# Cs 2214b -Assignment4

Student Name: MingCong, Zhou
Student Number: 250945414
Student Account: mzhou272

Assignment #4
Due: Mar. 26, 2017, by 23:55
Submission: on the OWL web site of the course

**Problem 1 (Counting tree leaves)** *[25 marks]* The set of leaves and the set of internal vertices of a full binary tree are defined recursively as follows:

**Basis step:** The root $r$ is a leaf of the full binary tree with exactly one vertex $r$. This tree has no internal vertices.

**Recursive step:** The set of leaves of the tree $T = T_1 \cdot T_2$ is the union of the sets of leaves of $T_1$ and $T_2$. The internal vertices of $T$ are the root $r$ of $T$ and the union of the set of internal vertices of $T_1$ and the set of internal vertices f $T_2$.

Use structural induction to prove that $\ell(T)$, the number of leaves of a full binary tree $T$, is 1 more than $i(T)$, the number of internal vertices of $T$.

Let P(n) denotes that there are n leaves on a binary tree T, and there are n-1 internal vertices of T.

Here is a "proof" that P(n) is true.

Basis Step:

P (1) is true because when there is only 1 node in the tree, the node is the only one leaf.

Additionally, there are no internal node consist by the definition of tree.

Inductive step:

The inductive hypothesis is: P(k) is true for all integers k with 0≤k≤n

Assume that the inductive hypothesis P(k) is true, we need to proof P(k+1) is true:

Given a binary tree T with k+1 vertex, T = T1·T2 means T1 and T2 are subtrees of T

Assume that T1 and T2 have k1 and k2 vertices respectively, it must be true that P(k1) and P(k2) hold as k1 + k2 = n + 1 and k1 < n +1 and k2 < n + 1

As P(k1) P(k2) give true, T1 has n1 internal vertices and n1 + 1 leaves, while T2 has n2 internal vertices and n2 + 1 leaves. (By the definition of P(n))

By recursive definition, T's internal vertices consist of the root vertex, the internal Vertices of T1(n1) and the internal vertices of T2(n2). Thus, T now has 1 + n1 + n2 internal vertices. Furthermore, T's leaves include leaves of T1 and T2. Therefore, T has 2 + n1 + n2 leaves.

2 + n1 + n2 has one more than 1 + n1 + n2. Therefore, the full binary tree has 1 more leaf than its internal vertices. P(k+1) is proved.

Conclusion:

As the basis step and the inductive step are true.

By Strong Induction, P(n) is true for all positive integers.

Therefore, there are n leaves on a binary tree T, and there are n-1 internal vertices of T.

**Problem 2 (Summation)** *[15 marks]* Use mathematical induction to show that

$$\Sigma_{j=0}^{2n}(2j+1) = (2n+1)^2,$$

for all positive integers $n$. Provide detailed justifications for your answer.

Let P(n) represent that it is true that $\sum^{2n}_{j=0} (2j+1) = (2n+1)^2$ for the integer n.

Here is a "proof" that P(n) is true for all positive integers.

Basis step:

P (1) is true, because:

$\sum^{2}_{j=0} (2j+1) = (2\cdot0+1) + (2\cdot1+1) + (2\cdot2+1) = 1 + 3 + 5 = 9$

Which $(2n+1)^2 = 3^2 = 9$

Inductive step:

The inductive hypothesis is: P(k) is true for an arbitrary fixed integer $k \geq 1$.

Assume that the inductive hypothesis P(k) is true, we need to proof:

$\sum^{2(k+1)}_{j=0} (2j+1) = [2 \cdot (k+1) +1]^2$

$\sum^{2k+2}_{j=0} (2j+1) = (2k+3)^2 = 4k^2 + 12k + 9$

$= 4n^2 + 4n + 4n + 4n + (1+3+5)$

$= (2k+1)^2 + [2(2k+1) +1] + [2(2k+2) +1]$

Under this assumption, as p(k) is true:

$\sum^{2k}_{j=0}(2j+1) = (2k+1)^2$

we add $[2(2k+1) +1] + [2(2k+2) +1]$ to both side of the equation, we are getting:

$\sum^{2k}_{j=0}(2j+1) + [2(2k+1) +1] + [2(2k+2) +1] = (2k+1)^2 + [2(2k+1) +1] + [2(2k+2) +1]$

Which is the same as:

$\sum^{2k+2}_{j=0}(2j+1) = (2k+1)^2 + [2(2k+1) +1] + [2(2k+2) +1]$

$\sum^{2(k+1)}_{j=0} (2j+1) = [2 \cdot (k+1) +1]^2$

Therefore, this complete the induction step.

$P(k) \rightarrow P(k+1)$ is true for all positive integers k.

Conclusion:

As the basis step and the inductive step are true.

by mathematical induction, p(n) is true for all positive integers.

Therefore, $\sum^{2n}_{j=0}(2j+1) = (2n+1)^2$ for all positive integers.

**Problem 3 (Counting binary strings)** *[20 marks]* Consider all bit strings of length 15.

1. How many begin with 00?
2. How many begin with 00 and end with 11?
3. How many begin with 00 or end with 10?
4. How many have exactly ten 1's?
5. How many have exactly ten 1's such as none of these 1's are adjacent to each other?

Provide detailed justifications for your answers.

1. How many begin with 00

   See 00 as fixed, and we have 13 positions remaining. Each position can either be 0 or 1, which means there are 2 possibilities per position. Therefore, by product rule there are $2^{13}$ different strings that begin with 00.

2. How many begin with 00 and end with 11?

   00 and 11 are fixed therefore, in the middle we have 11 positions that are needed to be fill with either 0 or 1. Each position has two possibilities, therefore by product rule there are $2^{11}$ different strings that begin with 00 and end with 11.

3. How many begin with 00 or end with 10?

   A: string that begin with 00

   B: string that end with 10

   The question is: A ∨ B, which "or" means Union

   $|A \cup B| = |A| + |B| - |A \cap B|$

   $= 2^{13} + 2^{13} - |A \cap B|$

   $|A \cap B|$ is the string that begin with 00 and end with 10

   00 and 10 are fixed, then each remaining 11 positions has 2 possibilities.

   Therefore, by product rule $|A \cap B|$ is $2^{11}$

   $|A \cup B| = 2^{13} + 2^{13} - 2^{11} = 2^{14} - 2^{11}$

4. How many have exactly ten 1's?

   Does not matter they are adjacent or not. Therefore, we only need to choose 10 space that are available for 1. Therefore, the answer is C (10,15), which is 3003 strings.

5. How many have exactly ten 1 that are none - adjacent to each other

   Each one of 1 has to separate by 0 zero. Ten 1 need 9 zero to separate them, and there are only 15 space available. Therefore, the probability of having ten non-adjacent 1 is 0, and there are 0 strings that satisfy the condition.

**Problem 4 (Counting permutations)** *[20 marks]* Solve the following count-ing problems:

1. How many permutations of the eight letters $A, B, C, D, E, F, G, H$ have $A$ in the second position?
2. How many permutations of the eight letters $A, B, C, D, E, F, G, H$ have $A$ in one of the first two positions?
3. How many permutations of the eight letters $A, B, C, D, E, F, G, H$ have the two vowels after the six consonants?
4. How many permutations of the eight letters $A, B, C, D, E, F, G, H$ neither begin nor end with $D$?
5. How many permutations of the eight letters $A, B, C, D, E, F, G, H$ do not have the vowels next to each other?

Provide detailed justifications for your answers.

1. How many permutations of the eight letters ABCDEFGH have A in the second position?

| 7 | A | 6 | 5 | 4 | 3 | 2 | 1 |
|---|---|---|---|---|---|---|---|

A is fixed. Every other position is minus one after one of them is selected.

Therefore, the permutations of this conditions are 7! = 5040

2. How many permutations of the 8 letters ABCDEFGH have A in one of the first two positions?

Either A is in col1 or in col2

If A is in col1 the permutation is 7!

If B is in col2 the permutations is 7! Too

Therefore, the total permutations of this condition are 2*7! = 10080

3. How many permutations of the 8 letters … have the 2 vowels after the six consonants?

Vowels: AE

Consonants: BCDFGH

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 1 | 2 |

Either A or E in the last two columns.

The permutations are 6! * 2 = 1440

4. How many permutations of the eight letters … neither begin nor end with D?

a) D cannot be at the first position and the last position.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 7 | 6 | 5 | 4 | 3 | 2 | 1 | 6 |

There are 7 elements other than D that can fit in the first position, and 6 of them are ready to set in the last.

The answer is: 7! * 6 = 5040 * 5 = 30240

5. How many permutations of the 8 letters … do not have the vowels next to each other?

All possible permutation minus the permutation that they are next to each other is the target permutation.

All possible permutation: 8!

The permutation that two vowels are next to each other:

AE is fixed, see them as one. Therefore, is 7!

EA is fixed, see them as one. Therefore is 7!

The total is 2*7!

8! – 2*7! = 40320 – 10080 = 30240

**Problem 5 (Counting triominos)** *[20 marks]* We saw in class that every $2^n \times 2^n$ board, with one square removed, could be covered with triominos. Determine a formula counting the number of triominos covering such a truncated $2^n \times 2^n$ board. Prove this formula by induction.

By observation, the number of triominos = $(2^n * 2^n) – 1 / 3$ for $n \geq 1$

Let T(n) represent the number# of triominos covering a $2^n * 2^n$ with one square removed.

Let P(n) denotes that $T(n) = (2^n * 2^n – 1) / 3$ which $n \geq 1$

Basis Step:

P (1) is true since $(2^n * 2^n – 1) \bmod 3 = 3 \bmod 3 = 0$

Inductive Step:

The inductive hypothesis is: P(k) is true for an arbitrary fixed integer $k \geq 1$.

Assume that the inductive hypothesis P (k) is true, we need to proof P(k+1):

Since by the inductive hypothesis, $0 = 2^k * 2^k – 1 \bmod 3$, it follows that:

$0 = 2^{k+1} * 2^{k+1} – 1 \bmod 3$

$= 2^k * 2^k * 2 * 2 -1 \bmod 3$

$= 4 * (2^k * 2^k) -1 \bmod 3$

$= 4 * (2^k * 2^k - 1) + 3 \bmod 3 = 0$

$2^k * 2^k – 1 \bmod 3$ is equal to $4 * (2^k * 2^k - 1) \bmod 3$.

Therefore, this complete the induction step.

P (k)→ P (k + 1) is true for all positive integers k.

Conclusion:

As the basis step and the inductive step are true.

By mathematical induction, P(n) is true for all $n \geq 1$.

Therefore, $T(n) = (2^n * 2^n – 1) / 3$, where T(n) is the number of triominos covering a $2^n * 2^n$ with one square removed.