Nonnegative Matrix Theory and its Applications to Communication Network Problems

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Outline

- Model
- Sum Rate Maximization
- Friedland-Karlin Inequalities & Minimax Theorem
- Global Optimization Algorithm
- Max-min Weighted SIR & Nonlinear Perron-Frobenius Theory
- Fast Polynomial-time Algorithms
- Conclusion

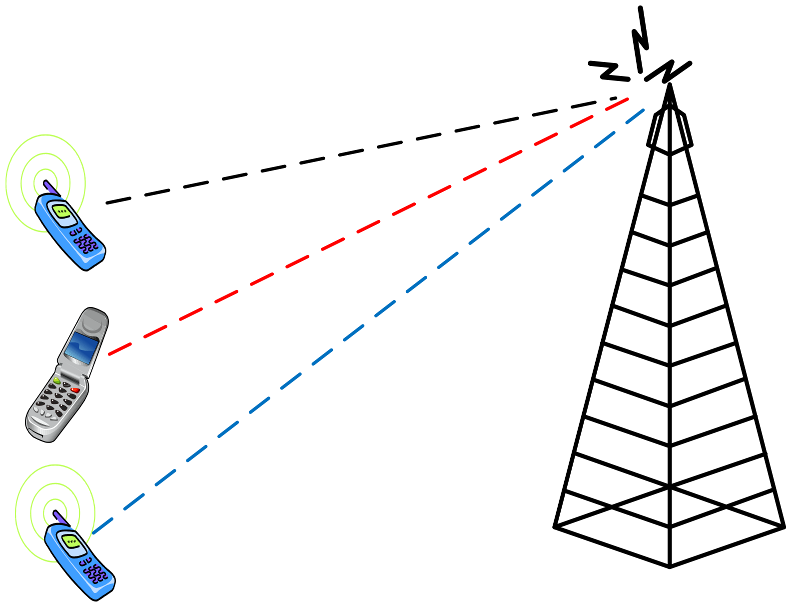
What makes a problem easy or hard

`... the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.`

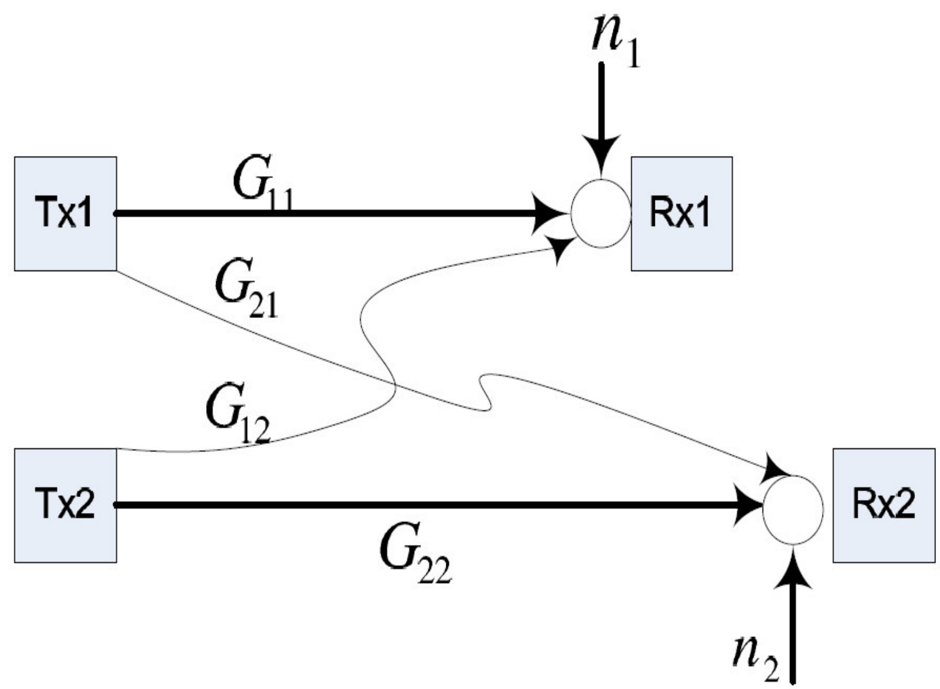
– *SIAM Review 1993*, R. Rockafellar

- Linear inequality theory & nonconvex integer programming (1947)
- Semidefinite matrix theory & nonconvex quadratic programming (1995)
- Nonnegative matrix theory & nonconvex cone programming (this talk)

Model



(a) Cellular wireless network



(b) Interference channel

Performance Metric

- Signal-to-Interference Ratio:

$$\mathsf{SIR}_l(\mathbf{p}) = \frac{G_{ll}p_l}{\sum_{j \neq l} G_{lj}p_j + n_l}$$

with G_{lj} the channel gains from transmitter j to receiver l and n_l the additive white Gaussian noise (AWGN) power at receiver l

- Attainable data rate (nats per channel use) is a function of SIR, e.g., Shannon capacity formula $r_l = \log(1 + \mathsf{SIR}_l)$

- Total power constraints $\mathbf{1}^\top \mathbf{p} \leq \bar{P}$

Interference Parameters

- Let $\mathbf{F}$ be a nonnegative matrix with entries:

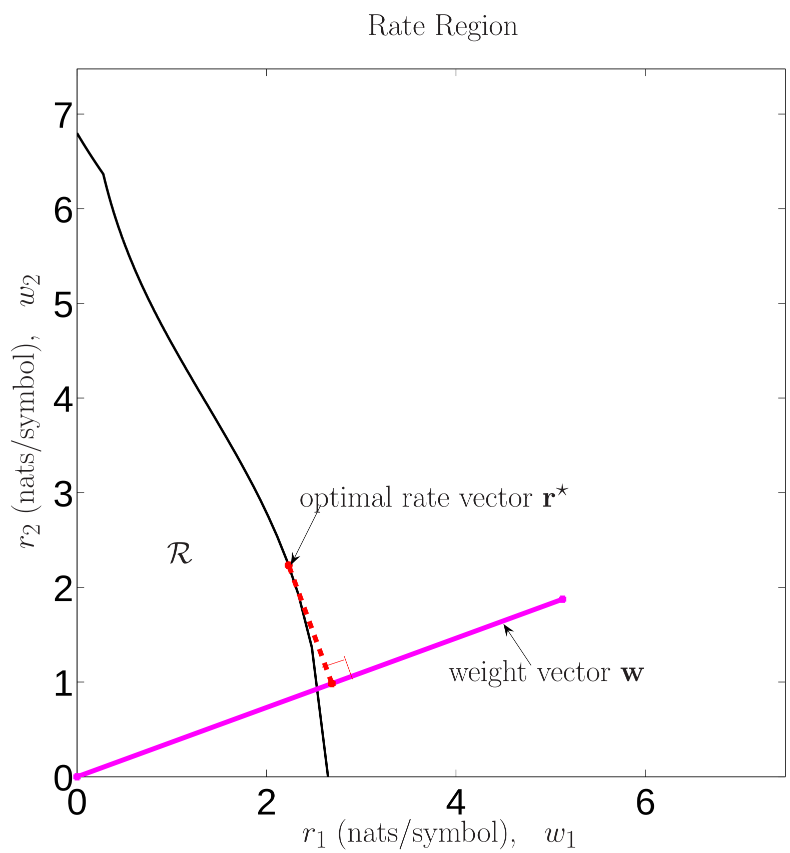
$$F_{ij} = \begin{cases} 0, & \text{if } i = j \\ \frac{G_{ij}}{G_{ii}}, & \text{if } i \neq j \end{cases}$$

and

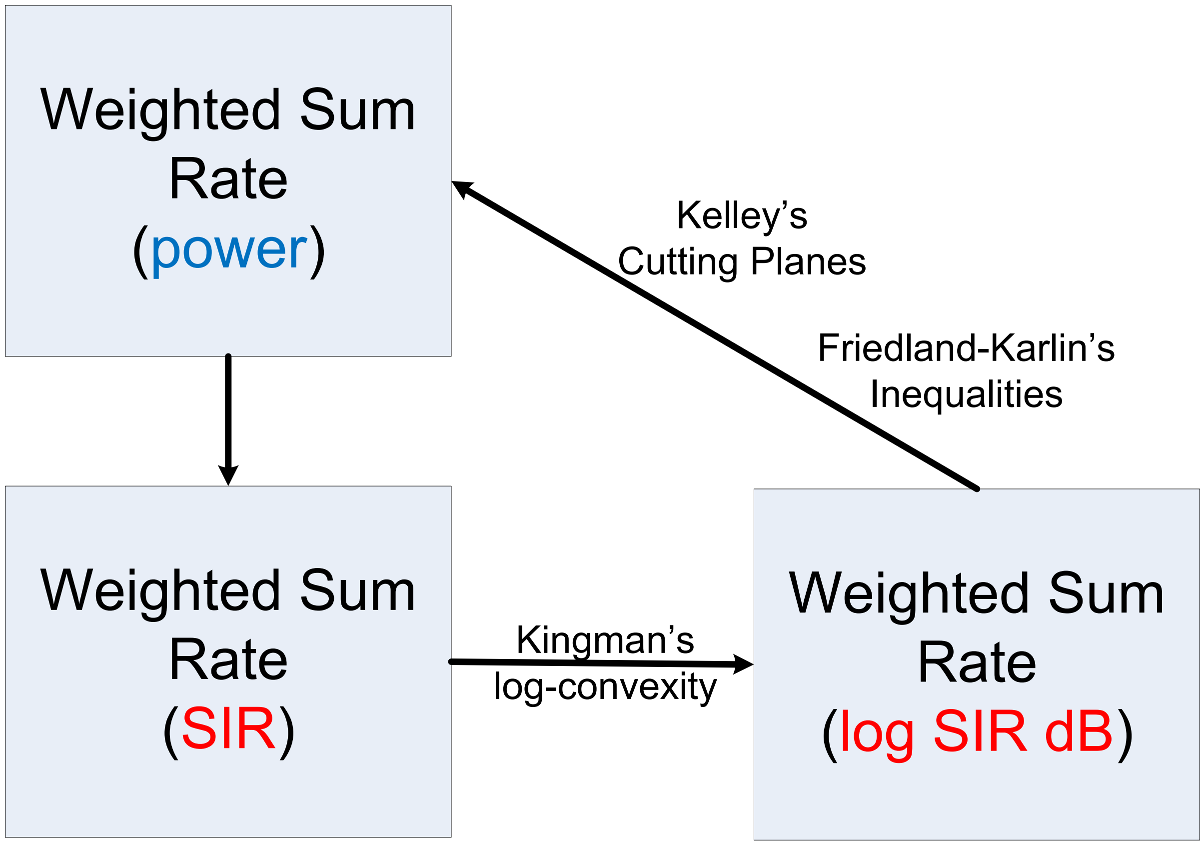
$$\mathbf{v} = \left(\frac{n_1}{G_{11}}, \frac{n_2}{G_{22}}, \ldots, \frac{n_L}{G_{LL}} \right)^\top .$$

Sum Rate Geometry Illustration

$$\begin{array}{rl} \text{maximize} & \sum_l w_l \log(1+\mathsf{SIR}_l(\mathbf{p})) \;=\; \sum_l w_l\, r_l \\ \text{subject to} & 0 \le p_l \le \bar{p}_l \quad \forall\, l, \\ \text{variables:} & p_l \quad \forall\, l. \end{array}$$



Solution Map



Friedland and Tan, *Nonnegative Matrix Inequalities and Applications to Multiuser Communication Problems*,
submitted to SIAM Journ. on Matrix Analysis & Applications, 2009

Sum Shannon Rate Global Optimization

- Nonlinear map between power $\mathbf{p}$ and SIR $\boldsymbol{\gamma} = \exp(\tilde{\boldsymbol{\gamma}})$:

$$\mathbf{p}^\star = (\mathbf{I} - \mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^\star))\mathbf{F})^{-1}\,\mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^\star))\mathbf{v} \qquad (1)$$

- Constraints of $\mathbf{p} \leq \bar{\mathbf{p}}$ into spectral radius constraints $\tilde{\boldsymbol{\gamma}}$

- Convert into convex maximization (dB domain):

$$\begin{array}{rl} \text{maximize} & \sum_l w_l \log(1 + \exp(\tilde{\gamma}_l)) \\ \text{subject to} & \log \rho(\mathbf{diag}(\exp(\tilde{\boldsymbol{\gamma}}))(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \leq 0 \quad \forall\, l, \\ \text{variables:} & \tilde{\gamma}_l, \quad \forall\, l. \end{array}$$

Nonnegative Matrix Theory: Minimax Theorem

- **Theorem 1.** *Friedland-Karlin inequality [$\mathbf{FriedlandKarlin'75}$]: For any irreducible nonnegative matrix $\mathbf{A}$,*

$$\prod_l \left((\mathbf{A}\mathbf{z})_l / z_l\right)^{x_l y_l} \geq \rho(\mathbf{A})$$

for all strictly positive $\mathbf{z}$, where $\mathbf{x}$ and $\mathbf{y}$ are the Perron and left eigenvectors of $\mathbf{A}$ respectively. Equality holds in (2) if and only if $\mathbf{z} = a\mathbf{x}$ for some positive a.

- Donsker-Varadhan's variational principle (1975):

$$\max_{\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^\top \boldsymbol{\lambda} = 1} \min_{\mathbf{p} \geq \mathbf{0}} \sum_l \lambda_l \frac{(\mathbf{A}\mathbf{p})_l}{p_l} = \min_{\mathbf{p} \geq \mathbf{0}} \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^\top \boldsymbol{\lambda} = 1} \sum_l \lambda_l \frac{(\mathbf{A}\mathbf{p})_l}{p_l}$$

- Extensions (see **[FriedlandTan'09]**)

Sum Shannon Rate Global Optimization

- Convex Maximization (dB domain):

$$\begin{array}{rl} \text{maximize} & \sum_l w_l \log(1 + \exp(\tilde{\gamma}_l)) \\ \text{subject to} & \log \rho(\mathsf{diag}(\exp(\tilde{\boldsymbol{\gamma}}))(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \leq 0 \quad \forall\, l, \\ \text{variables:} & \tilde{\gamma}_l, \quad \forall\, l. \end{array}$$

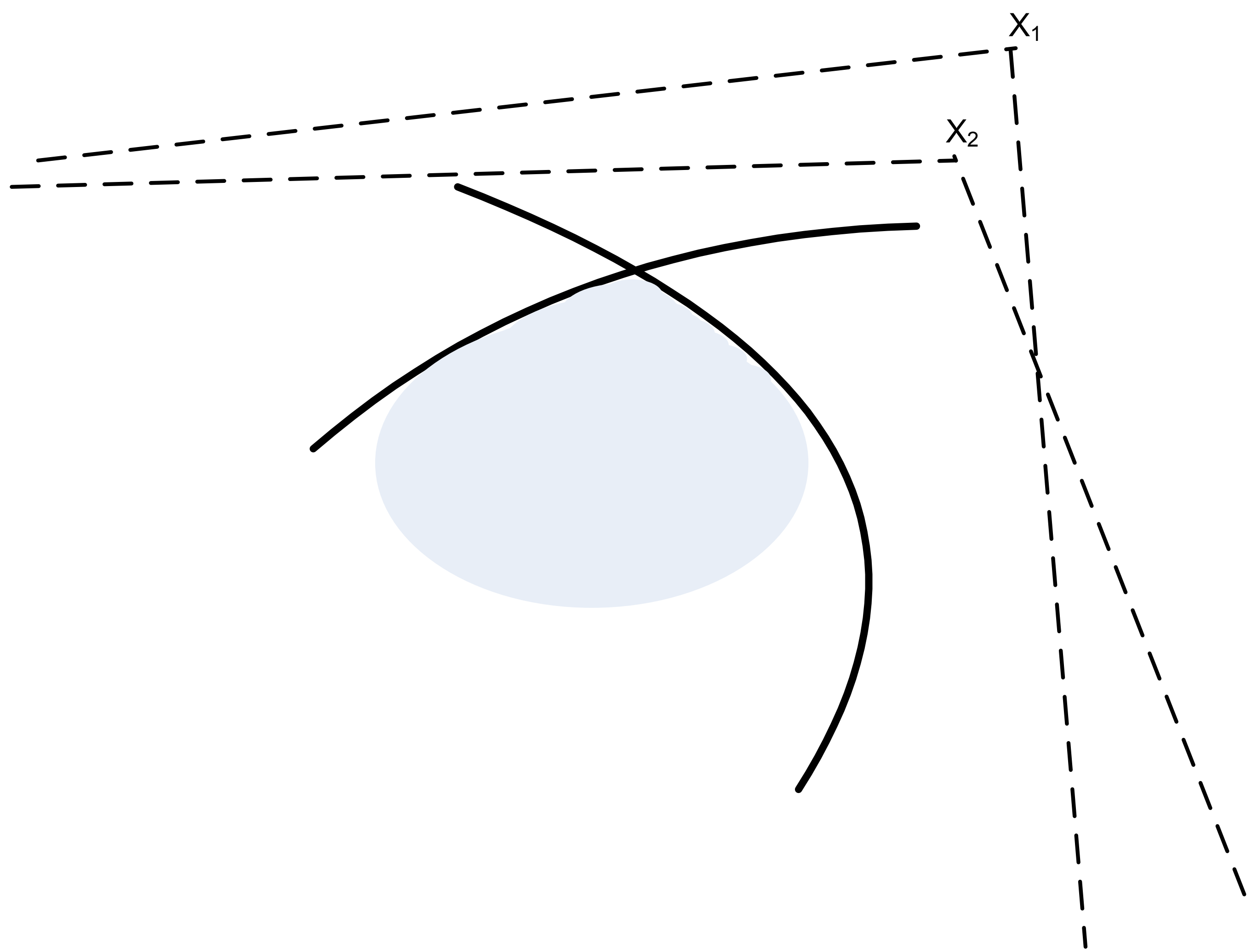
- Relaxation of the constraint set by the Friedland-Karlin Inequalities:

$$\prod_l \gamma_l^{x_l(\mathbf{A}) y_l(\mathbf{A})} \rho(\mathbf{A}) \leq \rho(\mathsf{diag}(\boldsymbol{\gamma})\mathbf{A})$$

$$\sum_l x_l(\mathbf{A}) y_l(\mathbf{A}) \tilde{\gamma}_l + \log \rho(\mathbf{A}) \leq \log \rho(\mathsf{diag}(\exp(\tilde{\boldsymbol{\gamma}}))\mathbf{A}) \quad \text{(dB domain)}.$$

- Outer approximation algorithm (Kelley's cutting planes)

Outer Approximation Algorithm Illustration



Sum Rate Global Optimization: Algorithm

Algorithm 1. [Sum Rate Outer Approximation Algorithm]

1. *Compute the vertices of the enclosing linear polyhedron* $D^{(0)}$*, described by the set of constraints:*

$$\sum_j (\mathbf{x}(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top) \circ \mathbf{y}(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))_j \tilde{\gamma}_j + \log \rho(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top) \leq 0,$$

and $\tilde{\gamma}_l \geq -K$ *for all* l*. Let* $V^{(0)}$ *be the set of vertices of* $D^{(0)}$*. Set* $k = 1$ *and go to Step 2.*

2. *Iteration* k*: Solve the problem:*

$$\begin{array}{rl} \textit{maximize} & \sum_l w_l \log(1 + e^{\tilde{\gamma}_l}) \\ \textit{subject to} & \tilde{\gamma}_l \in D^{(k-1)} \end{array}$$

by selecting $\max\left\{\sum_l w_l \log(1 + e^{\tilde{\gamma}_l}) : v \in V^{(k-1)}\right\}$*. Let* $\tilde{\boldsymbol{\gamma}}^k$ *be the optimizer to (2).*

3. *Compute*

$$\mathbf{p}^k = \Big(\mathbf{I} - \mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))\mathbf{F}\Big)^{-1} \mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))\mathbf{v}.$$

4. *If* $\mathbf{p}^k \le \bar{\mathbf{p}}$, *stop:* $\tilde{\boldsymbol{\gamma}}^k$ *is the solution to (2) and* $\mathbf{p}^k$ *is the solution to (1). Otherwise, let*

$$\begin{aligned} J^k \quad &= \{l : \log\rho(\mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top)) \\ &= \max\nolimits_{1\le j\le L} \log\rho(\mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))(\mathbf{F} + (1/\bar{p}_j)\mathbf{v}\mathbf{e}_j^\top))\} \end{aligned} \qquad (2)$$

and choose any $j^k \in J^k$.

5. *Compute the left eigenvector* $\mathbf{y}_{j^k}$ *and right (Perron) eigenvector* $\mathbf{x}_{j^k}$ *of* $\mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))(\mathbf{F} + (1/\bar{p}_{j^k})\mathbf{v}\mathbf{e}_{j^k}^\top)$. *Set*

$$\begin{aligned} G_{j^k}^k(\tilde{\boldsymbol{\gamma}}) \quad &= \log\rho(\mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))(\mathbf{F} + (1/\bar{p}_{j^k})\mathbf{v}\mathbf{e}_{j^k}^\top)) + \\ &\qquad \frac{[\exp(\tilde{\boldsymbol{\gamma}}^k)\circ\mathbf{x}_{j^k}\circ\mathbf{y}_{j^k}]^\top(\tilde{\boldsymbol{\gamma}} - \tilde{\boldsymbol{\gamma}}^k)}{\rho(\mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))(\mathbf{F} + (1/\bar{p}_{j^k})\mathbf{v}\mathbf{e}_{j^k}^\top))}. \end{aligned} \qquad (3)$$

6. *Set* $D^{(k)} = D^{(k-1)} \cap \{\tilde{\boldsymbol{\gamma}} : G^k_{j^k}(\tilde{\boldsymbol{\gamma}}) \leq 0\}$, $V^{(k)} = \{$*extreme points of* $D^{(k)}\}$.

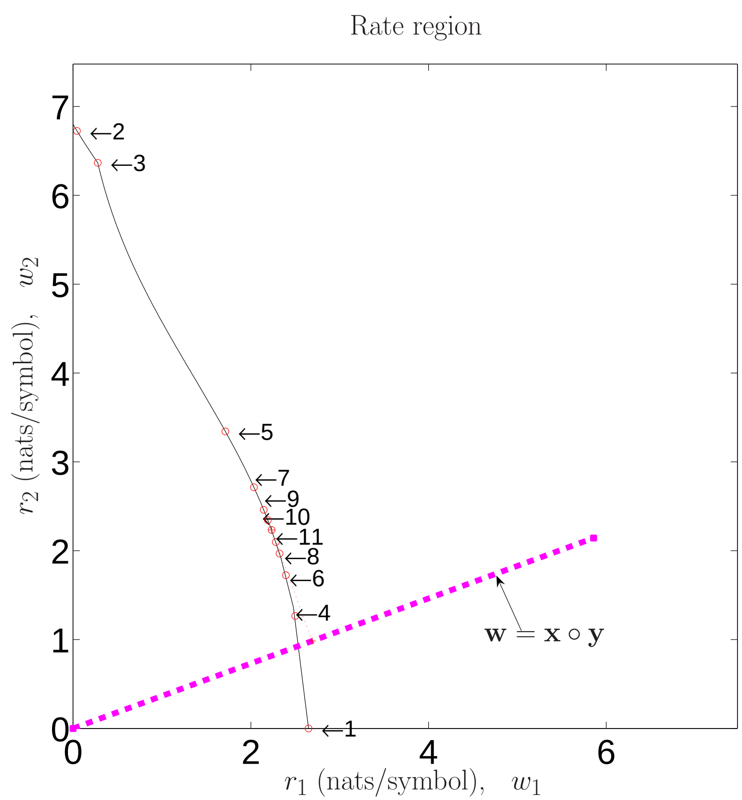
7. *Set* $k \leftarrow k+1$. *Go to Step 2.*

- Step 3 yields a feasible power vector $\hat{\mathbf{p}}^k$: $\hat{p}^k_l = \min\{p^k_l, \bar{p}_l\}$ for all l.

Global Optimizing Sum Rate: Examples

- $\lim_{k\to\infty} \min\left\{\left(\mathbf{I} - \mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))\mathbf{F}\right)^{-1}\mathrm{diag}(\exp(\tilde{\boldsymbol{\gamma}}^k))\mathbf{v}, \bar{p}_l\right\} = \mathbf{p}^\star$

- Fast convergence in numerical examples



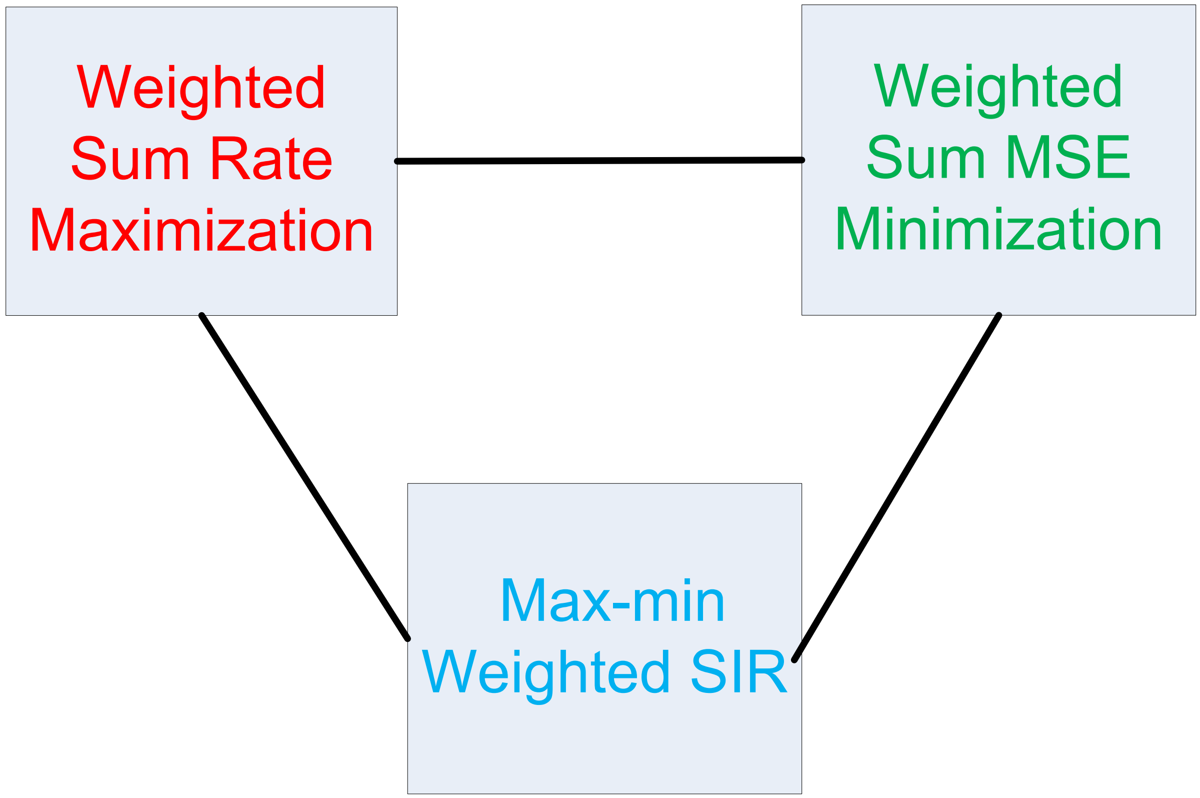
Global Optimizing Sum Rate: Examples

- Efficient and fast for small to medium-sized networks

Problem size	Maximal number of generated vertices	Number of iterations	CPU time (minutes)
2	15	12	0.062
4	139	760	4.1
6	14022	1238	83
8	283681	1968	468

Table 1: A comparison of the typical convergence and complexity statistics of Algorithm 1 with the problem size. The CPU time is computed based on an implementation on a 64-bit Sun/Solaris 10 (SunOS 5.10) computer.

Nonconvex Power Control Problems



Tan, Chiang and Srikant, IEEE INFOCOM 2009 & IEEE ISIT 2009

Max-min Weighted SIR

- $$\mathsf{SIR}_l(\mathbf{p}) = \frac{G_{ll}p_l}{\sum_{j\neq l} G_{lj}p_j + n_l} \qquad \mathbf{B} = \mathbf{F} + (1/\bar{P})\mathbf{v}\mathbf{1}^\top$$

- $$\begin{array}{ll} \text{maximize} & \min_l \dfrac{\mathsf{SIR}_l(\mathbf{p})}{\beta_l} \\ \text{subject to} & \mathbf{1}^\top \mathbf{p} \leq \bar{P}, \ \mathbf{p} \geq \mathbf{0}, \\ \text{variables:} & \mathbf{p}. \end{array}$$

- Optimal value: $1/\rho(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B})$ and optimal solution: $(\bar{P}/\mathbf{1}^\top\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B}))\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})\mathbf{B})$.

Nonlinear Perron-Frobenius Theory

- Find $(\check{\lambda}, \check{\mathbf{s}})$ in

$$\lambda \mathbf{s} = \mathbf{A}\mathbf{s} + \mathbf{b}, \quad \lambda \in \mathbb{R}, \quad \mathbf{s} \geq \mathbf{0}, \quad \|\mathbf{s}\| = 1,$$

  where $\mathbf{A}$ and $\mathbf{b}$ is a square irreducible nonnegative matrix and nonnegative vector, respectively and $\|\cdot\|$ a monotone vector norm.

- $(\check{\lambda}, \check{\mathbf{s}})$ is Perron-Frobenius eigenvalue-vector pair of $\mathbf{A} + \mathbf{b}\mathbf{c}_*^\top$, where

$$\mathbf{c}_* = \arg \max_{\|\mathbf{c}\|_*=1} \rho(\mathbf{A} + \mathbf{b}\mathbf{c}^\top),$$

  where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$, and $\check{\mathbf{s}} = (\mathbf{A}\check{\mathbf{s}} + \mathbf{b})/\|\mathbf{A}\check{\mathbf{s}} + \mathbf{b}\|$.

V. D. Blondel, L. Ninove and P. Van Dooren, *An affine eigenvalue problem on the nonnegative orthant*, Linear Algebra & its Applications, 2005

Nonlinear Perron-Frobenius Theory: Max-min SIR

- $$\mathsf{SIR}_l(\mathbf{p}^*) = \tau^*\beta_l \;\Rightarrow\; \frac{(p_l^*/\bar{P})}{\sum_{j\neq l} F_{lj}(p_l^*/\bar{P}) + (v_l/\bar{P})} = \tau^*\beta_l$$

  Let $\mathbf{s}^* = (1/\bar{P})\mathbf{p}^*$:

  $$(1/\tau^*)\mathbf{s}^* = \mathsf{diag}(\boldsymbol{\beta})\mathbf{F}\mathbf{s}^* + (1/\bar{P})\mathsf{diag}(\boldsymbol{\beta})\mathbf{v}, \quad \|\mathbf{s}\|_1 = 1$$

-
  - $s_l = p_l/\bar{p}_l$, $\mathbf{A} = \mathrm{diag}(\boldsymbol{\beta})\mathbf{F}$, $\mathbf{b} = (1/\bar{P})\,\mathrm{diag}(\boldsymbol{\beta})\mathbf{v}$ and $\lambda = 1/\tau^*$
  - $\|\cdot\| = \|\cdot\|_1 \longleftrightarrow \|\cdot\|_* = \|\cdot\|_\infty$ & $\mathbf{c}_* = \mathbf{1}$
  - $(\check{\lambda}, \check{\mathbf{s}})$ is the Perron-Frobenius eigenvalue and vector pair of $\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{P})\mathbf{v}\mathbf{1}^\top) = \mathsf{diag}(\boldsymbol{\beta})\mathbf{B}$

Max-min Weighted SIR: Algorithm

- **Algorithm 2. [Max-min Weighted** SIR**]**

  1. *Update power* $\mathbf{p}(k+1)$*:*

$$p_l(k+1) = \left(\frac{\beta_l}{\mathsf{SIR}(\mathbf{p}(k))}\right) p_l(k) \quad \forall\ l$$

  2. *Normalize* $\mathbf{p}(k+1)$*:* $\quad \mathbf{p}(k+1) = \mathbf{p}(k+1)/\mathbf{1}^\top \mathbf{p}(k+1) \cdot \bar{P}$

Approximation to Weighted Sum Rate Problem

- $$\begin{array}{rl} \text{maximize} & \sum_{l=1}^{L} w_l \log(1+\mathsf{SIR}_l(\mathbf{p})) \\ \text{subject to} & \sum_{l=1}^{L} p_l \le \bar{P}, \quad p_l \ge 0 \quad \forall\, l, \\ \text{variables:} & p_l \quad \forall\, l. \end{array} \tag{4}$$

- Quasi-inverse **[Wong54]**: $\mathbf{B}$ is a quasi-inverse of $\tilde{\mathbf{B}} \ge \mathbf{0}$ if $\mathbf{B} - \tilde{\mathbf{B}} = \mathbf{B}\tilde{\mathbf{B}} = \tilde{\mathbf{B}}\mathbf{B} \ge \mathbf{0}$

- $\rho(\tilde{\mathbf{B}}) = \frac{\rho(\mathbf{B})}{1+\rho(\mathbf{B})}$

- $\mathbf{x}(\tilde{\mathbf{B}}) = \mathbf{x}(\mathbf{B})$ & $\mathbf{y}(\tilde{\mathbf{B}}) = \mathbf{y}(\mathbf{B})$

Interference & SNR Regime

- Recall the matrix

$$\mathbf{B} = \mathbf{F} + (1/\bar{P})\mathbf{v}\mathbf{1}^\top$$

- (High SNR regime) $\tilde{\mathbf{B}}$ does not exist

  or any nonnegative matrix with a zero trace & positive off-diagonals

- (Low SNR regime) $\tilde{\mathbf{B}}$ always exists

  or any nonnegative matrix that is a dyad

- (Low interference/moderate SNR regime) $\tilde{\mathbf{B}}$ almost always exists

Approximation to Weighted Sum Rate Problem

Theorem 2. *If* $\tilde{\mathbf{B}} \geq \mathbf{0}$,

$$\sum_{l=1}^{L} w_l \log(1 + \mathsf{SIR}_l(\mathbf{p})) \leq \|\mathbf{w}\|_{\infty}^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})} \log(1 + 1/\rho(\mathbf{B})) \qquad (5)$$

for all feasible $\mathbf{p}$.

Equality is achieved if and only if $\mathbf{w} = \mathbf{x}(\mathbf{B}) \circ \mathbf{y}(\mathbf{B})$, *and* $\mathsf{SIR}_l(\mathbf{p}^\star) = (1/\rho(\mathbf{B}))\mathbf{1}\ \forall\, l$. *In this case,* $\mathbf{p}^\star = \mathbf{x}(\mathbf{B})$.

- Max-min SIR ($\boldsymbol{\beta} = \mathbf{1}$) as an "approximation" algorithm

$\boxed{\text{Simple Characterization via Nonnegative Matrix Theory}}$

“Charnes-Cooper” Trick plus Friedland-Karlin

- Maximizing objective (linear-fractional) function:

$$\max_{\mathbf{p}} \prod_{l} \left(((\mathbf{I}+\mathbf{F})\mathbf{p}+\mathbf{v})_l / (\mathbf{F}\mathbf{p}+\mathbf{v})_l\right)^{w_l}. \tag{6}$$

- Change of variable: $\mathbf{z} = (\mathbf{I}+\mathbf{B})\mathbf{p}$

- Transform maximization over $\mathbf{p}$ to $\mathbf{z}$ (when $\tilde{\mathbf{B}} \geq \mathbf{0}$):

$$\min_{\mathbf{z}} \prod_{l} \left(\frac{(\tilde{\mathbf{B}}\mathbf{z})_l}{z_l}\right)^{w_l} \tag{7}$$

- Friedland-Karlin inequalities on spectrum of $\tilde{\mathbf{B}}$ ($\sim B$)

Weighted Sum Rate Maximization: Exact Solution

Theorem 3. *If* $\tilde{\mathbf{B}} \geq \mathbf{0}$*, then the optimal solution to Sum Rate problem is given by* $\mathbf{p}^\star = (\mathbf{I} - \tilde{\mathbf{B}})\mathbf{z}^\star \succeq \mathbf{0}$*, where* $\mathbf{z}^\star$ *is given by*

$$z_l^\star = \frac{w_l}{\sum_j w_j \tilde{B}_{jl} / (\tilde{\mathbf{B}}\mathbf{z}^\star)_j} \tag{8}$$

for all l *and satisfies* $\mathbf{1}^\top \mathbf{z}^\star - \mathbf{1}^\top \tilde{\mathbf{B}}\mathbf{z}^\star = \bar{P}$*.*

Weighted Sum Rate Maximization: Algorithm

Algorithm 3. [Sum Rate Maximization Algorithm]

1. *Initialize an arbitrarily small* $\epsilon > 0$*.*

2. *Update auxiliary variable* $\mathbf{z}(k+1)$*:*

$$z_l(k+1) = \frac{w_l}{\sum_j w_j \tilde{B}_{jl}/(\tilde{\mathbf{B}}\mathbf{z}(k))_j} + \epsilon \quad \forall\, l. \tag{9}$$

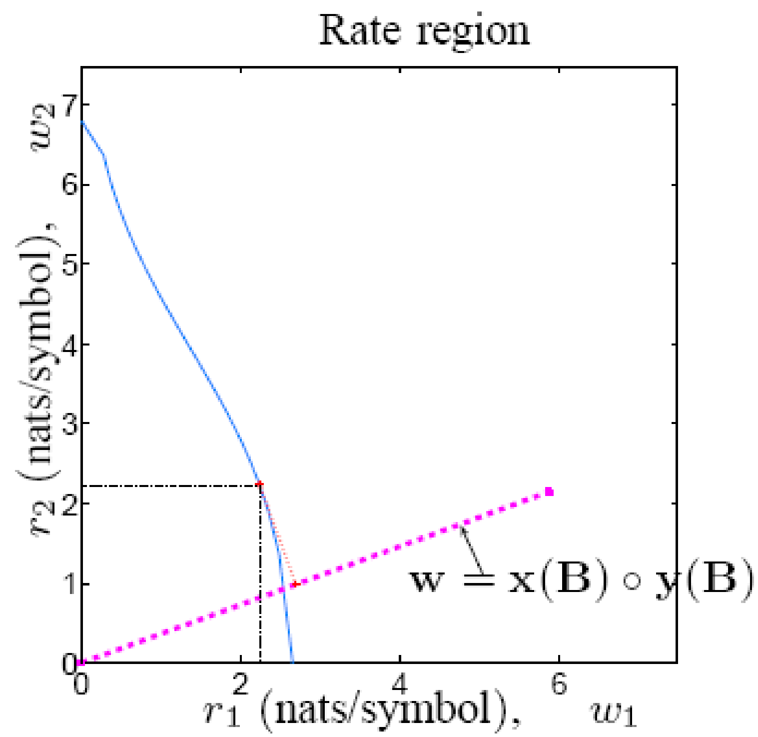
3. *Update* $\mathbf{p}(k+1)$*:*

$$p_l(k+1) = \frac{\mathsf{SIR}_l(\mathbf{p}(k))}{1 + \mathsf{SIR}_l(\mathbf{p}(k))} z_l(k+1) \quad \forall\, l. \tag{10}$$

4. *Normalize* $\mathbf{p}(k+1)$*:* $\mathbf{p}(k+1) \leftarrow \mathbf{p}(k+1) \cdot \bar{P}/(\mathbf{1}^\top \mathbf{p}(k+1))$*.*

Link Sum Rate Maximization & Max-min SIR

- Friedland-Karlin inequalities & Arithmetic-geometric mean inequality & $\mathbf{w} = \mathbf{x}(\mathbf{B}) \circ \mathbf{y}(\mathbf{B})$



New Link in Nonlinear Perron-Frobenius Theory & Optimization Theory

Nonlinear Perron-Frobenius Theory: Friedland-Karlin Minimax

- Minimax Theorem **[FriedlandKarlin'75]**:

$$\log \rho(\mathbf{A}) = \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^\top \boldsymbol{\lambda} = 1} \min_{\mathbf{p} \geq \mathbf{0}} \sum_l \lambda_l \log \frac{(\mathbf{A}\mathbf{p})_l}{p_l} = \min_{\mathbf{p} \geq \mathbf{0}} \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^\top \boldsymbol{\lambda} = 1} \sum_l \lambda_l \log \frac{(\mathbf{A}\mathbf{p})_l}{p_l}$$

- Nonlinear version ($f(\mathbf{p}) = \mathbf{A}\mathbf{p} + \mathbf{b}$):

$$\begin{aligned}
&\max_{\|\mathbf{c}\|_* = 1} \log \rho(\mathbf{A} + \mathbf{b}\mathbf{c}^\top) \\
&= \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^\top \boldsymbol{\lambda} = 1} \min_{\|\mathbf{p}\| = 1} \sum_l \lambda_l \log \frac{(\mathbf{A}\mathbf{p} + \mathbf{b})_l}{p_l} \\
&= \min_{\|\mathbf{p}\| = 1} \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{1}^\top \boldsymbol{\lambda} = 1} \sum_l \lambda_l \log \frac{(\mathbf{A}\mathbf{p} + \mathbf{b})_l}{p_l},
\end{aligned}$$

where optimal $\mathbf{p} = \mathbf{x}(\mathbf{A} + \mathbf{b}\mathbf{c}_*^\top)$ and $\boldsymbol{\lambda} = \mathbf{x}(\mathbf{A} + \mathbf{b}\mathbf{c}_*^\top) \circ \mathbf{y}(\mathbf{A} + \mathbf{b}\mathbf{c}_*^\top)$

Extension: Ideas for Fractional Programming

- Sum-of-ratios fractional program (positive $f(\mathbf{p})$, $g(\mathbf{p})$) **[FrenkSchaible05]**:

$$\min_{\mathbf{p}} \sum_l \frac{f_l(\mathbf{p})}{g_l(\mathbf{p})}. \tag{11}$$

- Domain transformation: $\mathbf{z} = g(\mathbf{p})$

- Transform optimization over $\mathbf{p}$ to $\mathbf{z}$ (when $g^{-1}(\mathbf{z}) \geq \mathbf{0}$):

$$\min_{\mathbf{z}} \sum_l \frac{(f(g^{-1}(\mathbf{z})))_l}{z_l} \tag{12}$$

- Link to minimax theorems in Nonlinear Perron-Frobenius theory when $f(g^{-1}(\mathbf{z}))$ is concave and monotone

Conclusion

- Nonconvex power control problem in CDMA networks and OFDM/ADSL channels
- Eigenvalue characterization enables efficient global optimization of Weighted Sum Rate Maximization
- Link nonconvex nonnegative cone programming and nonnegative matrix theory
  - Friedland-Karlin inequalities
  - Log-convexity of spectral radius
- Solving nonconvex problems is an interesting art!

Thank You

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