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Rumor Source Detection in Finite Graphs with Boundary Effects by Message-passing Algorithms

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***Abstract*—Finding information source in viral spreading has important applications such as to root out the culprit of a rumor spreading in online social networks. In particular, given a snapshot observation of the rumor graph, how to accurately identify the initial source of the spreading? In the seminal work by Shah and Zaman in 2011, this statistical inference problem was formulated as a maximum likelihood estimation problem and solved using a rumor centrality approach for graphs that are degree-regular. This however is optimal only if there are no boundary effects, e.g., the underlying number of susceptible vertices is countably infinite. In general, all practical real world networks are finite or exhibit complex spreading behavior, and therefore these boundary effects cannot be ignored. In this paper, we solve the constrained maximum likelihood estimation problem by a generalized rumor centrality for spreading in graphs with boundary effects. We derive a graph-theoretic characterization of the maximum likelihood estimator for degree-regular graphs with a single end vertex at its boundary and propose a message-passing algorithm that is near-optimal for graphs with more complex boundary consisting of multiple end vertices.**

I. Introduction

Shah et al in [1] formulated the problem of finding the culprit of a rumor spreading as a maximum likelihood estimation problem. In particular, a *rumor centrality* approach, a form of network centrality, solves this problem for degree-regular tree graphs *assuming that the underlying number of susceptible vertices is countably infinite*. This means that each infected vertex always has a susceptible vertex as its neighbor. The infected vertex with the most number of ways to spread to other vertices is the *rumor center* that coincides with the maximum likelihood estimate. This rumor centrality approach was subsequently extended to various problem settings, e.g., extension in [2]–[4] to random trees, extension in [4], [5] to constrained observations, extension in [6] to multiple source detection and extension in [7] to detection with multiple snapshot observations. It is shown in [8], [9] that the rumor center is equivalent to the graph centroid.

There is however a key limitation in the rumor centrality approach. *A main modeling assumption is that the underlying graph (i.e., number of susceptible vertices) is countably infinite.* First, this does not accurately model practical real-world networks where the number of underlying susceptible vertices is countably finite. For example, the current world's population is about seven billion. Second, we can incorporate even more complex dynamical spreading model. The *vertices without susceptible neighbors*, i.e., end vertices, in the rumor graph can model users who are disinterested in passing on the rumor to their neighbors. The end vertices thus introduce nontrivial boundary effects that cannot be ignored and this makes the constrained maximum likelihood estimation problem a much harder problem. In essence, the boundary effects limit the span of the dynamical spreading process, and in fact may increase the likelihood that vertices near the boundary are the culprit. Hence, the number of boundary end vertices and their location can significantly shape spreading and therefore the probabilistic inference performance. To be exact, existing algorithms in the literature, e.g., [1], [2], [5]–[7], are no longer optimal *even with the presence of a single end vertex* in degree-regular rumor graphs with boundary effects. Rumor source detection over a graph with boundary effects is an open problem, but one that is more realistic and also significantly generalizes all previous work [1], [2], [5]–[7].

A. Our Contributions

The main contributions are summarized as follows:

- We propose a *generalized rumor centrality* to solve the maximum likelihood estimation problem by taking into account the boundary effects of spreading.
- For a finite *degree-regular tree graph* with a single end vertex at the boundary, we derive a key result to analytically characterize the optimal maximum likelihood estimator.
- We extend our analysis to degree-regular tree graphs with multiple end vertices and propose message passing-based algorithms that narrow down the search and are near-optimal in performance.

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- The message passing algorithm gives enhanced and better performance as compared with a naive approach that merely uses the graph centroid for tree graphs.

II. Trees with a Single End Vertex

Let us consider the case when G is a regular tree that is finite, e.g., there are leaf vertices each with degree one. Now, consider the rumor graph $G_n \subseteq G$, where G_n only has a single *end vertex* that receives the rumor but does not spread the rumor further. In this section, we study how the influence of this single *end vertex* in G_n affects the maximum-likelihood estimation performance in finding the rumor culprit.

In particular, we compare this single end vertex special case with a naive prediction that assumes an underlying *infinite graph*. This illustrates that ignoring the boundary effect in the finite graph ultimately leads to a wrong estimate and thus requires an in-depth analysis and new rumor source detection algorithm design for the general case of *finite graphs*.

A. Impact of Boundary Effects On $P(G_n|v)$

Example 1. Consider G as a finite 3-regular tree and $G_5 \subseteq G$ as shown in Figure 1. Consider $P(G_5|v_1)$ and with a spreading order $\sigma : v_1 \to v_2 \to v_5 \to v_3 \to v_4$, we have $P(\sigma|v_1) = (1/3)\cdot(1/4)\cdot(1/3)\cdot(1/4)$. Had v_5 not been the end vertex, then $P(\sigma|v_1) = (1/3)\cdot(1/4)\cdot(1/5)\cdot(1/6)$. This demonstrates that the order at which the rumor spreads to the end vertex v_5 is important when computing $P(\sigma|v_1)$. Table I lists down all the spreading orders sorted according to the position of v_5. In particular, $P(G_5|v_1) = 34/720$. We also have $P(G_5|v_4) = P(G_5|v_3) = 7/720$ by symmetry, and $P(G_5|v_2) = 40/720$. Now, observe that v_1 is the *centroid*, but $P(G_5|v_1) < P(G_5|v_2)$, and thus $\hat{v}$ is not v_1.

Example 1 reveals some interesting properties of boundary effects due to even a single end vertex:

- $P(\sigma_i|v)$ increases with how soon the end vertex appears in σ_i (as ordered from left to right of σ_i).
- When there is at least one end vertex in G_n, then $P(G_n|v)$ is no longer proportional to $|M(v,G_n)|$.

This means that $P(\sigma_i|v)$ is no longer a constant for each i, and is dependent on the position of the *end vertex* in each spreading order. We proceed to compute $P(\sigma_i|v)$ as follows. For brevity of notation, let v_e be the end vertex and define

$$M_v^{v_e}(G_n,k) = \{\sigma | v_e \text{ is on the } k\text{th position of } \sigma\};$$
$$P_v^{v_e}(G_n,k) = P(\sigma|v), \text{ for } \sigma \in M_v^{v_e}(G_n,k),$$

where $M_v^{v_e}(G_n,k)$ is the set of all the spreading orders starting from v and with v_e at the kth position, and its size is the combinatorial object of interest:

$$m_v^{v_e}(G_n,k) = |M_v^{v_e}(G_n,k)|. \tag{1}$$

Let D be the distance (in terms of number of hops) from v to v_e. Then we have

$$|M(v,G_n)| = \sum_{k=D+1}^{n} m_v^{v_e}(G_n,k). \tag{2}$$

Now, (2) shows that $M(v,G_n)$ can be decomposed into $M_v^{v_e}(G_n,k)$ for $k = D+1, D+2, \ldots, n$. This decomposition allows us to handle the boundary effect due to the different position of the end vertex in each spreading order. Let $P_v^{v_e}(G_n,k)$ be the corresponding probability for each k. We can rewrite $P(G_n|v)$ for the finite tree graph as:

$$P(G_n|v) = \sum_{k=D+1}^{n} m_v^{v_e}(G_n,k)\cdot P_v^{v_e}(G_n,k). \tag{3}$$

Thus, the rumor source detection problem is to find the vertex $\hat{v}$ that solves

$$P(G_n|\hat{v}) = \max_{v_i\in G_n} P(v_i|G_n). \tag{4}$$

Since $P(G_n|v)$ is no longer proportional to $|M(v,G_n)|$, we now describe how to compute $P(G_n|v)$ in G_n over an underlying d-regular graph G. First, consider $P_v^{v_e}(G_n,k)$ and let $z_d(i) = (i-1)(d-2)$, then

$$P_v^{v_e}(G_n,k) = \prod_{i=1}^{k-1}\frac{1}{d+z_d(i)}\cdot\prod_{i=k-1}^{n-2}\frac{1}{d+z_d(i)-1}, \tag{5}$$

where the first factor of $P_v^{v_e}(G_n,k)$ in (5) is the probability that k vertices are infected once the rumor reaches the end vertex, (i.e., v_e is the kth vertex infected in G_n), and the second factor is the probability that all remaining $n-k$ vertices are infected thereafter. On the other hand, the value of $m_v^{v_e}(G_n,k)$ in (1) is dependent on the network topology, and thus there is no closed-form expression in general (though when G_n is a line, a closed-form expression for $m_v^{v_e}(G_n,k)$ is given in (6)). We now use this special case to demonstrate how the end vertex affects the probability $P(v|G_n)$.

B. Analytical Characterization of Likelihood Function

Suppose G is a finite degree-regular tree and G_n is a line graph with a single end vertex. Without loss of generality, suppose n is odd (to ensure a unique v_c) and $n = 2t+1$ for some t. Label all the vertices in G_n as shown in Figure 2. To compute $P(G_n|v_i)$ for $v_i \in G_n$, from (3) and (5), we already have $P_{v_i}^{v_e}(G_n,k)$, so we need to compute $m_{v_i}^{v_e}(G_n,k)$. The enumeration of $m_{v_i}^{v_e}(G_n,k)$ can be accomplished in polynomial-time complexity with a path-counting message-passing algorithm (see, e.g., Chapter 16 in [10]). In particular, we have a closed-form expression for $m_v^{v_e}(G_n,k)$ given by:

$$m_{v_i}^{v_e}(G_n,k) = \binom{k-2}{k-n+i-1}, \tag{6}$$

when $i \neq n$. Thus, we have the following analytical formula for $P(G_n|v_i)$:

$$P(G_n|v_i) = \begin{cases} \displaystyle\prod_{l=1}^{n-1}\frac{1}{z_d(l)+1}, & i = n; \\ \displaystyle\sum_{k=n-i+1}^{n}\binom{k-2}{k-n+i-1}\cdot P_{v_i}^{v_e}(G_n,k), & \text{otherwise,} \end{cases} \tag{7}$$

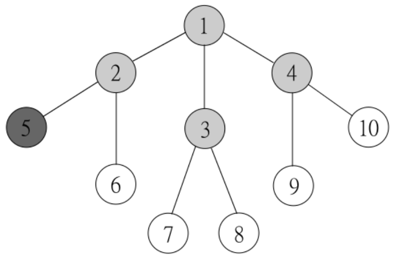
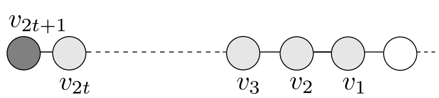


Fig. 1. Example of G as a finite 3-regular tree and G_n as a subtree with a single end vertex $v_e = v_5$. The maximum likelihood estimate $\hat{v}$ is v_2, while a naive application of the rumor centrality in [2], i.e., the centroid v_c of G_n, yields v_1.

TABLE I
NUMERICAL EXAMPLE OF $P(\sigma_i|G_5)$ USING G_5 IN FIGURE 1

σ_i	Spreading Order	$P(\sigma_i\|G_5)$	σ_i	Spreading Order	$P(\sigma_i\|G_5)$
σ_1	v_1, v_2, v_5, v_3, v_4	$\frac{1}{144}$	σ_7	v_1, v_2, v_3, v_4, v_5	$\frac{1}{360}$
σ_2	v_1, v_2, v_5, v_4, v_3	$\frac{1}{144}$	σ_8	v_1, v_2, v_4, v_3, v_5	$\frac{1}{360}$
σ_3	v_1, v_3, v_2, v_5, v_4	$\frac{1}{240}$	σ_9	v_1, v_3, v_2, v_4, v_5	$\frac{1}{360}$
σ_4	v_1, v_4, v_2, v_5, v_3	$\frac{1}{240}$	σ_{10}	v_1, v_3, v_4, v_2, v_5	$\frac{1}{360}$
σ_5	v_1, v_2, v_3, v_5, v_4	$\frac{1}{240}$	σ_{11}	v_1, v_4, v_2, v_3, v_5	$\frac{1}{360}$
σ_6	v_1, v_2, v_4, v_5, v_3	$\frac{1}{240}$	σ_{12}	v_1, v_4, v_3, v_2, v_5	$\frac{1}{360}$

Fig. 2. G_n as a line graph with a single end vertex $v_e = v_{2t+1}$.



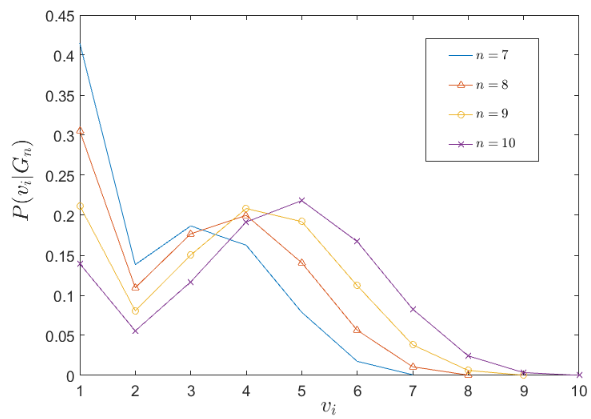


Fig. 3. $P(v|G_n)$, where G_n is a line graph with a single end vertex v_1 over an underlying 4-regular finite graph. Observe that v_1 in this figure is corresponding to v_{2t+1} in Figure 2.

where $P_{v_i}^{v_e}(G_n, k)$ is given in (5).

In (7), we suppose that n is odd. Using (7), let us numerically compute $P(G_n|v_i)$ for all v_i in Figure 3, where G is a 4-regular tree and G_n is a line graph with a single end vertex $v_e = v_n$ as boundary for different values of $n = 7, 8, 9, 10$. The x-axis is the vertex v_i where $i = 1, 2, \ldots, 10$, and the y-axis plots $P(v_i = v^\star|G_n)$. As shown in Figure 3, the influence due to the end vertex on $P(v_i = v^\star|G_n)$ dominates that of the centroid when $n = 7, 8, 9$. However, when $n = 10$, the influence due to the centroid v_c on $P(v_i|G_n)$ dominates that of the end vertex v_e.

Theorem 1. *Suppose G is a d-regular graph ($d > 2$) with finite order. If G_n is a line-graph with a single end vertex, then $\exists j$ such that $P(v_c|G_n) > P(v_e|G_n)$ when $n > j$.*

Remark: When n increases, the location of v_c in G_n converges to a neighborhood as n grows larger.

Example 2. To verify Theorem 1, we plot $P(v_i|G_n)$ for an example of a line G_n with G being a finite 4-regular graph in Figure 3. Clearly, we have $j = 9$.

Theorem 1 implies that, for any d-regular underlying graph, when G_n is a line with a single end vertex, the influence of the end vertex v_e on $P(v_i|G_n)$ decreases monotonically as n grows. In fact, this reduces to the special case in [2], when n goes to infinity asymptotically, i.e., $\hat{v}$ is the centroid v_c.

C. Optimality Characterization of Likelihood Estimate

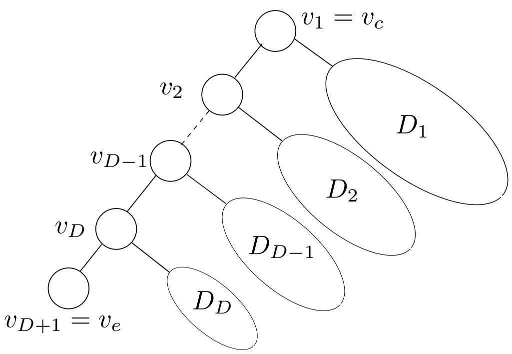
From (3), we observe that, in addition to the spreading order, the distance (number of hops) between the end vertex and v also affects the likelihood probability $P(G_n|v)$. Let v_c and v_p be two adjacent vertices in G_n with $|M(v_c, G_n)| > |M(v_p, G_n)|$. Suppose G_n has an end vertex v_e and assume that v_c is closer to v_e than v_p. Then, these two assumptions may lead us to $P(G_n|v_c) > P(G_n|v_p)$. We formalize this main optimality result that characterizes the probabilistic inference performance between any two adjacent vertices and the location of $\hat{v}$ in G_n with a single end vertex v_e.

Theorem 2. *Let G be a tree with finite order and $G_n \subseteq G$ be a subtree of G with a single end vertex $v_e \in G_n$, then the maximum likelihood estimator $\hat{v}$ with maximum probability $P(G_n|v)$ is located on the path from the centroid v_c to v_e.*

III. TREES WITH MULTIPLE END VERTICES

In this section, we consider the case when G_n has more than a single end vertex (naturally, this also means $d > 2$ in G ruling out the trivial case of G being a line). The key insight from the single end vertex analysis still holds: Once the rumor reaches an end vertex in G, $\hat{v}$ can be located near this very first infected end vertex. In addition, the algorithm design approach is to decompose the graph into subtrees to narrow the search for the maximum-likelihood estimate solution. To better understand the difficulty of solving the general case, we

Fig. 4. Theorem 2 illustrates a unique optimality feature that $\hat{v}$ is located on the path from v_c to v_e, i.e., $\hat{v} \in \{v_1, v_2, ..., v_{D+1}\}$. Denote the distance (number of hops) from v_c to v_e as D, and let P denote the path from v_c to v_e where $P = (v_1, v_2, ..., v_{D+1})$. We define $D_i = \{\ v \in G_n |$ the path from v to v_e containing $v_i\ \}$, i.e., D_i is a set of subtree vertices as illustrated.



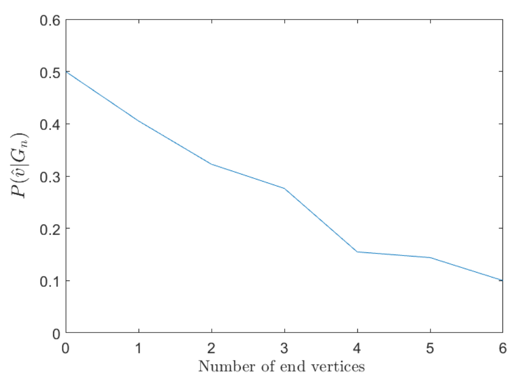


Fig. 5. A numerical plot of $P(\hat{v}|G_n)$ versus the number of end vertices by using the example in Figure 1.

start with a special case: The entire finite underlying network is infected, i.e., $G_n = G$, then $P(G_n|v) = 1/n$ for each vertex in G_n, as each vertex is equally likely to spread the rumor to all the other vertices in G to yield $G_n = G$. In this case, $P(\hat{v}|G_n)$ is exactly the minimum detection probability. For example, consider a 3-regular underlying graph G with 10 vertices. Figure 5 illustrates the detection probability as the number of end vertices in G_n increases with increasing n as the rumor spreads. This means that the problem is harder to solve when the number of end vertices increases.

A. Message-passing Algorithm

We propose a message-passing algorithm to find $\hat{v}$ on the finite regular tree G by leveraging the key insights derived in the previous sections. We summarize these features as follows:

1) If there is only a single end vertex v_e in G_n, then $\hat{v}$ is located on the path from v_c to v_e.
2) If $G_n = G$, then for all $v_i \in G_n$, $P(v_i|G_n) = 1/n$.
3) If G_n has q end vertices, then there exists an n' such that, if $n > n'$, then $P(v_c|G_n) > \max\limits_{1\le i\le k}\{P(v_{e_i}|G_n)\}$. Furthermore, n' increases as q increases.
4) If two vertices v_1 and v_2 are on the symmetric position of G_n, then $P(v_1|G_n) = P(v_2|G_n)$. For example, v_3 and v_4 are topologically symmetric in Figure 1.

In particular, Feature 1 is the optimality result pertaining to the decomposition of G_n into subtrees to narrow the search for $\hat{v}$. The subtree t_{ML} in G_n corresponds to first finding the decomposed subtree containing the centroid and the likelihood estimate needed for Theorem 2 to apply. Then, Features 3 and 4 identify $\hat{v}$ on a subtree t_{ML} of G_n as Theorem 2 only pinpoints the relative position of $\hat{v}$.

Algorithm 1 Message-passing algorithm to compute $\hat{v}$ for G_n with multiple end vertices

Input: G_n, $\kappa = \{\}$

Step 1: Compute the centroid v_c of G_n.

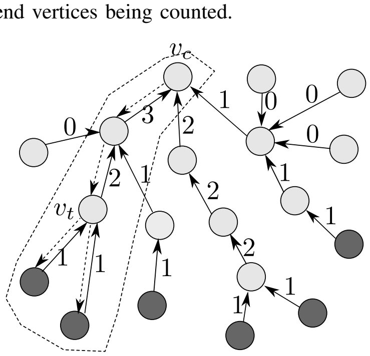
Step 2: Choose v_c as the root of a tree and use a message-passing algorithm to count the number of end vertices on each branch of this rooted tree.

Step 3: Starting from v_c, and at each hop choose the child with the maximum number of end vertices (if there were more children with the same maximal number of end vertices, then choose all of them). This tree traversal yields a subtree t_{ML} rooted at v_c.

Output: $\kappa = \{$parent vertices of leaves of $t_{ML}, v_c\}$

Algorithm 1 first finds the centroid v_c of G_n, and then determines the number of end vertices corresponding to each branch of the centroid v_c. The final step is to collect vertices on the subtree where $\hat{v}$ is, and this leads to a subtree of G_n denoted as t_{ML}. Observe that each step requires $O(n)$ computational time complexity. Observe that t_{ML} in a graph with multiple end vertices is akin to the *path* from the centroid to the end vertex in a rumor graph with a single end vertex in Section II. Finally, we obtain a set κ containing the parent vertices of the leaves of t_{ML} and v_c.

Fig. 6. An illustration of how Algorithm 1 works on a tree graph rooted at v_c with six end vertices (more shaded). Observed that v_c branches out to three subtrees. Here, t_{ML} is the subtree containing the five vertices within the dotted line. The numerical value on the edge indicates the message containing the number of end vertices being counted.



Now, let us use the example in Figure 6 to illustrate how Algorithm 1 runs. Let G_{19} be the network in Figure 6 with the six end vertices depicted as more shaded. Suppose v_c is determined by the end of Step 1. Then, Step 2 enumerates the number of end vertices at each branch of the subtrees connected to v_c, and these numbers are then passed iteratively from the leaves to v_c. These messages correspond to the numerical value on the edges in Figure 6. The message in Step 2 is an $upward$ (leaf-to-root) message. Step 3 is a message passing procedure from v_c back to the leaves, which is a $downward$ message, and the message is the maximum of number of end vertices in each branch. For example, the message from v_c to $child(v_c)$ is $\max\{1, 2, 3\}$ which is 3. Lastly, the second part of Step 3 collects those vertices whose $upward\ message = downward\ message$. For example, the left hand-side child of v_c is first added to t_{ML}, and then v_t is added to t_{ML}, and finally, the two leaves on the left hand-side is added to t_{ML}. Observe that t_{ML} must be connected.

B. Simulation Results for Finite d-regular Tree Networks

We simulate the rumor spreading in the degree regular tree network G for $d = 3, 4, 5, 6$ with $|G| = 1000$ and $|G_n| = 100$. For each d, we simulate a thousand times the spread of a rumor on G by picking $v^\star$ uniformly on G, and compare the average performance of Algorithm 1 and a naive heuristic that simply uses the rumor centrality approach in [8]. To fairly compare these two algorithms, say let $k = |\kappa|$ when Algorithm 1 yields a set with k vertices, then the naive heuristic finds a set of k vertices having the top k maximum $|M(v, G_n)|$ for all v of G_n. Obviously, k depends on the topology of G_n for each run of the simulation, and thus is not a constant in general over that thousand times. To quantify the performance of these two algorithms, let us define the error function of a set S:

$$\text{error}(S) = min\{\text{distance}(v, v^\star)|\forall v \in S\}.$$

This is simply the smallest number of hops between $v^\star$ and the nearest vertex in S. Figure 7 shows the distribution of these error hops for both algorithms when the underlying graph G is 4-regular. This illustrates that Algorithm 1 can make a good guess for $P(\text{error}(\kappa) \leq 1 \text{ hop}) > 0.70$ in most cases, but there are occasions when the error is large. Table II shows the average of the error (number of hops) between the estimate and $v^\star$ for a thousand simulation runs. We can observe that the average error decreases as d grows. The reason is that the number of infected vertices is fixed, and so as d becomes larger, the diameter of the graph becomes smaller, and moreover, the Top-k heuristic always chooses the set of vertex in the "center" of G_n. Hence, the average error decreases.

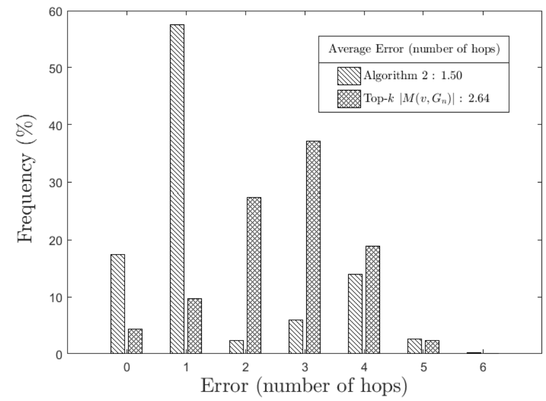


Fig. 7. Comparing the error distribution (in the number of hops) between Algorithm 1 and the top-k algorithm when G is a 4-regular finite tree.

TABLE II
AVERAGE ERROR (IN TERMS OF NUMBER OF HOPS) COMPARING ALGORITHM 1 AND TOP-k ALGORITHM WHEN G IS A d-REGULAR GRAPH, FOR $d = 3, 4, 5, 6$.

d	$\|\kappa\|$	Algorithm 1	Top-k Algorithm
3	6.34	1.44	3.26
4	5.65	1.50	2.64
5	4.05	1.48	2.36
6	3.72	1.40	2.32

IV. CONCLUSION

We proposed a generalized rumor centrality to solve the rumor source detection problem for degree-regular tree graphs with boundary effects. The boundary effect models finite underlying graphs or more complex spreading, and non-trivially affects the likelihood $P(G_n|v)$. For the special case of a single end vertex, we proved a unique optimality characterization that the *most probable source* lies on the path from the centroid to the end vertex. To tackle the general case with multiple end vertices, we proposed a message-passing algorithm to narrow down the search to a set of vertices containing the maximum likelihood estimate. We evaluated our algorithm to be near-optimal that can outperform the heuristic that naively ignores the boundary effects.

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