

Solving Inequalities and Proving Farkas's Lemma Made Easy

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Introduction

We consider the following problem: given a matrix $A = [a_{ij}]$ in $\mathbb{R}^{m \times n}$ and a column vector b in \mathbb{R}^m , find $x = (x_1, x_2, \dots, x_n)^T$ that satisfies the following linear system, or prove that no such vector x exists:

$$Ax \leq b, x \geq 0. \quad (1)$$

We illustrate a simple method for doing this with an example:

$$\begin{aligned} -x_1 - 2x_2 + x_3 &\leq -1 \\ x_1 - 3x_2 - x_3 &\leq 2 \\ -x_1 - 2x_2 + 2x_3 &\leq -2 \end{aligned} \quad (2)$$

with $x_i \geq 0$ ($i = 1, 2, 3$).

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We first convert this system of inequalities into a system of equations by introducing a new nonnegative slack variable for each inequality. Putting these variables on the left-hand side, and the others on the right-hand side we have the following system:

$$\begin{aligned}x_4 &= -1 + x_1 + 2x_2 - x_3 \\x_5 &= 2 - x_1 + 3x_2 + x_3 \\x_6 &= -2 + x_1 + 2x_2 - 2x_3\end{aligned}\tag{3}$$

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We call a system of equations such as (3) a *dictionary*. The variables on the left-hand side are called *basic*, and the variables on the right-hand side are called *cobasic*. We get a *basic solution* to the equations in (3) by setting all the cobasic variables to zero, which gives $x_4 = -1$, $x_5 = 2$, $x_6 = -2$. Unfortunately this is not a nonnegative solution. The algorithm proceeds as follows: it finds the smallest-indexed basic variable that is set to a negative value. In this case it is x_4 . In the equation for x_4 it identifies the cobasic variable with the smallest index that has a positive coefficient (in this case it is x_1), solves this equation for x_1 , and substitutes the result for x_1 in the other equations.

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This yields a new dictionary:

$$\begin{aligned}x_1 &= 1 - 2x_2 + x_3 + x_4 \\x_5 &= 1 + 5x_2 - x_4 \\x_6 &= -1 - x_3 + x_4\end{aligned}\tag{4}$$

The step we just performed is called a *pivot* operation. In (4), we first set the cobasic (i.e., right-hand) variables to zero and get the basic solution $x_1 = 1$, $x_5 = 1$, $x_6 = -1$. Again, we find the basic variable with the smallest index and negative value, namely, x_6 . In the equation for x_6 we find the smallest-indexed cobasic variable with a positive coefficient, here x_4 .

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We pivot by solving this equation for x_4 and substituting for x_4 in the other equations, obtaining the new dictionary:

$$\begin{aligned}x_1 &= 2 - 2x_2 + 2x_3 + x_6 \\x_4 &= 1 + x_3 + x_6 \\x_5 &= 0 + 5x_2 - x_3 - x_6\end{aligned}\tag{5}$$

We are now in luck. The basic solution is nonnegative, and its restriction to our original three variables gives a feasible solution to (2): $x_1 = 2$, $x_2 = 0$, $x_3 = 0$. So far so good.

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An immediate question raises itself: What happens if there is no solution to the original problem? Consider the following problem:

$$\begin{aligned} -x_1 + 2x_2 + x_3 &\leq 3 \\ 3x_1 - 2x_2 + x_3 &\leq -17 \\ -x_1 - 6x_2 - 23x_3 &\leq 19 \end{aligned} \tag{6}$$

with $x_i \geq 0$ ($i = 1, 2, 3$). We get an initial dictionary by introducing three slack variables and letting them be the basic variables:

$$\begin{aligned} x_4 &= 3 + x_1 - 2x_2 - x_3 \\ x_5 &= -17 - 3x_1 + 2x_2 - x_3 \\ x_6 &= 19 + x_1 + 6x_2 + 23x_3 \end{aligned} \tag{7}$$

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The algorithm proceeds as before by choosing the equation for x_5 and solving for x_2 :

$$\begin{aligned}x_2 &= 17/2 + (3/2)x_1 + (1/2)x_3 + (1/2)x_5 \\x_4 &= -14 - 2x_1 - 2x_3 - x_5 \\x_6 &= 70 + 10x_1 + 26x_3 + 3x_5\end{aligned}\tag{8}$$

Here we encounter something new. We select the equation for x_4 , as we should, but find that there is no cobasic variable with a positive coefficient. We rewrite this equation with all variables on the left-hand side, including those with zero coefficients, getting

$$2x_1 + 0x_2 + 2x_3 + \mathbf{1}x_4 + \mathbf{1}x_5 + \mathbf{0}x_6 = -14.\tag{9}$$

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We multiply each inequality in (6) by the coefficient of its corresponding slack variable

$$\begin{aligned} & \mathbf{1} \times (-x_1 + 2x_2 + x_3 \leq 3) \\ & + \mathbf{1} \times (3x_1 - 2x_2 + x_3 \leq -17) \\ & + \mathbf{0} \times (-x_1 - 6x_2 - 23x_3 \leq 19) \end{aligned} \tag{10}$$

and add the inequalities in (10) to get

$$2x_1 + 2x_3 \leq -14. \tag{11}$$

The above procedure yields an *inconsistent inequality*, which in fact is a counterexample to the claim that the original system is feasible, i.e., the algorithm furnishes a *proof of contradiction*.

Algorithm

We now have a complete description of the algorithm that we call the “b-rule” for solving problems of form (1):

Step 1: Introduce m slack variables x_{n+1}, \dots, x_{n+m} and use these as the basis (left-hand side) of an initial dictionary

$$x_{n+i} = b_i - \sum_{j=1}^n a_{ij}x_j \quad (i = 1, \dots, m) \quad (12)$$

Step 2: Set the cobasic (right-hand side) variables to zero. Find the smallest index of a basic (left-hand side) variables with a negative value. If there is none, terminate with a feasible solution.

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Step 3: Find the cobasic variable in the equation chosen in step 2 that has the smallest index and a positive coefficient. If there is none, terminate as the problem is infeasible, and the coefficients of the slack variables represent a *certificate of infeasibility*. Otherwise, solve this equation for the indicated cobasic variable, and substitute the result in all of the other equations. Go to step 2.

Further Remarks

- ▶ Can the algorithm cycle indefinitely or will it always terminate in a finite number of steps?
- ▶ Is the algorithm output (either feasible solution or certificate of infeasibility) sensitive to the parameters of the algorithm? For instance, choose which cobasic variable to pivot or choose values other than zero for cobasic variables at each iteration?
- ▶ What can be observed about the certificates of infeasibility output by the algorithm for different systems of inequalities?
- ▶ Is your implementation of the algorithm efficient and scalable?

[1] David Avis and Bohdan Kaluzny, Solving inequalities and proving Farkas's Lemma made easy. The American Mathematical Monthly, 111(2), pp. 152-157, 2004.