Max-min Weighted SIR for MIMO Downlink System: Optimality and Algorithms

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***Abstract*—Designing fast algorithms that adapt the transmit and receive power and beamformers to optimize performance for different users is important in wireless MIMO downlink systems. This paper studies the max-min weighted SIR problem in the downlink, where multiple users are weighted according to priority and are subject to a total power constraint. The difficulty of this nonconvex problem is compounded by the coupling in the transmit and receive beamformers, thereby making it hard to optimize in a distributed fashion. We first show that this problem can be optimally and efficiently computed using a fast algorithm when the channels are rank-one. The optimal transmit and receive power and beamformers are also derived analytically. We then exploit the MIMO uplink-downlink duality to adapt our algorithm to compute a local optimal solution for channels with general rank.**

I. Introduction

In bandwidth-limited wireless networks, the spatial diversity of networks arising from the use of antenna arrays at both the transmitter and the receivers can be exploited to mitigate the interference between multiple users and increase the overall performance. Beamforming is a spatial diversity technique used at both the transmitter and the receiver to increase the signal-to-interference-noise ratio (SIR) of each user when multiple users share a common bandwidth. We consider the joint optimization of transmit power and transmit beamformer at the base station and receive beamformers at the mobile users in a wireless Multiple-Input-Multiple-Output (MIMO) downlink system.

Two optimization goals in the literature include maximizing the minimum weighted SIR or minimizing the total transmit power subject to given SIR constraints [1]–[5]. In the Multiple-Input-Single-Output (MISO) special case, these two optimization problems have been widely studied, and can be solved using different techniques, e.g., semidefinite programming or nonnegative matrix theory [1], [6]–[9]. However, in the general MIMO case, the optimal solution to the total power minimization problem is known only for the case when all the channels are rank-one [1]. For channels with general rank, finding the optimal power and beamformers under these two optimization goals is still an open problem.

This paper studies the max-min weighted SIR problem in the MIMO downlink system. We show that when all the channels are rank-one, the optimal power and beamformers can be given analytically. In particular, both the transmit and receive beamformers are Linear Minimum Mean Squared Error (LMMSE) filters. We then propose a fast algorithm with geometric convergence rate that computes the optimal solution. For channels with general rank, there is an additional difficulty in the coupling between the transmit and receive beamformers that make it hard to optimize in a distributed manner. It is well-known that the uplink-downlink duality can be exploited to solve the highly-coupled downlink problem by an auxiliary decoupled uplink problem [4], [10]. We leverage the uplink-downlink duality to propose a fast algorithm for channels with general rank.

The following notations are used. Boldface upper-case letters denote matrices, boldface lowercase letters denote column vectors, italics denote scalars, and $\mathbf{u} \geq \mathbf{v}$ ($\mathbf{B} \geq \mathbf{F}$) denotes componentwise inequality between vectors $\mathbf{u}$ and $\mathbf{v}$ (matrices $\mathbf{B}$ and $\mathbf{F}$). The Perron-Frobenius eigenvalue of a nonnegative matrix $\mathbf{F}$ is denoted as $\rho(\mathbf{F})$, and the Perron (right) and left eigenvectors of $\mathbf{F}$ associated with $\rho(\mathbf{F})$ are denoted by $\mathbf{x}(\mathbf{F})$ and $\mathbf{y}(\mathbf{F})$ respectively. The diagonal matrix formed by the components of a vector $\mathbf{v}$ is denoted as $\mathrm{diag}(\mathbf{v})$. The superscripts $(\cdot)^\mathsf{T}$ and $(\cdot)^\dagger$ denotes transpose and complex conjugate transpose respectively and $(\cdot)^+$ denotes pseudo-inversion. The trace of a square matrix $\mathbf{Q}$ is denoted as $\mathrm{Tr}(\mathbf{Q})$. The set of positive real numbers is denoted by $\mathbb{R}_+$.

II. System Model

We consider a MIMO downlink system with N antennas at the transmitter and K users, the kth user equipped with M_k antennas, operating in a frequency-flat fading channel. The downlink channel can be modeled as a vector Gaussian broadcast channel given by

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x} + \mathbf{z}_k, \quad k = 1, \ldots, K, \tag{1}$$

where $\mathbf{y}_k \in \mathbb{C}^{M_k \times 1}$ is the received signal of the kth user, $\mathbf{H}_k \in \mathbb{C}^{M_k \times N}$ is the channel matrix between the transmitter and the kth user, $\mathbf{x} \in \mathbb{C}^{N \times 1}$ is the transmitted signal vector, and $\mathbf{z}_k \sim \mathcal{CN}(0, \mathbf{R}_k)$ is the circular symmetric Gaussian noise vector, with covariance $\mathbf{R}_k$, at the kth user. Without loss of generality, we assume $\mathbf{R}_k = \mathbf{I}$.

III. Optimization in Rank-One Channels

In this section, we optimize both power and beamformers when all channel matrices are rank-one, i.e., $\mathrm{Rank}(\mathbf{H}_k) =$

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$1\ \forall k$.[1] In this case, we can decompose the channel matrix of the kth user as follows:

$$\mathbf{H}_k = a_k \mathbf{h}_k \tilde{\mathbf{h}}_k^\dagger, \tag{2}$$

where $\mathbf{h}_k \in \mathbb{C}^{M_k \times 1}$ is the normalized receive spatial signature (i.e., $\mathbf{h}_k^\dagger \mathbf{h}_k = 1$) induced on the kth user by the transmitter, $\tilde{\mathbf{h}}_k \in \mathbb{C}^{N\times 1}$ is the normalized transmit spatial signature (i.e., $\tilde{\mathbf{h}}_k^\dagger \tilde{\mathbf{h}}_k = 1$) of the transmitter towards the kth user, and $a_k \in \mathbb{R}_+$ is the attenuation of the path. We assume a linear transmission and reception strategy. Since each channel matrix has rank one, each user can be served by only one data stream. Hence, the transmitted signal vector can be written in the form of $\mathbf{x} = \sum_{k=1}^K \mathbf{x}_k = \sum_{k=1}^K d_k \sqrt{p_k}\mathbf{u}_k$, where $\mathbf{x}_k \in \mathbb{C}^{N\times 1}$ is the signal intended for the kth user, $\mathbf{u}_k \in \mathbb{C}^{N\times 1}$ is the normalized transmit beamformer for the kth user, d_k is the information signal for the kth user, and $p_k \in \mathbb{R}_+$ is the transmit power for the kth user. At the kth user, the received signal is processed by a normalized receive beamformer $\mathbf{v}_k \in \mathbb{C}^{M_k\times 1}$. Let $\mathbf{p} = [p_1, \ldots, p_K]^\mathsf{T}$, $\mathbb{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_K]$, and $\mathbb{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_K]$. The SIR of the kth user can be written as

$$\mathrm{SIR}_k(\mathbf{p}, \mathbb{U}, \mathbf{v}_k) = \frac{p_k a_k^2 \left|\mathbf{v}_k^\dagger \mathbf{h}_k \tilde{\mathbf{h}}_k^\dagger \mathbf{u}_k\right|^2}{\sum\limits_{\substack{i=1\\ i\neq k}}^K p_i a_k^2 \left|\mathbf{v}_k^\dagger \mathbf{h}_k \tilde{\mathbf{h}}_k^\dagger \mathbf{u}_i\right|^2 + 1}. \tag{3}$$

Let $\boldsymbol{\beta} \in \mathbb{R}^{K\times 1}$ be a priority vector where β_k is the priority assigned by the network to the kth user. We are interested in the following max-min weighted SIR problem given by

$$\begin{array}{ll} \max\limits_{\mathbf{p},\mathbb{U},\mathbb{V}} & \min\limits_k \dfrac{\mathrm{SIR}_k(\mathbf{p}, \mathbb{U}, \mathbf{v}_k)}{\beta_k} \\ \text{subject to} & \mathbf{1}^\mathsf{T}\mathbf{p} \leq \bar{P}, \quad \mathbf{p} \geq \mathbf{0}, \\ & \mathbf{u}_k^\dagger \mathbf{u}_k = 1 \text{ and } \mathbf{v}_k^\dagger \mathbf{v}_k = 1 \quad \forall k. \end{array} \tag{4}$$

The following result gives a closed-form solution for the optimal $\mathbb{V}$ in (4).

Theorem 1: The optimal receive beamformers $\mathbf{v}_k^*$ in (4) are the normalized matched filters (scaled by arbitrary complex phases) corresponding to the receive spatial signature of the users. Also, for any feasible $\mathbf{p}$ and transmit beamformers $\mathbb{U}$, $\mathbf{v}_k^*$ can be obtained from the normalized LMMSE receiver as follows:

$$\mathbf{v}_k' = \left(\sum_{\substack{i=1\\ i\neq k}}^K p_i \mathbf{H}_k \mathbf{u}_i \mathbf{u}_i^\dagger \mathbf{H}_k^\dagger + \mathbf{I}\right)^+ \mathbf{H}_k \mathbf{u}_k \qquad \forall k, \tag{5}$$

$$\mathbf{v}_k^* = \frac{\mathbf{v}_k'}{\|\mathbf{v}_k'\|} \qquad \forall k. \tag{6}$$

Corollary 1: For asymptotically small signal-to-noise ratio (SNR), $\mathbf{v}_k^*$ is the matched filter corresponding to any feasible power and transmit beamformers $(\mathbf{p}, \mathbb{U})$.

[1]This model is valid when there is only one line-of-sight path between the base station and each user, and the distance separation is large compared to the antenna separation.

Corollary 2: For asymptotically large signal-to-noise ratio (SNR), $\mathbf{v}_k^*$ is the zero-forcing filter corresponding to any feasible power and transmit beamformers $(\mathbf{p}, \mathbb{U})$.

Remark 1: Theorem 1 implies that the optimal approach for the user with a rank-one channel is to steer his receive beamformer towards the transmitter. Also, this beamsteering approach is optimal for a rank-one user regardless of the channel conditions of the other users. This result was also observed empirically in [2]. We note that this result for the max-min weighted SIR problem also applies to the total power minimization problem in [1].

When the channels are rank-one, the optimal solution can be computed using a fast algorithm with a geometric convergence rate. Theorem 1 can be used to obtain the optimal receive beamformer for each user regardless of the current transmit power $\mathbf{p}$ and the current transmit beamformers $\mathbb{U}$. For each user, this optimal receive beamformer needs only be computed once at the start and then kept fixed. Thus, the problem degenerates into a MISO problem with a unique global optimal solution [8], and the distributed MISO joint beamforming and power control algorithm [9] can then be used to find the optimal $\mathbf{p}$ and $\mathbb{U}$.

IV. Optimization in Full Rank Channels

In this section, we optimize both power and beamformers when the channel is full rank. Each user can now be served by multiple data streams. Let the kth user be served by L_k data streams and let the total number of data streams transmitted be $S = \sum_{k=1}^K L_k$. In order to exploit the spatial multiplexing gain, we require the transmitted data streams to be linearly independent. Hence, we assume that $S \leq N$ and $L_k \leq \mathrm{Rank}(\mathbf{H}_k)$. The transmitted signal vector can be written as $\mathbf{x} = \sum_{k=1}^K \mathbf{x}_k = \sum_{k=1}^K \sum_{l=1}^{L_k} d_{kl}\sqrt{p_{kl}}\mathbf{u}_{kl}$, where $\mathbf{x}_k \in \mathbb{C}^{N\times 1}$ is the signal intended for the kth user, $\mathbf{u}_{kl} \in \mathbb{C}^{N\times 1}$ is the normalized transmit beamformer for the lth stream of the kth user, and d_{kl} and $p_{kl} \in \mathbb{R}_+$ are the information signal and transmit power respectively for that stream. The transmit covariance matrix is $\mathbf{Q}(\mathbf{p}, \mathbb{U}) = \mathbb{E}(\mathbf{x}\mathbf{x}^\dagger)$. At the kth receiver, the beamformer $\mathbf{v}_{kl} \in \mathbb{C}^{M_k\times 1}$ is used to decode its lth stream. Let $\mathbf{p} = [p_{11}, \ldots, p_{1L_1}, p_{21}, \ldots, p_{KL_K}]^\mathsf{T}$, $\mathbb{U} = [\mathbf{u}_{11}, \ldots, \mathbf{u}_{1L_1}, \mathbf{u}_{21}, \ldots, \mathbf{u}_{KL_K}]$, and $\mathbb{V} = [\mathbf{v}_{11}, \ldots, \mathbf{v}_{1L_1}, \mathbf{v}_{21}, \ldots, \mathbf{v}_{KL_K}]$. The SIR of each stream can be expressed as

$$\mathrm{SIR}_{kl}(\mathbf{p}, \mathbb{U}, \mathbf{v}_{kl}) = \frac{p_{kl}\left|\mathbf{v}_{kl}^\dagger \mathbf{H}_k \mathbf{u}_{kl}\right|^2}{\sum\limits_{i=1}^K \sum\limits_{\substack{j=1\\ (i,j)\neq(k,l)}}^{L_i} p_{ij}\left|\mathbf{v}_{kl}^\dagger \mathbf{H}_k \mathbf{u}_{ij}\right|^2 + 1}. \tag{7}$$

Let $\boldsymbol{\beta} = [\beta_{11}, \ldots, \beta_{1L_1}, \beta_{21}, \ldots, \beta_{KL_K}]^\mathsf{T}$ be a priority vector where β_{kl} is the priority assigned by the network to the lth data stream of the kth user. The max-min weighted SIR problem is given by

$$\begin{array}{ll} \max\limits_{\mathbf{p},\mathbb{U},\mathbb{V}} & \min\limits_{k,l} \dfrac{\mathrm{SIR}_{kl}(\mathbf{p}, \mathbb{U}, \mathbf{v}_{kl})}{\beta_{kl}} \\ \text{subject to} & \mathbf{Tr}(\mathbf{Q}(\mathbf{p}, \mathbb{U})) \leq \bar{P}, \quad \mathbf{p} \geq \mathbf{0}, \\ & \mathbf{u}_{kl}^\dagger \mathbf{u}_{kl} = 1 \text{ and } \mathbf{v}_{kl}^\dagger \mathbf{v}_{kl} = 1 \quad \forall k, l. \end{array} \tag{8}$$

Next, we concatenate all the S streams and re-index them by $s(k,l) = \sum_{i=1}^{k-1} L_i + l$. Let the sth stream be intended for the k_sth user. The SIR of the sth stream can then be written as

$$\mathrm{SIR}_s(\mathbf{p}, \mathbb{U}, \mathbf{v}_s) = \frac{p_s \left|\mathbf{v}_s^\dagger \mathbf{H}_{k_s} \mathbf{u}_s\right|^2}{\sum_{\substack{i=1\\ i\neq s}}^{S} p_i \left|\mathbf{v}_s^\dagger \mathbf{H}_{k_s} \mathbf{u}_i\right|^2 + 1}. \tag{9}$$

We define the matrix $\mathbf{G}(\mathbb{U},\mathbb{V}) \in \mathbb{R}^{S\times S}$, the (cross channel interference) matrix $\mathbf{F}(\mathbb{U},\mathbb{V}) \in \mathbb{R}^{S\times S}$, and the vector $\mathbf{n}(\mathbb{U},\mathbb{V}) \in \mathbb{R}^{S\times 1}$ with the following entries:

$$G_{si}(\mathbf{u}_i, \mathbf{v}_s) = \left|\mathbf{v}_s^\dagger \mathbf{H}_{k_s} \mathbf{u}_i\right|^2, \tag{10}$$

$$F_{si}(\mathbf{u}_i, \mathbf{u}_s, \mathbf{v}_s) = \begin{cases} 0, & \text{if} \quad s = i \\ \frac{G_{si}(\mathbf{u}_i,\mathbf{v}_s)}{G_{ss}(\mathbf{u}_s,\mathbf{v}_s)}, & \text{if} \quad s \neq i, \end{cases} \tag{11}$$

$$\mathbf{n}(\mathbb{U},\mathbb{V}) = \left(\frac{1}{G_{11}(\mathbf{u}_1,\mathbf{v}_1)}, \ldots, \frac{1}{G_{S,S}(\mathbf{u}_S,\mathbf{v}_S)}\right)^\mathsf{T}. \tag{12}$$

From (9), the $\mathrm{SIR}_s(\mathbf{p}, \mathbb{U}, \mathbf{v}_s)$ can be written as $p_s/(\mathbf{F}(\mathbb{U},\mathbb{V})\mathbf{p} + \mathbf{n}(\mathbb{U},\mathbb{V}))_s$. Since our data streams are independent, our sum power constraint can be expressed as $\mathbf{Tr}(\mathbf{Q}(\mathbf{p},\mathbb{U})) = \sum_{k=1}^{K}\sum_{l=1}^{L_k} p_{kl}\mathbf{Tr}\left(\mathbf{u}_{kl}\mathbf{u}_{kl}^\dagger\right) = \sum_{k=1}^{K}\sum_{l=1}^{L_k} p_{kl}\|\mathbf{u}_{kl}\|^2 = \sum_{k=1}^{K}\sum_{l=1}^{L_k} p_{kl} = \mathbf{1}^\mathsf{T}\mathbf{p} \leq \bar{P}$. Hence, (8) becomes

$$\begin{array}{ll} \max\limits_{\mathbf{p},\mathbb{U},\mathbb{V}} & \min\limits_{s} \frac{\mathrm{SIR}_s(\mathbf{p},\mathbb{U},\mathbf{v}_s)}{\beta_s} \\ \text{subject to} & \mathbf{1}^\mathsf{T}\mathbf{p} \leq \bar{P}, \quad \mathbf{p} \geq \mathbf{0}, \\ & \mathbf{u}_s^\dagger\mathbf{u}_s = 1 \text{ and } \mathbf{v}_s^\dagger\mathbf{v}_s = 1 \quad \forall s. \end{array} \tag{13}$$

Global optimization of (13) is an open problem because the nonconvexity and the mutual coupling of the transmit and receive beamformers make it difficult to jointly optimize the beamformers. However, a local optimal solution can be developed by alternate optimization (iteratively fix a group of variables and optimize the other variables).

First, consider optimizing only power in (13). By fixing the beamformers, i.e., $\mathbb{U} = \tilde{\mathbb{U}}$ and $\mathbb{V} = \tilde{\mathbb{V}}$, we have the following max-min weighted SIR problem:

$$\begin{array}{ll} \max\limits_{\mathbf{p}} & \min\limits_{s} \frac{\mathrm{SIR}_s(\mathbf{p},\tilde{\mathbb{U}},\tilde{\mathbf{v}}_s)}{\beta_s} \\ \text{subject to} & \mathbf{1}^\mathsf{T}\mathbf{p} \leq \bar{P}, \quad \mathbf{p} \geq \mathbf{0}, \end{array} \tag{14}$$

which was solved in [9]. For completeness, we will state the following theorem and algorithm from [9].

Theorem 2: The optimal objective and solution of (14) is given by $1/\rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\bar{P},\tilde{\mathbb{U}},\tilde{\mathbb{V}}))$ and $(\bar{P}/\mathbf{1}^\mathsf{T}\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\bar{P},\tilde{\mathbb{U}},\tilde{\mathbb{V}})))\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}(\bar{P},\tilde{\mathbb{U}},\tilde{\mathbb{V}}))$ respectively, where

$$\mathbf{B}(\bar{P},\tilde{\mathbb{U}},\tilde{\mathbb{V}}) = \mathbf{F}(\tilde{\mathbb{U}},\tilde{\mathbb{V}}) + (1/\bar{P})\mathbf{n}(\tilde{\mathbb{U}},\tilde{\mathbb{V}})\mathbf{1}^\mathsf{T}. \tag{15}$$

Algorithm 1: Max-min weighted SIR power control

1) Update power $\mathbf{p}(n+1)$:

$$p_s(n+1) = \left(\frac{\beta_s}{\mathrm{SIR}_s(\mathbf{p}(n),\tilde{\mathbb{U}},\tilde{\mathbf{v}}_s)}\right) p_s(n) \quad \forall s. \tag{16}$$

2) Normalize $\mathbf{p}(n+1)$:

$$\mathbf{p}(n+1) \leftarrow \mathbf{p}(n+1) \cdot \bar{P}/\mathbf{1}^\mathsf{T}\mathbf{p}(n+1). \tag{17}$$

Starting from any initial point $\mathbf{p}(0)$, $\mathbf{p}(n)$ in Algorithm 1 converges geometrically fast to the optimal solution of (14) given in Theorem 2 [9]. We now leverage Algorithm 1 to propose a fast algorithm for the MIMO downlink case.

Next, consider optimizing only the receive beamformers in (13) by fixing the power and transmit beamformers, i.e., $\mathbf{p} = \tilde{\mathbf{p}}$ and $\mathbb{U} = \tilde{\mathbb{U}}$. We have the following max-min weighted SIR problem given by

$$\begin{array}{ll} \max\limits_{\mathbb{V}} & \min\limits_{s} \frac{\mathrm{SIR}_s(\tilde{\mathbf{p}},\tilde{\mathbb{U}},\mathbf{v}_s)}{\beta_s} \\ \text{subject to} & \mathbf{v}_s^\dagger\mathbf{v}_s = 1 \quad \forall s. \end{array} \tag{18}$$

Lemma 1: The optimal receive beamformers in (18) are the normalized LMMSE receivers given by

$$\tilde{\mathbf{v}}_s'(\tilde{\mathbf{p}},\tilde{\mathbb{U}}) = \left(\sum_{\substack{i=1\\ i\neq s}}^{S} \tilde{p}_i \mathbf{H}_{k_s}\tilde{\mathbf{u}}_i\tilde{\mathbf{u}}_i^\dagger\mathbf{H}_{k_s}^\dagger + \mathbf{I}\right)^{+} \mathbf{H}_{k_s}\tilde{\mathbf{u}}_s, \tag{19}$$

$$\tilde{\mathbf{v}}_s^*(\tilde{\mathbf{p}},\tilde{\mathbb{U}}) = \frac{\tilde{\mathbf{v}}_s'(\tilde{\mathbf{p}},\tilde{\mathbb{U}})}{\left\|\tilde{\mathbf{v}}_s'(\tilde{\mathbf{p}},\tilde{\mathbb{U}})\right\|}. \tag{20}$$

Finally, we consider optimizing the transmit beamformers in (13). Because the transmit beamformers $\mathbb{U}$ in (13) are coupled across the different SIRs, we cannot decouple (13) into independent maximization problems by fixing $\mathbf{p}$ and $\mathbb{V}$ (which was the approach taken to solve (18)). However, by using the uplink-downlink duality theory in [10], we are able to express (13) in a form that enables us to decouple the problem. The uplink-downlink duality theory states that under a single sum power constraint and the same additive white noise for all users, the achievable SIR region of a downlink broadcast channel (BC) is identical to that of a dual uplink multiple access channel (MAC). The channel matrix associated with the dual MAC is the conjugate transposed channel matrix of the BC. Further, the transmit and receive beamforming vectors in the uplink are also the receive and transmit beamforming vectors in the downlink respectively. Specifically, there exists a set of uplink powers in the dual MAC that achieves the same set of SIR's as the BC, and vice versa. The uplink-downlink duality theory is thus used to convert the downlink problem into an uplink problem (the transmit beamforming optimization converts into a receive beamforming optimization). The uplink problem is easier to solve in a distributed fashion because the receive beamformers are decoupled.

Consider the dual MAC of (13). Let $\mathbf{q} = [q_1, \ldots, q_S]^\mathsf{T}$ be the uplink power, and $\mathbb{V}$ and $\mathbb{U}$ be the transmit and receive beamforming vectors respectively. The uplink SIR is given by

$$\hat{\mathrm{SIR}}_s(\mathbf{q},\mathbb{V},\mathbf{u}_s) = \frac{q_s\left|\mathbf{u}_s^\dagger\mathbf{H}_{k_s}^\dagger\mathbf{v}_s\right|^2}{\sum_{\substack{i=1\\ i\neq s}}^{S} q_i\left|\mathbf{u}_s^\dagger\mathbf{H}_{k_i}^\dagger\mathbf{v}_i\right|^2 + 1}. \tag{21}$$

We define the dual cross channel interference matrix $\hat{\mathbf{F}}(\mathbb{U},\mathbb{V}) \in \mathbb{R}^{S\times S}$ with the following entries:

$$\hat{F}_{si}(\mathbf{v}_i,\mathbf{v}_s,\mathbf{u}_s) = \begin{cases} 0, & \text{if} \quad s = i \\ \frac{G_{is}(\mathbf{u}_s,\mathbf{v}_i)}{G_{ss}(\mathbf{u}_s,\mathbf{v}_s)}, & \text{if} \quad s \neq i, \end{cases} \tag{22}$$

Hence, $\hat{\text{SIR}}_s(\mathbf{q},\mathbb{V},\mathbf{u}_s)$ can be written as $q_s/(\hat{\mathbf{F}}(\mathbb{U},\mathbb{V})\mathbf{q} + \mathbf{n}(\mathbb{U},\mathbb{V}))_s$. Let $\boldsymbol{\gamma} = [\text{SIR}_1(\mathbf{p},\mathbb{U},\mathbf{v}_1),\ldots,\text{SIR}_S(\mathbf{p},\mathbb{U},\mathbf{v}_S)]^\mathsf{T}$ and let $\hat{\boldsymbol{\gamma}} = [\hat{\text{SIR}}_1(\mathbf{q},\mathbb{V},\mathbf{u}_1),\ldots,\hat{\text{SIR}}_S(\mathbf{q},\mathbb{V},\mathbf{u}_S)]^\mathsf{T}$. (23) and (24) gives the uplink and downlink powers which achieves the same set of SIRs in the dual MAC and BC [10].

$$\mathbf{q}(\mathbf{p},\mathbb{U},\mathbb{V}) = (\mathbf{I} - \text{diag}(\boldsymbol{\gamma})\hat{\mathbf{F}}(\mathbb{U},\mathbb{V}))^{-1}\text{diag}(\boldsymbol{\gamma})\mathbf{n}(\mathbb{U},\mathbb{V}), \tag{23}$$

$$\mathbf{p}(\mathbf{q},\mathbb{U},\mathbb{V}) = (\mathbf{I} - \text{diag}(\hat{\boldsymbol{\gamma}})\mathbf{F}(\mathbb{U},\mathbb{V}))^{-1}\text{diag}(\hat{\boldsymbol{\gamma}})\mathbf{n}(\mathbb{U},\mathbb{V}). \tag{24}$$

Hence, by fixing $\mathbf{p} = \tilde{\mathbf{p}}$ and $\mathbb{V} = \tilde{\mathbb{V}}$, (13) can be written as

$$\begin{array}{ll} \max\limits_{\mathbb{U}} & \min\limits_{s} \dfrac{\hat{\text{SIR}}_s(\mathbf{q}(\tilde{\mathbf{p}},\mathbb{U},\tilde{\mathbb{V}}),\tilde{\mathbb{V}},\mathbf{u}_s)}{\beta_s} \\ \text{subject to} & \mathbf{u}_s^\dagger\mathbf{u}_s = 1 \quad \forall s. \end{array} \tag{25}$$

Since the transmit beamformers $\mathbb{U}$ in (25) are coupled through the uplink power $\mathbf{q}(\tilde{\mathbf{p}},\mathbb{U},\tilde{\mathbb{V}})$, (25) is not decoupled. Suppose we fix $\mathbb{U}$ in $\mathbf{q}$ to be $\tilde{\mathbb{U}}$, i.e., we partially fix $\mathbb{U}$ in (25) as follows:

$$\begin{array}{ll} \max\limits_{\mathbb{U}} & \min\limits_{s} \dfrac{\hat{\text{SIR}}_s(\mathbf{q}(\tilde{\mathbf{p}},\tilde{\mathbb{U}},\tilde{\mathbb{V}}),\tilde{\mathbb{V}},\mathbf{u}_s)}{\beta_s} \\ \text{subject to} & \mathbf{u}_s^\dagger\mathbf{u}_s = 1 \quad \forall s. \end{array} \tag{26}$$

Since $\mathbb{U}$ in (26) are decoupled, (26) can be easily solved.

Lemma 2: Let $\tilde{\mathbf{q}} = \mathbf{q}(\tilde{\mathbf{p}},\tilde{\mathbb{U}},\tilde{\mathbb{V}})$. The optimal beamformers in (26) are the normalized LMMSE receivers (of the dual MAC) given by

$$\tilde{\mathbf{u}}'_s(\tilde{\mathbf{q}},\tilde{\mathbb{V}}) = \left(\sum_{\substack{i=1\\i\neq s}}^{S} \tilde{q}_i\mathbf{H}_{k_i}^\dagger\tilde{\mathbf{v}}_i\tilde{\mathbf{v}}_i^\dagger\mathbf{H}_{k_i} + \mathbf{I}\right)^{+} \mathbf{H}_{k_s}^\dagger\tilde{\mathbf{v}}_s, \tag{27}$$

$$\tilde{\mathbf{u}}^*_s(\tilde{\mathbf{q}},\tilde{\mathbb{V}}) = \frac{\mathbf{u}'_s(\tilde{\mathbf{q}},\tilde{\mathbb{V}})}{\left\|\mathbf{u}'_s(\tilde{\mathbf{q}},\tilde{\mathbb{V}})\right\|}. \tag{28}$$

For any feasible $\tilde{\mathbb{U}}$, Lemma 2 implies that

$$\min_s \frac{\hat{\text{SIR}}_s(\mathbf{q}(\tilde{\mathbf{p}},\tilde{\mathbb{U}},\tilde{\mathbb{V}}),\tilde{\mathbb{V}},\tilde{\mathbf{u}}^*_s)}{\beta_s} \geq \min_s \frac{\hat{\text{SIR}}_s(\mathbf{q}(\tilde{\mathbf{p}},\tilde{\mathbb{U}},\tilde{\mathbb{V}}),\tilde{\mathbb{V}},\tilde{\mathbf{u}}_s)}{\beta_s}. \tag{29}$$

Hence, by fixing $\mathbb{U}$ in the dual uplink power alone and applying Lemma 2, we obtain a set of transmit beamformers $\tilde{\mathbb{U}}^*$ which give a greater or equal minimum weighted SIR as compared to the case of fully fixing $\mathbb{U}$. Next, we can use (24) to obtain $\mathbf{p}(\tilde{\mathbf{q}},\tilde{\mathbb{U}}^*,\tilde{\mathbb{V}})$ in the BC such that this new minimum weighted SIR is preserved in the BC. Hence, we have jointly optimized $\mathbf{p}$ and $\mathbb{U}$ to increase the minimum SIR. Note that if $\tilde{\mathbb{U}}$ is the exact solution to (26), then (26) and (25) are equivalent problems and we have equality in (29).

Corollary 3: The beamformers of any local optimal solution to (13) (including the global optimal solution) must be LMMSE filters.

Corollary 3 motivates the use of LMMSE filters for both the receive and transmit beamforming. Using Algorithm 1, Lemma 1, Lemma 2 and the uplink-downlink duality theory, we propose a local optimal algorithm for solving (13), which we give as Algorithm 2.

Algorithm 2: Max-min weighted SIR power control and beamforming

1) Base station computes auxiliary variables:
$$\mathbf{p}(n) = (\mathbf{I} - \text{diag}(\hat{\boldsymbol{\gamma}}(n))\mathbf{F}(\mathbb{U}(n),\mathbb{V}(n)))^{-1} \times\text{diag}(\hat{\boldsymbol{\gamma}}(n))\mathbf{n}(\mathbb{U}(n),\mathbb{V}(n)).$$
2) Base station updates auxiliary variables:
$$p_s(n+1) = \left(\frac{\beta_s}{\text{SIR}_s(\mathbf{p}(n),\mathbb{U}(n),\mathbb{V}(n))}\right)p_s(n) \quad \forall s.$$
3) Base station normalizes the auxiliary variable to obtain the downlink transmit power $\mathbf{p}(n+1)$:
$$\mathbf{p}(n+1) \leftarrow \mathbf{p}(n+1)\cdot\bar{P}/\mathbf{1}^\mathsf{T}\mathbf{p}(n+1).$$
4) Mobile users update receive beamformers $\mathbb{V}(n+1)$:
$$\mathbf{v}_s(n+1) = \left(\sum_{\substack{i=1\\i\neq s}}^{S} p_i(n+1)\mathbf{H}_{k_s}\mathbf{u}_i(n)\mathbf{u}_i^\dagger(n)\mathbf{H}_{k_s}^\dagger + \mathbf{I}\right)^{+}\mathbf{H}_{k_s}\mathbf{u}_s(n) \quad \forall s,$$
$$\mathbf{v}_s(n+1) \leftarrow \frac{\mathbf{v}_s(n+1)}{\|\mathbf{v}_s(n+1)\|} \quad \forall s.$$
5) Base station computes dual auxiliary variables:
$$\mathbf{q}(n) = (\mathbf{I} - \text{diag}(\boldsymbol{\gamma}(n+1))\hat{\mathbf{F}}(\mathbb{U}(n),\mathbb{V}(n+1)))^{-1} \times\text{diag}(\boldsymbol{\gamma}(n+1))\mathbf{n}(\mathbb{U}(n),\mathbb{V}(n+1)).$$
6) Base station updates dual auxiliary variables:
$$q_s(n+1) = \left(\frac{\beta_s}{\hat{\text{SIR}}_s(\mathbf{q}(n),\mathbb{V}(n+1),\mathbb{U}(n))}\right)q_s(n) \quad \forall s.$$
7) Base station normalizes the dual auxiliary variable to obtain the dual uplink power $\mathbf{q}(n+1)$ (this dual uplink power is also the transmit power of the mobile users):
$$\mathbf{q}(n+1) \leftarrow \mathbf{q}(n+1)\cdot\bar{P}/\mathbf{1}^\mathsf{T}\mathbf{q}(n+1).$$
8) Base station updates transmit beamformers $\mathbb{U}(n+1)$:
$$\mathbf{u}_s(n+1) = \left(\sum_{\substack{i=1\\i\neq s}}^{S} q_i(n+1)\mathbf{H}_{k_i}^\dagger\mathbf{v}_i(n+1)\mathbf{v}_i^\dagger(n+1)\mathbf{H}_{k_i} + \mathbf{I}\right)^{+}\mathbf{H}_{k_s}^\dagger\mathbf{v}_s(n+1) \quad \forall s,$$
$$\mathbf{u}_s(n+1) \leftarrow \frac{\mathbf{u}_s(n+1)}{\|\mathbf{u}_s(n+1)\|} \quad \forall s.$$

Next, we will show that Algorithm 2 converges to a local optimal solution of (13). By introducing an auxilliary variable τ in (13), we can write the Lagrangian of (13) as

$$\begin{aligned} \mathcal{L} &= \tau + \sum_{r=1}^{S}\lambda_r(p_r\left|\mathbf{v}_r^\dagger\mathbf{H}_{k_r}\mathbf{u}_r\right|^2 - \beta_r\tau\sum_{\substack{i=1\\i\neq r}}^{S}p_i\left|\mathbf{v}_r^\dagger\mathbf{H}_{k_r}\mathbf{u}_i\right|^2 - \beta_r\tau) \\ &\quad + \lambda\left(\bar{P} - \mathbf{1}^\mathsf{T}\mathbf{p}\right) + \sum_{r=1}^{S}\lambda'_r\left(\mathbf{u}_r^\dagger\mathbf{u}_r - 1\right) + \sum_{r=1}^{S}\lambda''_r\left(\mathbf{v}_r^\dagger\mathbf{v}_r - 1\right), \end{aligned}$$

where $\lambda \geq 0$ and $\lambda_r \geq 0\ \forall r$ are the dual variables. Since the inequality constraints are always tight at local optimality, a set of local optimal dual variables always exists. By taking the partial derivatives of $\mathcal{L}$ with respect to $\mathbf{v}_s$, $\mathbf{u}_s$, and p_s, and then eliminating λ_s''s and λ_s'''s, we can obtain the Karush-Kuhn-Tucker (KKT) conditions for local optimal solutions of (13).

Lemma 3: Any local optimal solution $(\mathbf{p}, \mathbb{U}, \mathbb{V})$ of (13) satisfies (30)-(35) for some non-negative λ and λ_i's.

$$\sum_{i=1}^{S} \lambda_i p_i \left|\mathbf{v}_i^\dagger \mathbf{H}_{k_i} \mathbf{u}_i\right|^2 = \frac{\mathrm{SIR}_s(\mathbf{p}, \mathbb{U}, \mathbf{v}_s)}{\beta_s} \quad \forall s, \tag{30}$$

$$\mathbf{1}^\mathsf{T}\mathbf{p} = \bar{P}, \tag{31}$$

$$\mathbf{u}_s^\dagger \mathbf{u}_s = 1 \quad \forall s, \tag{32}$$

$$\mathbf{v}_s^\dagger \mathbf{v}_s = 1 \quad \forall s, \tag{33}$$

$$\begin{aligned} &p_s \mathbf{H}_{k_s} \mathbf{u}_s \mathbf{u}_s^\dagger \mathbf{H}_{k_s}^\dagger \mathbf{v}_s \\ &-\mathrm{SIR}_s(\mathbf{p}, \mathbb{U}, \mathbf{v}_s) \left(\sum_{\substack{i=1 \\ i\neq s}}^{S} p_i \mathbf{H}_{k_s} \mathbf{u}_i \mathbf{u}_i^\dagger \mathbf{H}_{k_s}^\dagger + \mathbf{I} \right) \mathbf{v}_s = \mathbf{0}\ \forall s, \end{aligned} \tag{34}$$

$$\begin{aligned} &\frac{\lambda_s}{\lambda} \mathbf{H}_{k_s}^\dagger \mathbf{v}_s \mathbf{v}_s^\dagger \mathbf{H}_{k_s} \mathbf{u}_s \\ &- \left(\sum_{\substack{i=1 \\ i\neq s}}^{S} \frac{\lambda_i}{\lambda} \mathrm{SIR}_i(\mathbf{p}, \mathbb{U}, \mathbf{v}_i) \mathbf{H}_{k_i}^\dagger \mathbf{v}_i \mathbf{v}_i^\dagger \mathbf{H}_{k_i} + \mathbf{I} \right) \mathbf{u}_s = \mathbf{0}\ \forall s. \end{aligned} \tag{35}$$

Remark 2: From the uplink-downlink duality, a local optimal solution $(\mathbf{p}, \mathbb{U}, \mathbb{V})$ to the downlink problem and a local optimal solution $(\mathbf{q}, \mathbb{V}, \mathbb{U})$ to the dual uplink problem achieve the same weighted SIR. By relating (30)-(35) to the first-order conditions for local optimality of the dual uplink problem, it can be seen that $\frac{\lambda_i}{\lambda}\mathrm{SIR}_i(\mathbf{p}, \mathbb{U}, \mathbf{v}_i)$ is exactly the dual uplink power q_i.

Lemma 4: Algorithm 2 converges to a local optimal solution of (13), with a monotonically increasing $\min_s(\mathrm{SIR}_s(\mathbf{p}, \mathbb{U}, \mathbf{v}_s)/\beta_s)$.

Note that $\mathbf{q}$ is the exact uplink transmit power. Hence, in step 7, the transmitter must inform the receivers of the computed uplink transmit power.

Appendix

Proof of Theorem 1: Since $\mathrm{SIR}_k(\mathbf{p}, \mathbb{U}, \mathbf{v}_k)$ depends on $\mathbb{V}$ only through $\mathbf{v}_k$, we can rewrite (4) as

$$\begin{array}{ll} \max\limits_{\mathbf{p},\mathbb{U}} & \min\limits_{k} \max\limits_{\substack{\mathbf{v}_k \\ \mathbf{v}_k^\dagger \mathbf{v}_k = 1}} \dfrac{\mathrm{SIR}_k(\mathbf{p}, \mathbb{U}, \mathbf{v}_k)}{\beta_k} \\ \text{subject to} & \mathbf{1}^\mathsf{T}\mathbf{p} \leq \bar{P}, \quad \mathbf{p} \geq \mathbf{0}, \\ & \mathbf{u}_k^\dagger \mathbf{u}_k = 1 \quad \forall k. \end{array} \tag{36}$$

The optimal receive beamformers $\mathbf{v}_k^*$ are the matched filters corresponding to the receive spatial signature $\mathbf{h}_k$ of the antenna array. By substituting (2) into the normalized LMMSE filters (5) and (6), it can be shown that we will always obtain the desired matched filters. Hence, the normalized LMMSE filter is always optimal for any arbitrary $\mathbf{p}$ and $\mathbb{U}$. □

Proof of Lemma 1: Since $\mathrm{SIR}_s(\tilde{\mathbf{p}}, \tilde{\mathbb{U}}, \mathbf{v}_s)$ depends on $\mathbb{V}$ only through $\mathbf{v}_s$, we can exchange the $\max\limits_{\mathbb{V}} \min\limits_{s}$ operators in (18). Then, we can decouple (18) into S independent maximization problems, where the optimal $\tilde{\mathbf{v}}_s^*(\tilde{\mathbf{p}}, \tilde{\mathbb{U}})$'s are the normalized LMMSE filters. □

Proof of Corollary 3: If $\tilde{\mathbf{p}}$ and $\tilde{\mathbb{U}}$ in Lemma 1 correspond to a local optimal solution, then the receive beamformers of this local optimal solution must be LMMSE filters. The same analysis can be made with Lemma 2. □

Proof of Lemma 4: Since each step in Algorithm 2 increases the objective function of (13), the objective function is strictly monotonically increasing. The sum power constraint $\mathbf{1}^\mathsf{T}\mathbf{p}(n) \leq \bar{P}$ and the unity norm constraint on the beamformers implies the existence of a limiting value for $\mathrm{SIR}_s(\mathbf{p}(n), \mathbb{U}(n), \mathbb{V}(n))$ so the sequence must converge monotonically. By substituting the equations in Algorithm 2 into (30)-(35), it is easily shown that Algorithm 2, at convergence, satisfies all the first-order conditions in Lemma 3. □

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