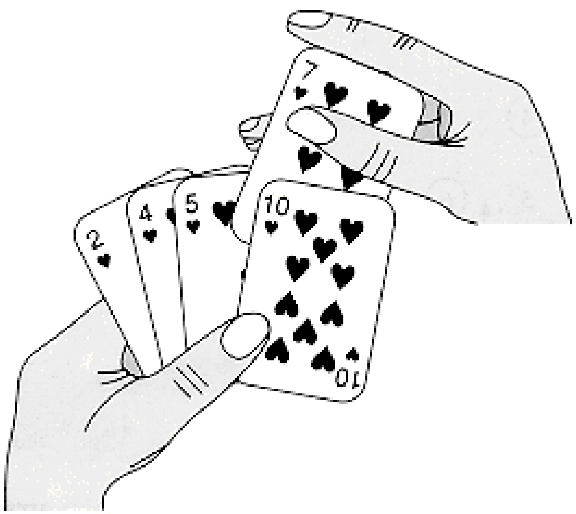
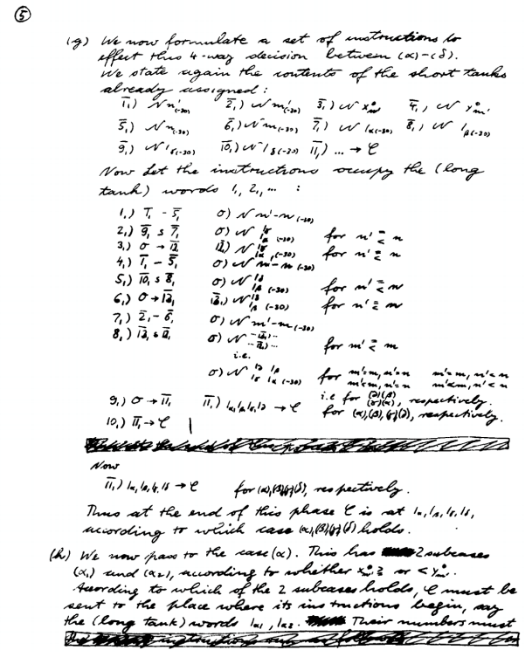
CS3334 Data Structures

Lecture 5: Divide-Conquer & Merge Sort




(g) We now formulate a set of instructions to effect this 4-way decision between (α)-(δ). We state again the contents of the short tanks already assigned:

Now let the instructions occupy the (long tank) words 1, 2, ... :

Thus at the end of this phase C is at ... according to which case (α),(β),(γ),(δ) holds.

(h) We now pass to the case (α). This has 2 subcases (α₁) and (α₂), according to whether ... According to which of the 2 subcases holds, C must be sent to the place where its instructions begin, say the (long tank) words ... Their numbers must

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How to Solve It?

George Pólya

1) First, you have to understand the problem.
2) After understanding, make a plan.
3) Carry out the plan.
4) Look back on your work. How could it be better?

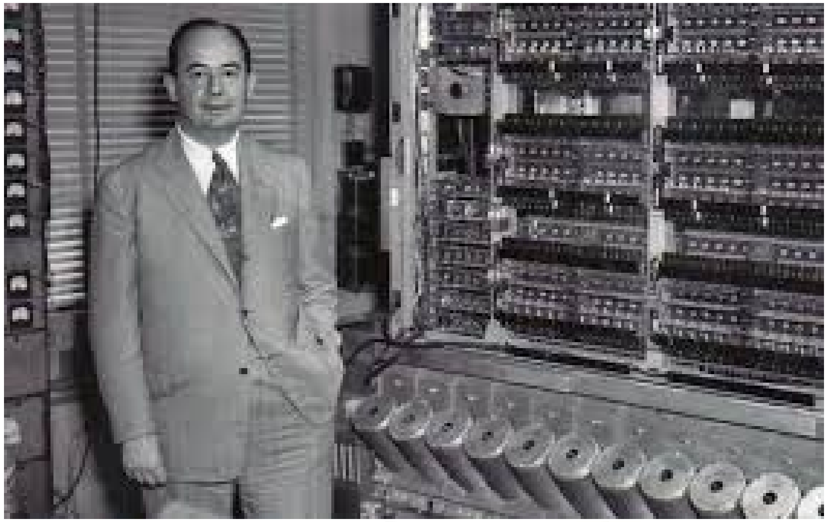
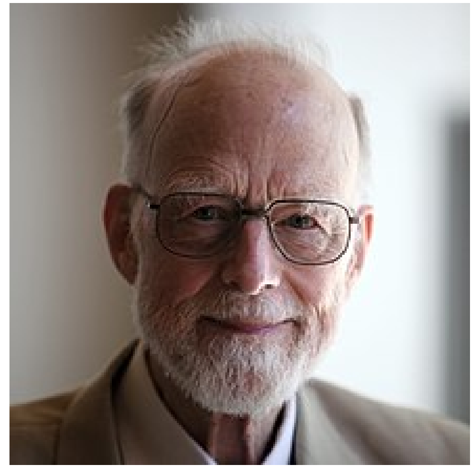
If this fails, Pólya advises: "If you can't solve a problem, then there is an easier problem you can solve: find it."
https://en.wikipedia.org/wiki/How_to_Solve_It

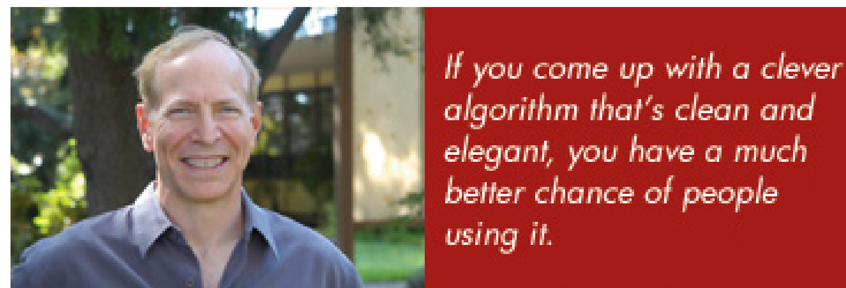
Clearly expound the idea of Divide-and-Conquer

Break down a problem into smaller ones you can handle/play with

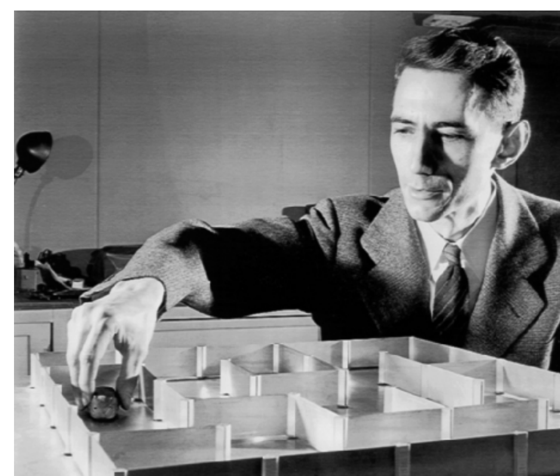
"Johnny (von Neumann) was the only student I was ever afraid of," George Pólya

Giants of Computer Science





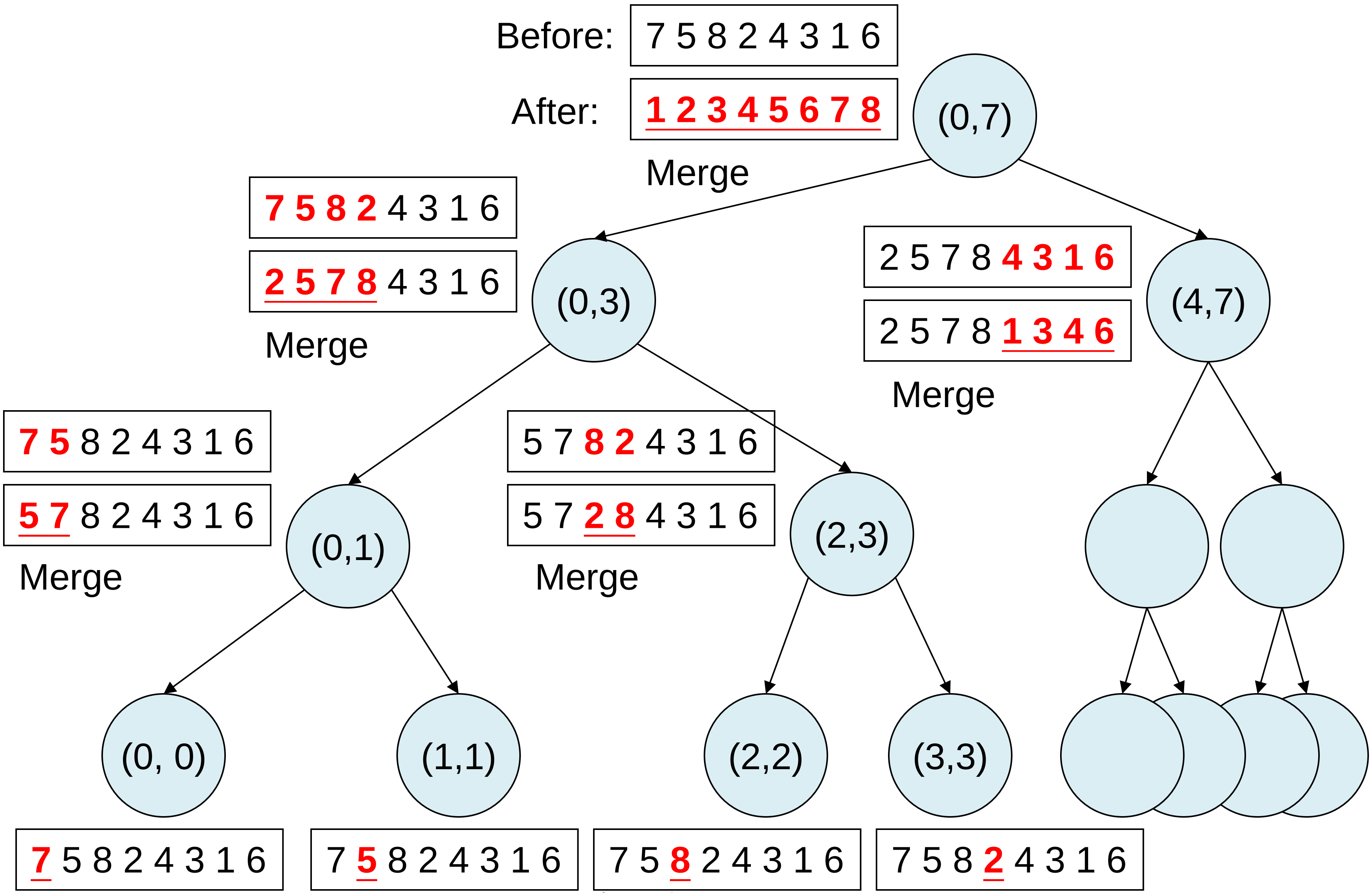
Idea: Divide and Conquer

- **Divide**: Divide a problem into smaller and simpler subproblems
- **Conquer**: Solve recursively each subproblem
- **Merge**: Combine solutions to subproblems to get the solution to original problem
- Examples of historical algorithms:
  - Gauss Fast Fourier Transform (1805)
    - Gauss discovered the algorithm at age 28
  - Merge Sort algorithm (1945)
    - John von Neumann may have discovered it by playing poker
  - Karatsuba algorithm (1960)
    - Karatsuba discovered the algorithm at age 23

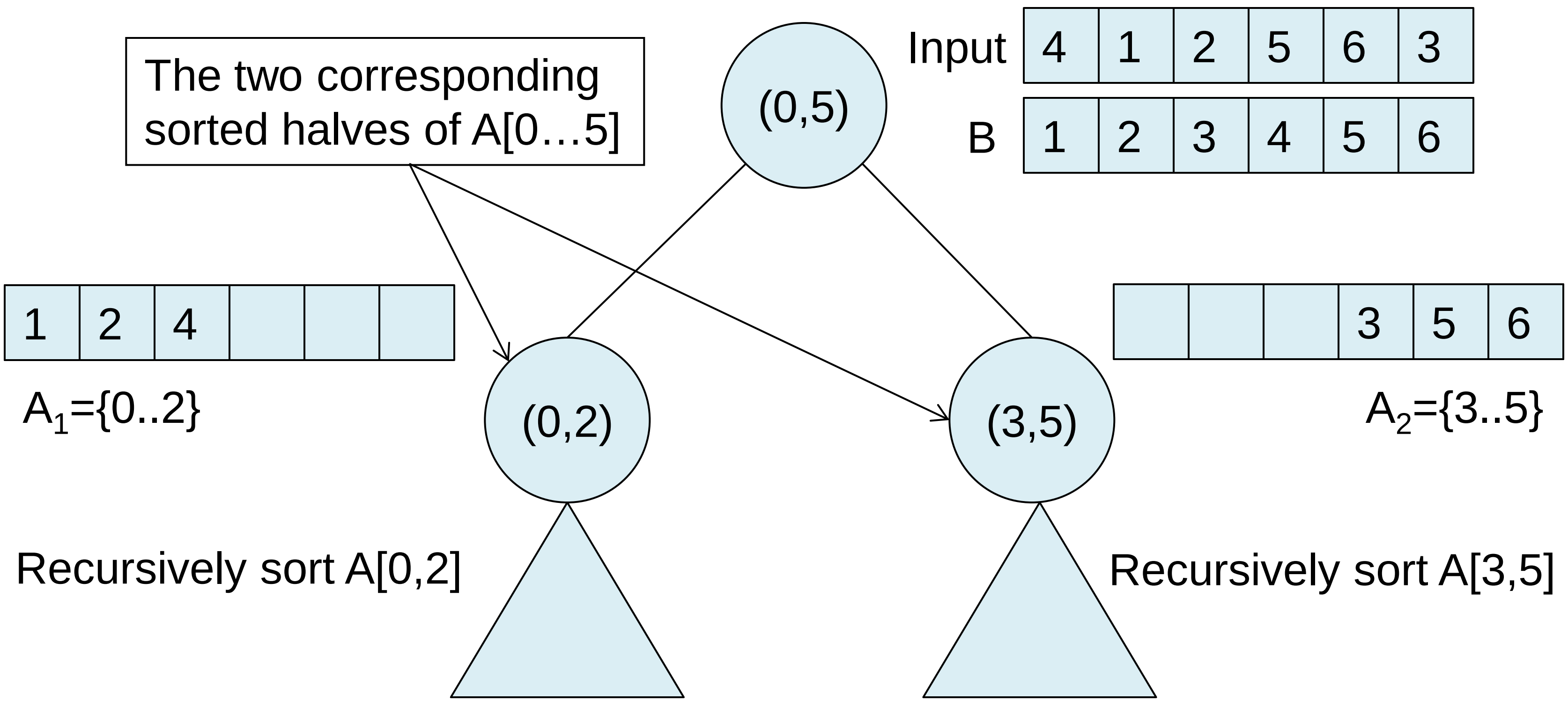
Merge Sort

- Given:
  - A[0..n-1]: array to be sorted
  - B[0..n-1]: buffer array (global variable)
  - Array of type `item`
- Apply idea of divide-and-conquer:
  - Recursively divide A[0..n-1] into two halves until each half has one element
  - Sort each half by merging its two corresponding sorted halves into one sorted list

An Illustration



Zoom-in to the Merge Step



"1" is the smallest element, so insert "1" to B[ ]

"2" is the smallest element, so insert "2" to B[ ]

"3" is the smallest element, so insert "3" to B[ ]

"4" is the smallest element, so insert "4" to B[ ]

A_1 becomes empty, so insert the remaining elements in A_2 to B[ ] one by one

Merge Sort (Code) (1/2)

```
void mergesort(item A[], int low, int up)
{
   if (low < up)
   {
      int m = (low+up)/2; //Integer division
      //Recursively divide A[0..n-1] into two halves
      //until each half has one element
      mergesort(A,low,m);
      mergesort(A,m+1,up);

      //Sort each half by merging its two corresponding
      //sorted halves into one sorted list
      int i=low, j=m+1, k=low;       //A1={i..m}; A2={j..up}
      while (i<=m && j<=up)          //Repeatedly append the
         if (A[i] < A[j])            //smallest element x in
            B[k++]=A[i++];           //A1 and A2 to B[] until
         else                        //A1 or A2 becomes empty
            B[k++]=A[j++];
```

Merge Sort (Code) (2/2)

```
      //A1 or A2 becomes empty
      while (i<=m)
         B[k++]=A[i++];
      while (j<=up)
         B[k++]=A[j++];

      for (k=low; k<=up; k++)
         A[k]=B[k];  //Copy back from B[] to A[]

   } //If low<up
}
//In main program
   mergesort(A,0,n-1);
```

Space Analysis

- Array B[0..n-1] requires O(n) space (in terms of words)
- Each recursive call requires constant number of variables for bookkeeping
- Let S'(n) = space complexity, excluding B[0..n-1]
- $S'(n) \leq S'(n/2) + c \leq \ldots = O(\log n)$
- Total space $S(n) = O(n) + S'(n)$
  $= O(n + \log n) = O(n)$

Time Analysis

Assume $n = 2^k$

$$T(n) \le 2T(n/2) + cn \quad \text{for some constant } c$$

$$2T(n/2) \le 2^2T(n/2^2) + cn$$

$$2^2T(n/2^2) \le 2^3T(n/2^3) + cn$$

$$\dots$$

$$+)\quad 2^{k-1}T(n/2^{k-1}) \le 2^kT(n/2^k) + cn$$

$$T(n) \le 2^kT(n/2^k) + k(cn)$$

$$= nT(1) + \log_2 n(cn)$$

$$= O(n\log n)$$

Each recursion
(1) selects elements to B[] and (2) copy elements from B[] to A[]

$2[2T(n/2^2)+c(n/2)]$
$=2[2T(n/2^2)]+2[c(n/2)]$
$=2^2T(n/2^2)+cn$

$2^2[2T(n/2^3)+c(n/2^2)]$
$=2^2[2T(n/2^3)]+2^2[c(n/2^2)]$
$=2^3T(n/2^3)+cn$

$n = 2^k$
$\log_2 n = \log_2 2^k$
$\log_2 n = k\log_2 2$
$\log_2 n = k$

Merge Sort Theorem

Suppose n is a power of 2. Consider the recurrence relation of the form

$T(n) = 2\,T(n/2) + n$ if $n>1$ and $T(1)=0$.

Then $T(n) = n \log_2 n$.

[Quiz] Give a proof by induction.

Recall the Tower of Hanoi recurrence relation we saw earlier.

Fundamental Bound

- Theorem: Any *comparison-based* sorting algorithm requires $\Omega(n\log n)$ time
- Examples of comparison-based sorting algorithms:
  - Bubble Sort, Insertion Sort: $O(n^2)$ time
  - Merge Sort, Heap Sort (next lecture): $O(n\log n)$ time
  - Quick Sort (next lecture): $O(n^2)$ worst case, $O(n\log n)$ average case
- What are their common features?

Comparison-based Sorting Algorithms

- Variables and values are classified into 2 types:
  - Key type: `item`

    Only comparison or assignment allowed, e.g.,
    - `if (A[mid]==x)` [binary search]
    - `if (A[i]<A[j])` [merge sort]
    - `swap(A[j-1],A[j])` [bubble & insertion sort]

    Cannot create new values of key type, e.g.,
    `pivot=(A[i]+A[i+1])/2` not allowed

    Array index cannot be value of key type, e.g.,
    `B[A[i]]=A[i]` or `C[A[i]]++` not allowed
  - Other types

    No restrictions on the operations

Decision Tree Model

- Every comparison-based sorting algorithm can be abstracted as a *decision tree*
- *Comparison* of the order of two items constitutes *a single bit (binary digit)*
- A decision tree captures the comparisons of input values while ignoring all other computations (e.g., updating a loop counter, swapping a pair of input values, etc.)

An Example of Decision Tree

- Next page shows a decision tree for sorting 3 elements, X, Y and Z
- Each internal node is labeled by a pair of input values:

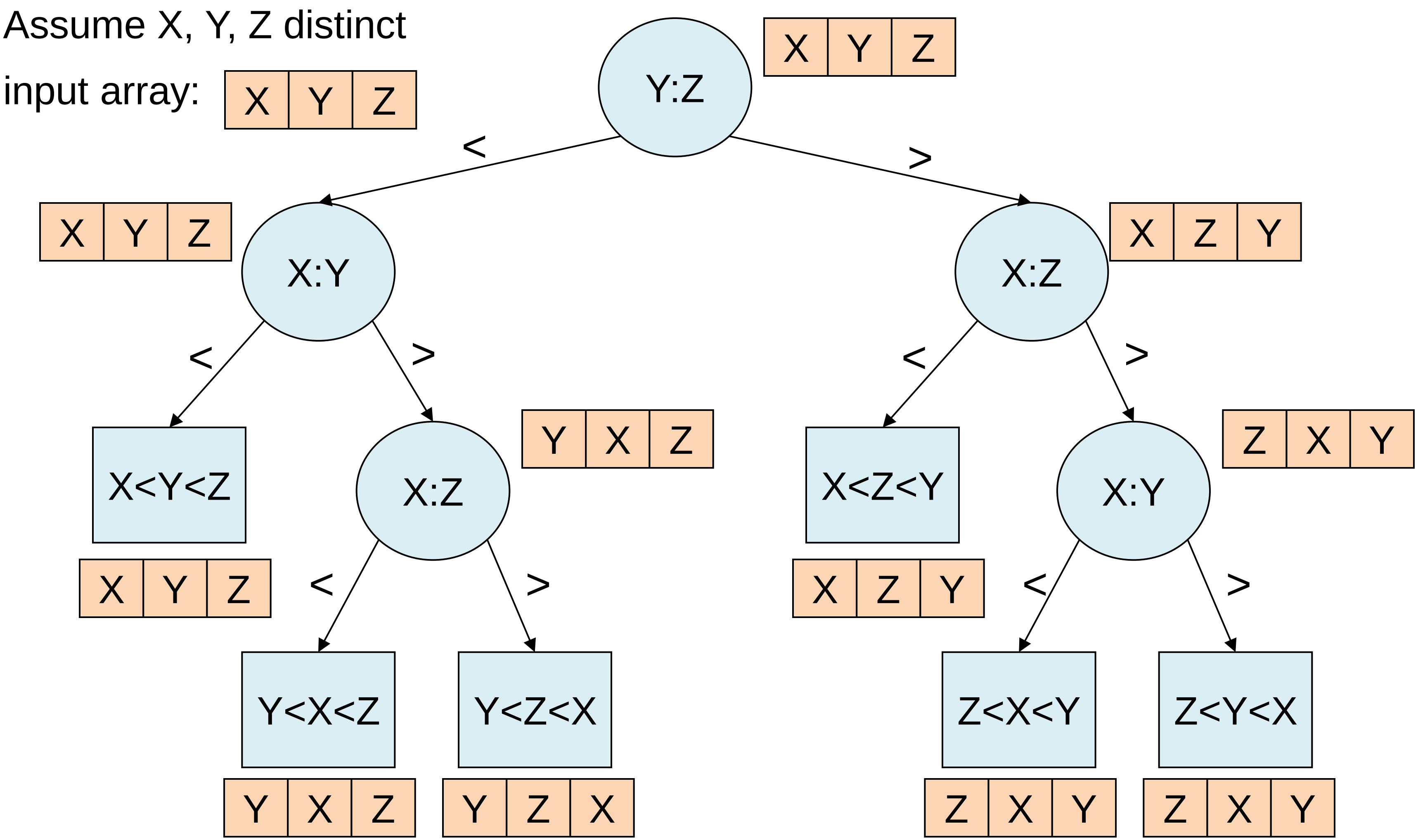
  a:b

- The two edges coming out of an internal node represent the 2 possible outcomes:

  a < b or a > b

- Each leave node is labeled by a permutation of the input values

A Decision Tree for n=3



How to Run a Decision Tree?

- Start from the root
- Repeat
  - Compare the two values mentioned in the current node (e.g., X:Y)
  - Follow the path according to outcome of the comparison to arrive at the next node (e.g., X<Y or Y>X)
- Until we arrive at a leaf node
- Report the permutation of the leaf as the answer

Worst-Case Running Time (1/4)

- How do we count the running time of a decision tree?
  - We only count the number of comparisons
  - E.g.: If the input is X<Z<Y, the decision tree in previous slide performs 2 comparisons
- What is the worst case running time of a decision tree?
  - **It is equal to the *height* of the tree, where *height* is the number of hops along the longest root-to-leaf path**

Worst-Case Running Time (2/4)

- Proof of Lower Bound:
  - Assume inputs are permutations of {1,…,n}
    - There are n! permutations.
  - Consider an arbitrary decision tree and calculate the *height* of the binary tree
    - Each tree has n! leaves, one for each permutation.
    - If the binary tree has height h, then it has ≤ 2^h leaves.
    - Thus, $2^h \geq n!$.

Worst-Case Running Time (3/4)

- So the tree must have height h large enough so that

$$2^h \geq n!$$

i.e., $h \geq \log_2(n!)$

$\geq \log_2[(n)(n\text{-}1)\ldots(n/2+1)(n/2)(n/2\text{-}1)\ldots(1)]$

$\geq \log_2[\underbrace{(n)(n\text{-}1)\ldots(n/2+1)}]$

(only keep the first n/2 terms)

$\geq \log_2[(n/2)(n/2)\ldots(n/2)]$

Reduce each term to n/2

$\geq \log_2(n/2)^{n/2}$

$\geq (n/2)\log_2(n/2)$

$\geq (n/2)(\log_2 n\text{-}\log_2 2)$

$\geq (n/2)(\log_2 n)\text{-}(n/2)(\log_2 2)$

$\geq (n/2)(\log_2 n)\text{-}n/2$

$= \Omega(n\log n)$

– Worst case time complexity

Worst-Case Running Time (4/4)

- Another proof using Stirling's Approximation (as n tends to infinity):

$$n! \geq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

How many trailing zeros does this number have?

(first stated in 1733 by Abraham de Moivre and later refined by James Stirling)

- So the tree must have height h large enough so that

$$\begin{aligned}\log_2(2^h) &\geq \log_2(n!)\\ &\geq \log_2\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)\\ &= \log_2(\sqrt{2\pi n}) + n\log_2 n - n\log_2 e\\ &= \Omega(n\log n)\end{aligned}$$

– What is the physical meaning of $\log_2(n!)$?

Counting Inversions

- Music site tries to match your song preferences with others.
  - You rank n songs.
  - Site consults database to find people with similar tastes → Recommender System

- Similarity metric: number of inversions between two rankings.
  - My rank: 1, 2, …, n.
  - Your rank: $a_1, a_2, \ldots, a_n$.
  - Songs i and j inverted if $i < j$, but $a_i > a_j$.

Songs

	A	B	C	D	E
Me	1	2	3	4	5
You	1	3	4	2	5

<u>Inversions</u>

3-2, 4-2

- Brute force: check all $\Theta(n^2)$ pairs i and j. [Quiz] Write down this algorithm

Counting Inversions: Divide-and-Conquer

- Divide-and-conquer.

1	5	4	8	10	2	6	9	12	11	3	7

Counting Inversions: Divide-and-Conquer

- Divide-and-conquer.
  - Divide: separate list into two pieces.

1	5	4	8	10	2	6	9	12	11	3	7

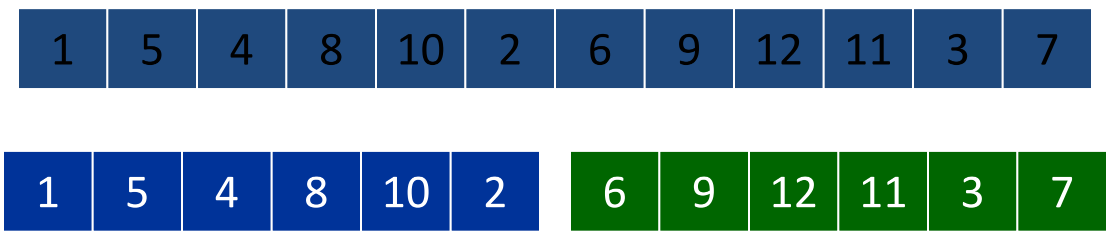
Divide: O(1).

1	5	4	8	10	2

6	9	12	11	3	7

Counting Inversions: Divide-and-Conquer

- Divide-and-conquer.
  - Divide: separate list into two pieces.
  - Conquer: recursively count inversions in each half.



Divide: O(1).

Conquer: 2T(n / 2)

5 blue-blue inversions

5-4, 5-2, 4-2, 8-2, 10-2

8 green-green inversions

6-3, 9-3, 9-7, 12-3, 12-7, 12-11, 11-3, 11-7

Counting Inversions: Divide-and-Conquer

– Divide: separate list into two pieces.

– Conquer: recursively count inversions in each half.

– Combine: count inversions where a_i and a_j are in different halves, and return sum of three quantities.

1	5	4	8	10	2	6	9	12	11	3	7

Divide: O(1).

1	5	4	8	10	2

5 blue-blue inversions

6	9	12	11	3	7

8 green-green inversions

Conquer: 2T(n / 2)

9 blue-green inversions

5-3, 4-3, 8-6, 8-3, 8-7, 10-6, 10-9, 10-3, 10-7

Combine: ???

Total = 5 + 8 + 9 = 22.

Counting Inversions: Combine

Combine: count blue-green inversions

- Assume each half is sorted.
- Count inversions where a_i and a_j are in different halves.
- Merge two sorted halves into sorted whole.

3	7	10	14	18	19	2	11	16	17	23	25
						6	3	2	2	0	0

13 blue-green inversions: 6 + 3 + 2 + 2 + 0 + 0

Count: O(n)

2	3	7	10	11	14	16	17	18	19	23	25

Merge: O(n)