Integration by partial functions

Case 1 :

Ex

Ex 2/ $\int \frac{x+1}{x^3+x^2-6x}\,dx$

Soln/ $x^3+x^2-6x = x(x^2+x-6)$
$= x(x-2)(x+3)$

$$\frac{x+1}{x(x-2)(x+3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$$

$$= \frac{A(x-2)(x+3) + Bx(x+3) + Cx(x-2)}{x(x-2)(x+3)}$$

$\therefore\ x+1 = A(x-2)(x+3) + \overset{Bx}{\cancel{Bx}}(x+3) + Cx(x-2)$

$x=0$: $1 = A(-2)(3) + 0 + 0 \Rightarrow A = -\frac{1}{6}$

$x=2$: $\cancel{1 = A(-2)(3) + 0 +}$
$3 = B(2)(5) \Rightarrow B = \frac{3}{10}$

$x=-3$: $-2 = 0 + 0 + C(-3)(-5) \Rightarrow C = \frac{2}{15}$

$$\int \frac{x+1}{x^3+x^2-6x}\,dx = -\frac{1}{6}\int\frac{dx}{x} + \frac{3}{10}\int\frac{dx}{x-2} + \frac{2}{15}\int\frac{dx}{x+3}$$

$$= -\frac{1}{6}\ln x + \frac{3}{10}\ln(x-2) + \frac{2}{15}\ln(x+3) + C$$

Case 2: Repeated linear factors.

$$\frac{1}{(ax+b)^n} = \frac{A}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} \dots \frac{A_n}{(ax+b)^n}$$

Ex/ find $\int \frac{3x+5}{x^3-x^2-x+1}\,dx$

Soln/ $x^3-x^2-x+1 =$

$$\begin{array}{r|l} & x^2 - 1 \\ x-1 & x^3 - x^2 - x + 1 \\ - & x^3 - x^2 \\ \hline & -x+1 \\ & -x+1 \\ \hline & \end{array}$$

$x^3-x^2-x+1 = (x-1)(x^2-1)$
$= (x-1)(x-1)(x+1)$
$= (x-1)^2(x+1)$

$$\frac{3x+5}{(x+1)(x-1)^2} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$= \frac{A(x-1)^2 + B(x-1)(x+1) + C(x+1)}{(x+1)(x-1)^2}$$

$$2x+5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

$x=1$: $8 = 0 + 0 + C(2) \Rightarrow C = 4$

$x=-1$: $2 = A(4) + 0 + 0 \Rightarrow A = \frac{1}{2}$

No other values;

$x=0$: $5 = \frac{1}{2} + B(-1) + 4(1)$

$\Rightarrow +\frac{1}{2} = -B$

$B = -1/2$

$$\int \frac{3x+5}{(x+1)(x-1)^2}\,dx = \int \frac{\frac{1}{2}}{x+1}\,dx - \frac{1}{2}\int \frac{dx}{x-1} + \int \frac{4}{(x-1)^2}\,dx$$

$$= \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x-1) - \frac{4}{x-1} + C$$

Ex/ $\displaystyle\int \frac{x^4 - x^3 - x - 1}{x^3 - x^2}$

$$\begin{array}{r|l} & x \\ \hline x^3 - x^2 & x^4 - x^3 - x - 1 \\ - & x^4 - x^3 \\ \hline & -x - 1 \end{array}$$

$$= x - \frac{(x+1)}{(x^3 - x^2)}$$

$$x^3 - x^2 = x^2(x-1)$$

$$\frac{x+1}{x^3 - x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$x + 1 = A(x)(x-1) + B(x-1) + C(x^2)$$

$x = 0$: $1 = -B$ $\therefore B = -1$

$x = 1$: $2 = 0 + 0 + C$ $\therefore C = 2$

$x = -1$: $A(-1)(-2) + -1(-2) + \overset{2}{C}(1)$

$= 2A + 4$ $\therefore A = -2$

$$\int \frac{x+1}{x^3 - x^2} dx = \int \frac{-2}{x} dx - \int \frac{dx}{x^2} + \int \frac{2}{x-1} dx$$

$$= -2\ln x + \frac{1}{x} + 2\ln(x-1) + C$$

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case 3: Distinct quadratic factors

$\cancel{case}$ $\dfrac{1}{ax^2+bx+c} \longrightarrow \dfrac{Ax+b}{ax^2+bx+c}$

Ex/ Find: $\displaystyle\int \frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2}$

$x^4 + 3x^2 + 2 \Rightarrow (x^2)^2 + 3x^2 + 2$

$\boxed{\text{for } y = x^2}$ $\Rightarrow y^2 + 3y + 2$

$(y+1)(y+2)$

$(x^2+1)(x^2+2)$

$\therefore x^4 + 3x^2 + 2 = (x^2+1)(x^2+2)$

$$\frac{x^3 + x^2 + x + 2}{(x^2+1)(x^2+2)} = \frac{Ax + B}{x^2+1} + \frac{Cx + D}{x^2+2}$$

$x^3 + x^2 + x + 2 = (Ax + B)(x^2 + 2) + (Cx + D)(x^2 + 1)$

$= Ax^3 + 2Ax + Bx^2 + 2B + Cx^3 + Cx + Dx^2 + D$

like terms

$= (A + C)x^3 + (B + D)x^2 + (2A + C)x + (2B + D)$

this x^3 : $1 = A + C$ $\Rightarrow C = 1$

minus x^2 : $1 = B + D$ $\quad 0 = A + 0 \Rightarrow A = 0$

this x : $1 = 2A + C$ $\quad 1 = B + 0 \Rightarrow B = 1$

x^0 : $2 = 2B + D$ $\quad \Rightarrow D = 0$

$$\int \frac{x^3+x^2+x+2}{(x^2+1)(x^2+2)}\,dx = \int \frac{1}{x^2+1}\,dx + \int \frac{x}{x^2+2}\,dx$$

$$= \tan^{-1} x$$

Ex/ ⓧ $\displaystyle\int \frac{x}{(x-1)(x^2+4)}\,dx$ $\left[\text{Ans: } \frac{1}{10} \ln\left[\frac{(x-1)^2}{x^2+4}\right] + \frac{2}{5} \tan^{-1}\left(\frac{x}{2}\right) + c\right.$

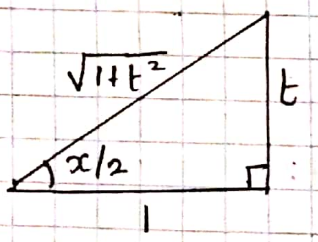
Case 4: Repeated quadratic factors:

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2x}{(ax^2+bx+c)^2} + \dots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Ex $\displaystyle\int \frac{2x^2+3}{(x^2+1)^2}\,dx$

The t-method

Rational expressions of $\sin x$ and $\cos x$ can be transformed to rational expressions of t by the change of variables:
$t = \tan\left(\frac{x}{2}\right)$



$\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$ and $\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$

so $\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$

$1 - \cos 2\theta = 2\sin\theta$
$1 - 2\sin^2\theta = \cos 2\theta$
Remember
$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$1 = \cos^2\theta + \sin^2\theta$

~~and~~ so

$$\cos x = 1 - \sin^2\left(\frac{x}{2}\right)$$

$$= 1 - \frac{2t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$$

and

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2 \cdot \frac{t}{\sqrt{1+t^2}} \Rightarrow \frac{2t}{1+t^2}$$

$$dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\left(1+\tan^2(x/2)\right)dx$$

$$= \frac{1}{2}(1+t^2)\,dx$$

$$\frac{2dt}{1+t^2} = dx$$

Ex/ Find $\int \frac{1}{1+\sin x}\,dx$

let $t = \tan\frac{x}{2}$ $\quad dx = \frac{2dt}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$

$$\int \frac{1}{1+\sin x}\,dx = \int \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$\int \frac{2\,du}{u^2}$

$$= \int \frac{2\,dt}{1+t^2+2t} = \int \frac{2dt}{t^2+2t+1} = \int \frac{2dt}{(t+1)^2} = \frac{-2}{t+1} + C$$

$$= \frac{-2}{\tan(x/2)+1} + C$$

Ex/ $\int \frac{1}{1+\cos x}\,dx$ $\quad \boxed{\text{Ans: } \tan(\tfrac{x}{2}) + C}$

Ex/ $\int \frac{1}{5+3\cos x}\,dx$ $\quad \boxed{\text{Ans: } \tfrac{1}{2}\tan^{-1}\left[\tfrac{1}{2}\tan x/2\right] + C}$

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First order ordinary differential equations

A differential equation is a relationship between an independent variable x and an independent variable y and one or more differential coefficient of y with respect to x

eg. $x^2 \frac{dy}{dx} + y \sin x = 0$

$$xy \frac{d^2y}{dx^2} + y\frac{dy}{dx} + e^x = 0$$

Solutions of differential equations

To solve a differential equation we have to find, defined a function for which the equation is true. This means that we have to manipulate the equations so as to eliminate all the differential coefficients and have a relationship between y and x

m1: by direct integration

If the equation can be arranged in the form $\frac{dy}{dx} = f(x)$ then it can be solved by direct integration

Ex/ $\frac{dy}{dx} = 3x^2 - 6x + 5$

Soln/ $\int \frac{dy}{dx}\, dx = \int (3x^2 - 6x + 5)\, dx$

$\int dy = x^3 - 3x^2 + 5x + C$

$y = x^3 - 3x^2 + 5x + C$

Ex/ Solve $x\frac{dy}{dx} = 5x^3 + 4$

Sol/ $\frac{dy}{dx} = \frac{5x^3 + 4}{x} = 5x^2 + 4/x$

$\int \frac{dy}{dx}\, dx = \int \left(5x^2 + \frac{4}{x}\right) dx$

$y = \frac{5x^3}{3} + 4\ln x + C$

M2 : By separating the variables

If the given equation is of the form $dy/dx = f(x, y)$ the variable y on the right hand side prevent solving by direct integration.

Ex/ Solve $\frac{dy}{dx} = \frac{2x}{y+1}$

Soln/ $dy(y+1) = dx(2x)$

$$\int (1+y)dy = \int 2x\,dx$$

$$y + \frac{y^2}{2} = x^2 + C$$

Ex/ Solve $\frac{dy}{dx} = (1+x)(1+y) = \frac{(1+x)}{(1+y)^{-1}}$, dy

$$\frac{dy}{dx} = \frac{(1+x)}{(1+y)^{-1}} \qquad dy(1+y)^{-1} = dx(1+x)$$

$$\Rightarrow \frac{dy}{1+y} = dx(1+x)$$

$$\int \frac{dy}{1+y} = \int dx(1+x)$$

$$\ln(1+y) = x + \frac{x^2}{2} + C$$

Ex/ $\frac{dy}{dx} = \frac{y^2 - xy^2}{x^2y - x^2}$

Soln : $\frac{dy}{dx} = \frac{y^2(1-x)}{x^2(y-1)} = \frac{y^2}{y-1} \cdot \frac{1-x}{x^2}$

$$\frac{y-1}{y^2}dy = \frac{1-x}{x^2}dx$$

$$\int \left(\frac{1}{y} - \frac{1}{y^2}\right)dy = \int \left(\frac{1}{x^2} - \frac{1}{x}\right)dx$$

$$\ln y + \frac{1}{y} = -\frac{1}{x} - \ln x + C$$