example II



Random Access Algorithm: Example III



Random Access Algorithm: Example IV



Throughput-Optimal Random Access

- $f = 1$ and deactivation function $g = g(Q_i(t))$
- Rajogopalan and Shah, Shin (2009)
  - $g(x) = 1/\log(1+x)$ throughput optimal
  - Proof requires knowledge of maximum queue size in network
- Ghaderi and Srikant (2010)
  - $g(x) = (1+x)^{-1/h(1+x)}$
  - $h(x)$ slowly increasing function e.g $\log\log(e+x)$
- However excessive backlogs and delays are produced!!

Remainder of Talk

- More aggressive/persistent schemes can improve delay
- No maximum stability guarantees
- Examine stability/instability via fluid limits
- Demonstrate instability in certain topologies

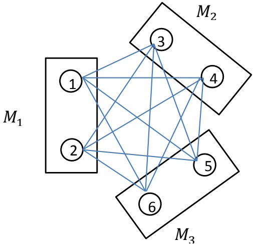
Fluid Limits

- R scale parameter, $\left(\frac{1}{R}\mathbf{Q}(Rt), \mathbf{U}(Rt)\right)_{t>0}$
- Continuous queue paths obtained by linear interpolation have "'skip free property" (see Meyn)
- $\|\mathbf{Q}(0)\| = R$, take limits as $R \to \infty$
- Sample paths of queues Lipschitz continuous, sequence is tight
- Fluid limits of queue process exist in $C[0, \infty)$

Fluid Limits

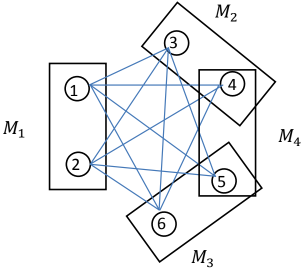
- R scale parameter, $\left(\frac{1}{R}\mathbf{Q}(Rt), \mathbf{U}(Rt)\right)_{t>0}$
- Continuous queue paths obtained by linear interpolation have "'skip free property" (see Meyn)
- $\|\mathbf{Q}(0)\| = R$, take limits as $R \to \infty$
- Sample paths of queues Lipschitz continuous, sequence is tight
- Fluid limits of queue process exist in $C[0, \infty)$
- Fast time scale with coarse granularity
- Fluid limits can be used to infer (in)stability

Example: Complete Partite Network



- Backoff $g(x) \doteq \frac{1}{(x+1)^\gamma}$
- Traffic - $(0.4, 0.4, 0.3, 0.2, 0.2, 0.3)\,\rho$
- Stable $\rho < 1, \;\; \gamma > 1$
- Nodes hold on to channel until backlog cleared
- Activity process does not leave component until all queues drained
- Random switching to alternate component
- Fluid limit reaches 0 in finite time $\Rightarrow$ Stability

Example: Network with Reduced Interference



- Removing one edge causes instability, $\forall \gamma > 1$
- Traffic - $(0.4, 0.4, 0.3, 0.2, 0.2, 0.3)\,\rho$
- M_4 visits occur i.o. use positive fraction of time
- $L(t)$ total queue length - explodes as $\rho \uparrow 1^-$

Fluid Limit Properties

- Fluid Limit is Random
- Piecewise Linear Paths because $\gamma > 1$
- No Idling

Fluid Limit Properties

- Fluid Limit is Random
- Piecewise Linear Paths because $\gamma > 1$
- No Idling
- Natural state: $\exists T > 0, Q_3(t) \geq Q_4(t),\ Q_5(t) \geq Q_6(t)$, equality only if both 0

Fluid Limit Properties

- Fluid Limit is Random
- Piecewise Linear Paths because $\gamma > 1$
- No Idling
- Natural state: $\exists T > 0, Q_3(t) \geq Q_4(t),\ \ Q_5(t) \geq Q_6(t)$, equality only if both 0
- M_1 visits occur i.o.

Fluid Limit Properties

- Fluid Limit is Random
- Piecewise Linear Paths because $\gamma > 1$
- No Idling
- Natural state: $\exists T > 0, Q_3(t) \geq Q_4(t),\ Q_5(t) \geq Q_6(t)$, equality only if both 0
- M_1 visits occur i.o.
- Switching out of $M_1 : \frac{3}{8} = \mathbb{P}\left\{M_2\right\} = \mathbb{P}\left\{M_3\right\}, \frac{1}{4} = \mathbb{P}\left\{M_4\right\}$

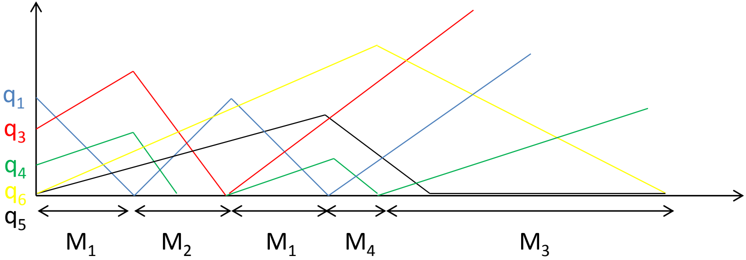
Fluid Limit Properties

- Fluid Limit is Random
- Piecewise Linear Paths because $\gamma > 1$
- No Idling
- Natural state: $\exists T > 0, Q_3(t) \geq Q_4(t),\ Q_5(t) \geq Q_6(t)$, equality only if both 0
- M_1 visits occur i.o.
- Switching out of M_1 : $\frac{3}{8} = \mathbb{P}\{M_2\} = \mathbb{P}\{M_3\}, \frac{1}{4} = \mathbb{P}\{M_4\}$
- Switching out of M_2 : $\overline{p} = \mathbb{P}\{M_1\} = 1 - \mathbb{P}\{M_3\}$

Fluid Limit Properties

- Fluid Limit is Random
- Piecewise Linear Paths because $\gamma > 1$
- No Idling
- Natural state: $\exists T > 0, Q_3(t) \geq Q_4(t),\ \ Q_5(t) \geq Q_6(t)$, equality only if both 0
- M_1 visits occur i.o.
- Switching out of $M_1 : \frac{3}{8} = \mathbb{P}\left\{M_2\right\} = \mathbb{P}\left\{M_3\right\}, \frac{1}{4} = \mathbb{P}\left\{M_4\right\}$
- Switching out of $M_2 : \overline{p} = \mathbb{P}\left\{M_1\right\} = 1 - \mathbb{P}\left\{M_3\right\}$
- Switching out of $M_4 : \mathbb{P}\left\{M_1\right\} = 0$

Sample Path Illustrating M_1 Cycles



Fluid Limit Instability

$L(t)$ grows faster than any sub linear function of t,

Theorem (Ghaderi, Borst, W.)

For any $m > 1$, there exists a constant $\rho^* < 1$ such that for all $\rho \in (\rho^*, 1)$,

$$\limsup_{T\to\infty} \mathbb{E}\left[\frac{T}{L^m(T)}\right] = 0$$

for any initial condition.

Two Cycle Argument

Prop

$\exists C > 0$ such that $\theta = 1 - (1-\rho)C$, $p = 1/12$. Over cycle pairs D_k, $k = 1, 2, \dots,$

(i) $\Delta T_k \leq C L_k$;

(ii) $L(t) \geq \theta L_k$ *for all* $t \in [T_k, T_{k+1}]$;

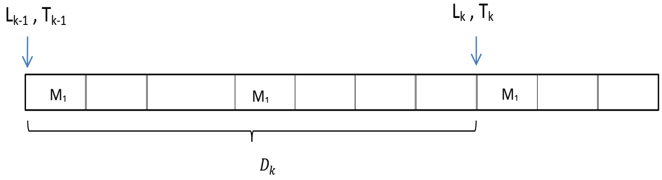
(iii) $\mathbb{P}\big(L_{k+1} - \theta L_k \geq \delta(\rho)\theta L_k | L_k\big) \geq p$,

with $\delta(\rho)$ a constant, depending on ρ, and $\delta(\rho) \uparrow \delta = \frac{1}{\beta^{\max}(1+\beta^{\max})(1+\alpha-\min\{\kappa_3,\kappa_6\})}$, as $\rho \uparrow 1$.

This proposition shows that for ρ close to 1 the load cannot significantly decrease over a pair of cycles and will increase by a substantial amount with non-zero probability.

Proof Sketch I: Preliminaries

M_1 cycles,



Definitions and 2-cycle arguments,

$$T_{N_t-1} < t \leq T_{N_t} \tag{1}$$

$$L(t) \geq \theta L_{N_t-1}, \ \theta \in (0,1] \tag{2}$$

$$T_{N_t} \leq T_{N_t-1} + CL_{N_t-1} \tag{3}$$

$$L_{N_t-1} \leq 2T_{N_t-1} \tag{4}$$

Proof Sketch II: Enough to bound $T_{N_t-1} L_{N_t-1}^{-m}$

$$\begin{aligned}
\limsup_{t\to\infty} \mathbb{E}\left[tL^{-m}(t)\right] &\overset{(2)}{\leq} \limsup_{t\to\infty} \mathbb{E}\left[T_{N_t}\theta^{-m} L_{N_t-1}^{-m}\right] \\
&\overset{(3)}{\leq} \theta^{-m}\left(\limsup_{t\to\infty} \mathbb{E}\left[T_{N_t-1} L_{N_t-1}^{-m}\right] + C \limsup_{t\to\infty} \mathbb{E}\left[L_{N_t-1}^{-m+1}\right]\right) \\
&\overset{(4)}{\leq} \theta^{-m}(1+2C) \limsup_{t\to\infty} \mathbb{E}\left[T_{N_t-1} L_{N_t-1}^{-m}\right].
\end{aligned}$$

Proof Sketch III: L_k^{-m} is a Geometric Supermartingale

$$\begin{aligned} \mathbb{E}\left[L_{k+1}^{-m} | \mathcal{F}_k\right] &\leq (1-p)(\theta L_k)^{-m} + p((\theta+\delta)L_k)^{-m} \\ &= \alpha_m L_k^{-m}, \end{aligned}$$

where,

$$\alpha_m := (1-p)\theta^{-m} + p(\theta+\delta)^{-m}, p = \frac{1}{12}$$

by two cycle argument, $\alpha_m < 1$, ρ close to 1
Hence a.s. convergence to 0 and . . . . . .

$$T_k \to \infty, \ a.s.$$

because $L_k \leq \rho T_k + 1 \leq T_k + 1$, so stopping times well defined . . .

Proof Sketch IV: $T_k L_k^{-m}$ *almost* a Geom. Supermart.

$$\begin{aligned}\mathbb{E}\left[T_k L_k^{-m} | \mathcal{F}_{k-1}\right] &\leq (T_{k-1} + CL_{k-1})\mathbb{E}\left[L_k^{-m} | \mathcal{F}_{k-1}\right] \\ &\leq (T_{k-1} + CL_{k-1})\alpha_m L_{k-1}^{-m} \\ &= \alpha_m T_{k-1} L_{k-1}^{-m} + \alpha_m C L_{k-1}^{-m+1}.\end{aligned}$$

$\epsilon_k \doteq C\alpha_m L_k^{-m+1}$, so that $\sum_{k=1}^{\infty} \mathbb{E}\left[\epsilon_k\right] \leq C\alpha_m \sum_{k=1}^{\infty} \alpha_{m-1}^k < \infty$ and hence $\lim_{k\to\infty} T_k L_k^{-m} = 0$ almost surely.
By induction,

$$\mathbb{E}\left[T_k L_k^{-m}\right] \leq \alpha^{k-1} C(1 + T^{(0)} + (k-1)\alpha),$$

for ρ sufficiently close to 1 $\rightarrow$ convergence in $\mathcal{L}_1$

Proof Sketch V: Stopped Sequence UI too

By Theorems of Rogers & Williams, (slightly adapted)

$T_k L_k^{-m}$ U.I sequence and then $T_{N_t} L_{N_t}^{-m}$ U.I family
Whence, given $\varepsilon > 0, \exists K_\varepsilon > 0$

$$\begin{aligned}\mathbb{E}\left[T_{N_t} L_{N_t}^{-m}\right] &\leq \sum_{k=1}^{\infty} \mathbb{E}\left[T_k L_k^{-m} \mathbb{1}_{\{N_t = k\}} \mathbb{1}_{\{T_{N_t} L_{N_t}^{-m} \leq K_\varepsilon\}}\right] + \varepsilon \\ &\leq K_\varepsilon \mathbb{P}\{N_t \leq D\} + \sum_{k=D+1}^{\infty} C k \alpha^{k-1} + \varepsilon,\ D \in \mathbb{N}\end{aligned}$$

Fixing ε and D, we find that

$$\limsup_{t\to\infty} \mathbb{E}\left[T_{N_t} L_{N_t}^{-m}\right] \leq (D+1)\frac{\alpha^D}{1-\alpha} + \varepsilon$$

so that $\limsup_{t\to\infty} \mathbb{E}\left[T_{N_t} L_{N_t}^{-m}\right] = 0$ for ρ sufficiently close to 1

- A theorem of Meyn (Ann. Prob. 1995), establishes a criteria for transience based on a fluid limit
- The idea is that the fluid limit may be used to determine a Lypuanov function
- Strict growth of the fluid limit is required
- Only growth in expectation our case
- Nevertheless . . . . . .

Instability Theorem

Theorem (Meyn)

Suppose that for a Markov chain $\{X(n); n = 1, 2, \ldots\}$ *with state space* S*, there exist positive functions* $W(\cdot)$ *and* $\Delta(\cdot)$ *on* S*, and a positive constant* c_0*, such that*

$$\mathbb{E}\left[W(X(n+1))|\mathcal{F}_n\right] \leq W(X(n)) - \Delta(X(n)),$$

whenever $X(n) \in S_{c_0} = \{x \in S : W(x) \leq c_0\}$*, with* $\mathcal{F}_n := \sigma(X(0), X(1), \ldots, X(n))$*. Then for all* $x \in S$*,*

$$\mathbb{P}_x\left\{\sum_{n=0}^{\infty} \Delta(X(n)) < \infty\right\} \geq 1 - W(x)/c_0.$$

Brief Sketch I:Definition of $\mathcal{W}$

Define, $W(x) \doteq \mathbb{E}\left[\mathcal{W} | Q(0) = x\right]$, where $\mathcal{W}$ is defined as

$$\mathcal{W} := \sum_{n=0}^{\|Q(0)\| T} \left[1 + \|Q(0)\| + a\|Q(n)\|\right]^{-m}$$

for some positive constants a and T to be determined later and $m > 1$.
Note that, $W(Q(0) = x, U(0) = u) = W(x)$,
Interpret this as an approximation to,

$$\|x\|^{m+1} W(x) \approx \mathbb{E}_{\hat{x}} \left\{ \int_0^T (1 + aL_{\hat{x}}(t))^{-m} dt \right\} = V(q_{\hat{x}}(t))$$

equality when $\|x\| \to \infty$, and $\hat{x} = \frac{x}{\|x\|}$ is initial state

Brief Sketch II: Drift Break Up

θ^1 backward shift operator

$$\theta^1 \mathcal{W} - \mathcal{W} = A + B + C$$

where

$$\begin{aligned}
A &\doteq -[1 + \|Q(0)\| + a\|Q(n)\|]^{-m}, \\
B &\doteq \sum_{n=1}^{\|Q(0)\|T} \left\{ [1 + \|Q(1)\| + a\|Q(n)\|]^{-m} \right. \\
&- \left. [1 + \|Q(0)\| + a\|Q(n)\|]^{-m} \right\}, \\
C &= \sum_{n=\|Q(0)\|T+1}^{\|Q(1)\|T} [1 + \|Q(1)\| + a\|Q(n)\|]^{-m}.
\end{aligned}$$

The term 'A' provides the negative drift, see Meyn

Brief Sketch III: Final Bound

After some manipulation ......,

$$\limsup_{\|x\|\to\infty} \|x\|^m \mathbb{E}_{\hat{x}} \left\{\theta^1 \mathcal{W} - \mathcal{W}\right\} \leq -(1+a)^{-m} +$$
$$m\mathbb{E}_{\hat{x}} \left\{ \int_0^\infty (1 + a\tilde{L}(s))^{-m-1} \mathrm{d}s \right\} + \mathbb{E}_{\hat{x}} \left\{ T(1 + a\tilde{L}(T))^{-m} \right\}$$

which bounds the drift in terms of integrals of the fluid process
Using earlier bounds,

$$m\mathbb{E}_{\hat{x}} \left\{ \int_0^\infty (1 + a\tilde{L}(s))^{-m-1} \mathrm{d}s \right\} \leq mBC(a\theta)^{-m-1} \frac{1}{1-\alpha}$$

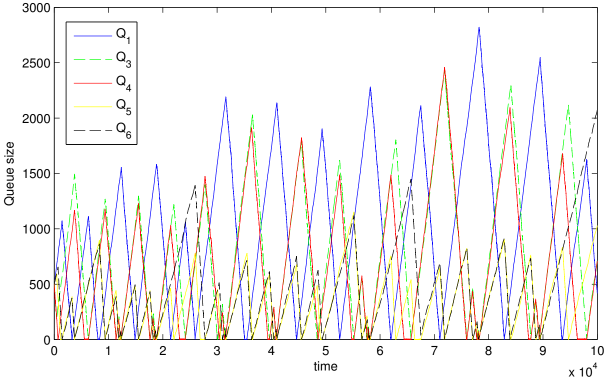
and take a large enough
Also,

$$\mathbb{E}_{\hat{x}} \left\{ T[1 + a\tilde{L}(T)]^{-m} \right\} \leq a^{-m} \mathbb{E}_{\hat{x}} \left\{ TL^{-m}(T/b) \right\} \tag{5}$$

but RHS bounded as $\limsup_{T\to\infty} \mathbb{E}_{\hat{x}} \left\{ TL^{-m}(T) \right\} = 0$, for $\rho \in (\rho^*, 1]$

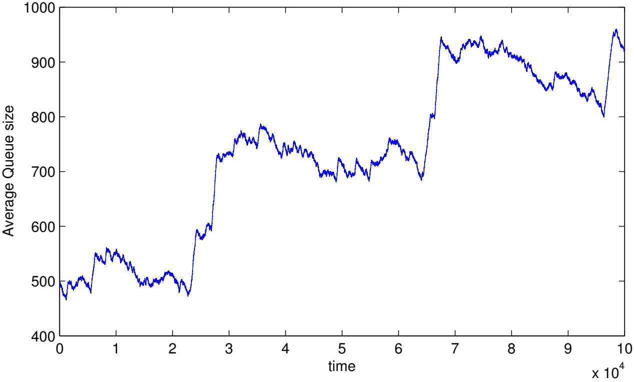
Sample Paths for PreLimit Queues

- Initial queue size $Q_\ell(0) = 500, \gamma = 2, \rho = 0.97$
- Backlogs cleared - queues exhibit qualitative behaviour of the fluid limit



Sample Path for Mean Queue Length

- Saw Tooth Growth pattern in node average queue size
- Large Increments immediately after M_4 visits (one per M_1 cycle)
- Slight downward trend otherwise



Conclusions

- Aggressive activation functions have piecewise linear (oscillatory) limits
- Instability for $\gamma > 1$ in a broad class of interference networks
- Same applies for random capture algorithms

Conclusions

- Aggressive activation functions have piecewise linear (oscillatory) limits
- Instability for $\gamma > 1$ in a broad class of interference networks
- Same applies for random capture algorithms
- Under Investigation
- Instability for any $\gamma > 1/K$ for network sizes of order K
- Sharpen fluid limit instability/stability criteria

Thanks!

- P. Robert, *Stochastic Networks and Queues (Stochastic Modeling and Applied Probability)*, Springer Verlag 2008.
- P. Billingsley, *Weak Convergence of Probability Measures*, Wiley–Interscience, 1968.
- S. N. Ethier, T. Kurtz *Markov Processes: Characterization and Convergence*, Wiley Interscience, 1986.
- Sean P. Meyn, *Transience of MultiClass Queueing Networks via Fluid Limit Models*, Annals of Probability, 1992.
- J.L.Dai, *On Positive Harris Recurrence of MultiClass Queueing Networks: A Unified Approach via Fluid Limits*, Annals of Applied Probability, Vol. 5, No. 1 pp 49- 77, 1995.
- W. Whitt, *Weak Convergence of Probability Measures on the Function Space* $C[0,\infty)$, Technical Report No. 120, Stanford University, April 1969.
- L.C.G. Rogers and D. Williams, *Diffusion Processes and Martingales: Foundations*, Wiley.