# To Prove or to Disprove: Information Inequalities Prover, Automated Reasoning by Convex Optimization & Software-as-a-Service

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Information expression: linear combination of information measures involving finite number of random variables.

Information inequality: $f \geq c$, where $f$ is an information expression and $c$ is a constant.

# Information Theoretic Inequalities

Some information inequalities are always true, e.g.,

$$H(A,B,C) \geq I(A;B)$$

while others may not, e.g.,

$$H(A,B,C) \geq I(A;B) + I(B;C)$$

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Given an information theoretic inequality, is there an algorithm to prove or disprove it automatically?

# The ITIP Framework: Information-Theoretic Inequality Prover by Linear Programming

# Information Expressions in Canonical Form

Any information expression can be represented as a linear combination of entropies and joint-entropies. e.g.:

$$
\begin{aligned}
H(A|B) &= H(A,B) - H(B) \\
I(A;B) &= H(A) + H(B) - H(A,B) \\
I(A;B|C) &= H(A,C) + H(B,C) - H(A,B,C) - H(C)
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e.g., for $n = 3$, we have:

$$H(A),\ H(B),\ H(C),\ H(A,B),\ H(A,C),\ H(B,C),\ H(A,B,C)$$

# Information Expressions in Canonical Form

This inspires us to express information inequalities in form of $b^T h \geq 0$, where $h \in \mathbb{R}^k$. e.g.

$$H(A|B) \geq I(A;B) \Longrightarrow 2H(A,B) - 2H(B) - H(A) \geq 0$$

| H(A) | H(B) | H(AB) |
|---|---|---|
| -1 | -2 | 2 |

$$\Longrightarrow b^T h \geq 0,$$

where

$$b = \begin{bmatrix} -1 & -2 & 2 \end{bmatrix}^T$$

# Elemental Inequalities

A valid $h$ has to ensure nonnegativity of all information measures.

e.g., for $n = 2$, we must have

$$\begin{aligned} H(A|B) &= H(A,B) - H(B) && \geq 0 \\ H(B|A) &= H(A,B) - H(A) && \geq 0 \\ I(A;B) &= H(A) + H(B) - H(A,B) && \geq 0 \end{aligned}$$

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This is equivalent to $Dh \geq 0$, where

$$D = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

# Elemental Inequalities

For $n$ random variables, there are

$$m = n + \binom{n}{2} 2^{n-2}$$

elemental inequalities to satisfy, thus $D \in \mathbb{R}^{m \times n}$

# Linear Programming Formulation

From the above discussion, we notice that to prove or disprove a given information inequality, we can calculate the lower bound of $b^T h$. This can be formulated into a simple LP:

$$\begin{aligned} \min \quad & p = b^T h \\ \text{s.t.} \quad & Dh \geq 0 \end{aligned}$$

If $p^* \geq 0$, we know the corresponding inequality is always true, otherwise we conclude that it does not always hold.

# Proof/Disproof Construction

Wait a minute... What this does is just verification, not proving. What we want is human-readable proof/disproof.

# Proof/Disproof Construction

Consider the dual problem:

$$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & b - D^T\lambda = 0 \\ & \lambda \geq 0 \end{aligned}$$

Assume the problem is feasible. Let the optimal solution be $\lambda^*$, we know $b = D^T\lambda^*$

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Notice that, for **ANY** feasible $h$, we have

$$b^T h = \lambda^{*T} D h \geq 0$$

This is our proof!

# Toy Example

Let's prove $2H(A,B) >= H(A) + H(B) \implies b = \begin{bmatrix} -1 & -1 & 2 \end{bmatrix}^T$,

$$D = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad \begin{array}{c} H(A|B) = H(A,B) - H(B) \geq 0 \\ H(B|A) = H(A,B) - H(A) \geq 0 \\ I(A;B) = H(A) + H(B) - H(A,B) \geq 0 \end{array}$$

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Solve that $\lambda^* = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$

Proof:

$$\begin{aligned} & 2H(A,B) - H(A) - H(B) \\ &= (H(A,B) - H(B)) + (H(A,B) - H(A)) \\ &= H(A|B) + H(B|A) \geq 0 \end{aligned}$$

# Disproof Construction

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There exists a subset of entries of $h^*$ that's $\infty \implies$ Add an upper bound to $h$

e.g. let $H(X_1, X_2, \cdots, X_n) = 1$.

$$\begin{aligned} \min \quad & b^T h \\ \text{s.t.} \quad & Dh \geq 0 \\ & e^T h = 1 \end{aligned}$$

, where $e = \left[0, 0, \cdots, 1\right]^T$

$$\begin{aligned} \max \quad & -\mu \\ \text{s.t.} \quad & b - D^T \lambda + \mu e = 0 \\ & \lambda \geq 0 \end{aligned}$$

# The Computational Challenge

# $D$ Matrix Size

$D \in \mathbb{R}^{m \times k}$, where $m = n + \binom{n}{2} 2^{n-2}$ and $k = 2^n - 1$

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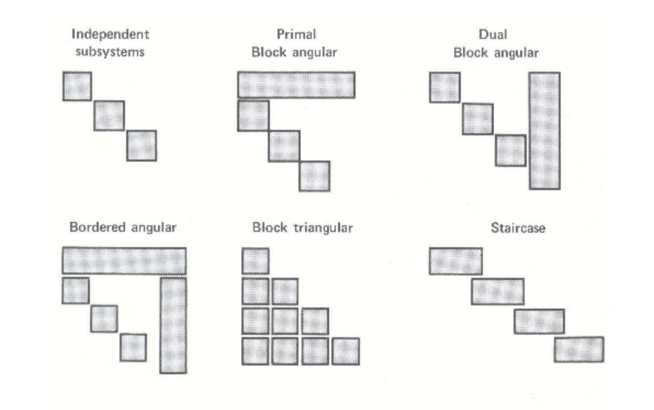
| n | m | k | sparsity |
|---|---|---|---|
| 2 | 3 | 3 | 0.222222 |
| 5 | 85 | 31 | 0.878558 |
| 10 | 11530 | 1023 | 0.996095 |
| 15 | 860175 | 32767 | 0.999878 |
| 20 | 49807380 | 1048575 | 0.999996 |

# $D$ Matrix Spatial Structure

One of the standard tricks to deal with large-scale LPs is to look for spatial structures in the constraint matrix ($D$ in our case) and decompose the LP into a number of smaller LPs.



# $D$ Matrix Spatial Structure

Unfortunately this cannot be done for our $D$ matrix.

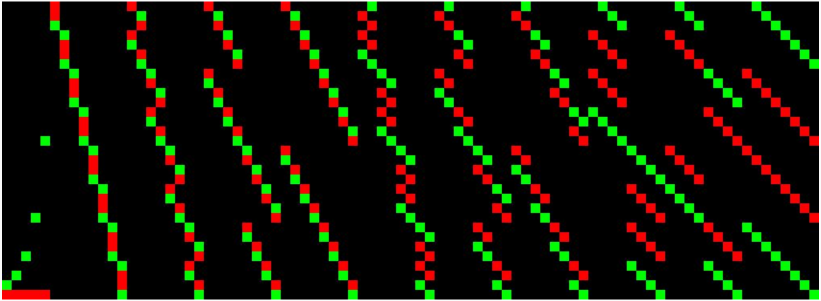


**Figure 1:** Sparsity map for $D^T$ when $n = 5$

No spatial structure that we can utilize :(

# Degeneracy

It's easy to see that in the dual problem

$$\begin{aligned} \max \quad & 0 \\ \text{s.t.} \quad & b - D^T\lambda = 0 \\ & \lambda \geq 0, \end{aligned}$$

there's an infinite number of optimal solution $\lambda^*$, which means the problem is highly degenerate.

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The simplex method would struggle on degenerate problems.

Can we use numerical algorithms (e.g. interior-point method)? Yes, but there's a catch...

# An ADMM-Based Scalable Solution

# Generic ADMM

Consider the alternating direction method of multipliers (ADMM) to solve the following optimization problem:

$$\begin{aligned} \min \quad & f(x) + g(y) \\ \text{s.t.} \quad & Ax + By = c, \end{aligned}$$

with $\rho$-augmented Lagrangian:

$$L_\rho = f(x) + g(y) + \lambda^T(Ax + By - c) + \frac{\rho}{2}||Ax + By - c||^2,$$

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a generic ADMM algorithm is given as follows:

**ALGORITHM 2:** Generic ADMM Algorithm

**repeat**

1. $x$-update: $x^{k+1} = \arg\min\{L_\rho(x, y^k, \lambda^k)\}$
2. $y$-update: $y^{k+1} = \arg\min\{L_\rho(x^{k+1}, y, \lambda^k)\}$
3. $\lambda$-update: $\lambda^{k+1} = \lambda^k + \rho(Ax^{k+1} + By^{k+1} - c)$

**until** *Stopping criteria is met*;

# Problem Reformulation

Consider the following reformulation:

$$\begin{aligned} \min \quad & b^T h \\ \text{s.t.} \quad & 0 \leq Ah \leq 1, \end{aligned}$$

where $A = \begin{bmatrix} D \\ I \end{bmatrix}$.

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We can just solve this one problem!

# Problem Reformulation

Now we reformulate the problem to ADMM form by adding slack variables $u$ and $v$:

$$\begin{aligned} \min \quad & b^T h \\ \text{s.t.} \quad & Bh + z = c, \end{aligned}$$

where $B = \begin{bmatrix} A \\ A \end{bmatrix}, z = \begin{bmatrix} -u \\ v \end{bmatrix}, c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, u, v \geq 0$

# ADMM Subproblem Computation

The $\rho$-augmented Lagrangian of the problem is:

$$L_\rho = b^T h + \lambda^T (Bh + z - c) + \frac{\rho}{2} ||Bh + z - c||^2$$

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This is an unconstrained QP, and we have a closed-form solution:

$$h^{k+1} = -\frac{1}{\rho}(B^T B)^{-1}(b + B^T \lambda^k + \rho B^T z^k - \rho B^T c).$$

It's easy to prove that $(B^T B)^{-1}$ exists.

# ADMM Subproblem Computation

$z$-update:

$$z^{k+1} = \arg\min L_\rho(h^{k+1}, z, \lambda^k)$$

Recall that $z = \begin{bmatrix} -u & v \end{bmatrix}^T$, so we can split this subproblem into $u$-update and $v$-update:

$$u^{k+1} = \arg\min\{L_\rho(h^{k+1}, u, \lambda^k) | u \geq 0\}$$
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They are constrained QPs, but luckily their KKT systems can be solved directly, giving us closed-form solutions:

$$u^{k+1} = (Ah^{k+1} + \frac{1}{\rho}\lambda_u^k)_+$$
$$v^{k+1} = (1 - Ah^{k+1} - \frac{1}{\rho}\lambda_v^k)_+$$

# ADMM Algorithm

**ALGORITHM 3:** ITIP Algorithm

**repeat**

1. $h$-update: $h^{k+1} = -\frac{1}{\rho}(B^T B)^{-1}(b + B^T \lambda^k + \rho B^T z^k - \rho B^T c)$
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The advantages:

1. Have simple closed-form solution for every subproblem (linear algebraic structures enable scalable and distributed computation, CPU-GPU configuration, cloud computing)

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The disadvantage:

- Need to “crossover” to obtain “elegent” proofs/disproofs

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In fact,

$$\begin{aligned} & H(XYZ) - H(X|YZ) - H(Y|XZ) - H(Z|XY) \\ =& 0.8(I(X;Y) + I(X;Z|Y) + I(Y;Z|X)) + \\ & 0.1(I(X;Z) + I(X;Y|Z) + I(Y;Z|X)) + \\ & 0.1(I(Y;Z) + I(X;Z|Y) + I(X;Y|Z)) \quad (4) \end{aligned}$$

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"Beauty is the first test: there is no permanent place in this world for ugly mathematics." - G. H. Hardy

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4. ”The proof is trivial and is left as an exercise.” - Annoymous

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The beauty of mathematics contains a few parameters: It is short.

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The beauty of mathematics contains a few parameters: It is short. It is unexpected. It enriches people. It is applicable to other problems.
- Ron Aharoni (Mathematics poetry and beauty)

# “Shortest” Proof by Automated Reasoning

Consider the “Shortest” Proof problem:

$$\begin{aligned} \min \quad & \|\lambda\|_0 \\ \text{s.t.} \quad & b - D^T\lambda = 0 \\ & \lambda \geq 0 \end{aligned}$$

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Automated reasoning to generate short computer proofs via recent convex relaxation ideas for sparse recovery in compressed sensing:

$$\begin{aligned} \min \quad & \|\lambda\|_1 \\ \text{s.t.} \quad & b - D^T\lambda = 0 \\ & \lambda \geq 0 \end{aligned}$$

This is a linear program that often yields the shortest proof.

# A Scalable Software-as-a-Service

# The ITIP Web Service (WiP)

`https://itip.algebragame.app`

# Q&A

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