Correspondence

On Maximum-Likelihood SINR Estimation of MPSK in a Multiuser Fading Channel

Siamak Sorooshyari, Chee Wei Tan, *Member, IEEE*, and H. Vincent Poor, *Fellow, IEEE*

***Abstract*—In this paper, maximum-likelihood (ML) signal-to-interference-plus-noise ratio (SINR) estimation is derived for a multiuser fading channel with additive Gaussian noise and M-ary phase-shift keying (MPSK) signaling for the users. This problem is analyzed in light of works that consider a single-user channel in performing ML signal-to-noise ratio estimation. This work is motivated by the importance of a user's knowledge of its SINR in resource-allocation decisions, particularly power control and rate adaptation.**

***Index Terms*—Fading channels, M-ary phase-shift keying (MPSK) modulation, maximum-likelihood (ML) estimation.**

I. Introduction

In a multiuser wireless network, a user's signal-to-interference-plus-noise ratio (SINR) is critical in governing resource-allocation decisions at the physical layer and at higher protocol layers. The importance of SINR estimation is driven by the necessity of accounting for the interference caused by the increasing number of wireless devices transmitting in the spectrum. In this paper, a maximum-likelihood (ML) estimator is derived for discrete-time estimation of the SINR under flat-fading conditions with additive thermal noise. The motivation behind ML estimation stems from its analytical optimality in maximizing a likelihood function and providing relatively tractable computation. A key component of our analysis is deriving the distribution of a user's received faded signal consisting of the information-bearing signal, cochannel interference, and thermal noise. This is followed by evaluation of a likelihood function in deriving the ML estimate. A one-step ML estimator is presented based on the Newton–Raphson method and the method-of-moments (MOM) estimator.

The analysis in this paper is general in considering an arbitrary number of interferers, M-ary phase-shift keying (MPSK) modulation, arbitrary mean and variance for the complex channel gains, and arbitrary power users. For the scenario in which each user experiences Rayleigh fading, a Cramer–Rao lower bound (CRLB) on estimator variance is derived for a user's interference-plus-noise component. It should be stressed that our work is different from prior SINR estimation works, such as [1] and [2], which restrict attention to code-division multiple-access and time-division multiple-access systems, respectively.

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S. Sorooshyari is with Alcatel-Lucent, Murray Hill, NJ 07974 USA (e-mail: siamak_sorooshyari@yahoo.com).

C. W. Tan is with the City University of Hong Kong, Kowloon, Hong Kong (e-mail: cheewtan@cityu.edu.hk).

H. V. Poor is with Princeton University, Princeton, NJ 08544 USA (e-mail: poor@princeton.edu).

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II. System Model and Statistical Preliminaries

For a user with index k, we define the complex scalar $G_k = \sqrt{P_k} B_k A_k = \sqrt{P_k} B_k A_{R,k} + j\sqrt{P_k} B_k A_{I,k}$ as comprising a real constant P_k and independent complex variables B_k and A_k, with $A_{R,k} = \Re\{A_k\}$ and $A_{I,k} = \Im\{A_k\}$. In effect, G_k conveys the amplitude and phase of the kth user's received signal, with P_k designating its transmit power. The complex random variable A_k is assumed to have distribution $A_k \sim CN(E[A_k], Var[A_k])$, where $E[A_k] = E[A_{R,k}] + jE[A_{I,k}]$ and $Var[A_k] = Var[A_{R,k}] + Var[A_{I,k}]$, with $Var[A_{R,k}] = Var[A_{I,k}]$. The complex random variable $B_k = B_{R,k} + jB_{I,k}$ has $E[|B_k|^2] = 1$ and probability distribution $P[B_k = b_{k,i}] = 1/M$ for $i = 1, \ldots, M$. The variables $A_{R,k}$, $A_{I,k}$, $B_{R,k}$, and $B_{I,k}$ are assumed to be mutually independent. Separating G_k into its real and imaginary components yields

$$\begin{aligned} G_k &= (\sqrt{P_k} B_{R,k} A_{R,k} - \sqrt{P_k} B_{I,k} A_{I,k}) \\ &\quad + j(\sqrt{P_k} B_{R,k} A_{I,k} + \sqrt{P_k} B_{I,k} A_{R,k}) \\ &= Z_k + jW_k \end{aligned}$$

where we have introduced the real random variables $Z_k \triangleq \Re\{G_k\} = \sqrt{P_k} B_{R,k} A_{R,k} - \sqrt{P_k} B_{I,k} A_{I,k}$ and $W_k \triangleq \Im\{G_k\} = \sqrt{P_k} B_{R,k} A_{I,k} + \sqrt{P_k} B_{I,k} A_{R,k}$. Conditioning on the event $B_k = b_{k,i}$ with $b_{k,i} = b_{R,k,i} + jb_{I,k,i}$ yields the complex random variable

$$\begin{aligned} G_{k,i} \triangleq \sqrt{P_k} b_{k,i} A_k = (\sqrt{P_k} b_{R,k,i} A_{R,k} - \sqrt{P_k} b_{I,k,i} A_{I,k}) \\ + j(\sqrt{P_k} b_{R,k,i} A_{I,k} + \sqrt{P_k} b_{I,k,i} A_{R,k}) \end{aligned}$$

with the corresponding designations

$$\begin{aligned} Z_{k,i} &\triangleq \Re\{G_{k,i}\} = \sqrt{P_k}(b_{R,k,i} A_{R,k} - b_{I,k,i} A_{I,k}) \\ W_{k,i} &\triangleq \Im\{G_{k,i}\} = \sqrt{P_k}(b_{R,k,i} A_{I,k} + b_{I,k,i} A_{R,k}). \end{aligned} \quad (1)$$

We proceed to evaluate the mean and variance of $Z_{k,i}$ and $W_{k,i}$. Omitting the algebra and using the fact that $\sqrt{b_{R,k,i}^2 + b_{I,k,i}^2} = 1$ for MPSK modulation, we obtain $E[Z_{k,i}] = \sqrt{P_k} b_{R,k,i} E[A_{R,k}] - \sqrt{P_k} b_{I,k,i} E[A_{I,k}]$, $Var[Z_{k,i}] = P_k Var[A_{R,k}]$, $E[W_{k,i}] = \sqrt{P_k} b_{R,k,i} E[A_{I,k}] + \sqrt{P_k} b_{I,k,i} E[A_{R,k}]$, and $Var[W_{k,i}] = P_k Var[A_{I,k}]$. We note

$$\begin{aligned} &E[Z_{k,i} W_{k,i}] \\ &\quad = P_k \left(b_{R,k,i}^2 E[A_{R,k}] E[A_{I,k}] + b_{R,k,i} b_{I,k,i} E\left[A_{R,k}^2\right] \right. \\ &\qquad \left. - b_{I,k,i}^2 E[A_{I,k}] E[A_{R,k}] - b_{R,k,i} b_{I,k,i} E\left[A_{I,k}^2\right] \right) \end{aligned}$$

thus yielding $E[Z_{k,i} W_{k,i}] - E[Z_{k,i}] E[W_{k,i}] = 0$. The variates $Z_{k,i}$ and $W_{k,i}$ in (1) will be jointly Gaussian since the quantities multiplying $A_{R,k}$ and $A_{I,k}$ are constants, with us having conditioned on the event $B_{R,k} = b_{R,k,i}$, $B_{I,k} = b_{I,k,i}$. The fact that $E[Z_{k,i} W_{k,i}] - E[Z_{k,i}] E[W_{k,i}] = 0$ indicates that the variates $Z_{k,i}$ and $W_{k,i}$ are uncorrelated. Since uncorrelated Gaussian random variables are also independent, we conclude that $Z_{k,i}$ and $W_{k,i}$ are independent random variables. Thus, we have $f_{Z_{k,i}, W_{k,i}}(z, w) = f_{Z_{k,i}}(z) f_{W_{k,i}}(w)$, with

$$\begin{aligned} Z_{k,i} &\sim N\Big(\sqrt{P_k}\,(b_{R,k,i} E[A_{R,k}] - b_{I,k,i} E[A_{I,k}])\,, \\ &\qquad P_k Var[A_{R,k}] \Big) \\ W_{k,i} &\sim N\Big(\sqrt{P_k}\,(b_{R,k,i} E[A_{I,k}] + b_{I,k,i} E[A_{R,k}])\,, \\ &\qquad P_k Var[A_{I,k}] \Big). \end{aligned} \quad (2)$$

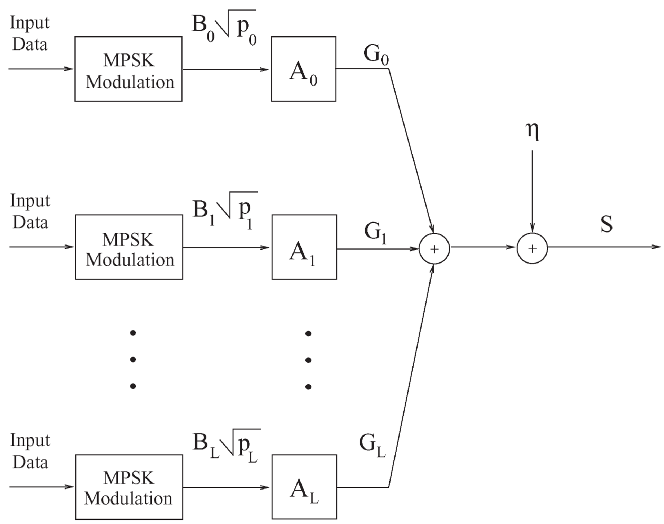

Fig. 1. Considered system model. All quantities except the transmit powers $\{P_k\}$ are complex.

We are now ready to examine the scenario with multiple users with the desired user's signal arbitrarily assigned the index $k = 0$. We restrict our attention to the model in Fig. 1. Thus, the received signal S will consist of the desired user's signal, complex thermal receiver noise η, and the signals of L interferers with indices $k=1,\ldots,L$. Specifically, we have

$$\begin{aligned} S = S_R + jS_I &= \sum_{k=0}^{L} G_k + \eta \\ &= \left(\sum_{k=0}^{L} Z_k + \Re\{\eta\}\right) + j\left(\sum_{k=0}^{L} W_k + \Im\{\eta\}\right) \end{aligned} \tag{3}$$

where the last equality follows from the definition of G_k. For $L+1$ users, we define the event

$$\mathsf{B}(i,j,\ldots,m) : B_0 = b_{0,i}, B_1 = b_{1,j}, \ldots, B_L = b_{L,m} \tag{4}$$

with the users' MPSK symbol indices $i,j,\ldots,m \in \{1,\ldots,M\}$ and $b_{k,i} = b_{R,k,i} + jb_{I,k,i}$. We also define the event space of the users' data sequence as the set $\beta = \{\mathsf{B}(i,j,\ldots,m)\}$, with cardinality M^{L+1}. We shall represent the elements of the set β via the index $\phi = 1,2,\ldots,M^{L+1}$, with each ϕ value representing a unique $L+1$-tuple given by (4). With such indexing, the set of events $\{\mathsf{B}(i,j,\ldots,m)\}$ can be represented via the set $\{\Phi = \phi : \phi = 1,2,\ldots,M^{L+1}\}$ with a one-to-one mapping between the events and indices. We define the conditional[1] random variables

$$\begin{aligned} S_{R|\phi} &= \sum_{k=0}^{L} Z_{k,\phi} + \Re\{\eta\} \\ S_{I|\phi} &= \sum_{k=0}^{L} W_{k,\phi} + \Im\{\eta\}. \end{aligned} \tag{5}$$

With $\Re\{\eta\}$ and $\Im\{\eta\}$ being independent and each having distribution $N(0, Var[\eta])$ from (2), we derive the conditional distributions

$$\begin{aligned} S_{R|\phi} &\sim N\left(\sum_{k=0}^{L} \sqrt{P_k}\left(b_{R,k,\phi}E[A_{R,k}] - b_{I,k,\phi}E[A_{I,k}]\right),\right. \\ &\qquad \left.\sum_{k=0}^{L} P_k Var[A_{R,k}] + Var[\eta]\right) \\ S_{I|\phi} &\sim N\left(\sum_{k=0}^{L} \sqrt{P_k}\left(b_{R,k,\phi}E[A_{I,k}] + b_{I,k,\phi}E[A_{R,k}]\right),\right. \\ &\qquad \left.\sum_{k=0}^{L} P_k Var[A_{I,k}] + Var[\eta]\right). \end{aligned} \tag{6}$$

[1]We have conditioned on the specific event $\Phi = \phi$ for $\phi \in \{1,2,\ldots,M^{L+1}\}$.

From (6), it is interesting to note that the $L+1$ users' data sequences will not affect the variance of the conditioned variables $S_{R|\phi}$ and $S_{I|\phi}$. It can be observed from (5) and the fact that $Z_{k,\phi}$, $W_{k,\phi}$, $\Re\{\eta\}$, and $\Im\{\eta\}$ are mutually independent that the conditioned variables $S_{R|\phi}$ and $S_{I|\phi}$ will be independent, and thus, $f_{S_{R|\phi},S_{I|\phi}}(z,w) = f_{S_{R|\phi}}(z)f_{S_{I|\phi}}(w)$. We now derive the distribution $f_{S_R,S_I}(z,w)$ by unconditioning upon the set of events $\{\mathsf{B}(i,j,\ldots,m)\}$. Since $P[\Phi = \phi] = 1/M^{L+1}$, we have

$$\begin{aligned} f_{S_R,S_I}(z,w) &= \sum_{\phi=1}^{M^{L+1}} f_{S_{R|\phi},S_{I|\phi}}(z,w)P[\Phi=\phi] \\ &= \frac{1}{M^{L+1}} \sum_{\phi=1}^{M^{L+1}} f_{S_{R|\phi}}(z) f_{S_{I|\phi}}(w) \end{aligned} \tag{7}$$

which denotes a mixture comprising the summation of M^{L+1} bivariate Gaussian probability density functions (pdfs) with parameters given by (6). Expanding (7), we obtain

$$\begin{aligned} &f_{S_R,S_I}(z,w) \\ &\quad= \frac{1}{M^{L+1}2\pi\sqrt{Var[S_{R|\phi}]Var[S_{I|\phi}]}} \\ &\qquad\times \sum_{\phi=1}^{M^{L+1}} \exp\left(-\frac{\left(z - E[S_{R|\phi}]\right)^2 + \left(w - E[S_{I|\phi}]\right)^2}{2Var[S_{I|\phi}]}\right) \\ &\quad= \frac{1}{M^{L+1}2\pi\left(\sum_{k=0}^{L} P_k Var[A_{R,k}] + Var[\eta]\right)} \\ &\qquad\times \sum_{\phi=1}^{M^{L+1}} \exp\left(-\frac{\left(z - E[S_{R|\phi}]\right)^2 + \left(w - E[S_{I|\phi}]\right)^2}{2\left(\sum_{k=0}^{L} P_k Var[A_{R,k}] + Var[\eta]\right)}\right) \end{aligned} \tag{8}$$

where we have used the fact that $Var[S_{R|\phi}] = Var[S_{I|\phi}] = \sum_{k=0}^{L} P_k Var[A_{R,k}] + Var[\eta]$ is not dependent on the users' data sequence and made the designation

$$\begin{aligned} E[S_{R|\phi}] &= \sum_{k=0}^{L} \sqrt{P_k}\left(b_{R,k,\phi}E[A_{R,k}] - b_{I,k,\phi}E[A_{I,k}]\right) \\ E[S_{I|\phi}] &= \sum_{k=0}^{L} \sqrt{P_k}\left(b_{R,k,\phi}E[A_{I,k}] + b_{I,k,\phi}E[A_{R,k}]\right). \end{aligned} \tag{9}$$

A probability mixture model is a distribution that is a convex combination of other distributions. It can be seen that (8) consists of a convex combination of bivariate Gaussian random variables, with the data-sequence probabilities $\{P[\Phi = \phi] : \phi = 1,2,\ldots,M^{L+1}\}$ representing the weights given to each of the individual M^{L+1} pdfs.

III. Signal-to-Inferereference-Plus-Noise Ratio Estimation

The general invariance principle of ML estimation [3] was used in [4] to estimate signal-to-noise ratio in terms of ML estimates of the signal and noise components. With the presence of interfering users, the same methodology can be used to decompose the SINR estimation process into two parts: 1) ML estimation of a signal component and 2) ML estimation of an interference-plus-noise component. We shall carry out the analysis for the user with index zero. The SINR of this user is defined as

$$SINR_0 = \frac{P_0 Var[A_{R,0}]}{\sum_{k=1}^{L} P_k Var[A_{R,k}] + Var[\eta]}. \tag{10}$$

The ML estimate of the SINR is defined as

$$\widehat{SINR}_0 = \frac{\widehat{X} Var[A_{R,0}]}{\widehat{Y}} \tag{11}$$

with $X \triangleq P_0$ and $Y \triangleq \sum_{k=1}^{L} P_k Var[A_{R,k}] + Var[\eta]$. Implicit to our analysis is a common assumption that a user's receiver has knowledge of the variance of the channel gain from its associated transmitter $Var[A_{R,0}]$ but does not have knowledge of the transmit power of its associated transmitter P_0.

A. ML SINR Estimation

We can obtain $\widehat{X}$ and $\widehat{Y}$ by evaluating the likelihood function of the joint distribution of the real and complex components of the received signal in (3), i.e., by solving $\partial f_{S_R,S_I}(z,w)/\partial X = 0$ and $\partial f_{S_R,S_I}(z,w)/\partial Y = 0$ for X and Y. We present the assignment $\Gamma_R(X,\phi) \triangleq E[S_{R|\phi}]$ and $\Gamma_I(X,\phi) \triangleq E[S_{I|\phi}]$ for the conditional means in (9) to stress their dependence on X and independence of Y. By evaluating $\partial f_{S_R,S_I}(z,w)/\partial X = 0$ and $\partial f_{S_R,S_I}(z,w)/\partial Y = 0$, we conclude that the desired parameters $\widehat{X}$ and $\widehat{Y}$ in (11) are obtained by solving the two equations

$$\begin{aligned} &-\sum_{\phi=1}^{M^{L+1}} \exp\left(-\frac{(z-\Gamma_R(X,\phi))^2 + (w-\Gamma_I(X,\phi))^2}{2\,(XVar[A_{R,0}]+Y)}\right) \\ &\quad + \frac{1}{2\,(XVar[A_{R,0}]+Y)} \\ &\quad \times \sum_{\phi=1}^{M^{L+1}} \exp\left(-\frac{(z-\Gamma_R(X,\phi))^2 + (w-\Gamma_I(X,\phi))^2}{2\,(XVar[A_{R,0}]+Y)}\right) \\ &\quad \times \left((z-\Gamma_R(X,\phi))^2 + (w-\Gamma_I(X,\phi))^2\right) = 0 \end{aligned} \tag{12}$$

and

$$\begin{aligned} &\sum_{\phi=1}^{M^{L+1}} \exp\left(-\frac{(z-\Gamma_R(X,\phi))^2 + (w-\Gamma_I(X,\phi))^2}{2\,(XVar[A_{R,0}]+Y)}\right) \\ &\quad + \frac{1}{2\,(XVar[A_{R,0}]+Y)} \\ &\quad \times \sum_{\phi=1}^{M^{L+1}} \exp\left(-\frac{(z-\Gamma_R(X,\phi))^2 + (w-\Gamma_I(X,\phi))^2}{2\,(XVar[A_{R,0}]+Y)}\right) \\ &\quad \times \left(\frac{\Omega(X,\phi)}{\sqrt{X}}\,(XVar[A_{R,0}]+Y) - (z-\Gamma_R(X,\phi))^2 \right. \\ &\qquad\quad \left. - (w-\Gamma_I(X,\phi))^2\right) = 0 \end{aligned} \tag{13}$$

for X and Y (representing the ML estimates of P_0 and the interference plus noise, respectively). We have made the designation

$$\begin{aligned} \Omega(X,\phi) \triangleq &-(z-\Gamma_R(X,\phi))[b_{R,0,\phi}E[A_{R,0}] - b_{I,0,\phi}E[A_{I,0}]] \\ &-(w-\Gamma_I(X,\phi))\,[b_{R,0,\phi}E[A_{I,0}] + b_{I,0,\phi}E[A_{R,0}]]\,. \end{aligned}$$

Note that the preceding variables z and w correspond to the measured values of S_R and S_I, respectively. We follow the preceding findings with a discussion of the information that is necessary for a user to conduct ML estimation of its SINR. As stated via (11), we have presumed that a receiver will have knowledge of the variance of the channel gain from its associated transmitter. From (9), (12), and (13), we observe that a user will require knowledge of the means of the interferers' channel gains and the transmit powers of the interferers. A user does not need knowledge pertaining to the variance of the interferers' channel gains. In the case of user 0, the necessary information corresponds to $Var[A_{R,0}]$, $\{E[A_{R,k}], E[A_{I,k}] : k \neq 0\}$, and $\{P_k : k \neq 0\}$. From a theoretical perspective, our assumption that first-order statistics are more readily available than higher order statistics, such as the channel gain variances, is consistent with prior SINR estimation works, such as [1] and [7]. From an engineering perspective, the acquisition of such information is heavily dependent on the specific system. For instance, this information would be easily obtained in the deployment of ML SINR estimation in the uplink of a cellular system.

B. Asymptotically Efficient SINR Estimation

The Newton–Raphson method can be used to iteratively solve the nonlinear equations (12) and (13) for $\widehat{X}$ and $\widehat{Y}$. However, we wish to consider a less computationally complex estimator, which does not require an iterative solution of nonlinear equations. It is possible to attain an asymptotically efficient estimator with a single step of the Newton–Raphson method and a starting value $\tilde{\theta}$ that has the property of being $\sqrt{N}$-consistent with X and Y.[2] Consider the problem of estimating the parameter vector $\theta = [X, Y]^T$. A sequence of estimators $\tilde{\theta}(n)$ will be $\sqrt{N}$-consistent to θ if $\sqrt{N}(\tilde{\theta}(n) - \theta)$ is bounded in probability. Given a $\sqrt{N}$-consistent estimator $\tilde{\theta}(n)$, the estimator sequence

$$\delta(n) = \tilde{\theta}(n) - \left(\mathbf{F}'\left(\tilde{\theta}(n)\right)\right)^{-1} \mathbf{F}\left(\tilde{\theta}(n)\right) \tag{14}$$

will be asymptotically efficient. We let $\tilde{\theta}(n) = [\tilde{X}(n), \tilde{Y}(n)]^T$ denote a $\sqrt{N}$-consistent estimator for X and Y. The estimator given by $\delta(n)$ will meet the CRLB in the asymptotic sense. In (14), the vector $\mathbf{F} = [f_1(X,Y), f_2(X,Y)]^T$, with $f_1(X,Y)$ and $f_2(X,Y)$ given by the left-hand side of (12) and (13), respectively. The matrix $\mathbf{F}'$ denotes the Jacobian matrix. We shall refer to the designations shown at the bottom of the next page in our discussion. We have

$$\left(\mathbf{F}'\left(\tilde{\theta}(n)\right)\right)^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{15}$$

with the individual components given by

$$\begin{aligned} a \triangleq \frac{\partial f_1}{\partial X} = &-\sum_{\phi=1}^{M^{L+1}} \frac{\partial N_\phi}{\partial X} - \frac{Var[A_{R,0}]}{2\,(XVar[A_{R,0}]+Y)^2} \\ &\times\left(\sum_{\phi=1}^{M^{L+1}} N_\phi \left((z-\Gamma_R(X,\phi))^2 + (w-\Gamma_I(X,\phi))^2\right)\right) \\ &+ \frac{1}{2\,(XVar[A_{R,0}]+Y)} \\ &\times\left(\sum_{\phi=1}^{M^{L+1}} \frac{\partial N_\phi}{\partial X} \left((z-\Gamma_R(X,\phi))^2 + (w-\Gamma_I(X,\phi))^2\right)\right. \end{aligned}$$

[2]This result and its corresponding proof are formally stated via [6, Th. 7.3.3].

$$+ N_\phi \left(2\left(z - \Gamma_R(X,\phi)\right)\left(\frac{\partial \Gamma_R(X,\phi)}{\partial X}\right) + 2\left(w - \Gamma_I(X,\phi)\right)\left(-\frac{\partial \Gamma_I(X,\phi)}{\partial X}\right)\right)\Bigg) \quad (16)$$

$$b \triangleq \frac{\partial f_1}{\partial Y} = -\sum_{\phi=1}^{M^{L+1}} \frac{\partial N_\phi}{\partial Y} - \frac{1}{2\left(XVar[A_{R,0}]+Y\right)^2} \times \left(\sum_{\phi=1}^{M^{L+1}} N_\phi \left(\left(z-\Gamma_R(X,\phi)\right)^2 + \left(w-\Gamma_R(X,\phi)\right)^2\right)\right) + \frac{1}{2\left(XVar[A_{R,0}]+Y\right)} \times \left(\sum_{\phi=1}^{M^{L+1}} \frac{\partial N_\phi}{\partial Y}\left(\left(z-\Gamma_R(X,\phi)\right)^2 + \left(w-\Gamma_R(X,\phi)\right)^2\right)\right) \quad (17)$$

$$c \triangleq \frac{\partial f_2}{\partial X} = \sum_{\phi=1}^{M^{L+1}} \frac{\partial N_\phi}{\partial X} - \frac{Var[A_{R,0}]}{2\left(XVar[A_{R,0}]+Y\right)^2} \times \left(\sum_{\phi=1}^{M^{L+1}} N_\phi \left(\frac{\Omega(X,\phi)}{\sqrt{X}}\left(XVar[A_{R,0}]+Y\right) - \left(z-\Gamma_R(X,\phi)\right)^2 - \left(w-\Gamma_R(X,\phi)\right)^2\right)\right) + \frac{1}{2\left(XVar[A_{R,0}]+Y\right)}\left(\sum_{\phi=1}^{M^{L+1}} N_\phi \frac{\partial D}{\partial X} + D\frac{\partial N_\phi}{\partial X}\right) \quad (18)$$

with the assignments

$$D = \frac{\Omega(X,\phi)}{\sqrt{X}}\left(XVar[A_{R,0}]+Y\right) - \left(z-\Gamma_R(X,\phi)\right)^2 - \left(w-\Gamma_I(X,\phi)\right)^2$$

$$\frac{\partial D}{\partial X} = \frac{\partial F}{\partial X}\left(XVar[A_{R,0}]+Y\right) + \left(\frac{\Omega(X,\phi)}{\sqrt{X}}\right)\left(Var[A_{R,0}]\right) + 2\left(z-\Gamma_R(X,\phi)\right)\left(\frac{\partial \Gamma_R(X,\phi)}{\partial X}\right) + 2\left(w-\Gamma_I(X,\phi)\right)\left(\frac{\partial \Gamma_I(X,\phi)}{\partial X}\right)$$

$$\frac{\partial F}{\partial X} = \frac{\left(\frac{\partial \Omega(X,\phi)}{\partial X}\right)\left(\sqrt{X}\right) - \left(\Omega(X,\phi)\right)\left(\frac{1}{2\sqrt{X}}\right)}{X}$$

$$\frac{\partial \Omega(X,\phi)}{\partial X} = \frac{1}{2\sqrt{X}}\left(b_{R,0,\phi}E[A_{R,0}] - b_{I,0,\phi}E[A_{I,0}]\right)^2 + \frac{1}{2\sqrt{X}}\left(b_{R,0,\phi}E[A_{I,0}] + b_{I,0,\phi}E[A_{R,0}]\right)^2.$$

Finally

$$d \triangleq \frac{\partial f_2}{\partial Y} = \sum_{\phi=1}^{M^{L+1}} \frac{\partial N_\phi}{\partial Y} + \frac{C}{2\left(XVar[A_{R,0}]+Y\right)} - \frac{1}{2\left(XVar[A_{R,0}]+Y\right)^2} \times \left(\sum_{\phi=1}^{M^{L+1}} N_\phi \left(\left(\frac{\Omega(X,\phi)}{\sqrt{X}}\right)\left(XVar[A_{R,0}]+Y\right) - \left(z-\Gamma_R(X,\phi)\right)^2 - \left(w-\Gamma_I(X,\phi)\right)^2\right)\right) \quad (19)$$

with the assignment

$$C = \sum_{\phi=1}^{M^{L+1}} \frac{\partial N_\phi}{\partial Y}\left(\left(\frac{\Omega(X,\phi)}{\sqrt{X}}\right)\left(XVar[A_{R,0}]+Y\right) - \left(z-\Gamma_R(X,\phi)\right)^2 - \left(w-\Gamma_I(X,\phi)\right)^2\right) + N_\phi\left(\frac{\Omega(X,\phi)}{\sqrt{X}}\right). \quad (20)$$

We shall use the MOM estimator [6] as the starting value $\tilde{\theta}(n)$. We now derive the MOM estimator for X and Y and subsequently prove that the MOM estimates are $\sqrt{N}$-consistent.[3] We define $\widehat{\mu}_{S_R} = (1/N)\sum_{n=1}^{N} S_R(n)$ and $\widehat{\mu}_{S_I} = (1/N)\sum_{n=1}^{N} S_I(n)$ as the sample means after N observations of the channel. From (6), we have

$$f_{S_R}(z) = \sum_{\phi=1}^{M^{L+1}} f_{S_{R|\phi}}(z) P[\Phi=\phi] = \frac{1}{M^{L+1}}\sum_{\phi=1}^{M^{L+1}} f_{S_{R|\phi}}(z) = \frac{\sum_{\phi=1}^{M^{L+1}} \exp\left(-\frac{\left(z-\Gamma_R(X,\phi)\right)^2}{2\left(XVar[A_{R,0}]+Y\right)}\right)}{M^{L+1}\sqrt{2\pi\left(XVar[A_{R,0}]+Y\right)}} \quad (21)$$

[3]It should be noted that MOM estimators are not unique since one can use moments of any order. We use the first moment here.

$$N_\phi = \exp\left(-\frac{\left(z-\Gamma_R(X,\phi)\right)^2 + \left(w-\Gamma_I(X,\phi)\right)^2}{2\left(XVar[A_{R,0}]+Y\right)}\right)$$

$$\frac{\partial N_\phi}{\partial Y} = N_\phi\left(\frac{\left(z-\Gamma_R(X,\phi)\right)^2 + \left(w-\Gamma_I(X,\phi)\right)^2}{2\left(XVar[A_{R,0}]+Y\right)^2}\right)$$

$$\frac{\partial N_\phi}{\partial X} = N_\phi\left(\frac{T}{2\left(XVar[A_{R,0}]+Y\right)} + \frac{Var[A_{R,0}]\left(\left(z-\Gamma_R(X,\phi)\right)^2 + \left(w-\Gamma_I(X,\phi)\right)^2\right)}{2\left(XVar[A_{R,0}]+Y\right)^2}\right)$$

$$\frac{\partial \Gamma_R(X,\phi)}{\partial X} = \frac{1}{2\sqrt{X}}\left(b_{R,0,\phi}E[A_{R,0}] - b_{I,0,\phi}E[A_{I,0}]\right)$$

$$\frac{\partial \Gamma_I(X,\phi)}{\partial Y} = \frac{1}{2\sqrt{X}}\left(b_{R,0,\phi}E[A_{I,0}] + b_{I,0,\phi}E[A_{R,0}]\right)$$

$$T = 2\left(z-\Gamma_R(X,\phi)\right)\left(\frac{\partial \Gamma_R(X,\phi)}{\partial X}\right) + 2\left(w-\Gamma_I(X,\phi)\right)\left(\frac{\partial \Gamma_I(X,\phi)}{\partial X}\right)$$

and derive $E[S_R]=\sum_{\phi=1}^{M^{L+1}}\Gamma_R(X,\phi)/2\pi M^{L+1}(XVar[A_{R,0}]+Y)$ by using the relation $\int_{-\infty}^{\infty} x\exp(-a(x-b)^2)dx = b\sqrt{\pi/a}$. A corresponding result can be derived for $E[S_I]$. Specifically, we have $E[S_I]=\sum_{\phi=1}^{M^{L+1}}\Gamma_I(X,\phi)/2\pi M^{L+1}(XVar[A_{R,0}]+Y)$, where, as throughout this work, we have assumed that $Var[A_{R,0}] = Var[A_{I,0}]$. We now have the two equations

$$\widehat{\mu}_{S_R} = \frac{\sum_{\phi=1}^{M^{L+1}}\Gamma_R(X,\phi)}{2\pi M^{L+1}\left(XVar[A_{R,0}]+Y\right)}$$

$$\widehat{\mu}_{S_I} = \frac{\sum_{\phi=1}^{M^{L+1}}\Gamma_I(X,\phi)}{2\pi M^{L+1}\left(XVar[A_{R,0}]+Y\right)}$$

which must be solved for X and Y to obtain the MOM estimates $\widehat{X}_{\mathrm{MOM}}$ and $\widehat{Y}_{\mathrm{MOM}}$, respectively. An attractive aspect of the MOM is the tractability in providing a solution for the quantities to be estimated. We obtain the solution

$$\begin{aligned}\widehat{X}_{\mathrm{MOM}} &= \left(\frac{\widehat{\mu}_{S_I}\alpha_1-\widehat{\mu}_{S_R}\alpha_2}{\widehat{\mu}_{S_I}\alpha_{1,0}-\widehat{\mu}_{S_R}\alpha_{2,0}}\right)^2\\ \widehat{Y}_{\mathrm{MOM}} &= \left(-\frac{M^{L+1}}{2\pi}\right)\\ &\quad\times\left(\frac{\alpha_{2,0}\alpha_1-\alpha_{1,0}\alpha_2}{H}\right.\\ &\qquad\left.+\frac{2\pi M^{L+1}Var[A_{R,0}]\left(\widehat{\mu}_{S_R}\alpha_2-\widehat{\mu}_{S_I}\alpha_1\right)^2}{H^2}\right)\end{aligned} \tag{22}$$

with the following designations:

$$\begin{aligned}H &= \widehat{\mu}_{S_I}\alpha_{1,0}-\widehat{\mu}_{S_R}\alpha_{2,0}\\ \alpha_1 &= \sum_{k=1}^{L}\sqrt{P_k}\sum_{\phi=1}^{M^{L+1}} b_{R,k,\phi}E[A_{R,k}]-b_{I,k,\phi}E[A_{I,k}]\\ \alpha_2 &= \sum_{k=1}^{L}\sqrt{P_k}\sum_{\phi=1}^{M^{L+1}} b_{R,k,\phi}E[A_{I,k}]+b_{I,k,\phi}E[A_{R,k}]\\ \alpha_{1,0} &= \sum_{\phi=1}^{M^{L+1}} b_{R,0,\phi}E[A_{R,0}]-b_{I,0,\phi}E[A_{I,0}]\\ \alpha_{2,0} &= \sum_{\phi=1}^{M^{L+1}} b_{R,0,\phi}E[A_{I,0}]+b_{I,0,\phi}E[A_{R,0}].\end{aligned} \tag{23}$$

It remains to be verified that our MOM estimator $\tilde{\theta}(n) = [\widehat{X}_{\mathrm{MOM}}, \widehat{Y}_{\mathrm{MOM}}]^T$ is $\sqrt{N}$-consistent with X and Y.

Proposition ([6, Th. 5.2.3]): Suppose that $U(n)$ and $V(n)$ are $\sqrt{N}$-consistent estimators of the quantities A and B. Let $f(u,v)$ and $g(u,v)$ be two real-valued functions of two variables, which exist at the point $u = A$, $v = B$. Suppose that $\partial f/\partial u$, $\partial f/\partial v$, $\partial g/\partial u$, and $\partial g/\partial v$, evaluated at the point $u = A$, $v = B$, also exist. Then, $f(U(n),V(n))$ and $g(U(n),V(n))$ will be $\sqrt{N}$-consistent estimators of $f(A,B)$ and $g(A,B)$. ■

It is known that the sample means $\widehat{\mu}_{S_R}$ and $\widehat{\mu}_{S_I}$ are $\sqrt{N}$-consistent estimators of the associated expected values $E[S_R]$ and $E[S_I]$, respectively [6, Th. 2.4.1]. We designate the sequences $U(n) = \widehat{\mu}_{S_R}$ and $V(n) = \widehat{\mu}_{S_I}$ and designate the functions $f(U(n),V(n)) = \widehat{X}_{\mathrm{MOM}}(\widehat{\mu}_{S_R},\widehat{\mu}_{S_I})$ and $g(U(n),V(n)) = \widehat{Y}_{\mathrm{MOM}}(\widehat{\mu}_{S_R},\widehat{\mu}_{S_I})$. It can be readily verified that f, g, $\partial f/\partial u$, $\partial f/\partial v$, $\partial g/\partial u$, and $\partial g/\partial v$ exist at the point $A = E[S_R]$, $B = E[S_I]$. Thus, the MOM estimators given by (22) are $\sqrt{N}$-consistent estimators of X, Y. The preceding formulation provides a connection between ML estimation, the Newton–Raphson method, and the MOM estimator since the assignment $\tilde{\theta}(n) = [\widehat{X}_{\mathrm{MOM}}, \widehat{Y}_{\mathrm{MOM}}]^T$ in (14) will yield the estimator $\delta(n)$, which meets the CRLB in the asymptotic sense. It should be noted that the MOM estimator in (22) is not valid in the scenario in which each user experiences Rayleigh fading. This is because we have derived the MOM estimator based on the first moment and $E[S_R] = E[S_I] = 0$ for this special case, causing the MOM equations to reduce to $(1/N)\sum_{n=1}^{N} S_R(n) = 0$ and $(1/N)\sum_{n=1}^{N} S_I(n) = 0$. It will be shown in the next section that evaluation of the ML estimate for the case in which all users experience Rayleigh fading obviates the necessity for asymptotic analysis or the computation of MOM estimates.

C. ML SINR Estimation With Rayleigh Fading

We comment on the complexity of obtaining $\widehat{X}$ and $\widehat{Y}$ by stressing that (12) and (13) are rather general and that simpler expressions exist for specific cases. For instance, for quadrature phase-shift keying modulation, we have $M = 4$ [e.g., the constellation points are $(b_{R,k,\phi} = 1, b_{I,k,\phi} = 0)$, $(b_{R,k,\phi} = 0, b_{I,k,\phi} = 1)$, $(b_{R,k,\phi} = -1, b_{I,k,\phi} = 0)$, $(b_{R,k,\phi} = 0, b_{I,k,\phi} = -1)$), and $b_{I,k,\phi}$, $b_{R,k,\phi} \in \{-1, 0, 1\}$ in (9)]. According to (9), the case in which each user experiences Rayleigh fading will have $\Gamma_R(X,\phi)=\Gamma_I(X,\phi)=\Omega(X,\phi)=0$ in (12) and (13), for which we obtain the solution $\widehat{X} = P_0$, $\widehat{Y} = -P_0Var[A_{R,0}] + (S_R^2 + S_I^2)/2$. This indicates that, in the absence of a line of sight, the receiver of user 0 is unable to obtain an ML estimate of its corresponding transmitter's power, and the in-phase and quadrature portions of the measured interference contribute equally to the ML estimate of the interference. Thus, in the case in which each user undergoes Rayleigh fading, ML estimation of the interference is possible, provided that a user's receiver has knowledge of its transmitter's power. We wish to examine the variance of the ML estimator of the interference for the scenario of each user experiencing Rayleigh fading. We note that, for $E[S_{R|\phi}] = E[S_{I|\phi}] = 0$, (8) will reduce to

$$f_{S_R,S_I}(z,w) = \frac{1}{\pi v}\exp\left(-\frac{z^2+w^2}{v}\right) \tag{24}$$

with the following assignment introduced for notational convenience: $v \triangleq 2(\sum_{k=0}^{L} P_kVar[A_{R,k}]+Var[\eta])=2(P_0Var[A_{R,0}]+Y)$. Thus, we have the marginal distribution $f_{S_R}(z) = (1/\sqrt{\pi v})\exp(-(z^2/v))$ and obtain $E[S_R^2] = v/2$ to arrive at

$$\begin{aligned}Var[\widehat{Y}] &= Var\left[\frac{S_R^2+S_I^2}{2}\right] = \frac{v^2}{4}\\ &= \left(P_0Var[A_{R,0}]+Y\right)^2.\end{aligned} \tag{25}$$

We investigate whether the estimator is unbiased or not by examining $E[\widehat{Y}] = -P_0Var[A_{R,0}] + (E[S_R^2] + E[S_I^2])/2$. From $E[S_R^2] = v/2$, it follows that $E[\widehat{Y}] = -P_0Var[A_{R,0}] + v/2 = Y$, which states that the ML estimate $\widehat{Y}$ is an unbiased estimator of the interference power experienced by a user. The distribution of $\widehat{Y}$ follows from noting via (6) and (7) that S_R and S_I will be independent and identically distributed zero-mean Gaussian variates when $E[A_{R,k}] = E[A_{I,k}] = 0$. This leads to $S_R^2 + S_I^2$ having an exponential distribution, and hence

$$f_{\widehat{Y}}(x) = \frac{2}{v}\exp\left(-\frac{2\left(x+P_0Var[A_{R,0}]\right)}{v}\right).$$

We now derive the CRLB in the scenario in which each user experiences Rayleigh fading. From (24), we obtain $\partial\ln f_{S_R,S_I}(z,w)/\partial Y = (z^2 + w^2 - v)/(1/2)v^2$ and use $E[S_R^2] = v/2$ and the fact that

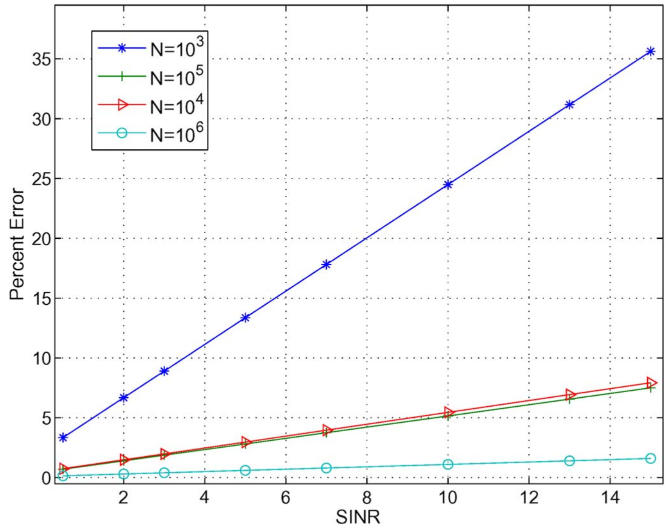

Fig. 2. Investigation of the accuracy of ML estimation of a user's interference $\widehat{Y}$ for the scenario of all users experiencing Rayleigh fading. The percent error was calculated according to $(abs(\widehat{Y}-Y)/Y)(100)$.

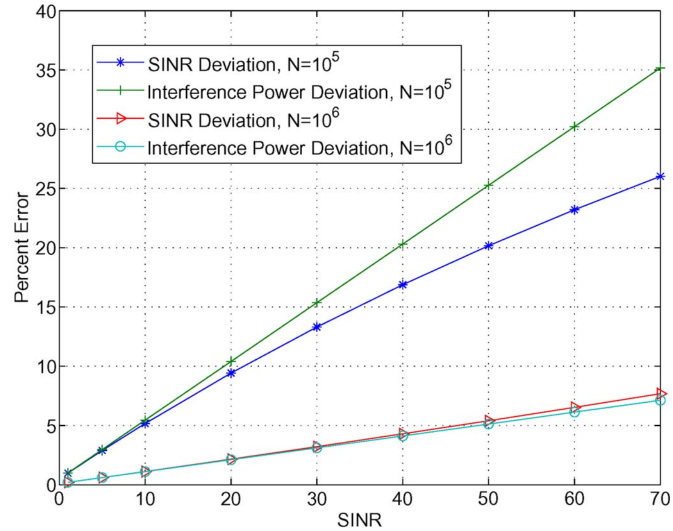

Fig. 3. Investigation of the accuracy of ML estimation of a user's interference $\widehat{Y}$ for the scenario of all users experiencing Rayleigh fading. The deviation in the ML estimate of the corresponding SINR value is included as a reference. The percent error was calculated via $(abs(\widehat{Y}-Y)/Y)(100)$ for the interference power deviation and via $(abs(\widehat{SINR}-SINR)/SINR)(100)$ for the SINR deviation.

$E[S_R^4]=E[S_I^4]=3v^2/4$ to obtain $E[(\partial \ln f_{S_R,S_I}(z,w)/\partial Y)^2]=1/(P_0Var[A_{R,0}]+Y)^2$. The preceding analysis leads to

$$\begin{aligned} Var[\widehat{Y}] &\geq \left(E\left[\left(\frac{\partial \ln f_{S_R,S_I}(z,w)}{\partial Y}\right)^2\right]\right)^{-1} \\ &= (P_0Var[A_{R,0}]+Y)^2 \end{aligned} \tag{26}$$

as a lower bound on the attainable unbiased estimator variance [5]. Comparing (25) and (26), we note that the ML estimate attains the CRLB and is therefore the minimum variance unbiased (MVU) estimator in this scenario. Such a property is expected since the pdf in (24) belongs to the exponential family of distributions [5]. It is also observed that the variance of the ML estimate increases with increasing levels of signal ($P_0Var[A_{R,0}]$) and interference (Y). We generalize our results to N observations of the channel by examining the sequence $S(n)=S_R(n)+jS_I(n)=\sum_{k=0}^{L}G_k(n)+\eta(n)$ for $n=0,1,\ldots,N$ prior to making the ML estimate of the SINR. In the specific case in which each user experiences Rayleigh fading, it can be shown that $\widehat{X}=P_0$ and $\widehat{Y}=1/(2N)\sum_{n=1}^{N}S_R^2(n)+S_I^2(n)-P_0Var[A_{R,0}]$ will satisfy the ML criterion. Our findings for a single channel observation are generalized to the N-observation scenario by noting that user 0 is unable to obtain an ML-optimal estimate of its corresponding transmitter's power, and the ML estimate $\widehat{Y}$ has a variance of $(P_0Var[A_{R,0}]+Y)^2/N$ and is an unbiased estimator of the interference witnessed by a user. Furthermore, the CRLB is generalized to $Var[\widehat{Y}]\geq(P_0Var[A_{R,0}]+Y)^2/N$, which confirms that the ML estimator is the MVU estimator.

The deviation between Y and $\widehat{Y}$ is investigated via simulation in Figs. 2 and 3. In Fig. 2, we have fixed Y via the normalization of $Y=1$ and varied $P_0Var[A_{R,0}]$ to obtain various SINR values. In Fig. 3, we have fixed X via the normalization of $X=1$ and adjusted the interference power to obtain various SINR values. The simulation results illustrate an increase in the percent error with increasing SINR levels. This is attributed to $Var[\widehat{Y}]$ in (25) being an increasing function of $P_0Var[A_{R,0}]$. We observe that an increased number of samples will be required to obtain an accurate SINR estimate at higher SINR values. This result indicates that a wireless user operating in a high SINR regime will need to wait longer to obtain a reliable SINR estimate. Finally, Figs. 2 and 3 support the general notion that the performance of an ML estimator is attractive only when sufficiently large numbers of data samples are available [3].

IV. Conclusion

Analytical expressions have been derived for ML SINR estimation in a multiuser fading channel with additive Gaussian noise and MPSK signaling. An asymptotically efficient estimator has been presented based on a single step of the Newton–Raphson method and the MOM estimator as the starting value. In the case of users undergoing Rayleigh fading, we have presented an explicit expression for ML estimation of a user's interference and derived the CRLB on estimator variance. The simulation results for the case of Rayleigh fading show a higher error rate between the estimated and true SINR with increasing SINR levels.

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References

[1] D. Ramakrishna, N. B. Mandayam, and R. D. Yates, "Subspace-based SIR estimation for CDMA cellular systems," *IEEE Trans. Veh. Technol.*, vol. 49, no. 5, pp. 1732–1742, Sep. 2000.
[2] M. Turkboylari and G. L. Stüber, "An efficient algorithm for estimating the signal-to-interference ratio in TDMA cellular systems," *IEEE Trans. Commun.*, vol. 46, no. 6, pp. 728–731, Jun. 1998.
[3] S. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1993.
[4] Y. Chen and N. C. Beaulieu, "Maximum likelihood estimation of SNR using digitally modulated signals," *IEEE Trans. Wireless Commun.*, vol. 6, no. 1, pp. 210–219, Jan. 2007.
[5] H. V. Poor, *An Introduction to Signal Detection and Estimation*, 2nd ed. New York: Springer-Verlag, 1994.
[6] E. L. Lehmann, *Elements of Large-Sample Theory*. New York: Springer-Verlag, 2004.
[7] Y. Chen and N. C. Beaulieu, "NDA estimation of SINR for QAM signals," *IEEE Commun. Lett.*, vol. 9, no. 8, pp. 688–690, Aug. 2005.