Assignment-1 AA

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1.

$S_4 = \{ e, (1,2), (1,3), (14), (23), (24), (34),$
$(123), (124), (132), (134), (142), (143),$
$(234), (243), (12)(34), (13)(24), (14)(23),$
$(1234), (1243), (1324), (1342), (1423),$
$(1432) \}$

Order $= 4! = 24 =$ no. of elements in S_4

Abelian groups are those that satisfy commutativity.

$(123)(124) \in S_4$

$(12)(13)(12)(14) = (24)$

$(124)(123) \in S_4$

$(12)(14)(12)(13) = e$

Hence not abelian

c
b a b
ab ad
d c a b d a
4 3 2 1
a d c b
a d c b
1 4 3 2
d c b
a b c d
b a c d
d a c b
a d c b
c d a b
3 4 1 2
b a c d
a b c d
a b c d
1 2 3 4

Assignment

Order of elements

$|e| = 1$

$(12)(13)(14)(23)(24)(34)$

↳ order is 2 as repeating the same transposition twice gives us the original sequence.

$(123)(124)(132)(134)(142)(143)(234)$

(243)

↳ order is 3

Applying (123) on abcd
$(12)(13)$

abcd
↓
bacd
↓
cabd

again, acbd
↓
bcad

again,
↓
cbad
↓
abcd, our original string

$(12)(34)$

$(13)(24)$

$(14)(23)$

↳ order is 2

Applying $(12)(34)$ on abcd

badc

again, abcd, our original string

(1234)

(1243)

(1324)

(1342)

(1423)

(1432)

↳ orders of 4

Applying (1234) on abcd

(12)(13)(14) bacd

↓

cabd

↓

dabc

again, adbc

↓

~~ba~~ bdac

↓

cdab

↓

again, dcab

↓

~~adcb~~ acdb

↓ ↓

~~bdca~~ bcda

↓ ↓

again, dbca cbda

↓ ↓

cbda dbca

↓ ↓

abdc abcd, original string

↓

again, badc

↓

dabc

↓

cabd

2.

1. $H_1 = \{e, (123), (134), (1243), (1432)\}$

$(123)(134) = (12)(13)(13)(14)$

$= (12)(14) = (124)$

$\notin H_1$

Not a subgroup as not closed

2. $H_2 = \{e, (1234), (1432), (13)(24)\}$

Cayley's table

e	(1234)	(1432)	$(13)(24)$
(1234)	$(13)(24)$	e	(1432)
(1432)	e	$(13)(24)$	(1234)
$(13)(24)$	(1432)	(1234)	e

→ Hence closed

→ Identity $= e$

→ Associative by Q1.

→ Inverse exists as seen by cayley's table

Hence Subgroup

Lagranges theorem → $|H_2| \mid |S_4|$

$4 \mid 24$ ✓ verified.

3. $H_3 = \{e, (12)(34), (14)(32), (13)(24)\}$

Cayley's table

e	$(12)(34)$	$(14)(32)$	$(13)(24)$
$(12)(34)$	e	$(13)(24)$	$(14)(32)$
$(14)(32)$	$(13)(24)$	e	$(12)(34)$
$(13)(24)$	$(14)(32)$	$(12)(34)$	e

→ closed

→ Identity e

→ Assowative

→ Inven exists by cayley's table

Subgroup ✓

Lagrange's $|H_3| \mid |S_4| = 4|24$ ✓ verifie a

3. 1. Not a subgroup

2. $\{e, (13)(24), (1234), (1432)\}$

$e e e^{-1} = e \in H_2$

~~(12)(34)~~

$(12) e (12) \in H_2$

$(13) e (13) \in H_2$

$(14) e (14) \in H_2$

$(23) e (23) \in H_2$

$(24) e (24)$,,

$(34) e (34)$,,

$(124) e (142)$,,

$(123) e (132)$,,

$(132) e (123)$,,

$(134) e (143)$,,

$(142) e (124)$,,

$(143) e (134)$,,

$(234) e (243)$,,

$(243) e (234)$,,

$(12)(34) e (12)(34)$,,

$(13)(24) e (13)(24)$,,

$(14)(23) e (14)(23)$,,

$(1234) e (1432)$,,

$(1243) e (1342)$,,

$(1324) e (1423)$,,

$(1342) e (1243)$,,

$(1423) e (1324)$,,

$(1432) e (1234)$,,

For $h \in H_2$

~~$g e g^{-1} = e$~~

~~$g g^{-1} = e \in H_2$~~

$e h e^{-1} = h \in H_2$

$e (13)(24) e$,,

$(12)(13)(24)(12) = (14)(23) \notin H_2$ hence not normal subgroup

d c

b a

1 4 3 2

a
c b d a
3 2 1 4

9. $H_2 = \{e, (12)(34), (14)(32), (13)(24)\}$

For $h \in H_2$

$ehe^{-1} = h \in H_2$

$e(12)(34)e \in H_2$

$\cancel{(12)}\cancel{(12)}(34)(12) = (12)(34) \in H_2$

~~Hence not a normal subgroup~~

$(13)(12)(34)(13) = (14)(23) \in H_2$

$(14)(12)(34)(14) = (13)(24) \in H_2$.

Same for $(23), (24), (3,4)$.

$(123)(12)(34)(132) = (14)(23) \in H_2$

$(124)(12)(34)(142) = (13)(24)$ ''

$(132)(12)(34)(123) = (13)(24)$ ''

$(134)(12)(34)(143) = (14)(23)$ ''

$(142)(12)(34)(124) = (14)(23)$ ''

$(143)(12)(34)(134) = (13)(24)$ ''

$(234)(12)(34)(243) = (13)(24)$ ''

$(243)(12)(34)(234) = (14)(23)$ ''

$(12)(34)\ (12)(34)\ (12)(34) = (12)(34)$ ''

$(13)(24)\ (12)(34)\ (13)(24) = (12)(34)$ ''

$(14)(23)\ (12)(34)\ (14)(23) = (12)(34)$ ''

(1234) '' $(1432) = (14)(23)$ ''

(1243) '' $(1342) = (13)(24)$ ''

(1324) $(1423) = (12)(34)$ ''

(1342) '' $(1243) = (13)(24)$ ''

(1423) $(1324) = (12)(34)$ ''

(1432) '' $(1234) = (14)(23)$ ''

(12)	(14)(32)	(12)	$\in H_2$
(13)	"	(13)	$\in H_2$
(14)	"	(14)	

Same fo (23), (24), (34)

(123)	(14)(32)	(132)	= (13)(24) $\in H_2$
(124)	(14)(32)	(142)	= (12)(34)
(132)	(14)(32)	(123)	= (12)(34)
(134)	(14)(32)	(143)	= (13)(24)
(142)	(14)(32)	(124)	= (13)(24)
(143)	(14)(32)	(134)	= (12)(34)
(234)	(14)(32)	(243)	= (12)(34)
(243)	(14)(32)	(234)	= (13)(24)
(12)(34)	"	(12)(34)	= (14)(23)
(13)(24)	"	(13)(24)	= (14)(23)
(14)(23)	"	(14)(23)	= (14)(23)
(1234)	"	(1432)	= (12)(34)
(1243)	"	(1342)	= (14)(23)
(1324)	"	(1423)	= (13)(24)
(1342)	"	(1243)	= (14)(23)
(1423)	"	(1324)	= (13)(24)
(1432)	"	(1234)	= (12)(34)

$\in H_2$

(12)	(13)(24)	(12)	= (14)(23) $\in H_2$
(13)	(13)(24)	(13)	= (13)(24) "
(14)	"	(14)	= (12)(34) "

Sam fo (23), (24), (34)

(123)	(13)(24)	(132)	= (12)(34)
(124)	"	(142)	= (14)(23)
(132)	"	(123)	= (14)(23)
(134)	"	(143)	= (12)(34)
(142)	"	(124)	= (12)(34)
(143)	"	(134)	= (14)(23)
(234)	"	(243)	= (14)(23)
~~(243)~~	"	(234)	= (12)(34
(12)(34)	"	(12)(34)	= (13)(24)
(13)(24)	"	(13)(24)	= (13)(24)
(14)(23)	"	(14)(23)	= (13)(24)
(1234)	"	(1432)	= (13)(24)
(1243)	"	(1342)	= (13)(24
(1324)	"	(1423)	= (12)(34)
(1342)	"	(1243)	= (14)(23)
(1423)	"	(1324)	= (12)(34)
(1432)	"	(12)4)	= (13)(24)

H_2

⇒ H2 IS NORMAL

4. Cyclic subgroup of S_4 of orders 2, 3, 4

↳ $G = \{g^n \mid n \in \mathbb{Z}\}$

↳ generator

For order = 2

H = Any permutation of order 2

$= \{(24)^n, n \in \mathbb{Z}\}$ is a subgroup of order 2.

For order = 3

$H = \{(123)^n, n \in \mathbb{Z}\}$ subgroup of order 3.

For order = 4

$H = \{(1234)^n, n \in \mathbb{Z}\}$ subgroup of order 4

A group G of order $|G| = n$ is cyclic iff it has an element of order n.

The respective subgroups are

$\{e, (24)\}$

$\{e, (123), (132)\}$

$\{e, (1234), (13)(24), (1432)\}$

a b c d
b a c d

5. Taking the subgroup $H = \{e, (24)\}$
left cosets

$e()H = \{(), (24)\}$
$(12)H = \{(12), (124)\}$
$(13)H = \{(13), (13)(24)\}$
$(14)H = \{(14), (124)\}$
$(23)H = \{(23), (234)\}$
$(24)H = \{(24), e\}$
$(34)H = \{(34), (243)\}$
$(123)H = \{(123), (1423)\}$
$(124)H = \{(124), (14)\}$
$(132)H = \{(132), (1342)\}$
$(134)H = \{(134), (1324)\}$
$(142)H = \{(142), (12)\}$
$(143)H = \{(1243), (143)\}$
$(234)H = \{(234), (23)\}$
$(243)H = \{(243), (34)\}$
$(12)(34)H = \{(12)(34), (1432)\}$
$(13)(24)H = \{(13)(24), (13)\}$
$(14)(23)H = \{(14)(23), (1234)\}$
$(1234)H = \{(14)(23), (1234)\}$
$(1243)H = \{(1243), (143)\}$
$(1324)H = \{(1324), (134)\}$
$(1342)H = \{(1342), (132)\}$
$(1423)H = \{(1423), (123)\}$
$(1432)H = \{(1432, (12)(34)\}$

B2

12 distinct combinations

marked by a tick

$\frac{|G|}{|H|} = \frac{24}{12} = 2$. ✓ verified

B

6.

Left coset partitions into equivalent classes

$b \equiv_{\lambda} a \pmod{H}$

$\Rightarrow S_4 = \bigcup_{a \in S_4} aH$ and $S_4 = \bigcup_{a \in S_4} aK$

For $(H \cap K)$, $S_4 = \bigcup_{a \in S_4} a(H \cap K)$

E

8) T.P. $[G:H\cap K] \leq [G:H][G:K]$

~~[illegible]~~

$H\cap K \subseteq H$

$H \cap K \subseteq K$

$H\cap K \subseteq G$ as $H \subseteq G$ and $K \subseteq G$

T.P. $H\cap K$ is a subgroup

Let $x, y \in (H\cap K) \Rightarrow x,y \in H$ and $x,y \in K$

$xy^{-1} \in H$ and $xy^{-1} \in K$

$\Rightarrow xy^{-1} \in H\cap K$

$\Rightarrow H\cap K \leq G$

As G is finite, left cosets of H and K are finite and ~~of the~~ so ~~left coset of~~ $H\cap K$ is also finite.

3 4

20, $S_4 = \bigcup_{a\in S_4} a(H\cap K) = \bigcup_{a\in S_4} aH = \bigcup_{a\in S_4} aK$

$\Rightarrow \bigcup a(H\cap K) = \bigcup aH \cap \bigcup aK$

$|S_4/H\cap K| = |S_4/H \cap S_4/K|$

$\Rightarrow |S_4/H\cap K| = |S_4/H|\,|S_4/K|$

$\Rightarrow [G:H\cap K] \leq [G:H][G:K]$

b)

using $[G:X] = [G:Y][Y:X]$

$$[G:H\cap K] = [G:H][H:H\cap K]$$
$$= [G:K][K:H\cap K]$$

Hence

$$[G:H] \mid [G:H\cap K]$$
$$[G:K] \mid [G:H\cap K]$$

$$\operatorname{lcm}([G:K][G:H]) \le [G:H\cap K] \le [G:H][G:K]$$

If $[G:K]$ and $[G:H]$ are co-prime

$$\operatorname{lcm}(\;''\;) = [G:H][G:K]$$

Hence

$$[G:H\cap K] = [G:K][G:H]$$