UWO CS2214 Mar. 18, 2019

Assignment #3
Due: Mar. 26, 2017, by 23:55
Submission: on the OWL web site of the course

Format of the submission. You must submit a **single** file which must be in **PDF** format. All other formats (text or Miscrosoft word format) will be **ignored** and considered as **null**. You are strongly encouraged to type your solutions using a text editor. To this end, we suggest the following options:

1. Miscrosoft word and convert your document to PDF
2. the typesetting system LaTeX; see `https://www.latex-project.org/` and `https://en.wikipedia.org/wiki/LaTeX#Example` to learn about LaTeX; see `https://www.tug.org/begin.html` to get started
3. using a software tool for typing mathematical symbols, for instance `http://math.typeit.org/`
4. using a Handwriting recognition system such as those equipping tablet PCs

Hand-writing and scanning your answers is allowed but not encouraged:

1. if you go this route please use a scanning printer and **do not take a picture of your answers with your phone**,
2. if the quality of the obtained PDF is too poor, your submission will be **ignored** and considered as **null**.

Problem 1 (Counting tree leaves) *[25 marks]* The set of leaves and the set of internal vertices of a full binary tree are defined recursively as follows:

Basis step: The root r is a leaf of the full binary tree with exactly one vertex r. This tree has no internal vertices.

Recursive step: The set of leaves of the tree $T = T_1 \cdot T_2$ is the union of the sets of leaves of T_1 and T_2. The internal vertices of T are the root r of T and the union of the set of internal vertices of T_1 and the set of internal vertices f T_2.

Use structural induction to prove that $\ell(T)$, the number of leaves of a full binary tree T, is 1 more than $i(T)$, the number of internal vertices of T.

Solution 1 We shall prove that, for an arbitrary full binary tree T, its number of leaves $\ell(T)$ satisfies the property $\mathcal{P}(T)$ below:

$$\ell(T) = i(T) - 1.$$

Basis step: The root r is a leaf and has no internal vertices, that is, $\ell(T) = 1$ and $i(T) = 0$, hence it satisfies $\ell(T) = i(T) - 1$.

Recursive step: Let $T = T_1 \cdot T_2$ be a full binary tree built from two full binary trees T_1, T_2. We shall prove that if $\mathcal{P}(T_1)$ and $\mathcal{P}(T_2)$ both hold, then so does $\mathcal{P}(T)$. So, let us assume that $\mathcal{P}(T_1)$ and $\mathcal{P}(T_2)$ both hold. By definition of $\ell(T)$, we have:

$$\ell(T) = \ell(T_1) + \ell(T_2).$$

By induction hypothesis, we have:

$$\ell(T_1) = i(T_1) - 1 \quad \text{nad} \quad \ell(T_2) = i(T_2) - 1$$

By definition of $i(T)$, we have:

$$i(T) = i(T_1) + i(T_2) + 1$$

Putting everything together:

$$\begin{array}{rcl} \ell(T) & = & \ell(T_1) + \ell(T_2) \\ & = & i(T_1) - 1 + i(T_2) - 1 \\ & = & i(T) - 1. \end{array}$$

Hence, we have proved that $\mathcal{P}(T)$ holds.

Therefore, we have proved by induction that for all binary trees we have the number of leaves is 1 more than the number of internal vertices.

Problem 2 (Summation) *[15 marks]* Use mathematical induction to show that

$$\Sigma_{j=0}^{2n}(2j+1) = (2\,n+1)^2,$$

for all positive integers n. Provide detailed justifications for your answer.

Solution 2 We shall prove for an arbitrary positive integer n the property $P(n)$ below holds:

$$\Sigma_{j=0}^{2n}(2j+1) = (2\,n+1)^2.$$

Basis step: For $n = 1$, we have

$$\Sigma_{j=0}^{2}(2j+1) = 1+3+5 = 9 = (2+1)^2.$$

Hence the property $P(n)$ holds for $n = 1$.

Recursive step: Let us prove that for all $k \geq 1$ if $P(k)$ holds then so does $P(k+1)$. So let $k \geq 1$, let assume that $P(k)$ holds, that is,

$$\Sigma_{j=0}^{2k}(2j+1) = (2\,k+1)^2,$$

and let us prove that $P(k+1)$ holds as well, that is:

$$\Sigma_{j=0}^{2k+2}(2j+1) = (2\,k+3)^2,$$

We have:

$$\begin{array}{rcl} \Sigma_{j=0}^{2(k+1)}(2j+1) & = & \Sigma_{j=0}^{2k}(2j+1) \quad + \quad 2(2k+1)+1 \quad + \quad 2(2k+2)+1 \\ & = & (2\,k+1)^2 + 8k + 8. \end{array}$$

Since $(2\,k+3)^2 - (2\,k+1)^2 = 8k+8$, we deduce that $P(k+1)$ holds indeed.

Therefore, we have proved by induction that for all positive integer n, the property $P(n)$ holds.

Problem 3 (Counting binary strings) *[20 marks]* Consider all bit strings of length 15.

1. How many begin with 00?
2. How many begin with 00 and end with 11?
3. How many begin with 00 or end with 10?
4. How many have exactly ten 1's?
5. How many have exactly ten 1's such as none of these 1's are adjacent to each other?

Provide detailed justifications for your answers.

Solution 3 For every bit string $b_1 b_2 \cdots b_{15}$ each of the bits $b_1, b_2, \ldots, b_{15}$ can take two values, namely 0 or 1. Applying the product rule, the sum rule and the subtraction rule,

1. the number of bit strings $b_1 b_2 \cdots b_{15}$ beginning with 00 is 2^{13},
2. the number of bit strings $b_1 b_2 \cdots b_{15}$ beginning with 00 and ending with 11 is 2^{11},
3. the number of bit strings $b_1 b_2 \cdots b_{15}$ beginning with 00 or ending with 10 is $2^{13} + 2^{13} - 2^{11}$,

4. the number of bit strings $b_1 b_2 \cdots b_{15}$ with exactly ten 1's is $\binom{15}{10}$, that is, the number of ways of choosing 10 bits among $b_1, b_2, \ldots, b_{15}$,
5. the number of bit strings $b_1 b_2 \cdots b_{15}$ having exactly ten 1's such as none of these 1's are adjacent to each other is zero. Indeed, in order to separate each of these ten 1's from the others, we would need (at least) nine 0's.

Problem 4 (Counting permutations) *[20 marks]* Solve the following counting problems:

1. How many permutations of the eight letters A, B, C, D, E, F, G, H have A in the second position?
2. How many permutations of the eight letters A, B, C, D, E, F, G, H have A in one of the first two positions?
3. How many permutations of the eight letters A, B, C, D, E, F, G, H have the two vowels after the six consonants?
4. How many permutations of the eight letters A, B, C, D, E, F, G, H neither begin nor end with D?
5. How many permutations of the eight letters A, B, C, D, E, F, G, H do not have the vowels next to each other?

Provide detailed justifications for your answer.

Solution 4

1. Choose a letter to be the first one and then choose a permutation of the remaining six: $7 \times 6!$.
2. Choose where to place A, then choose a permutation of the remaining seven: $2 \times 7!$.
3. Choose a permutation of the consonants, then a choose a permutation of the vowels: $6! \times 2!$.
4. Choose a place for D, then choose a permutation of the remaining seven: $6 \times 7!$.
5. $7 \times 2! \times 6!$ do have the vowels next to each other, so $8! - 7 \times 2! \times 6!$ do not have the vowels next to each other.

Problem 5 (Counting triominos) *[20 marks]* We saw in class that every $2^n \times 2^n$ board, with one square removed, could be covered with triominos. Determine a formula counting the number of triominos covering such a truncated $2^n \times 2^n$ board. Prove this formula by induction.

Solution 5

Basis step: if $n = 1$, then $2^n \times 2^n - 1 = 3$ and a single triomino suffices

Recursive step: Let $t(n)$ be the number of triomino needed to cover a truncated $2^n \times 2^n$ board. We want to express $t(n+1)$ as a function of $t(n)$. So, consider a truncated $2^{n+1} \times 2^{n+1}$ board. Removing one square from one the four quadrants and removing three squares forming a triomino from the other three yields:

$$t(n+1) = 4t(n) + 1.$$

This suggests:

$$t(n) = \frac{4^n - 1}{3},$$

which is easy to verify by induction.