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Assignment 2

(i)

$Z(G) = \{g \in G \mid gx = xg \ \forall x \in G\}$

To show $Z(G)$ is a normal subgroup -

A subgroup $H \trianglelefteq G$ iff $ghg^{-1} \in H \ \forall g \in G$ & $h \in H$

Proving subgroup:

For $b \in Z(G) \ \forall g \in G$

$gb = bg$

$g = bgb^{-1}$

$b^{-1}g = gb^{-1} \Rightarrow b^{-1} \in Z(G)$

for $a \in Z(G)$

$ga = ag$

$gab^{-1} = agb^{-1}$

$gab^{-1} = ab^{-1}g$

Hence $ab^{-1} \in Z(G)$

For $a \in Z(G) \ \forall g \in G$

$ga = ag$

$gag^{-1} = a \in Z(G)$ Hence normal subgroup

If $G/Z(G)$ is cyclic then G is abelian

$G/Z(G) = \langle g Z(G) \rangle$

For $a \in G$,

$aZ(G) = g^i Z(G) \rightarrow$ by $(g_1 Z(G))(g_2 Z(G)) = (g_1 g_2) Z(G)$

$\Rightarrow (g Z(G))^m = g^m Z(G)$

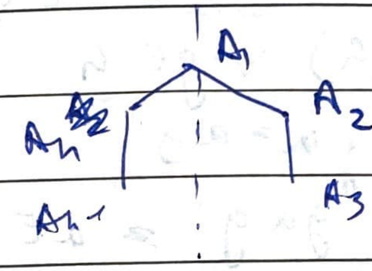
Same for $b \in G$

$ab = g^i Z(G)^1 g^j Z(G)^2$

$= g^{i+j} Z(G)^1 Z(G)^2$

$= g^j Z(G)^2 g^i Z(G)^1 = ba$ Hence abelian

2, taking an n sided polygon P



flipping it $\rightarrow$ A_2 to A_n
(f) $\quad$ A_3 to A_{n-1} etc

rotating it $\rightarrow$
(r)

flipping again $\rightarrow$ A_2 to A_n
(f) $\quad$ A_3 to A_{n-1}

$$f r^i f = r^{n-i}$$

Let $r \in Z(D_n)$ and $f \in D_n$

$$fr = rf$$

$$fr^i = r^i f \qquad (f^{-1} = f)$$

$$r^{n-i} = r^i \qquad i = n-i$$

$$i = \frac{n}{2} \qquad \Rightarrow i = \frac{n}{2}$$

also, $r^n = r^{2i} = e$

$\Rightarrow 2i = 0$

$\Rightarrow i = 0$

$\Rightarrow Z(D_n) = \begin{cases} e & , n \text{ odd} \\ r^{n/2}, e & n \text{ even} \end{cases}$

$180°$

$n = 2i$

$\frac{n}{2}$

3. $|G| = n$

p

Subgroup of index p is normal

Cauchy's theorem, $a \in G$ s.t. $|a| = p$

Let there be an f where G acts on left cosets of H by left multiplication.

$$f(g, xH) = g.xH \quad , \quad g, x \in G$$

This leads to a homomorphism from $G \to S_p$ where $|S_p| = p!$

↳ permutation set

The Kernel is in H

$G/K \cong S_p$

$|G/K| \mid |S_p| \Rightarrow |G/K| \mid p!$

$|G/K| \mid |G| \Rightarrow |G/K| \mid p$

$|G/K| = [G:K] = [G:H][H:K] = p[H:K]$

$\Rightarrow [H:K] = 1$

$\Rightarrow K = H$

As Kernel is normal, H is also normal

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4. $H \trianglelefteq G$, $K \trianglelefteq G$

$H \cap K = 1$

$G = HK$

T.P. $G \cong H \times K$ $\qquad f : G \to H \times K$

Proving injective,

Consider $g \in G$

$g = h_1 k_1 = h_2 k_2$

$h_1 k_1 = h_2 k_2$

$h_2^{-1} h_1 k_1 = k_2$

$h_2^{-1} h_1 = e \Rightarrow h_1 = h_2$ $\qquad$ (as $H \cap K = 1$)

$h_1 = h_2 k_2 k_1^{-1}$

$\Rightarrow k_2 k_1^{-1} = e$

$k_2 = k_1$

Hence injective

$|G| = |HK| = \dfrac{|H| . |K|}{|H \cap K|}$ $\to 1$

$|H \times K| = |H||K|$

Hence order of both is same and f is also injective

Hence surjective proved so f is bijective

$\Rightarrow G \cong H \times K$

It is a group homomorphism as $f(g_1 g_2) = (h_1 \times k_1) . (h_2 \times k_2) = f(g_1) f(g_2)$

5. $|G| = p^{\alpha}$ ($\alpha \in \mathbb{N}$, p prime)

If H is a normal subgroup of G, prove $H \cap Z(G) \neq \{e\}$

If $|G| = p^{\alpha}$ then $Z(G) \neq \{e\}$

$H = p^m$ as $|H| \mid |G|$

~~By~~ similarly, $Z(H) \neq \{e\}$

$\cancel{|Z(G)| = \sum_{a \in Z(G)} |Cl(a)|}$

$C_G(a) = \{g \in G \mid ga = ag\}$

$\cancel{a \in Z(G) \text{ iff } C_G(a) = G}$

$\cancel{Z(G) =}$

$Z(G) = \bigcap_{a \in G} C_G(a)$ as $H \trianglelefteq G$, $C_H(a) \leq C_G(a)$

Hence $Z(G) \cap \bigcap_{a \in H} C_H(a) \neq \{e\}$

as $\bigcap_{a \in H} C_H(a) \subseteq H$, $H \cap Z(G) \neq \{e\}$