Solutions of the exercises on Propositional and Predicate Logic

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Exercises on slide 20

Exercise 1

Show $[p \wedge (p \to q)] \to q$ is a tautology.

Solution

Let us make a truth table for this proposition:

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

All truth values of $[p \wedge (p \to q)] \to q$ in the truth table are true no matter what the truth values of its simple components. So this proposition is a tautology by definition.

Exercise 2

Show $(p \to q) \leftrightarrow (\bar{q} \to \bar{p})$ is a tautology.

Solution

Let us make a truth table for this proposition:

p	q	$\bar{q}$	$\bar{p}$	$p \to q$	$\bar{q} \to \bar{p}$	$(p \to q) \leftrightarrow (\bar{q} \to \bar{p})$
T	T	F	F	T	T	T
T	F	T	F	F	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T

All truth values of $(p \to q) \leftrightarrow (\bar{q} \to \bar{p})$ in the truth table are true no matter what the truth values of its simple components. So this proposition is a tautology by definition.

Exercise 3

Show $[\bar{q} \wedge (p \rightarrow q)] \rightarrow \bar{p}$ is a tautology.

Solution

Let us make a truth table for this proposition:

p	q	$\bar{q}$	$\bar{p}$	$p \rightarrow q$	$\bar{q} \wedge (p \rightarrow q)$	$[\bar{q} \wedge (p \rightarrow q)] \rightarrow \bar{p}$
T	T	F	F	T	F	T
T	F	T	F	F	F	T
F	T	F	T	T	F	T
F	F	T	T	T	T	T

All truth values of $[\bar{q} \wedge (p \rightarrow q)] \rightarrow \bar{p}$ in the truth table are true no matter what the truth values of its simple components. So this proposition is a tautology by definition.

Exercise 4

Why can no simple proposition be a tautology?

Solution

It is because a simple proposition is a declarative statement which is either true or false by definition, so it is not necessarily always true. For example a simple proposition *Sun is shining* is not always true.

Exercise on slide 25

What about the correctness of the argument $(p \rightarrow q) \wedge (r \rightarrow \bar{p}) \wedge r \vdash \bar{q}$?

Solution

Let us try to use inference rules:
1. r (premise)
2. $r \rightarrow \bar{p}$ (premise)
3. $\bar{p}$ (from 1 and 2 using Modus Ponens)
4. ?

Further application of the inference rules will not prove the correctness of the argument. So let us make a truth table for $((p \rightarrow q) \wedge (r \rightarrow \bar{p}) \wedge r) \rightarrow \bar{q}$

p	q	r	$p \rightarrow q$	$\bar{p}$	$r \rightarrow \bar{p}$	$(p \rightarrow q) \wedge (r \rightarrow \bar{p}) \wedge r$	$\bar{q}$	$((p \rightarrow r) \wedge (r \rightarrow \bar{p}) \wedge r) \rightarrow \bar{q}$
T	T	T	T	F	F	F	F	T
T	T	F	T	F	T	F	F	T
T	F	T	F	F	F	F	T	T
T	F	F	F	F	T	F	T	T
F	T	T	T	T	T	T	F	F
F	T	F	T	T	T	F	F	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	F	T	T

As it follows from the truth table, $((p \to q) \wedge (r \to \bar{p}) \wedge r) \to \bar{q}$ is not a tautology, so the argument $(p \to q) \wedge (r \to \bar{p}) \wedge r \vdash \bar{q}$ is not valid. In particular a counter example for it is when p, q and r are false, true and true correspondently.

Exercises on slide 34

Exercise 1

Translate the following into symbolic form:
(i) Everybody likes him
(ii) Somebody cried out for help and called the police
(iii) Nobody can ignore her

Solution

(i) $(\forall x)L(x)$, where $L(x)$ - x *likes him*.
(ii) $(\exists x)[H(x) \wedge P(x)]$, where $H(x)$ - x *cried out for help* and $P(x)$ - x *called the police*
(iii) $\sim (\exists x)I(x)$ or $(\forall x)[\sim I(x)]$, where $I(x)$ - x *can ignore her*
UoD for these examples are all human beings.

Exercise 2

Find an UoD and two unary predicates $P(x)$ and $Q(x)$ such that $(\forall x)[P(x) \to Q(x)]$.

Solution

UoB - all human beings.
$P(x)$ - x *is a student* and $Q(x)$ - x *is intelligent*. Whenever a human being is a student, he is intelligent.

Exercise 3

Find an UoD and two unary predicates $P(x)$ and $Q(x)$ such that $(\exists x)[P(x) \wedge Q(x)]$ is false but $(\exists x)P(x) \wedge (\exists x)Q(x)$ is true.

Solutions proposed by students

a) UoB - all human beings.
$P(x)$ - x *has blue eyes* and $Q(x)$ - x *has black eyes*.
There exist people with blue eyes and with black eyes, but one cannot have blue and black eyes at the same time.

b) UoB - all cars.
$P(x)$ - x *has 4 wheels* and $Q(x)$ - x *has 6 wheels*.

c) UoB - all integers.
$P(x)$ - $x > 5$ and $Q(x)$ - $x < 3$.

Exercise 4

Show that $(\forall x)P(x) \vdash (\exists x)P(x)$

Solution

1. $(\forall x)P(x)$ (premise)
2. $P(a)$ for some a from UoD (from 1 using Universal Specification)
3. $(\exists x)P(x)$ (from 2 using Existential Generalisation)

Exercise 5

Given the premises $(\exists x)P(x)$ and $(\forall x)[P(x) \to Q(x)]$ give a series of steps concluding that $(\exists x)Q(x)$

Solution

1. $(\exists x)P(x)$ (premise)
2. $P(a)$ for some a from UoD (from 1 using Existential Specification)
3. $(\forall x)[P(x) \to Q(x)]$ (premise)
4. $P(a) \to Q(a)$ for some a from UoD (from 3 using Universal Specification)
5. $Q(a)$ for some a from UoD (from 2 and 4 using Modus Ponens)
6. $(\exists x)Q(x)$ (from 5 using Existential Generalisation).

Exercise on slide 37

Show $\sim (\forall x)(\exists y)P(x, y) \equiv (\exists x)(\forall y)[\sim P(x, y)]$ using the logical equivalence above and using the fact that logically equivalent propositions can be interchanged in a compound proposition

Solution

$\sim (\forall x)(\exists y)P(x, y) \equiv (\exists x)(\forall y)[\sim P(x, y)]$ if and only if $\sim (\forall x)(\exists y)P(x, y) \leftrightarrow (\exists x)(\forall y)[\sim P(x, y)]$ is a tautology. And $\sim (\forall x)(\exists y)P(x, y) \leftrightarrow (\exists x)(\forall y)[\sim P(x, y)]$ is a tautology if and only if $\sim (\forall x)(\exists y)P(x, y) \to (\exists x)(\forall y)[\sim P(x, y)]$ and $(\exists x)(\forall y)[\sim P(x, y)] \to [\sim (\forall x)(\exists y)P(x, y)]$ are tautologies.

We will use the facts that $\sim (\forall x)F(x) \equiv (\exists x)[\sim F(x)]$ and $\sim (\exists x)F(x) \equiv (\forall x)[\sim F(x)]$ for any $F(x)$.

Let us show that whenever $\sim (\forall x)(\exists y)P(x, y)$ is true then $(\exists x)(\forall y)[\sim P(x, y)]$ is true.

Let $\sim (\forall x)(\exists y)P(x,y)$ is true, and let $Q(x) = (\exists y)P(x,y)$ then $\sim (\forall x)Q(x)$ is true and $\sim (\forall x)Q(x) \equiv \exists x[\sim Q(x)]$, so $\exists x[\sim Q(x)]$ is true.

Using Existential Specification if $\exists x[\sim Q(x)]$ is true, then $\sim Q(a)$ is true for some a in UoD.

But $\sim Q(a) =\sim (\exists y)P(a,y)$ and $\sim (\exists y)P(a,y) \equiv (\forall y)[\sim P(a,y)]$, so $(\forall y)[\sim P(a,y)]$ is true for some a in UoD. Using Existential Generalisation $(\exists x)(\forall y)[\sim P(x,y)]$ is true.

So $\sim (\forall x)(\exists y)P(x,y) \rightarrow (\exists x)(\forall y)[\sim P(x,y)]$ is a tautology.

And let us show that whenever $(\exists x)(\forall y)[\sim P(x,y)]$ is true then $\sim (\forall x)(\exists y)P(x,y)$ is true.

Let $(\exists x)(\forall y)[\sim P(x,y)]$ is true, and let $Q(x) = (\forall y)[\sim P(x,y)]$, so $(\exists x)Q(x)$ is true. Then using Existential Specification $Q(a)$ is true for some a in UoD.

$Q(a) = (\forall y)[\sim P(a,y)]$ and $(\forall y)[\sim P(a,y)] \equiv\sim (\exists y)P(a,y)$, so $\sim (\exists y)P(a,y)$ is true for some a in UoD.

Let $L(a) = (\exists y)P(a,y)$, then $\sim L(a)$ is true for some a in UoD. Using Existential Generalisation $(\exists x)[\sim L(x)]$ is true. But $(\exists x)[\sim L(x)] \equiv\sim (\forall x)L(x)$, and so $\sim (\forall x)L(x)$ is true. Hence $\sim (\forall x)(\exists y)[P(x,y)]$ is true.

So $(\exists x)(\forall y)[\sim P(x,y)] \rightarrow [\sim (\forall x)(\exists y)P(x,y)]$ is a tautology.

By this we have proved that $\sim (\forall x)(\exists y)P(x,y) \leftrightarrow (\exists x)(\forall y)[\sim P(x,y)]$ is a tautology. Therefore $\sim (\forall x)(\exists y)P(x,y) \equiv (\exists x)(\forall y)[\sim P(x,y)]$.

Exercise 1.1.8 page 13

Given the tree propositions p, q and r, construct truth tables for:
(i) $(p \wedge q) \rightarrow \bar{r}$
(ii) $(p \underline{\vee} r) \wedge \bar{q}$
(iii) $p \wedge (\bar{q} \vee r)$
(iv) $p \rightarrow (\bar{q} \vee \bar{r})$
(v) $(\overline{p \vee q}) \leftrightarrow (r \vee p)$.

Solution

					(i)			(ii)		(iii)		(iv)
p	q	r	$\bar{r}$	$p \wedge q$	$(p \wedge q) \to \bar{r}$	$p \underline{\vee} r$	$\bar{q}$	$(p \underline{\vee} r) \wedge \bar{q}$	$\bar{q} \vee r$	$p \wedge (\bar{q} \vee r)$	$\bar{q} \vee \bar{r}$	$p \to (\bar{q} \vee \bar{r})$
T	T	T	F	T	F	F	F	F	T	T	F	F
T	T	F	T	T	T	T	F	F	F	F	T	T
T	F	T	F	F	T	F	T	F	T	T	T	T
T	F	F	T	F	T	T	T	T	T	T	T	T
F	T	T	F	F	T	T	F	F	T	F	F	T
F	T	F	T	F	T	F	F	F	F	F	T	T
F	F	T	F	F	T	T	T	T	T	F	T	T
F	F	F	T	F	T	F	T	F	T	F	T	T

						(v)
p	q	r	$p \vee q$	$\overline{p \vee q}$	$r \vee p$	$(\overline{p \vee q}) \leftrightarrow (r \vee p)$
T	T	T	T	F	T	F
T	T	F	T	F	T	F
T	F	T	T	F	T	F
T	F	F	T	F	T	F
F	T	T	T	F	T	F
F	T	F	T	F	F	T
F	F	T	F	T	T	T
F	F	F	F	T	F	F

Exercise 1.2 page 15

Determine whether each of the following is a tautology, a contradiction or neither:
1. $p \to (p \vee q)$
2. $(p \to q) \wedge (\bar{p} \vee q)$
3. $(p \vee q) \leftrightarrow (q \vee p)$
4. $(p \wedge q) \to p$
5. $(p \wedge q) \wedge (\overline{p \vee q})$
6. $(p \to q) \to (p \wedge q)$
7. $(\bar{p} \wedge q) \wedge (p \vee \bar{q})$
8. $(p \to \bar{q}) \vee (\bar{r} \to p)$

Solution

						1	2	3	4
p	q	$p \vee q$	$p \to q$	$\bar{p} \vee q$	$p \wedge q$	$p \to (p \vee q)$	$(p \to q) \wedge (\bar{p} \vee q)$	$(p \vee q) \leftrightarrow (q \vee p)$	$(p \wedge q) \to p$
T	T	T	T	T	T	T	T	T	T
T	F	T	F	F	F	T	F	T	T
F	T	T	T	T	F	T	T	T	T
F	F	F	T	T	F	T	T	T	T
						tautology	neither	tautology	tautology

				5			7		6
p	q	$p \wedge q$	$\overline{p \vee q}$	$(p \wedge q) \wedge (\overline{p \vee q})$	$\bar{p} \wedge q$	$p \vee \bar{q}$	$(\bar{p} \wedge q) \wedge (p \vee \bar{q})$	$p \rightarrow q$	$(p \rightarrow q) \rightarrow (p \wedge q)$
T	T	T	F	F	F	T	F	T	T
T	F	F	F	F	F	T	F	F	T
F	T	F	F	F	T	F	F	T	F
F	F	F	T	F	F	T	F	T	F
				contradiction			contradiction		neither

					8
p	q	r	$p \rightarrow \bar{q}$	$\bar{r} \rightarrow p$	$(p \rightarrow \bar{q}) \vee (\bar{r} \rightarrow p)$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	F	T
					tautology

Exercise 1.3 page 19

1. Prove that $(p \rightarrow q) \equiv (\bar{p} \vee q)$
2. Prove that $(p \wedge q)$ and $(\overline{p \rightarrow \bar{q}})$ are logically equivalent propositions.
3. Prove that $(\overline{p \underline{\vee} q}) \equiv (p \underline{\vee} \bar{q})$

Solution

To prove task 1,2 and 3 we must show that $(p \rightarrow q) \leftrightarrow (\bar{p} \vee q)$, $(p \wedge q) \leftrightarrow (\overline{p \rightarrow \bar{q}})$ and $(\overline{p \underline{\vee} q}) \leftrightarrow (p \underline{\vee} \bar{q})$ are tautologies.

				1				2
p	q	$p \rightarrow q$	$\bar{p} \vee q$	$(p \rightarrow q) \leftrightarrow (\bar{p} \vee q)$	$p \wedge q$	$p \rightarrow \bar{q}$	$\overline{p \rightarrow \bar{q}}$	$(p \wedge q) \leftrightarrow (\overline{p \rightarrow \bar{q}})$
T	T	T	T	T	T	F	T	T
T	F	F	F	T	F	T	F	T
F	T	T	T	T	F	T	F	T
F	F	T	T	T	F	T	F	T

					3
p	q	$p \underline{\vee} q$	$\overline{p \underline{\vee} q}$	$p \underline{\vee} \bar{q}$	$\overline{p \underline{\vee} q} \leftrightarrow (p \underline{\vee} \bar{q})$
T	T	F	T	T	T
T	F	T	F	F	T
F	T	T	F	F	T
F	F	F	T	T	T

As it follows from the truth tables above, $(p \rightarrow q) \leftrightarrow (\bar{p} \vee q)$, $(p \wedge q) \leftrightarrow (\overline{p \rightarrow \bar{q}})$ and $(\overline{p \underline{\vee} q}) \leftrightarrow (p \underline{\vee} \bar{q})$ are tautologies.

Exercise 1.4.3 page 27

Test the validity of the following arguments.

3. James is either a policeman or a footballer. If he is a policeman, then he has big feet. James has not got big feet so he is a footballer.

Solution

Let p, q and r be:
p: James is a policeman
q: James is a footballer
r: James has big feet.

Then the argument will be: $(p\underline{\vee}q) \wedge (p \to r) \wedge \bar{r} \vdash q$, where $(p\underline{\vee}q)$, $(p \to r)$ and $\bar{r}$ are premises and q is a conclusion.
An alternative argument will be $(p\underline{\vee}q) \wedge (p \to r) \vdash (\bar{r} \to q)$, where $(p\underline{\vee}q)$ and $(p \to r)$ are premises and $(\bar{r} \to q)$ is a conclusion. It is because $((a \wedge b) \to c) \equiv (a \to (b \to c))$ for any a, b and c. Let us check the validity of the first argument by building a truth table for $(p\underline{\vee}q) \wedge (p \to r) \wedge \bar{r} \to q$

p	q	r	$p\underline{\vee}q$	$p \to r$	$\bar{r}$	$(p\underline{\vee}q) \wedge (p \to r) \wedge \bar{r}$	$(p\underline{\vee}q) \wedge (p \to r) \wedge \bar{r} \to q$
T	T	T	F	T	F	F	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	F	T
T	F	F	T	F	T	F	T
F	T	T	T	T	F	F	T
F	T	F	T	T	T	T	T
F	F	T	F	T	F	F	T
F	F	F	F	T	T	F	T

As it follows from the truth table above, $(p\underline{\vee}q) \wedge (p \to r) \wedge \bar{r} \to q$ is a tautology, so the argument $(p\underline{\vee}q) \wedge (p \to r) \wedge \bar{r} \vdash q$ is valid.

Exercise 1.5.4 page 36

Consider the following predicates:
$P(x, y) : x > y$
$Q(x, y) : x \leq y$
$R(x) : x - 7 = 2$
$S(x) : x > 9$
If the universe of discourse is the real numbers, give the truth value of each of the following propositions:
(i)$(\exists x)R(x)$
(ii)$(\forall y)[\sim S(y)]$
(iii)$(\forall x)(\exists y)P(x, y)$
(iv)$(\exists y)(\forall x)Q(x, y)$

(v)$(\forall x)(\forall y)[P(x,y) \vee Q(x,y)]$
(vi)$(\exists x)S(x) \wedge \sim (\forall x)R(x)$
(vii)$(\exists y)(\forall x)[S(y) \wedge Q(x,y)]$
(viii)$(\forall x)(\forall y)[\{R(x) \wedge S(y)\} \rightarrow Q(x,y)]$

Solution

(i) T, $\exists x, x = 9$, that $R(x)$ is true
(ii) F, counter example $y = 10$
(iii) T, for any real number always exists another real number that is less then it.
(iv) F, there is no such real number that is grater or equal to all other real numbers.
(v) T, any two real numbers x and y are either $x > y$ or $x \leq y$.
(vi) T, there exist real numbers that are grater than 9, and not all real numbers are equal to 9
(vii) F, there is no such real number that is grater or equal to all other real numbers, even if this number is grater than 9.
(viii) T, this follows from the fact that $(\forall x)R(x)$ is false. Therefore $(\forall x)(\forall y)[R(x) \wedge S(y)]$ is also false, so $(\forall x)(\forall y)[\{R(x) \wedge S(y)\} \rightarrow Q(x,y)]$ is true.