New stochastic sketching methods for Big Data Ridge Regression

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Abstract

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1.2.1 General case

$Z = AI_C^T(I_C A I_C^T)^{-1} I_C A$

$\rho = 1 - \lambda_{min}(A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}})$.
$A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}} = \sum_i p_i A^{\frac{1}{2}} I_{C_i}^T (I_{C_i} A I_{C_i}^T)^{-1} I_{C_i} A^{\frac{1}{2}}$

for any $i \in \{1, \dots, n\}$, $A^{\frac{1}{2}} I_{C_i}^T (I_{C_i} A I_{C_i}^T)^{-1} I_{C_i} A^{\frac{1}{2}}$ is a projection matrix and then its eigenvalues are a nonempty subset of $\{0, 1\}$.

Since λ_{max} is convex, we obtain that :

$0 \leqslant \lambda_{min}(A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}}) \leqslant \lambda_{max}(A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}}) \leqslant \sum_i p_i \lambda_{max}(A^{\frac{1}{2}} I_{C_i}^T (I_{C_i} A I_{C_i}^T)^{-1} I_{C_i} A^{\frac{1}{2}}) \leqslant 1.$

$\mathbf{C} = (I_{C_1}^T, \dots, I_{C_r}^T)$.

$A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}} = (A^{\frac{1}{2}} \mathbf{C} D)(D \mathbf{C}^T A^{\frac{1}{2}})$ where $D = \text{diag}(\sqrt{p_1}(I_{C_1} A I_{C_1}^T)^{-\frac{1}{2}}, \dots, \sqrt{p_r}(I_{C_r} A I_{C_r}^T)^{-\frac{1}{2}})$

Proposition 1.2.1

$\lambda_{min}(A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}}) \geqslant \binom{n-1}{s-1} \dfrac{\lambda_{min}(A)}{\lambda_{max}(A)} \min_i p_i$

Proof :

$\lambda_{min}(A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}}) \geqslant \lambda_{min}(\mathbf{C}^T A \mathbf{C}) \lambda_{min}(D^2)$
$\lambda_{min}(D^2) = \min_i \dfrac{p_i}{\lambda_{max}(I_{C_i} A I_{C_i}^T)} \geqslant \min_i \dfrac{p_i}{\lambda_{max}(I_{C_i}^T I_{C_i}) \lambda_{max}(A)} \geqslant \min_i \dfrac{p_i}{\lambda_{max}(A)}$, since for any $i \in \{1, \dots, n\}$, for any x in $\mathbb{R}^n$ $\left\langle I_{C_i}^T I_{C_i} x \,\middle|\, x \right\rangle = \|I_{C_i} x\|^2 \leqslant \|x\|^2$ and then $\lambda_{max}(I_{C_i}^T I_{C_i}) \leqslant 1$.

Therefore, $\lambda_{min}(A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}}) \geqslant \min_i p_i \dfrac{\lambda_{min}(\mathbf{C}^T A \mathbf{C})}{\lambda_{max}(A)} = \min_i p_i \dfrac{\lambda_{min}(A) \lambda_{min}(\mathbf{C}\mathbf{C}^T)}{\lambda_{max}(A)}$.

$\mathbf{C}\mathbf{C}^T = \sum_i I_{C_i}^T I_{C_i} = \binom{n-1}{s-1} I_n$ and then we obtain that :

$\lambda_{min}(A^{-\frac{1}{2}}E[Z]A^{-\frac{1}{2}}) \geqslant \binom{n-1}{s-1} \dfrac{\lambda_{min}(A)}{\lambda_{max}(A)} \min_i p_i$

1.2.2 Uniform case

For any i, $p_i = \dfrac{1}{\binom{n}{s}}$ is the uniform probability of choosing s rows uniformly on $\{1, \ldots, n\}$, knowing that s is the sketchsize.

Proposition 1.2.2

$\lambda_{min}(A^{-\frac{1}{2}}E[Z]A^{-\frac{1}{2}}) \geqslant \dfrac{s}{n} \dfrac{\lambda_{min}(A)}{\lambda_{max}(A)}$

2. *Hadamard Sketches*

2.1 Algorithm

2.2 Convergence rate (draft)

$Z = AS^T(SAS^T)^{-1}SA$

$\rho = 1 - \lambda_{min}(A^{-\frac{1}{2}}E[Z]A^{-\frac{1}{2}})$
$S_i = I_{C_i}H.$

Proposition 2.2.1

$$\lambda_{min}(A^{-\frac{1}{2}}E[Z]A^{-\frac{1}{2}}) \geqslant \frac{s}{n}\frac{\lambda_{min}(A)}{\lambda_{max}(A)}$$

Proof :

$\tilde{A} = \frac{1}{n}HAH^T = \frac{H}{\sqrt{n}}A\frac{H^T}{\sqrt{n}}.$

$$A^{-\frac{1}{2}}E[Z]A^{-\frac{1}{2}} = E[A^{\frac{1}{2}}S^T(SAS^T)^{-1}SA^{\frac{1}{2}}] = \sum_i p_i A^{\frac{1}{2}}H^T I_{C_i}^T(I_{C_i}HAH^T I_{C_i}^T)^{-1}I_{C_i}HA^{\frac{1}{2}}$$

$$= A^{\frac{1}{2}}H^T\frac{1}{n}E[I_C^T(I_C\tilde{A}I_C^T)^{-1}I_C]HA^{\frac{1}{2}} = H^{-1}\tilde{A}^{\frac{1}{2}}E[I_C^T(I_C\tilde{A}I_C^T)^{-1}I_C]\tilde{A}^{\frac{1}{2}}(\frac{H^T}{n})^{-1} = H^{-1}\tilde{A}^{\frac{1}{2}}E[I_C^T(I_C\tilde{A}I_C^T)^{-1}I_C]\tilde{A}^{\frac{1}{2}}H.$$

Hence :

$\rho = 1 - \lambda_{min}(A^{-\frac{1}{2}}E[Z]A^{-\frac{1}{2}}) = 1 - \lambda_{min}(\tilde{A}^{\frac{1}{2}}E[I_C^T(I_C\tilde{A}I_C^T)^{-1}I_C]\tilde{A}^{\frac{1}{2}})$

We recognize the convergence rate in the Randomized Newton Method and then, denoting by $\rho_{Newton}(M)$ the convergence rate of the Newton method associated with the definite positive matrix M, we obtain that :
$\rho = 1 - \lambda_{min}(\tilde{A}^{\frac{1}{2}}E[I_C^T(I_C\tilde{A}I_C^T)^{-1}I_C]\tilde{A}^{\frac{1}{2}}) = 1 - (1 - \rho_{Newton}(\tilde{A})) = \rho_{Newton}(\tilde{A}) = \rho_{Newton}(A)$, since A and $\tilde{A}$ have the same eigenvalues.

3. *Count-min Sketches*

3.1 Algorithm

3.2 Convergence rate

S is constructed as follows :
For every $i \in \{1, \ldots, n\}$, l is chosen uniformly on $\{1, \ldots, n\}$ and ϵ uniformly on $\{-1, 1\}$, then S is updated in his l^{th} row as :
$S(l, :) := S(l, :) + \epsilon\, e_i^T$, where e_i^T is the i^{th} coloumn of the identity matrix.

4. *Conclusion*

References

[1] ROBERT GOWER AND PETER RICHTARIK, Randomized iterative methods for linear systems, SIAM, (2015).