UWO CS2214 April 1st, 2019

Assignment #3
Due: April 9, 2019, by 23:55
Submission: on the OWL web site of the course

Format of the submission. You must submit a **single** file which must be in **PDF** format. All other formats (text or Miscrosoft word format) will be **ignored** and considered as **null**. You are strongly encouraged to type your solutions using a text editor. To this end, we suggest the following options:

1. Miscrosoft word and convert your document to PDF
2. the typesetting system LaTeX; see `https://www.latex-project.org/` and `https://en.wikipedia.org/wiki/LaTeX#Example` to learn about LaTeX; see `https://www.tug.org/begin.html` to get started
3. using a software tool for typing mathematical symbols, for instance `http://math.typeit.org/`
4. using a Handwriting recognition system such as those equipping tablet PCs

Hand-writing and scanning your answers is allowed but not encouraged:

1. if you go this route please use a scanning printer and **do not take a picture of your answers with your phone**,
2. if the quality of the obtained PDF is too poor, your submission will be **ignored** and considered as **null**.

Problem 1 (Relations) *[25 marks]*

1. Show that the relation

$$R = \{(x, y) |\ (x - y) \text{ is an even integer}\}$$

is an equivalence relation on the set $\mathbb{R}$ of real numbers.

2. Show that the relation

$$\boxed{R = \{((x_1, y_1), (x_2, y_2))\ (x_1 < x_2) \ \text{ or } \ ((x_1 = x_2) \ \text{ and } \ (y_1 \leq y_2))\}}$$

is a total ordering relation on the set $\mathbb{R} \times \mathbb{R}$.

Solution 1

1. (a) R is reflexive, since for all $x \in \mathbb{R}$, we have $x - x = 0$ which is even, hence for all $x \in \mathbb{R}$, we have $(x, x) \in R$.

(b) R is symmetric, since for all $x, y \in \mathbb{R}$, if $x - y \equiv 0 \mod 2$ holds then so does $y - x \equiv 0 \mod 2$, that is, if $(x, y) \in R$ holds then so does $(y, x) \in R$.

(c) R is transitive, since for all $x, y, z \in \mathbb{R}$, if $x - y \equiv 0 \mod 2$ and $y - x \equiv 0 \mod 2$ both hold then so does $x - z = (x-y)+(y-z) \equiv 0 \mod 2$, that is, if $(x, y) \in R$ and $(y, z) \in R$ both hold then so does $(x, z) \in R$.

Therefore, R is an equivalence relation.

2. (a) R is reflexive, since for all $(x_1, y_1) \in \mathbb{R} \times \mathbb{R}$, we have $((x_1 = x_1)$ and $y_1 \leq y_1$, that is, for all $(x_1, y_1) \in \mathbb{R} \times \mathbb{R}$ we have $((x_1, y_1), (x_1, y_1)) \in R$.

   (b) R is anti-ymmetric, since for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R} \times \mathbb{R}$, if $((x_1, y_1), (x_2, y_2)) \in R$ and $((x_2, y_2), (x_1, y_1)) \in R$ both hold then neither $x_1 < x_2$ nor $x_2 < x_1$ holds but both $((x_1 = x_2)$ and $y_1 \leq y_2)$ and $((x_2 = x_1)$ and $y_2 \leq y_1)$ hold, which implies $(x_1, y_1) = (x_2, y_2)$.

   (c) R is transitive. To prove this consider $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R} \times \mathbb{R}$ such that $((x_1, y_1), (x_2, y_2)) \in R$ and $((x_2, y_2), (x_3, y_3)) \in R$ both hold. We shall prove that $((x_1, y_1), (x_3, y_3)) \in R$ also holds. Four cases must be inspected:
      i. $x_1 < x_2$ and $x_2 < x_3$,
      ii. $x_1 < x_2$ and $x_2 = x_3$ and $y_2 \leq y_3$,
      iii. $x_1 = x_2$ and $y_1 \leq y_2$ and $x_2 < x_3$,
      iv. $x_1 = x_2$ and $y_1 \leq y_2$ and $x_1 = x_2$ and $y_2 \leq y_3$,

      which respectively imply:
      i. $x_1 < x_3$,
      ii. $x_1 < x_3$,
      iii. $x_1 < x_3$,
      iv. $x_1 = x_3$ and $y_1 \leq y_3$,

      that is $((x_1, y_1), (x_3, y_3)) \in R$.

3. Therefore, R is an ordering relation on the set $\mathbb{R} \times \mathbb{R}$.

4. R is a total ordering relation on the set $\mathbb{R} \times \mathbb{R}$. Indeed, for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R} \times \mathbb{R}$, we have
   (a) either $x_1 < x_2$ (in which case $((x_1, y_1), (x_2, y_2)) \in R$ holds),
   (b) or $(x_1 = x_2$ and $y_1 \leq y_2$ (in which case $((x_1, y_1), (x_2, y_2)) \in R$ holds),
   (c) or $(x_1 = x_2$ and $y_1 > y_2$ (in which case $((x_2, y_2), (x_1, y_1)) \in R$ holds),
   (d) or $x_1 > x_2$ (in which case $((x_2, y_2), (x_1, y_1)) \in R$ holds).

Problem 2 (Basic probability calculations) *[25 marks]* In a roulette, a wheel with 38 numbers is spun. Of these, 18 are red, and 18 are black. The other two numbers, which are neither black nor red, are 0 and 00. The probability that when the wheel is spun it lands on any particular number is 1/38.

1. What is the probability that the wheel lands on a red number?
2. What is the probability that the wheel lands on a black number twice in a row?
3. What is the probability that the wheel lands on 0 or 00?
4. What is the probability that in five spins the wheel never lands on either 0 or 00?

Provide detailed justifications of your answers.

Solution 2

1. There are 18 favorable outcomes for a total of $18 + 18 + 2$ possible outcomes. Hence the probability is $18/(18 + 18 + 2)$ is $9/19$.
2. There are 18×18 favorable outcomes for a total of $(18 + 18 + 2)^2$ possible outcomes. Hence the probability is $18/(18+18+2)$ is $81/361$.
3. There are 2 favorable outcomes for a total of $18 + 18 + 2$ possible outcomes. Hence the probability is $2/(18 + 18 + 2)$ is $2/19$.
4. We use Bernoulli trials formula. We see "wheel lands on 0 or 00" as a success. The number of trials is $n = 5$, the probability of a success is $p = 2/19$ and the desired number of success if $k = 1$. Thus the requested probability is

$$\binom{n}{k} p^k (1-p)^{n-k} = 5(2/19)^1(17/19)^4 = \frac{835210}{2476099}.$$

Note that the question is unclear: it should say "at least one success" or "exactly one success". We have considered the second option. The first one can be handled easily as

$$1 - \binom{n}{0} p^0 (1-p)^{n-0} = 1 - (17/19)^5.$$

Problem 3 (Bayes theorem) *[25 marks]* Suppose that 8% of all bicycle racers use steroids, that a bicyclist who uses steroids tests positive for steroids 96% of the time, and that a bicyclist who does not use steroids tests positive for steroids 9% of the time (this is a "false positive" test result).

1. What is the probability that a randomly selected bicylist who tests positive for steroids actually uses steroids?

2. What is the probability that a randomly selected bicyclist who tests negative for steroids did not use steroids?

Solution 3 Let S be the event that the "a bicyclist uses steroids" Let E be the event "a test is positive". What we know from the problem statement is:

$$P(S) = 8/100, P(\overline{S}) = 92/100, P(E|S) = 96/100, P(E|\overline{S}) = 9/100.$$

From there, we deduce:

$$P(\overline{E}|\overline{S}) = 1 - P(E|\overline{S}) = 91/100, P(\overline{E}|S) = 1 - P(E|S) = 4/100.$$

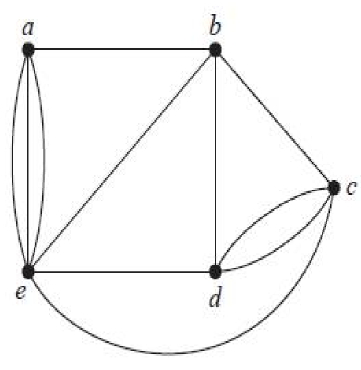
1. We use Bayes' formula:

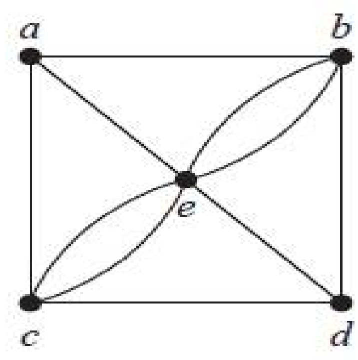
$$\begin{aligned} P(S|E) &= \frac{P(E|S)p(S)}{p(E|S)p(S)+p(E|\overline{S})p(\overline{S})} \\ &= \frac{96/100\times 8/100}{96/100\times 8/100+9/100\times 92/100} \\ &= \frac{64}{133}. \end{aligned}$$

2. We use Bayes' formula:

$$\begin{aligned} P(\overline{S}|\overline{E}) &= \frac{P(\overline{E}|\overline{S})p(\overline{S})}{p(\overline{E}|\overline{S})p(\overline{S})+p(\overline{E}|S)p(S)} \\ &= \frac{91/100\times 92/100}{91/100\times 92/100+4/100\times 8/100} \\ &= \frac{2093}{2101} \end{aligned}$$

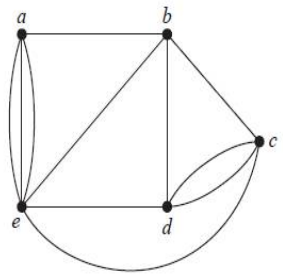
Problem 4 (Graphs) *[25 marks]* For each of the following two graphs, determine whether or not it has an Euler circuit. Justify your answers. If the graph has an Euler circuit, use the algorithm described in class to find it, including drawings of intermediate subgraphs.





Solution 4

1. First, consider the graph on the left. Every node has an even degree, hence there exists an Euler circuit, say from a to a. We use the algorithm seen in class. Observe that the following circuits partition of the set of the edges:

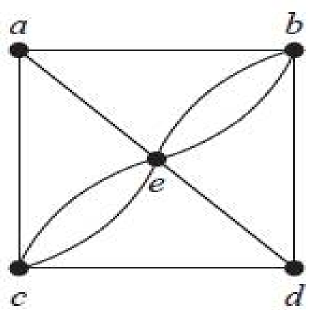


$$(a, e, a), (a, b, d, e, a), (b, c, e, b), (c, d, c).$$

From there we deduce am Euler circuit from a to a:

$$(a, b, c, d, c, e, b, d, e, a, e, e).$$

2. Second, consider the graph on the right. Every node has an even degree, except a and d. Hence there exists an Euler path from a to d. Removing the edge (a, f) we build an Euler circuit around a. Using the algorithm, we have the following circuits partition of the set of the edges:



$$(a, b, d, c, a), (b, e, c, e, b).$$

From there we deduce am Euler circuit from a to a:

$$(a, b, e, c, e, b, d, c, a)$$

and an Euler path from a and d:

$$(a, b, e, c, e, b, d, c, a, d)$$