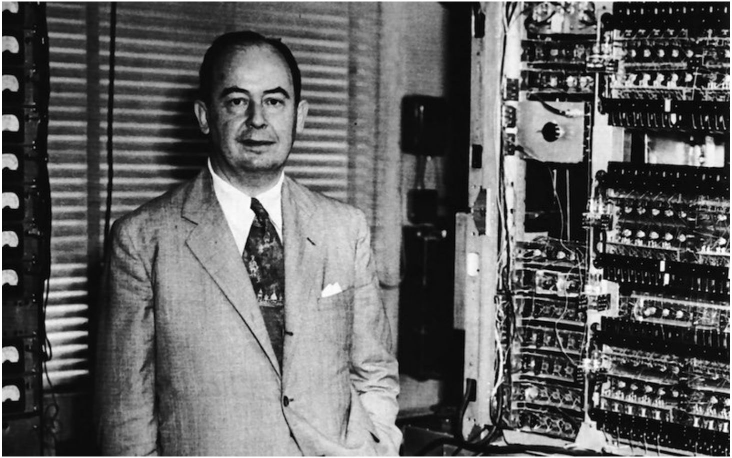
Creative Mathematics and Computer Science Learning for Gifted Students

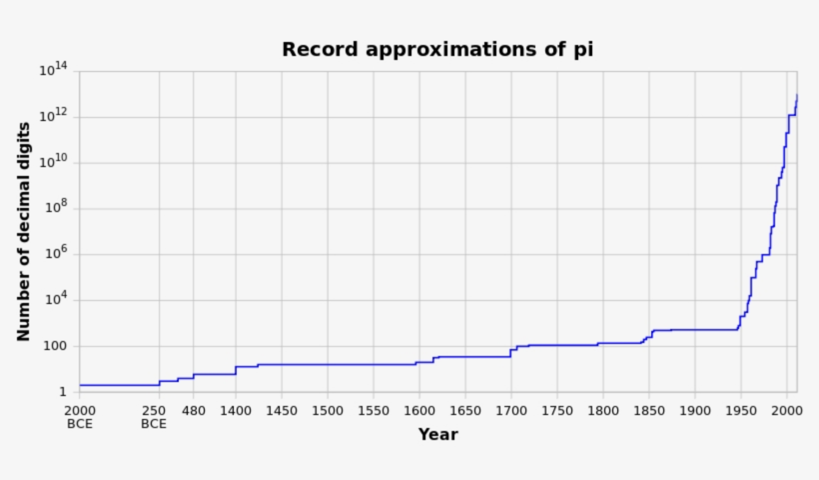
Dr. Tan, Chee Wei

John von Neumann: The Power of Computer [1]



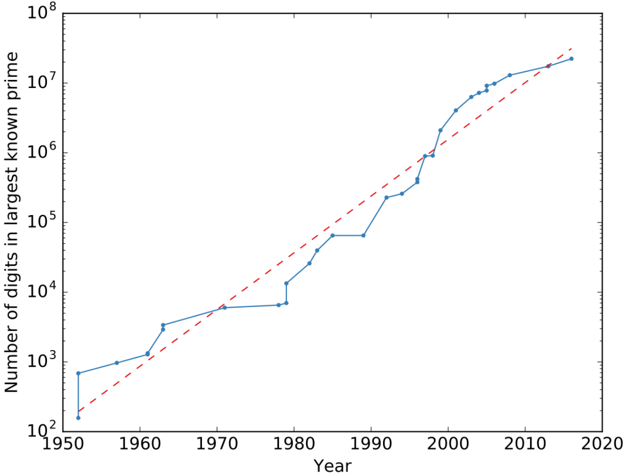
[1]World's first electronic computer, ENIAC, was created in the basement of Fuld Hall at the Institute for Advanced Study in 1945 by John von Neumann who also proposed determining the value of π to many decimal places with a view toward obtaining a statistical measure of the randomness of the distribution of the digits - a feat accomplished by the IAS machine yielding 2,037 digits of π in seventy hours

Approximating π: The Power of Computation



Approximating π is a fascinating story of mathematics and computer science, from Archimedes to Aryabhata to Euler to Gauss and to ENIAC

The Largest Prime Number: The Power of Computation



Finding the largest prime number is a fascinating story of mathematics and computer science, for there is an infinitude of prime numbers and its use

Proofs and Discourse: The Power of Communications

Teacher: "... and now I want to prove this theorem."
Pupil: "Why bother to prove it, teacher? I take your word for it."

Professor: „... Und nun will ich Ihnen diesen Lehrsatz jetzt auch beweisen."
Junge: „Wozu beweisen, Herr Professor? Ich glaub' es Ihnen so."

Figure 1.

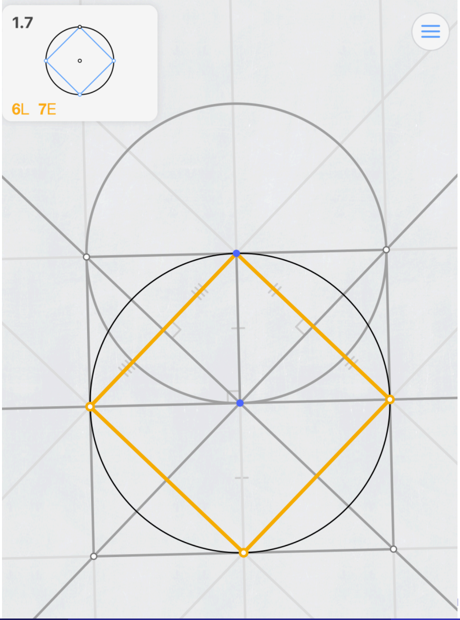
From *The Mischief Book* by Wilhelm Busch, as translated by Abby Langdon Alger and published by R. Worthington in 1880, reproduced from *Max and Moritz, from the Pen of Wilhelm Busch*, edited and annotated by H. Arthur Klein, Dover Publications, 1962.

- Ability to develop proofs and discourse is about *communications*
- Unique mentor role at Julia Robinson Mathematics Festivals (JRMF)
- Strogatz Prize on Mathematics Communications Competition

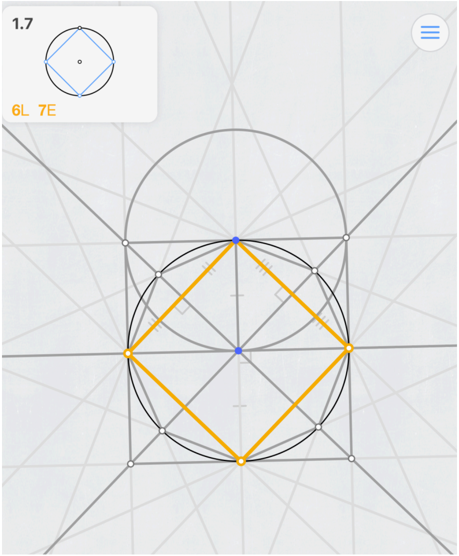
Overview Jan 14 & Jan 28, 2021

- Learning Geometry through Euclidea App
- Introduction and some strategies to solving problems in Euclidea
- *There is no royal road to geometry*
- Euclidea will guide you through challenging but rewarding territory. Each step is a separate task that requires reflection and preparation. As long as you do not give up, you will succeed.
- See if you can find as many *construction proofs* as possible and critique on each of your attempt
- Pythagoras' Theorem: $x^2 + y^2 = z^2$ where x and y are the base and height of a right-angled triangle
- Assignment 1: Complete Alpha Level by January 28
- Must finish all Assignments and a Project by April to get Certificate of Achievement

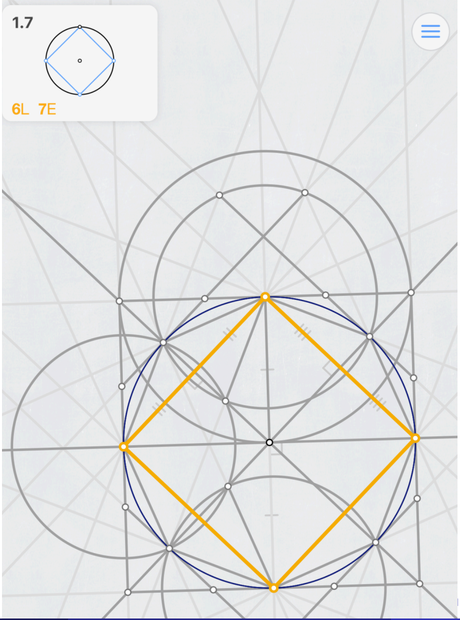
$2\sqrt{2} < \pi < 4$



Doubling the Polygon: Square to Octagon



$2^2\sqrt{2-\sqrt{2}} < \pi < 2^3\sqrt{3-2\sqrt{2}}$



Mathematics and Computation of Pi

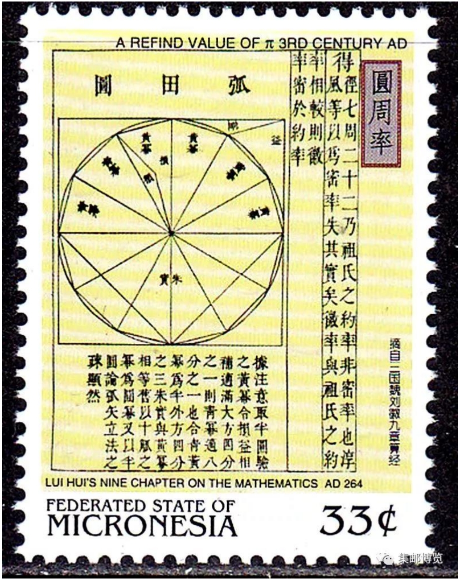
- An earliest example of Infinite Product and Nested Radical in Mathematics
- Viete's Formula for Pi

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

- Interpret left-hand side as *Ratio of Area of the Inscribed Square to Area of the Circle* (i.e., Euclidea level 1.7)
- Interpret right-hand side as *iterative infinite constructions of inner inscribed polygons*: first-term is ratio of Square to Octagon; second term is ratio of Octagon to Hexadecagon (16-gon) etc
- Can you use Viete's infinite product to deduce

$$\pi = \prod_{n \to \infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots}}}$$

Area is Stronger than Perimeter: Liu Hui's Dodecagon

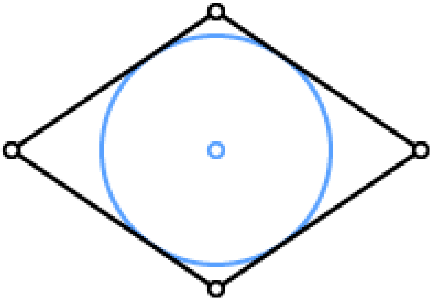


Euclidea Level 1.7 + 2.3

Creative Construction via Euclidea Level 1.7

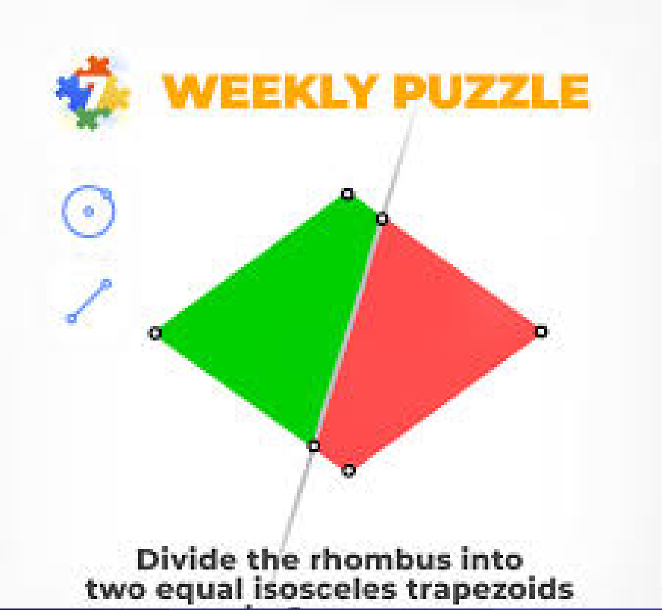
Assignment 2 by February 25:

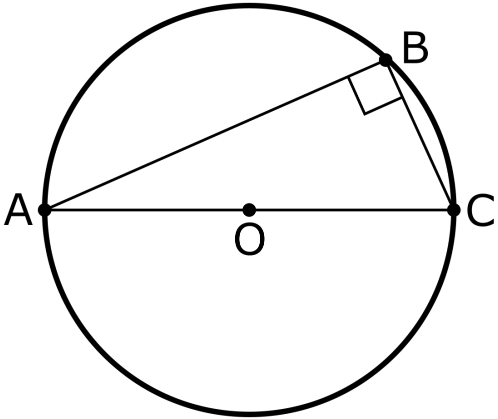
- Complete Euclidea Beta Level
- Construct Archimedes's Octagon (both inner and outer) using Euclidea Level 1.7 in as few elementary steps as possible (was 36 E)
- Construct Liu Hui's Dodecagon (inner) by using Euclidea Level 1.7 plus Level 2.3 and hence deduce some inequalities of π
- (Bonus) Write a computer program using Vietes formula to determine the polygon needed to compute π accurate up to n decimal places
- Must finish all Assignments and a Project by April to get Certificate of Achievement



What is the radius of a circle inscribed in a rhombus with diagonals of length 10 and 24? (A) 4 (B) $\frac{58}{13}$ (C) $\frac{60}{13}$ (D) 5 (E) 6

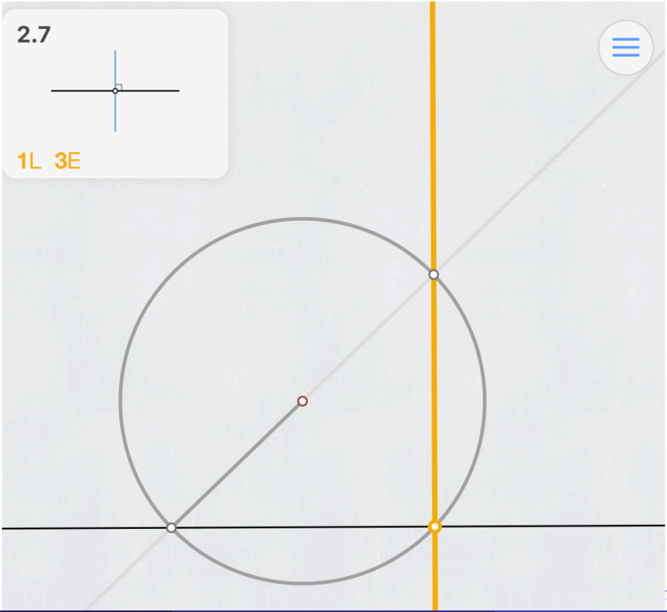
Euclidea Weekly Puzzle



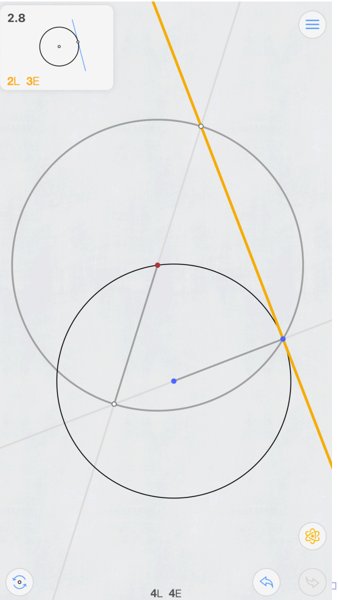


Thales's Theorem: The diameter of a circle always subtends a right angle (Revisit Euclidea Level 1.7 with 7E moves!)

Euclidea Level 2.7 and Thales's Theorem



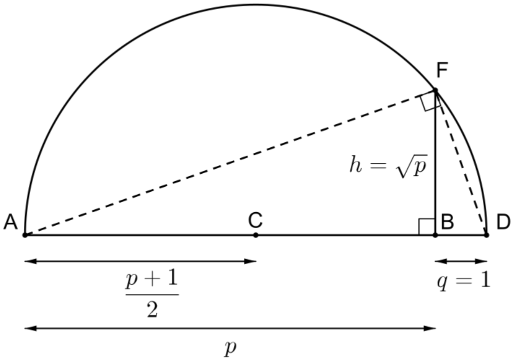
Thales's Theorem in Euclidea Level 2.8



Euclidea Level 2.8: 3 E-moves (was 4) New Record in 2015

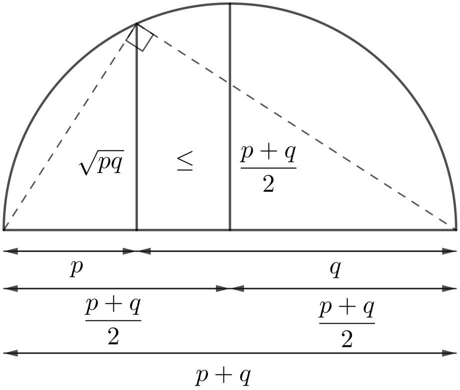
Click Here to See Proof at "Cut The Knot Math"!

Geometric Mean Theorem



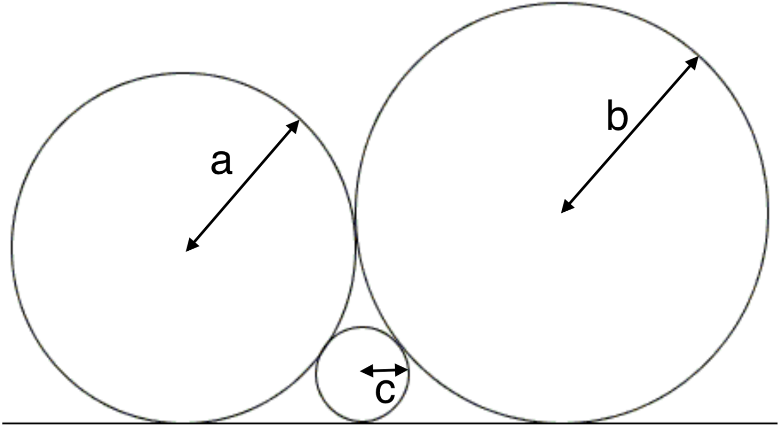
The geometric mean theorem is about similarity of right triangles: The length of the altitude on the hypotenuse of the right triangle is the geometric mean of the two line segments it creates on the hypotenuse. An application is to compute square roots!

Arithmetic-Geometric Mean (AGM) Inequality

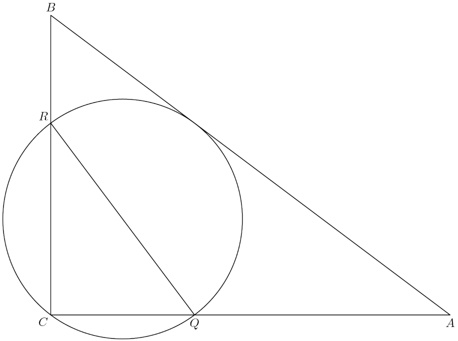


The Arithmetic-Geometric Mean Inequality states that for any set of nonnegative real numbers, the arithmetic mean of the set is greater than or equal to the geometric mean of the set.

AGM and Geometric Inequalities

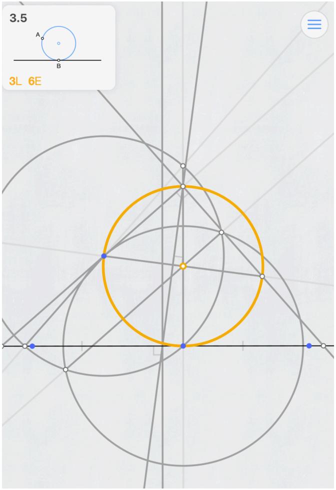


$$a + b \geq 2\sqrt{ab}$$

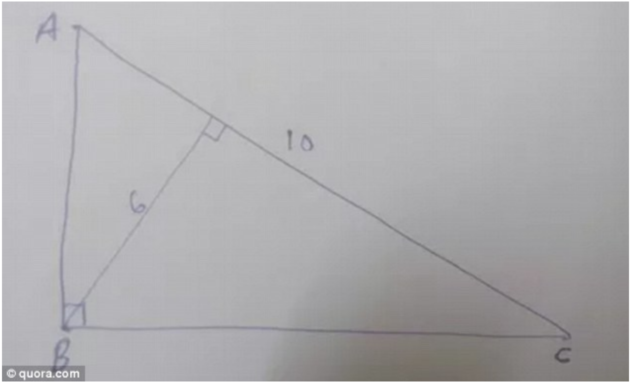


In $\triangle ABC, AB = 10$ $AC = 8$ and $BC = 6$. Circle P is the *circle with smallest radius* which passes through C and is tangent to AB. Let Q and R be the points of intersection, distinct from C, of circle P with sides AC and BC, respectively. The length of segment QR is

(A) 4.75 **(B)** 4.8 **(C)** 5 **(D)** $4\sqrt{2}$ **(E)** $3\sqrt{3}$



Think Geometric Mean Theorem and the fact that the two interior right-angled triangles of a right-angled triangle are similar.



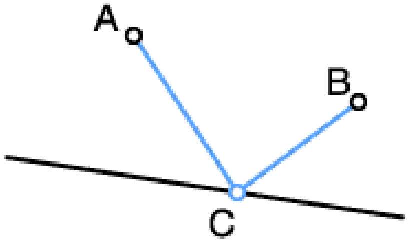
What is the area of this right triangle ($|AC| = 10$)? Try solving first before click to read

Overview Mar 11 & Mar 25

Assignment 3 by March 31:

- Let's complete Euclidea Gamma and Delta Levels!
- Optimization problems in geometry
- Geometry and Art: Japanese Sangaku
- Ptolemy Theorem, and Proving the Pythagoras Theorem
- Assignment 3: What mathematical knowledge have you learned using Euclidea?
  - Option 1: Theorem-driven, e.g., Thales Theorem, Ptolemy Theorem, Euler's theorem
  - Option 2: Problem-solving-driven, e.g., problems in AHSME, IB, *Sangaku*
  - Create novel Euclidea-like puzzles in your project
- Must finish all Assignments and a Project by April to get Certificate of Achievement

Optimization Problem 1: Heron's Problem



$$|AC| + |CB| \to \min$$

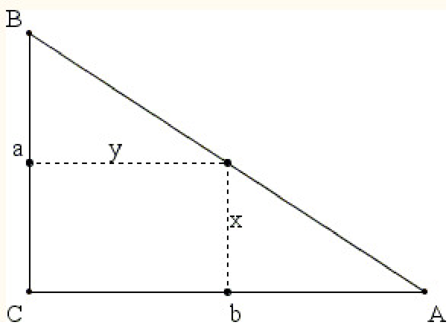
Construct a point C on the given line and segments AC and BC such that the sum of their length is minimal.

Japan and Temple Geometry, Scientific American, 1998, Click to Read

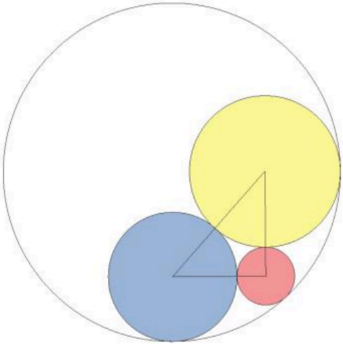
Sangaku: AGM Inequalities as Art

Sacred Mathematics: Japanese Temple Geometry by Fukagawa Hidetoshi and Tony Rothman, 2008, Click to Read

Optimization Problem 2: Sangaku

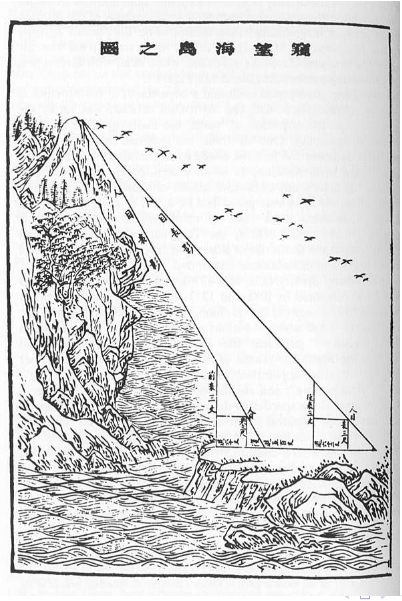


Find the location of the point on the hypotenuse AB that maximizes the area of the rectangle, Sangaku in 1806 by Hotta Sensuke written on a tablet hung in the Gikyosha shrine of Niikappugun, Hokkaido

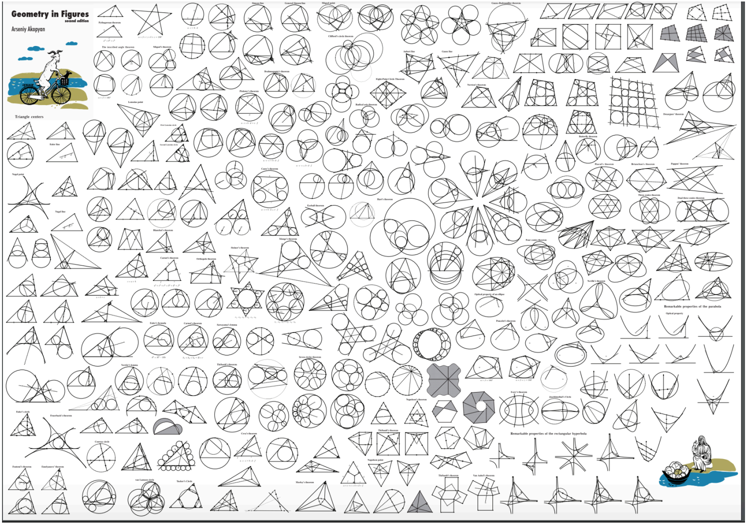


(Tochigi, 1853) Suppose that the centers of three circles, each touching the other two externally, lie at the vertices of a right triangle, and that a fourth circle touches all three internally. Prove that the largest diameter is the sum of the other three, Click to Read

Liu Hui’s Triangles and Surveying Applications



Geometry in Figures



Russian Mathematical Museum, Adyghe Republic, Russia

Gauss–Legendre Algorithm (Happy π Day!)

Step 1. Initial value setting: $a_0 = 1, \quad b_0 = \frac{1}{\sqrt{2}}, \quad t_0 = \frac{1}{4}, \quad p_0 = 1.$

Step 2. Repeat till the difference of a_n and b_n is within desired accuracy:

$$a_{n+1} \leftarrow \frac{a_n + b_n}{2},$$

$$b_{n+1} \leftarrow \sqrt{a_n b_n},$$

$$t_{n+1} \leftarrow t_n - p_n(a_n - a_{n+1})^2,$$

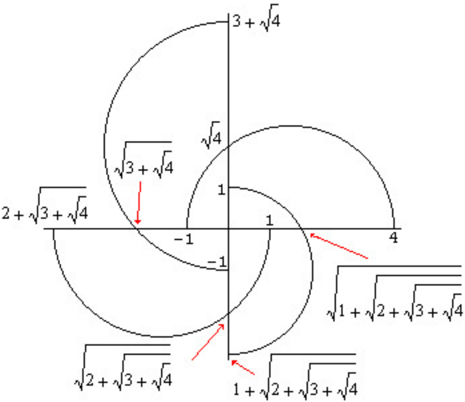
$$p_{n+1} \leftarrow 2p_n.$$

Step 3. Approximately, $\pi \approx \frac{(a_{n+1}+b_{n+1})^2}{4t_{n+1}}$.

Ramanujan: The Man Who Knew Infinity

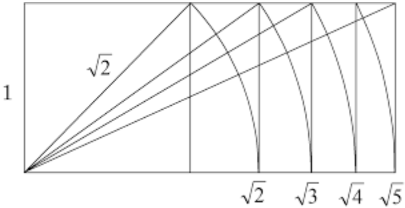
Ramanujan, Modular Equations, and Approximations to Pi or How to Compute One Billion Digits of Pi

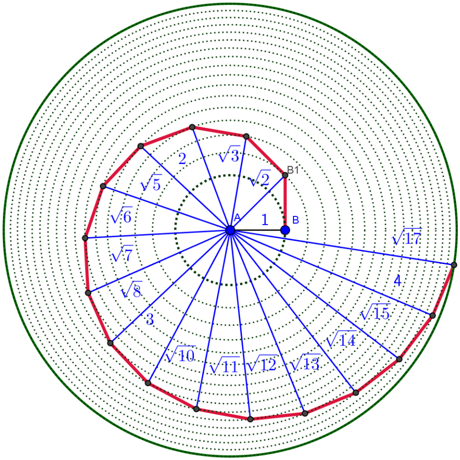
Constructible Numbers



Click here for Constructible Number at "Cut The Knot Math"!

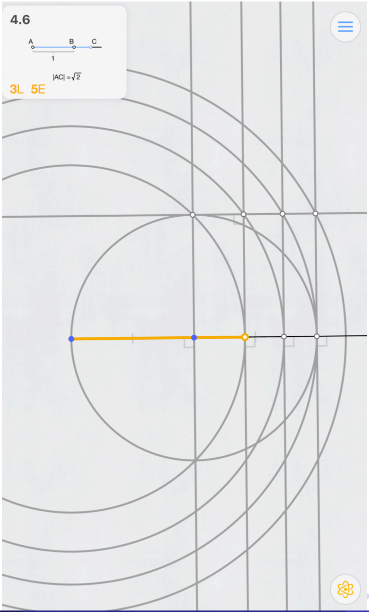
Spiral of Theodorus



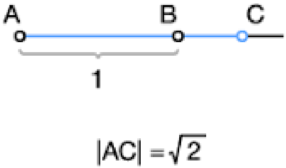


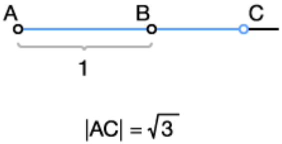
What's the cost of Spiral of Theodorus (cf. Euclidea 4.6 and 4.7)?

Square Root Spiral in Euclidea

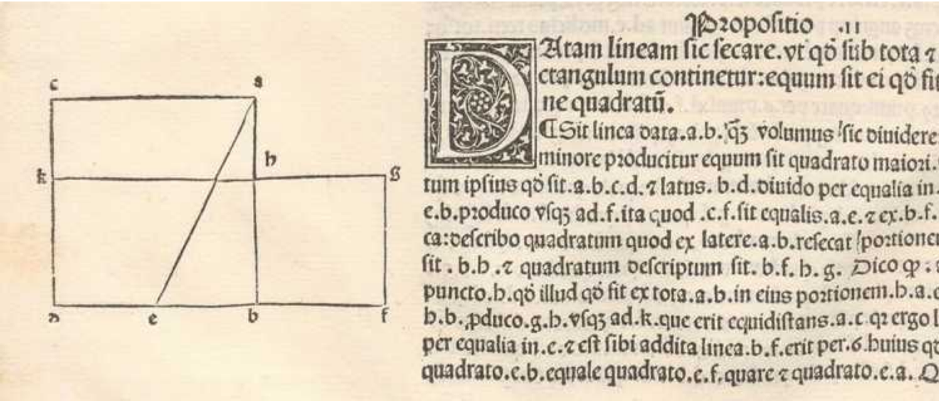


Approximations of π and Euclidea Levels 4.6 and 4.7





Archimedes' constant $\approx$ Pythagoras' constant $+$ Theodorus' constant



Propositio .11
Datam lineam sic secare. vt qđ sub tota ⁊
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ne quadratū.
¶Sit linea data. a.b. q̄ volumus sic diuidere
minore producitur equum sit quadrato maiori.
tum ipsius qđ sit. a.b.c.d. ⁊ latus. b.d. diuido per equalia in.
e.b. produco vsq; ad. f. ita quod. c.f. sit equalis. a.e. ⁊ ex. b.f.
ca: describo quadratum quod ex latere. a.b. resecat portione
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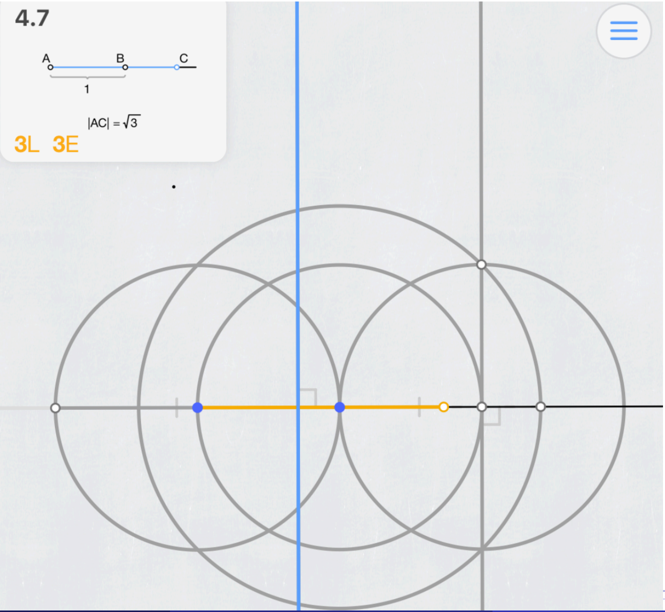
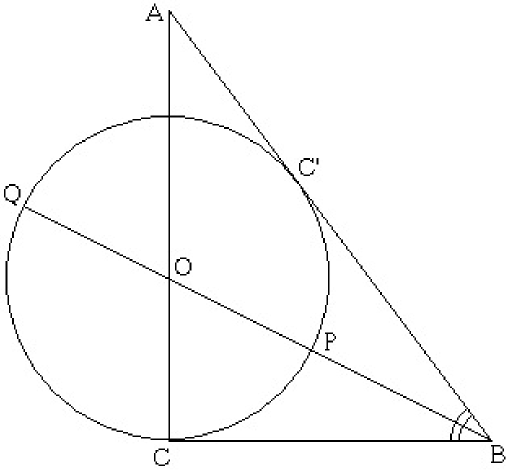
puncto. b. qđ illud qđ fit ex tota. a.b. in eius portionem. b.a.
b.h. pduco. g.h. vsq; ad. k. que erit equidistans. a.c. q2 ergo
per equalia in. e. ⁊ est sibi addita linea. b.f. erit per. 6. huius q
quadrato. e.b. equale quadrato. e.f. quare ⁊ quadrato. e.a.

Extract from Euclid's "Elements" (1482 edition): When, as the whole line is to the greater segment, so is the greater to the lesser.

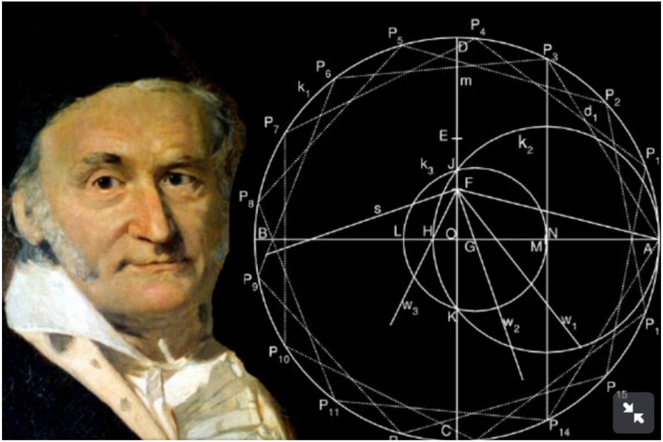
Can You build The Golden Ratio in 10E or less (was 11E)?



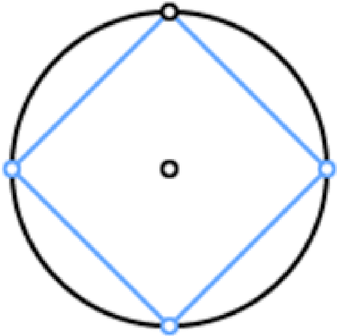
The Golden Ratio in 3-4-5 Pythagorean Triangle

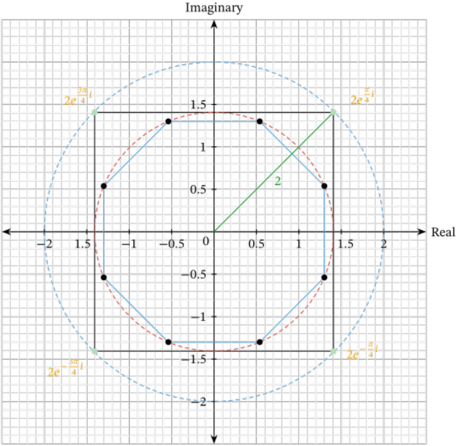


Click here to find the Golden Ratio on Cut The Knot Math! Revisit the 1978 AHSME Question now. What do you observe?



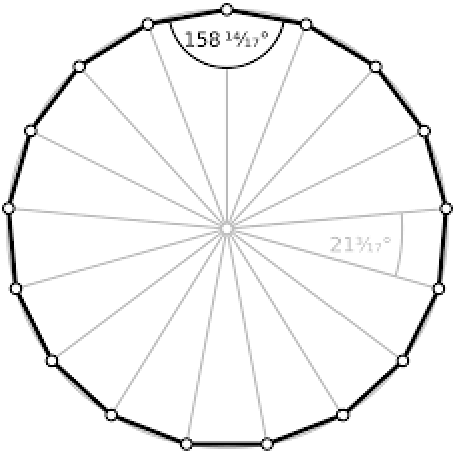
Carl Friedrich Gauss (30 April 1777 – 23 February 1855)





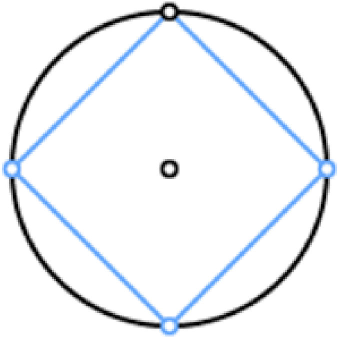
$$x^4 = 1$$

Gauss's Heptadecagon



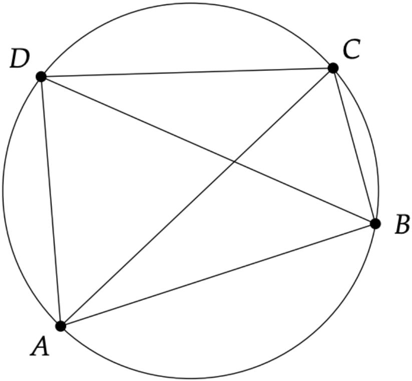
$$x^{17} = 1$$

Optimization Problem 3: Largest Inscribed Quadrilateral



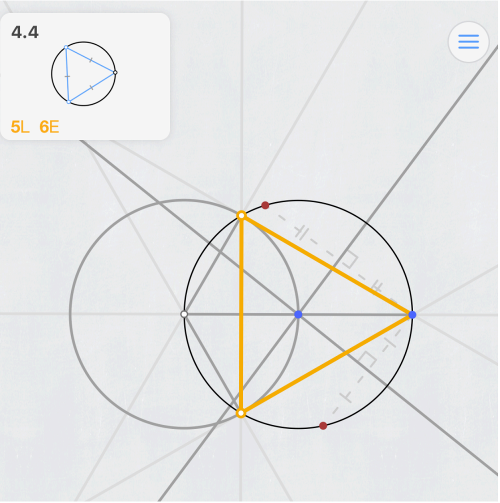
Show that among all quadrilaterals of a given perimeter the square has the largest area. What is the minimum cost to construct the optimal solution?

Ptolemy Theorem



If $ABCD$ is a cyclic quadrilateral, then $|AB||CD| + |AD||BC| = |AC||BD|$.
Click Here to See proof at "Cut The Knot Math"!

Euclidea Level 4.4 and Ptolemy Theorem



Click Here to See proof at "Cut The Knot Math"!

Euclidea Level 10.8 and Ptolemy Theorem in Inscribed Pentagon

Click Here to See *Sangaku construction* at "Cut The Knot Math". Now use Ptolemy Theorem to find the Golden Ratio!

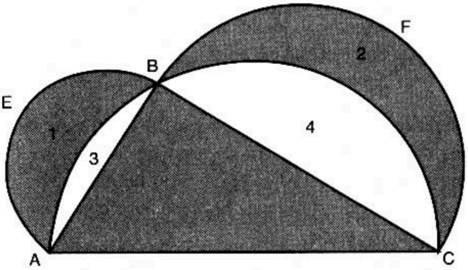
Overview Apr 8 & Apr 29 (Graduation)

- Let's complete Euclidea Gamma and Delta Levels!
- What Newton, Gauss and Euler tell us about geometry
- Circle packing, Sphere packing and Shannon's theorem
- Project (Video and Writeup) using Euclidea
- Create problems that are low threshold high ceiling (mathematics festival type) in writeup
- Graduation Day cum Math Festival on April 29 (to celebrate Claude Shannon's Day)
- Must finish all Assignments and your Project by April to get Certificate of Achievement

ttp://www.algebragamification.com/jrmf/

National Math Festival, Julia Robinson Mathematics Festival (JRMF)

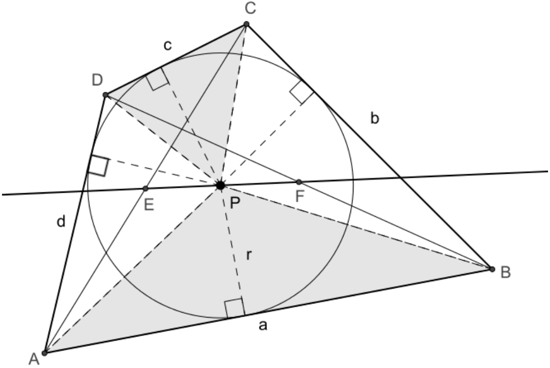
Prove that the total area of the regions 1 and 2 below equals the area of the triangle.



The unshaded area is part of the semi-circle in which the triangle is inscribed, and AB and BC are diameters for the other two semi-circles.

Click here for a sample of geometry problems in JRMF. Can you create similar type of problems?

Newton's Theorem in Euclidea Level 2.10 and 3.9

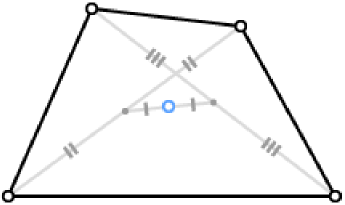


In every tangential quadrilateral other than a rhombus, the center of the incircle lies on the *Newton line*.

All triangles can have an incircle (Try it via Euclidea Level 2.2!), but not all quadrilaterals do. When does a tangential quadrilateral exist?

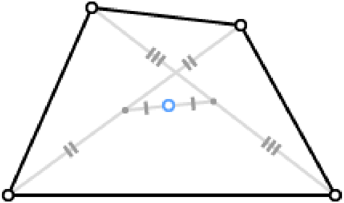
Read Euler, Read Euler, He is the Master of Us All

Leonhard Euler (15 April 1707 – 18 September 1783)



Connect the midpoints of any quadrilateral and you get a parallelogram. Prove it!
Furthermore, the *center of the parallelogram* is the *center of quadrilateral*, and the area of the parallelogram is half of that of the quadrilateral.

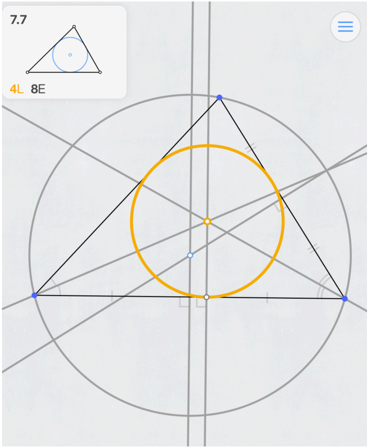
Euclidea Level 3.9 and Euler Quadrilateral Theorem



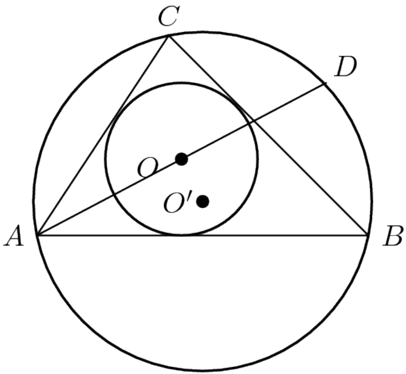
Euler–Pythagoras theorem: For a convex quadrilateral with sides a, b, c, d, diagonals e and f, and g being the line segment connecting the midpoints of the two diagonals, the following equations holds:

$$a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + 4g^2$$

Euler's Theorem and Euclidea 4.4 and 7.7



The distance between the center of a circumcircle of radius R and that of the incircle of radius r of a triangle is $d = \sqrt{R(R-2r)}$.

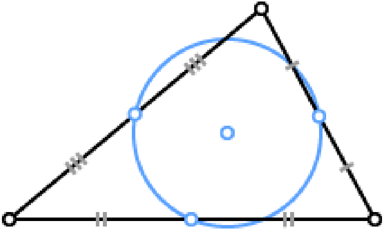


Triangle ABC is inscribed in a circle with center O'. A circle with center O is inscribed in triangle ABC. AO is drawn, and extended to intersect the larger circle in D. Then we must have:

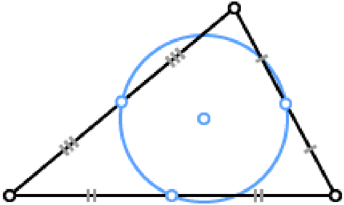
(A) $CD = BD = O'D$ (B) $AO = CO = OD$ (C) $CD = CO = BD$

(D) CD=OD=BD (E) O'B=O'C=OD

Euler Circle and Euler Line: Euclidea Level 6.9



Euler Circle also known as The Nine-point circle

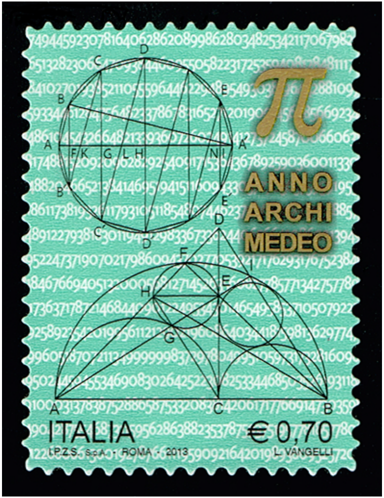


A circle is a happy thing to be–
Think how the joyful perpendicular
Erected at the kiss of tangency
Must meet my central point, my avator.
And lovely as I am, yet only 3 Points are needed to determine me.
By Christopher Morley (1890-1957)

Summer Triangle: Finding Your True North

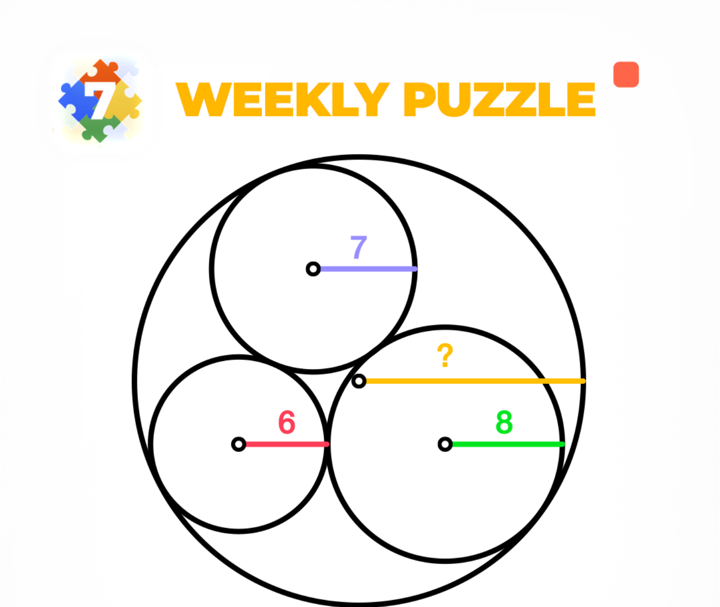
The Summer Triangle with the Milky Way (NASA 1996)

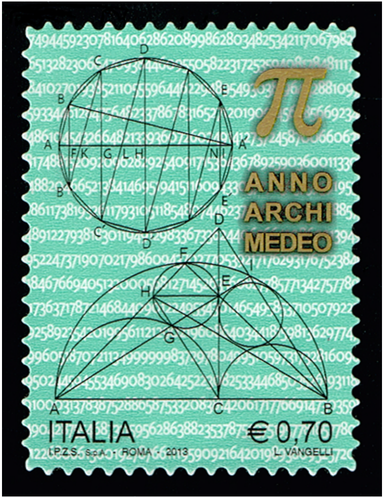
Descartes Theorem and Archimedes's Circles



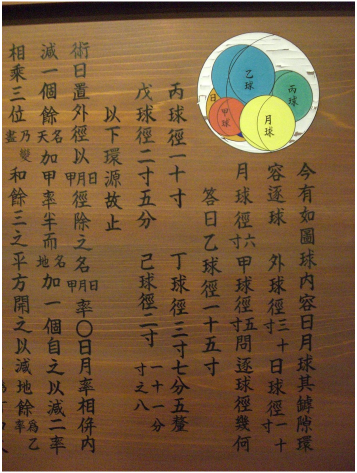
Arbelos and Euclidea Level 14.5

Euclidea "Weekly Puzzle" No. 83, Mar 2021





Arbelos and Level 14.5



1822 tablet at Samukawa Shrine in Kanagawa Prefecture, which predates F. Soddy's Hexlet theorem by more than a century

Happy Birthday Claude Shannon!

Sphere Packing made ubiquitous by Shannon in 1948!

References

- Number Theory in Problem Solving by Konrad Pilch, 2016
- Ideas in Geometry by Alison Ahlgren and Bart Snapp, 2010
- Solving Mathematical Problems by Terence Tao, 2006

- Please scan the QR code and complete the form as part of this course: https://forms.gle/Lhkh46HYD1ExqfVz6