Derivative of the Exponential Map

We want to find the derivative of the rotation matrix $R(v) = \exp(v_\times)$ with respect to the rotation vector v. The result will be a 3rd order tensor. To avoid dealing with tensor notation, we will look at the effect the rotation has on an arbitrary vector X: $F(v) = R(v)X$. The function F is a function from a vector space to a vector space, and hence its derivative is a 3×3 matrix.

To calculate $\mathrm{D}_v F$ we use a trick where we replace a function by the integral of its derivative with respect to an auxiliary variable s:

$$\begin{aligned}
\mathrm{D}_v F &= \exp(v_\times) \left(\exp(-v_\times) \mathrm{D}_v \exp(v_\times) X\right) \\
&= \exp(v_\times) \int_0^1 \frac{d}{ds} \left[\exp(-s v_\times) \mathrm{D}_v \exp(s v_\times) X\right] ds
\end{aligned}$$

we apply the product rule to the derivative with respect to s:

$$= \exp(v_\times) \int_0^1 \left[-v_\times \exp(-s v_\times) \mathrm{D}_v \exp(s v_\times) X + \exp(-s v_\times) \mathrm{D}_v \left(v_\times \exp(s v_\times) X\right)\right] ds$$

and the product rule to the derivative with respect to v:

$$= \exp(v_\times) \int_0^1 \exp(-s v_\times) \left(\mathrm{D}_v v_\times\right) \exp(s v_\times) X ds$$

At this point we need to calculate the derivative of $v_\times$: for any vector Z, we have

$$(\mathrm{D}_v v_\times) Z = \mathrm{D}_v (v \times Z) = -\mathrm{D}_v (Z_\times v) = Z_\times \mathrm{D}_v v = -Z_\times$$

Applying this to to the derivative of F:

$$\begin{aligned}
\mathrm{D}_v F &= -\exp(v_\times) \int_0^1 \exp(-s v_\times) \left(\exp(s v_\times) X\right)_\times ds \\
&= -\int_0^1 \exp((1-s) v_\times) \left(\exp(s v_\times) X\right)_\times ds
\end{aligned}$$

For any rotation matrix R, we have $R Z_\times = (RZ)_\times R$, and so:

$$\begin{aligned}
&= -\left(\exp(v_\times) X\right)_\times \int_0^1 \exp((1-s) v_\times) ds \\
&= -\left(R(v) X\right)_\times \int_0^1 R(tv) dt \\
&= -\left(R(v) X\right)_\times T(v)
\end{aligned}$$

The integral $T(v) = \int_0^1 \exp(s v_\times) ds$ can be computed using the Rodrigues' formula:

$$\begin{aligned}
T(v) &= \int_0^1 \exp(s v_\times) ds \\
&= \int_0^1 \left(\mathrm{Id} + \frac{\sin(sa)}{sa} s v_\times + \frac{1-\cos(sa)}{(sa)^2} (s v_\times)^2\right) ds \\
&= \int_0^1 \left(\mathrm{Id} + \frac{\sin(sa)}{a} v_\times + \frac{1-\cos(sa)}{a^2} v_\times^2\right) ds \\
&= \mathrm{Id} + \frac{1-\cos(a)}{a^2} v_\times + \frac{1}{a^2} \left(1 - \frac{\sin(a)}{a}\right) v_\times^2
\end{aligned}$$