ASSIGNMENT [X] ON [COURSE NAME]		
Student's Code	AIMS African Institute for Mathematical Sciences SENEGAL	Deadline
[Your Code]		**[Date, Time]**
January 3, 2024		2019-2020
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1 Solution of Exercise 1

Let V and W be two vectors spaces over a field K and $T : V \longrightarrow W$ be a linear map. Show that:

a) $T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i T(x_i)$ where $a_i \in K, \forall i = 1, 2, ..., n (x_1, ..., x_n) \in V$, To show this, let's use the linearity property of the linear transformation T. A linear transformation T satisfies two properties: additivity and homogeneity.

1. **Additivity:** $T(u + v) = T(u) + T(v)$ for all vectors u,v in the domain V.
2. **Homogeneity:** $T(cu) = cT(u)$ for all vectors u in the domain V and all scalars c in the field K.

Now, let's prove the given statement: Given: $(a_1, ..., a_n) \in K$ and $(a_1, ..., a_n) \in V$, we want to show that: $T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i T(x_i)$ We can use induction to prove this statement. The base case is n = 1, where the statement is trivially true.
Base case (n=1): $T(a_1 x_1) = a_1 T(x_1)$ This holds true by linearity property of T.
Inductive Step : Assume that the statement holds for n=k, i.e.,
$T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i T(x_i)$

Now, consider $n = k + 1$: $(\sum_{i=1}^{k+1} a_i x_i) = T(\sum_{i=1}^{k+1} a_i x_i + a_{k+1} x_{k+1})$

By the additivity property of T, this is equal to:

$T(\sum_{i=1}^{k} a_i x_i) + T(a_{k+1} x_{k+1})$

Now, apply the inductive assumption: $\sum_{i=1}^{k} a_i T(x_i) + T(a_{k+1} x_{k+1})$
Now, by the homogeneity property, $T(a_{k+1} x_{k+1}) = a_{k+1} T(x_{k+1})$:

$\sum_{i=1}^{k} a_i T(x_i) + a_{k+1} T(x_{k+1})$

b) if $(v_1, ..., v_n) \in V$ is a dependent family of vectors then $(T(v_1), ..., T(v_n))$ is also dependent, if $(v_1, ..., v_n) \in V$ is a dependent family of vectors then, here exist scalars $c_1, ..., c_n$, not all zero, such that:

$c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0$ Now, let's apply the linear transformation T to both sides of this equation:

$T(c_1 v_1 + c_2 v_2 + ... + c_n v_n) = T(0)$

Using the linearity property of T, we get:

$c_1 T(v_1) + c_2 T(v_2) + ... + c_n T(v_n) = 0$

This shows that $(T(v_1), ..., T(v_n))$.

So, the dependence of the original family of vectors is preserved under the linear transformation T.

c) if, moreover, T is one to one and $(v_1, ..., v_n) \in V$ is an independent family of vectors then $(T(v_1), ..., T(v_n))$ is also independent,

Let's prove this: Suppose $(v_1, ..., v_n)$ is an independent family of vectors in V. This means that the equation

$c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0$ implies that all the coefficients $c_1, c_2, ..., c_n$ are zero.

Now, consider the image of this equation under the linear transformation T:

$T(c_1 v_1 + c_2 v_2 + ... + c_n v_n) = T(0)$

Using the linearity property of T, this becomes:

$c_1 T(v_1) + c_2 T(v_2) + ... + c_n T(v_n) = 0$

Since T is injective, the only way the linear combination $c_1 T(v_1) + c_2 T(v_2) + ... + c_n T(v_n)$ can equal zero is if all the coefficients $c_1, c_2, ..., c_n$ are zero. This follows from the injectivity of T.

Therefore, $(T(v_1), ..., T(v_n)$ is an independent family of vectors in the range of T.

d) if, moreover, T is onto and $S \subset V$ is a spanning set for V then $T(S)$ is spanning set for W.

Let's prove this:

1. $S \subset V$ **is a spanning set for V:** This means that every vector $v \in V$ can be expressed as a linear combination of vectors in S. Mathematically, for any $v \in V$, there exist scalars $c_1, c_2, ..., c_n$ and vectors $s_1, s_2, ..., s_n \in S$ such that:

$v = c_1 s_1 + c_2 s_2 + ... + c_n s_n$

2. T **is onto (surjective):**

   This means that every vector in the codomain W is the image of at least one vector in V under T. Now, consider an arbitrary vector $w \in W$. Since T is onto, there exists a vector $v \in V$ such that $T(v) = w$. By the spanning property of $S \in V$, we can express v as a linear combination of vectors in S:

   $v = c_1 s_1 + c_2 s_2 + ... + c_n s_n$

   Now, apply T to both sides: $w = T(v) = c_1 T(s_1) + c_2 T(s_2) + ... + c_n T(s_n)$

   This shows that w can be expressed as a linear combination of vectors in $T(S)$. Since w was arbitrary, it follows that $T(S)$ is a spanning set for W.

   Therefore, if T is onto and $S \subset V$ is a spanning set for V, then $T(S)$ is a spanning set for W.

2 Solution of Exercise 2

Let P_2 be the set of polynomials of degree less or equal to two and

$$\begin{aligned} T : \mathbb{P}_2 &\longrightarrow \mathbb{P}_2 \\ f &\longmapsto f' \end{aligned}$$

be a linear map. Consider the family:

$$\beta_{\mathbb{P}_2} = \{2 + x, 1 + x, \tfrac{1}{2}x^2\}$$

a) Prove that $\beta_{\mathbb{P}_2}$ is a basis of $\mathbb{P}_2$. The first thing to do is prove that the vectors in $\beta_{\mathbb{P}_2}$ are linearly independent. We want to show that for scalars, c_1, c_2, c_3, if $c_1(2 + x) + c_2(1 + x) + c_3(\frac{1}{2}x^2) = 0, \quad then$
   $c_1 = c_2 = c_3 = 0$

   $c_1(2 + x) + c_2(1 + x) + c_3(\frac{1}{2}x^2) = 0$

   Expand this equation:

   $2c_1 + c_2 + \frac{1}{2}c_3 x^2 + (c_1 + c_2)x = 0$

   The only way for this polynomial to be identically zero for all x is if each coefficient is zero.
   So, we set each coefficient to zero:

$$\begin{cases} 2c_1 + c_2 &= 0 \\ c_1 + c_2 &= 0 \\ \frac{1}{2}c_3 &= 0 \end{cases}$$

The solutions to this system are $c_1 = c_2 = c_3 = 0$, which means that the vectors in $\beta_{\mathbb{P}_2}$ are linearly independent.

The second condition is to show that $\beta_{\mathbb{P}_2}$ Spanning set: Any polynomial f in $\mathbb{P}_2$ can be expressed as a linear combination of vectors in $\beta_{\mathbb{P}_2}$.

It is not necessary to prove the second condition, because $\beta_{\mathbb{P}_2}$ dimension is equal to three and vectors of $\beta_{\mathbb{P}_2}$ are linearly independent, then we can conclude that $\beta_{\mathbb{P}_2}$ spans $\mathbb{P}_2$.

b) Determine the matrix M_T of the map T with respect to the basis to the basis $\beta_{\mathbb{P}_2}$.

To find the matrix representation of f with respect to the canonical basis $\gamma = e_1, e_2, e_3$ where $e_1 = (1, 0, 0), e_2 = (0, 1, 0) and e_3 = (0, 0, 1)$ 1), we apply f to each vector in γ. $T(e_1) = (2, 1, 0) = 2e_1 + e_2$
$T(e_2) = (1, 1, 0) = e_1 + e_2$
$T(e_3) = (0, 0, \frac{1}{2}) = \frac{1}{2}e_3$
Therefore, the matrix M_T representing f with respect to γ is:

$$M_T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

3 Solution of exercice 3

Is the set of matrices over a field K with m rows and n columns, denoted by $M_{m\times n}(\mathbb{K})$, a vector space? Justify your answer clearly

To show this, we need to verify the ten axioms of a vector space. Here's a brief justification for each axiom:

1. Closure under Addition:

    . If A and B are matrices in $M_{m\times n}(\mathbb{K})$, then $A + B$ is also in $M_{m\times n}(\mathbb{K})$ because the sum of two mxn matrices is still an mxn matrix.

2. Associativity of Addition:

    . Matrix addition is associative, i.e., $(A + B) + C = A + (B + C)$ for any matrices A, B and C in $M_{m\times n}(\mathbb{K})$.

3. Existence of an Additive Identity:

- The zero matrix (O), where all entries are zero, serves as the additive identity. For any matrix A in $M_{m\times n}(\mathbb{K})$, we have $A+0=A$

4. Existence of Additive Inverses:

   - For any matrix in $M_{m\times n}(\mathbb{K})$,, the additive inverse ?A exists, and $A+(-A)=0$.

5. Closure under Scalar Multiplication:

   - If A is a matrix in $M_{m\times n}(\mathbb{K})$ and c is a scalar in K, then cA is also in $M_{m\times n}(\mathbb{K})$ because scalar multiplication of a matrix by a scalar produces another matrix of the same size.

6. Compatibility of Scalar Multiplication with Field Multiplication:

   - Scalar multiplication is compatible with field multiplication, i.e., $(cd)A=c(dA)$ for any scalars c and d in K and any matrix $A\in M_{mxn}(\mathbb{K})$.

7. Identity Element for Scalar Multiplication:

   - The identity element for scalar multiplication is 1 in the field $K.1A=A$ for any matrix $A\in M_{mxn}(\mathbb{K})$

8. Distributivity of Scalar Multiplication with Respect to Vector Addition:

   - Scalar multiplication distributes over vector addition, i.e., $c(A+B)=cA+cB$ for any scalar c and matrices $AandB\in M_{mxn}(\mathbb{K})$. $K.1A=A$ for any matrix $A\in M_{mxn}(\mathbb{K})$

9. Distributivity of Scalar Multiplication with Respect to Field Addition:

   - Scalar multiplication distributes over field addition, i.e., $(c+d)A=cA+dA$ for any scalars c and d in for any scalar c and d $\in K$ and any matrix $A\in M_{mxn}(\mathbb{K})$.

10. Compatibility of Scalar Multiplication with Matrix Multiplication:

    - Scalar multiplication is compatible with matrix multiplication, i.e., $(cd)A=c(dA)$ for any scalars c and d in for any scalar c and d $\in K$ and any matrix $A\in M_{mxn}(\mathbb{K})$ for any scalars c and d $\in K$ and any matrix $.A\in M_{mxn}(\mathbb{K})$.

Since all ten axioms are satisfied, $M_{mxn}(\mathbb{K})$ is indeed a vector space over the field K