Convex Functions and Sets

Chee Wei Tan

Convex Optimization and its Applications to Computer Science

Outline

- Convex sets and examples
- Separating and supporting hyperplanes
- Convex functions and examples
- Conjugate functions

Convex Set

Set C is a convex set if the line segment between any two points in C lies in C, ie, if for any $x_1, x_2 \in C$ and any $\theta \in [0, 1]$, we have

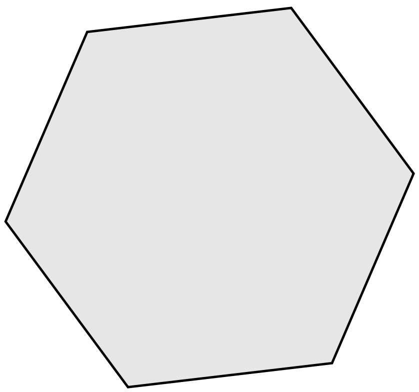
$$\theta x_1 + (1 - \theta)x_2 \in C$$

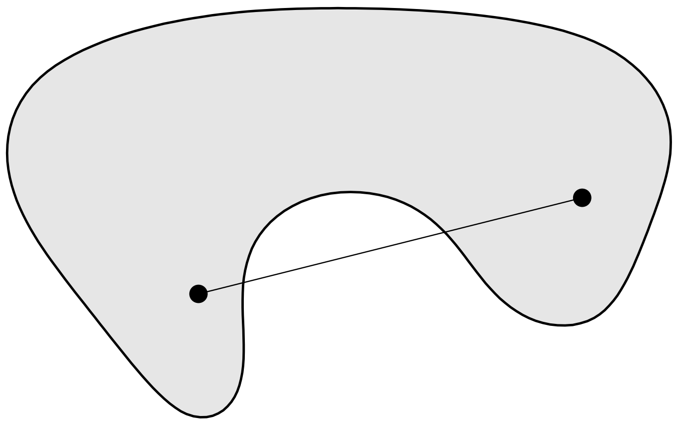
Convex hull of C is the set of all convex combinations of points in C:

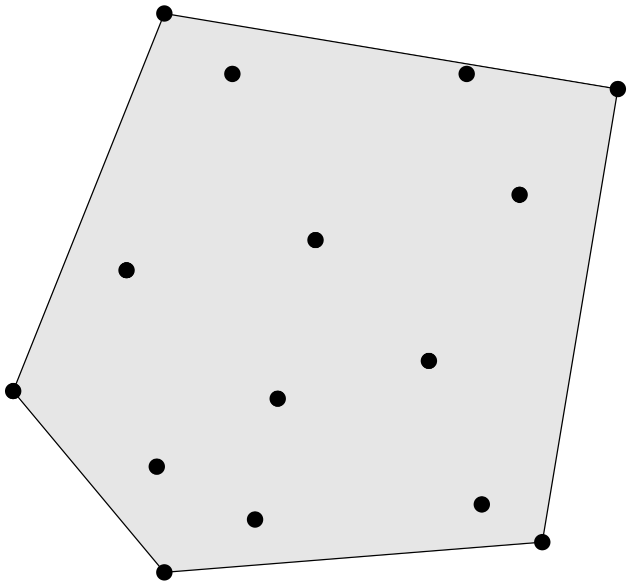
$$\left\{ \sum_{i=1}^{k} \theta_i x_i | x_i \in C, \theta_i \geq 0, i = 1, 2, \dots, k, \sum_{i=1}^{k} \theta_i = 1 \right\}$$

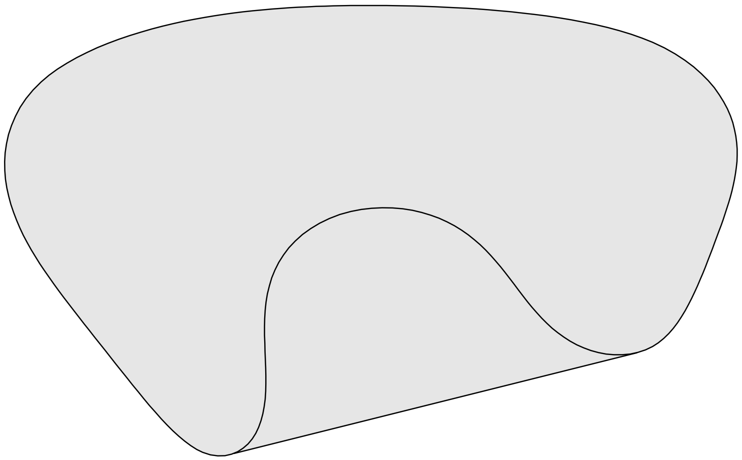
Can generalize to infinite sums and integrals

Examples





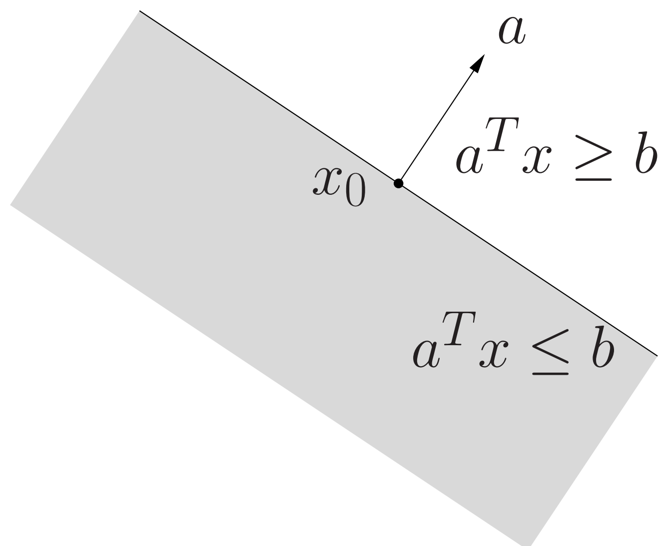




Examples of Convex Sets

- Hyperplane in $\mathbf{R}^n$ is a set: $\{x | a^T x = b\}$ where $a \in \mathbf{R}^n, a \neq 0, b \in \mathbf{R}$

  Divides $\mathbf{R}^n$ into two halfspaces: eg, $\{x | a^T x \leq b\}$ and $\{x | a^T x > b\}$



- Polyhedron is the solution set of a finite number of linear equalities and inequalities (intersection of finite number of halfspaces and hyperplanes)

Examples of Convex Sets

- Euclidean ball in $\mathbf{R}^n$ with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

Verify its convexity by triangle inequality

- Generalize to ellipsoids:

$$\mathcal{E}(x_c, P) = \left\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\right\}$$

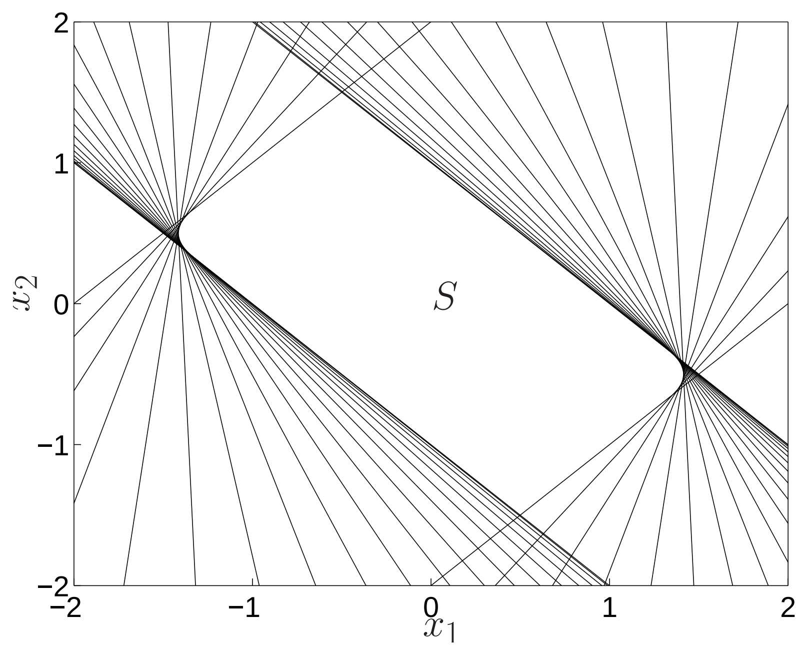
P: symmetric and positive definite. Lengths of semi-exes of $\mathcal{E}$ are $\sqrt{\lambda_i}$ where λ_i are eigenvalues of P

Convexity-Preserving Operations

- Intersection.

- Example: $S = \left\{x \in \mathbf{R}^m \,\middle|\, |p(t)| \leq 1 \text{for} |t| \leq \frac{\pi}{3}\right\}$ where $p(t) = \sum_{k=1}^{m} x_k \cos kt$.

- Since $S = \bigcap_{|t| \leq \frac{\pi}{3}} S_t$, where $S_t = \left\{x \,\middle|\, -1 \leq (\cos t, \ldots, \cos mt)^T x \leq 1\right\}$, S is convex



Convexity-Preserving Operations

- Linear-fractional functions: $f : \mathbf{R}^n \to \mathbf{R}^m$:

$$f(x) = \frac{Ax+b}{c^T x + d}, \quad \mathbf{dom}\, f = \{x | c^T x + d > 0\}$$

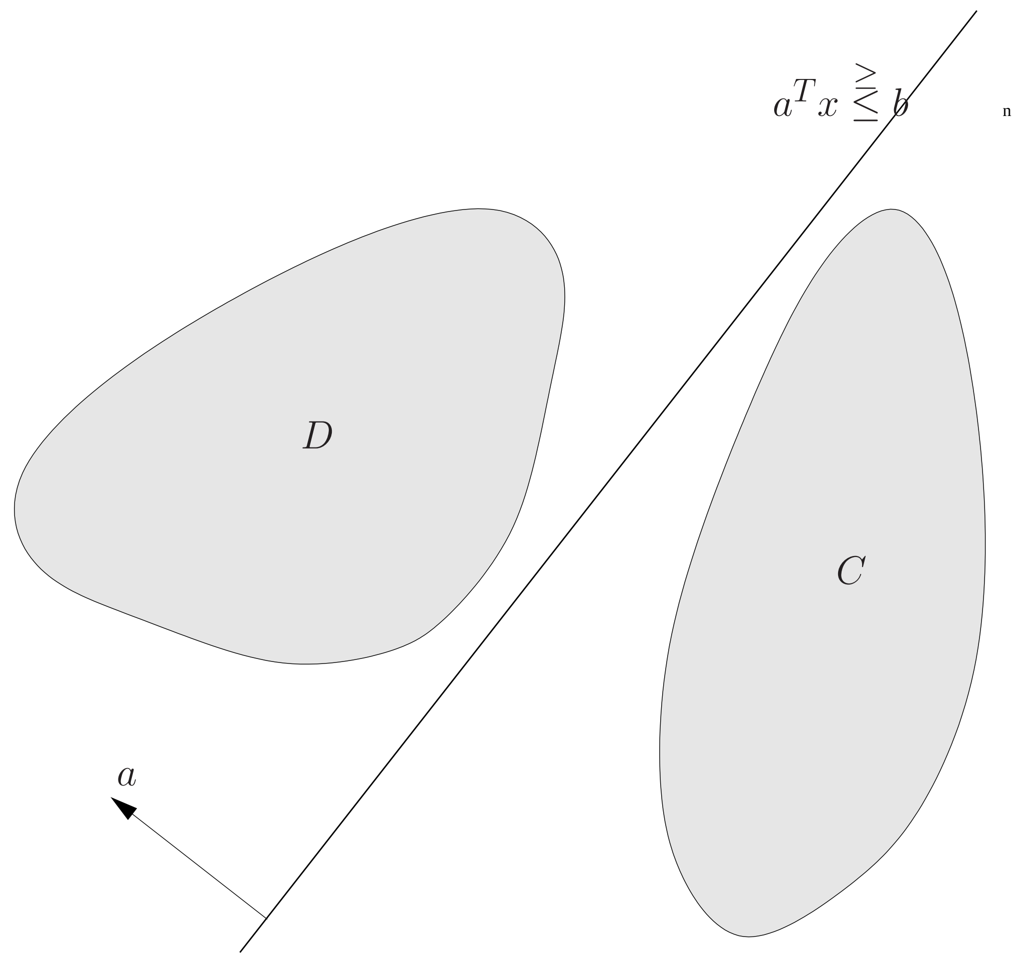
- If set C in $\mathbf{dom}\, f$ is convex, image $f(C)$ is also convex set

- Example: $p_{ij} = \mathbf{Prob}(X = i, Y = j), q_{ij} = \mathbf{Prob}(X = i | Y = j)$. Since

$$q_{ij} = \frac{p_{ij}}{\sum_k p_{kj}},$$

  if C is a convex set of joint prob. for (X, Y), the resulting set of conditional prob. of X given Y is also convex

Separating Hyperplane Theorem



- C and D: non-intersecting convex sets, $i.e.$, $C \bigcap D = \phi$. Then there exist $a \neq 0$ and b such that $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$.

Separating Hyperplane Theorem: Application

- Theorem of alternatives for strict linear inequalities:

$$Ax \prec b$$

are infeasible if and only if there exists $\lambda \in \mathbf{R}^m$ such that

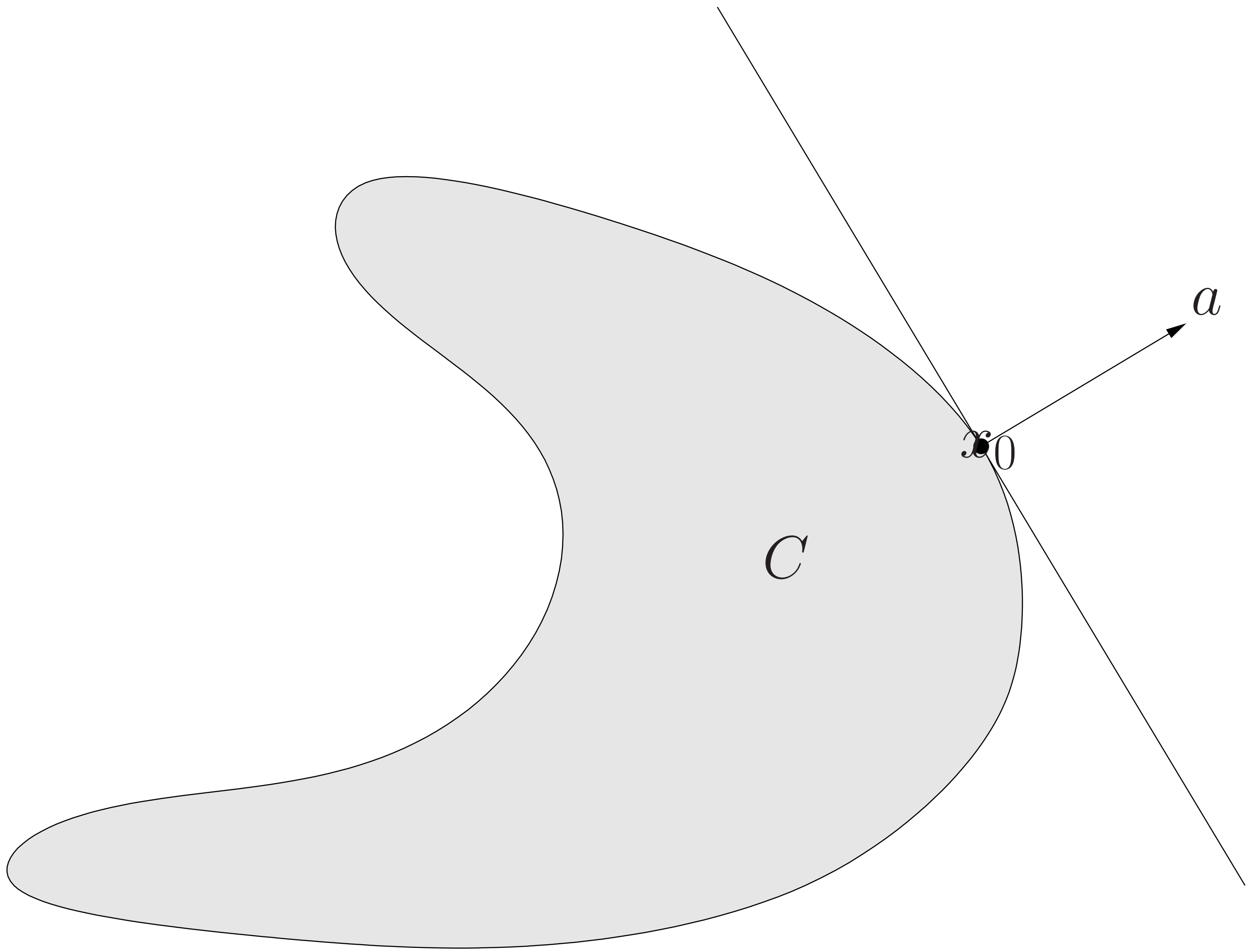
$$\lambda \neq 0, \;\; \lambda \succeq 0, \;\; A^T\lambda = 0, \;\; \lambda^T b \leq 0.$$

Supporting Hyperplane Theorem

- Given a set $C \in \mathbf{R}^n$ and a point x_0 on its boundary, if $a \neq 0$ satisfies $a^T x \leq a^T x_0$ for all $x \in C$, then $\{x | a^T x = a^T x_0\}$ is called a supporting hyperplane to C at x_0

- For any nonempty convex set C and any x_0 on boundary of C, there exists a supporting hyperplane to C at x_0, i.e., there is a vector $a \in R^n, a \neq 0$, such that

$$\sup_{z \in C} \; a^T z \leq a^T x_0.$$


a
x_0
C

Convex Functions

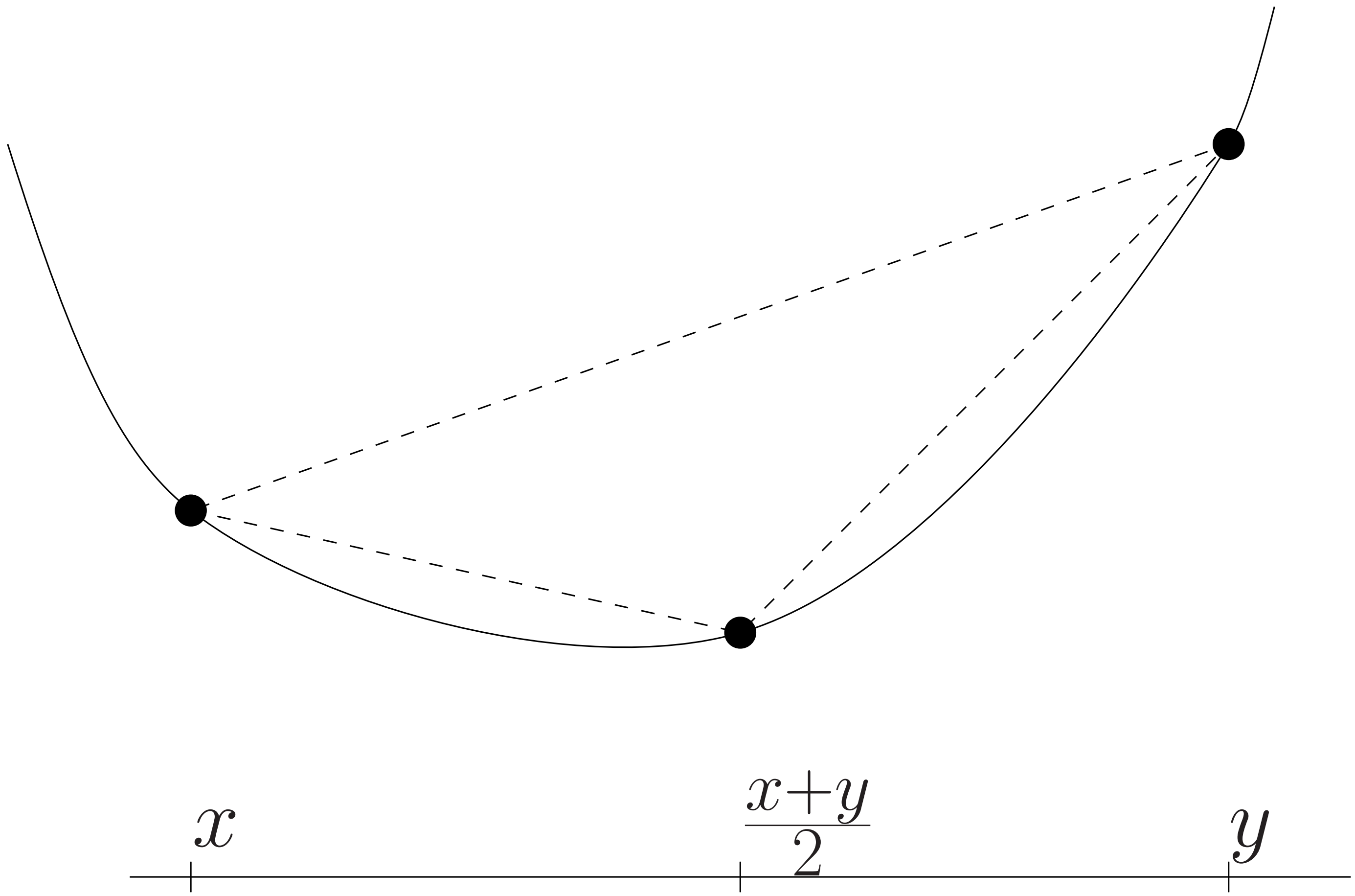
$f : \mathbf{R}^n \to \mathbf{R}$ is a convex function if $\mathbf{dom}\, f$ is a convex set and for all $x, y \in \mathbf{dom}\, f$ and $\theta \in [0, 1]$, we have

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

f is strictly convex if strict inequality above for all $x \neq y$ and $0 < \theta < 1$

f is concave if $-f$ is convex

Affine functions are convex and concave

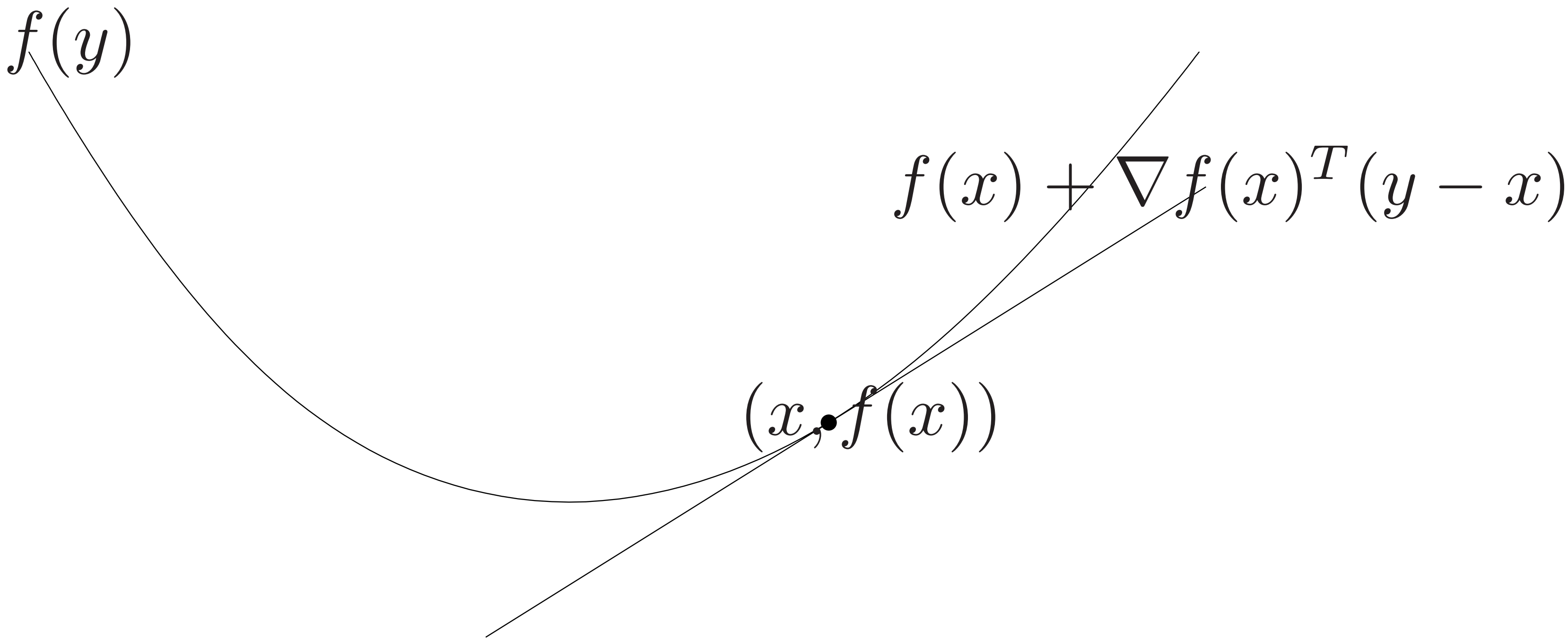

x
$\frac{x+y}{2}$
y

Conditions of Convex Functions

1. For differentiable functions, f is convex iff

$$f(y) - f(x) \geq \nabla f(x)^T (y - x)$$

for all $x, y \in \mathbf{dom}\, f$, and $\mathbf{dom}\, f$ is convex



- $f(y) \geq \tilde{f}_x(y)$ where $\tilde{f}_x(y)$ is first order Taylor expansion of $f(y)$ at x.

- Local information (first order Taylor approximation) about a convex function provides global information (global underestimator).

- If $\nabla f(x) = 0$, then $f(y) \geq f(x)$, $\forall y$, thus x is a global minimizer of f

Conditions of Convex Functions

2. For twice differentiable functions, f is convex iff

$$\nabla^2 f(x) \succeq 0$$

for all $x \in \mathbf{dom}\, f$ (upward slope) and $\mathbf{dom}\, f$ is convex

3. f is convex iff for all $x \in \mathbf{dom}\, f$ and all v,

$$g(t) = f(x + tv)$$

is convex on its domain $\{t \in \mathbf{R} | x + tv \in \mathbf{dom}\, f\}$

Examples of Convex or Concave Functions

- e^{ax} is convex on **R**, for any $a \in$ **R**
- x^a is convex on $\mathbf{R}_{++}$ when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$
- $|x|^p$ is convex on **R** for $p \geq 1$
- $\log x$ is concave on $\mathbf{R}_{++}$
- $x \log x$ is strictly convex on $\mathbf{R}_{++}$
- Every norm on $\mathbf{R}^n$ is convex
- $f(x) = \log \sum_{i=1}^{n} e^{x_i}$ is convex on $\mathbf{R}^n$

Convexity-Preserving Operations

- $f = \sum_{i=1}^{n} w_i f_i$ convex if f_i are all convex and $w_i \geq 0$

- $g(x) = f(Ax + b)$ is convex iff $f(x)$ is convex

- $f(x) = \max\{f_1(x), f_2(x)\}$ convex if f_i convex, *e.g.*, sum of r largest components is convex

- $f(x) = h(g(x))$, where $h : \mathbf{R}^k \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}^k$.

  If $k = 1$: $f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$. So

  f is convex if h is convex and nondecreasing and g is convex, or if h is convex and nonincreasing and g is concave ...

- $g(x) = \inf_{y \in C} f(x, y)$ is convex if f is convex and C is convex

Conjugate Function

Given $f : \mathbf{R}^n \to \mathbf{R}$, conjugate function $f^* : \mathbf{R}^n \to \mathbf{R}$ defined as:

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$

with domain consisting of $y \in \mathbf{R}^n$ for which the supremum is finite

- $f^*(y)$ always convex: it is the pointwise supremum of a family of affine functions of y
- Fenchel's inequality: $f(x) + f^*(y) \geq x^T y$ for all x, y (by definition)
- $f^{**} = f$ if f is convex and closed
- Useful for Lagrange duality theory

Examples of Conjugate Functions

- $f(x) = ax + b$, $f^*(a) = -b$
- $f(x) = -\log x$, $f^*(y) = -\log(-y) - 1$ for $y < 0$
- $f(x) = e^x$, $f^*(y) = y \log y - y$
- $f(x) = x \log x$, $f^*(y) = e^{y-1}$
- $f(x) = \frac{1}{2} x^T Q x$, $f^*(y) = \frac{1}{2} y^T Q^{-1} y$ (Q is positive definite)
- $f(x) = \log \sum_{i=1}^n e^{x_i}$, $f^*(y) = \sum_{i=1}^n y_i \log y_i$ if $y \succeq 0$ and $\sum_{i=1}^n y_i = 1$ ($f^*(y) = \infty$ otherwise)

Log-convex and Log-concave Functions

$f : \mathbf{R}^n \to \mathbf{R}$ is log-convex if $f(x) > 0$ and $\log f$ is convex:

$$f(\theta x + (1-\theta)y) \leq f(x)^\theta f(y)^{1-\theta} \;\text{ for }\; 0 \leq \theta \leq 1$$

A log-convex function is convex (Prove it)

$f : \mathbf{R}^n \to \mathbf{R}$ is log-concave if $f(x) > 0$ and $\log f$ is concave

Many probability distributions are log-concave: Cumulative distribution function of Gaussian density, Multivariate normal distribution, Exponential distribution, Uniform distribution, Wishart distribution

Summary

- Definitions of convex sets and convex functions
- Convexity-preserving operations
- Global information from local characterization: Supporting Hyperplane Theorem
- Convexity is the watershed between 'easy' and 'hard' optimization problems. Recognize convexity. Utilize convexity.

Reading assignment: Chapters 1, 2 and 3 of textbook.