New stochastic sketching methods for Big Data Ridge Regression

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July, 2017

Abstract

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1.2.1 General case

Throughout the computations, we denote by $Z = AI_C^T(I_C A I_C^T)^{-1} I_C A$. That is a quantity that intervenes in the computation of the convergence rate.

The convergence rate is defined by $\rho = 1 - \lambda_{min}(A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}})$.

By defiition, $A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}} = \sum_i p_i A^{\frac{1}{2}} I_{C_i}^T (I_{C_i} A I_{C_i}^T)^{-1} I_{C_i} A^{\frac{1}{2}}$

for any $i \in \{1, \dots, n\}$, $A^{\frac{1}{2}} I_{C_i}^T (I_{C_i} A I_{C_i}^T)^{-1} I_{C_i} A^{\frac{1}{2}}$ is a projection matrix and then its eigenvalues are a nonempty subset of $\{0, 1\}$.

Since λ_{max} is convex, we obtain that :

$$0 \leqslant \lambda_{min}(A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}}) \leqslant \lambda_{max}(A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}}) \leqslant \sum_i p_i \lambda_{max}(A^{\frac{1}{2}} I_{C_i}^T (I_{C_i} A I_{C_i}^T)^{-1} I_{C_i} A^{\frac{1}{2}}) \leqslant 1.$$

Denote by $\mathbf{C} = (I_{C_1}^T, \dots, I_{C_r}^T)$.

$A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}} = (A^{\frac{1}{2}} \mathbf{C} D)(D \mathbf{C}^T A^{\frac{1}{2}})$ where $D = \text{diag}(\sqrt{p_1}(I_{C_1} A I_{C_1}^T)^{-\frac{1}{2}}, \dots, \sqrt{p_r}(I_{C_r} A I_{C_r}^T)^{-\frac{1}{2}})$

Proposition 1.2.1

$$\lambda_{min}(A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}}) \geqslant \binom{n-1}{s-1} \frac{\lambda_{min}(A)}{\lambda_{max}(A)} \min_i p_i$$

Proof :

$\lambda_{min}(A^{-\frac{1}{2}} E[Z] A^{-\frac{1}{2}}) \geqslant \lambda_{min}(\mathbf{C}^T A \mathbf{C}) \lambda_{min}(D^2)$

$\lambda_{min}(D^2) = \min_i \dfrac{p_i}{\lambda_{max}(I_{C_i} A I_{C_i}^T)} \geqslant \min_i \dfrac{p_i}{\lambda_{max}(I_{C_i}^T I_{C_i}) \lambda_{max}(A)} \geqslant \min_i \dfrac{p_i}{\lambda_{max}(A)}$, since for any $i \in \{1, \dots, n\}$, for any x in $\mathbb{R}^n$ $\left\langle I_{C_i}^T I_{C_i} x \,|\, x \right\rangle = \|I_{C_i} x\|^2 \leqslant \|x\|^2$ and then $\lambda_{max}(I_{C_i}^T I_{C_i}) \leqslant 1$.

Therefore, $\lambda_{min}(A^{-\frac{1}{2}}E[Z]A^{-\frac{1}{2}}) \geqslant \min_i p_i \dfrac{\lambda_{min}(\mathbf{C}^T A\mathbf{C})}{\lambda_{max}(A)} = \min_i p_i \dfrac{\lambda_{min}(A)\lambda_{min}(\mathbf{C}\mathbf{C}^T)}{\lambda_{max}(A)}$.

$\mathbf{C}\mathbf{C}^T = \sum_i I_{C_i}^T I_{C_i} = \binom{n-1}{s-1} I_n$ and then we obtain that :

$\lambda_{min}(A^{-\frac{1}{2}}E[Z]A^{-\frac{1}{2}}) \geqslant \binom{n-1}{s-1} \dfrac{\lambda_{min}(A)}{\lambda_{max}(A)} \min_i p_i$

1.2.2 Uniform case

For any i, $p_i = \dfrac{1}{\binom{n}{s}}$ is the uniform probability of choosing s rows uniformly on $\{1, \ldots, n\}$, knowing that s is the sketch size. That leads towards that corollary of **Proposition** 1.2.1 :

Corollary 1.2.2

$$\lambda_{min}(A^{-\frac{1}{2}}E[Z]A^{-\frac{1}{2}}) \geqslant \frac{s}{n} \frac{\lambda_{min}(A)}{\lambda_{max}(A)}$$

2. *Randomized orthonormal systems*

This type of randomized system is well-suited for big data regression, thanks to the efficiency of matrix multiplication used in this method.
When the dimension of our matrix A is n, we denote by H_n the Hadamard matrix (well defined if the dimension of the problem n is a power of 2) defined recursively as :

$H_p = ..$ and $H_1 = 1.$

The Hadamard sketch consists of choosing a sketch matrix $S \in \mathcal{M}_{s,n}$ where s is called the sketch size of the problem, as follows :
we sample s $i.i.d.$ rows of the form $s^T = e_j^T H_n D$ with probability $\frac{1}{n}$ for $j = 1, \ldots, n$,where $(e_j)_j$ forms a canonical base of $\mathbb{R}^n$, and $D = diag(\nu)$ is a diagonal matrix of $i.i.d.$ Rademacher variables $\nu \in \{-1, 1\}^n$.

2.1 Algorithm

2.2 Convergence rate (draft)

Now we denote by $Z = AS^T(SAS^T)^{-1}SA$, where S is our Hadamard random matrix.

The convergence rate is then $\rho = 1 - \lambda_{min}(A^{-\frac{1}{2}}E[Z]A^{-\frac{1}{2}})$
Notice that $S_i = I_{C_i}HD.$ where $(C_i)_i$ are uniform random subsets of $\{1, \ldots, n\}$ of size s, as defined in the $Randomized\ Newton$ section 1.

Let's condition on the Rademacher diagonal matrix D.

Define by $\tilde{A}_D = \frac{H}{\sqrt{n}}DAD\frac{H^T}{\sqrt{n}}$. We obtain that :

$$\begin{aligned}
A^{-\frac{1}{2}}E[Z|D]A^{-\frac{1}{2}} &= E[A^{\frac{1}{2}}S^T(SAS^T)^{-1}SA^{\frac{1}{2}}|D] \\
&= \sum_i p_i A^{\frac{1}{2}}DH^T I_{C_i}^T(I_{C_i}HDADH^T I_{C_i}^T)^{-1}I_{C_i}HDA^{\frac{1}{2}} \\
&= A^{\frac{1}{2}}DH^T E[I_C^T(I_C\tilde{A}I_C^T)^{-1}I_C]HDA^{\frac{1}{2}} \\
&= nDH^{-1}\tilde{A}^{\frac{1}{2}}E[I_C^T(I_C\tilde{A}I_C^T)^{-1}I_C]\tilde{A}^{\frac{1}{2}}n(H^T)^{-1}D \\
&= DH^T\tilde{A}^{\frac{1}{2}}E[I_C^T(I_C\tilde{A}I_C^T)^{-1}I_C]\tilde{A}^{\frac{1}{2}}HD.
\end{aligned}$$

<u>(following to be changed)</u>
Hence :

$\rho = 1 - \lambda_{min}(A^{-\frac{1}{2}}E[Z]A^{-\frac{1}{2}}) = 1 - \lambda_{min}(\tilde{A}^{\frac{1}{2}}E[I_C^T(I_C\tilde{A}I_C^T)^{-1}I_C]\tilde{A}^{\frac{1}{2}})$

We recognize the convergence rate in the Randomized Newton Method and then, denoting by $\rho_{Newton}(M)$ the convergence rate of the Newton method associated with the definite positive matrix M, we obtain that :
$\rho = 1 - \lambda_{min}(\tilde{A}^{\frac{1}{2}} E[I_C^T(I_C\tilde{A}I_C^T)^{-1}I_C]\tilde{A}^{\frac{1}{2}}) = 1 - (1 - \rho_{Newton}(\tilde{A})) = \rho_{Newton}(\tilde{A}) = \rho_{Newton}(A)$, since A and $\tilde{A}$ have the same eigenvalues.

Proposition 2.2.1

$$\lambda_{min}(A^{-\frac{1}{2}}E[Z]A^{-\frac{1}{2}}) \geqslant \frac{s}{n}\frac{\lambda_{min}(A)}{\lambda_{max}(A)}$$

3. *Count-min Sketches*

3.1 Algorithm

3.2 Convergence rate

S is constructed as follows :
For every $i \in \{1, \ldots, n\}$, l is chosen uniformly on $\{1, \ldots, n\}$ and ϵ uniformly on $\{-1, 1\}$, then S is updated in his l^{th} row as :
$S(l, :) := S(l, :) + \epsilon\, e_i^T$, where e_i^T is the i^{th} coloumn of the identity matrix.

4. *Conclusion*

References

[1] ROBERT GOWER AND PETER RICHTARIK, Randomized iterative methods for linear systems, SIAM, (2015).