The Risk Substitution Rate as a Fundamental Risk Aversion Measure

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Abstract

The risk substitution rate is a measure of the willingness to substitute m^{th}- for n^{th}-degree risk increases and can represent $(n/m)^{th}$-degree Ross more risk aversion (Liu and Meyer 2013). Alternatively, generalizations of the Arrow-Pratt risk premium, Pratt's probability premium, and Jindapon and Neilson's paying for risk reductions approach can represent that risk aversion order (Liu and Neilson 2019). This note extends the latter measures to the attitudes determining the risk impact on utility derivatives for additive and multiplicative risk and compares them to the risk substitution rate. Across all those decision contexts, the risk substitution rate not only provides a homogenous interpretation for the other measures, but yields itself important quantifications. Accounting for those relations may be helpful when studying such preferences.

1 Introduction

Liu and Meyer's (2013) $(n/m)^{th}$-degree risk substitution rate measures a decision maker's willingness to substitute an m^{th}- for an n^{th}-degree increase in additive risk. Comparing these rates for two decision makers can express $(n/m)^{th}$-degree Ross more risk aversion: the more $(n/m)^{th}$-degree Ross risk-averse has the higher rate. In a complementary result, Liu and Neilson (2019) show that, when appropriately generalized, the Arrow-Pratt risk premium, Pratt's (1964) probability premium, and Jindapon and Neilson's (2007) paying for risk reductions

(PRR) approach can represent this risk aversion order equivalently.[1] In those representations, the risk substitution rate has actually a central role. For, it quantitatively coincides with the generalized risk premium and the generalized probability premium, and it quantifies the willingness to commit to the risk tradeoff determining the PRR.[2] The closeness between the risk substitution rate and those measures is of interest when measuring those preferences. Thus, the choice-based measures in Liu and Neilson provide different complementary ways to quantify risk substitution rates.

This note gauges the risk substitution rate's central role in comparative risk aversion more generally by analyzing two generic decision contexts: direct risk effects on utility derivatives and multiplicative risk. Direct risk effects on utility derivatives are often decisive for reactions to risk.[3] Decisions under multiplicative risk deserve a special attention for their distinct character. By their capacity to mitigate or increase risk exposure directly, such decisions involve countervailing effects, and behavioral reactions may run counter established intuition.[4]

The main result is that, for appropriate definitions, the closeness between risk substitution rate and Liu and Neilson measures from EU comparisons under additive risk extends to all those decision contexts. Hence, across all those contexts, the adapted measures from Liu and Neilson can predict the corresponding risk substitution rate.

Two caveats arise when comparing the attitudes toward multiplicative risk impacts on utility derivatives. First, the adapted Liu and Neilson measures are restricted to uniform

[1] The original definitions of Arrow-Pratt risk premium and probability premium refer to the willingness to trade a first-degree risk increase against second-degree risk, $(n, m) = (2, 1)$. Jindapon and Neilson refer to the trade-off between first- and n^{th}-degree risk increases, $(n, m) = (n, 1)$.

[2] Liu and Neilson mention in passing the rate's identity with the generalized risk and probability premia, but, like Liu and Meyer, they do not elaborate on its role in relation to PRR. Both papers only focus on comparisons of plain expected utility and additive risk.

[3] For example, precautionary saving or labor depend on the risk impact on future marginal utility (Eeckhoudt and Schlesinger 2008, Netzer and Scheuer 2007). The risk impact on the second utility derivative determines a decision maker's willingness to change such a decision due to a risk increase (Heinzel 2021). In a similar vein, the background risk effect on derived risk aversion hinges on intrapersonal comparisons of risk attitudes of consecutive degrees (Wang and Li 2014).

[4] Multiplicative risks are widespread. Classic examples include saving with risky return, labor/leisure decisions with risky wage rate, portfolio choice with a risky asset, partial insurance, and production with risky prices or output. Bostian and Heinzel (2020) find that a prudent decision maker, who expands saving in response to risk on exogenous future income, will typically reduce saving in response to return risk.

(positive or negative) impacts of n^{th}- and m^{th}-degree risk increases, like risk substitution rates (see Heinzel 2021). Second, in order to coincide with a risk substitution rate, risk premia need to be defined as path-dependent in the spirit of Machina and Neilson (1987). This definition departs from the conventional willingness-to-pay approach to risk premia, such as Bostian and Heinzel's (2018) multiplicative precautionary premium.

Section 2 provides basic concepts and results to analyze the risk comparative statics of utility derivatives under additive risk. Section 3 adapts the Liu and Neilson measures and establishes the central role of risk substitution rates for interpersonal comparisons of the attitudes determining the impact of additive risk increases on utility derivatives. Section 4 develops a similar analysis for decisions under multiplicative risk. Section 5 concludes.

2 Preliminaries

To define more n^{th}-degree risk, let $F(z) = F^{[1]}(z)$ and $G(z) = G^{[1]}(z)$ be the cumulative distribution functions (CDFs) of two random variables $\tilde{z}_l$ and $\tilde{z}_h$ with finite support in $[z_a, z_b]$ and equal start and end points, $F(z_a) = G(z_a) = 0$ and $F(z_b) = G(z_b) = 1$. Denote higher-order CDFs $F^{[k]}(z) = \int_a^z F^{[k-1]}(t)\,dt$ for $k = 2, 3, \ldots$, and similarly for $G(z)$.

Definition 1 (Ekern 1980) *$G(z)$ has more n^{th}-degree risk than $F(z)$ if*

$$
\begin{aligned}
G^{[k]}(z_b) &= F^{[k]}(z_b) \quad \text{for } k = 1, \ldots, n \\
G^{[n]}(z) &\geq F^{[n]}(z) \quad \text{for all } z \in [z_a, z_b] \text{ and ``}>\text{'' holding for some } z \in (z_a, z_b).
\end{aligned}
$$

Such a risk increase implies an n^{th}-degree stochastic dominance shift from F to G, letting the first $n-1$ moments unchanged. I will also denote risk in its low state $\tilde{z}_l$, associated with F, and risk in its high state $\tilde{z}_h$, associated with G. With an analogous definition, I will consider CDF $H(z)$ of random variable $\tilde{z}_m$ that has more m^{th}-degree risk than $F(z)$.

Ekern shows then that for an additive risk $\tilde{\varepsilon}$ to wealth x, in short $\tilde{x} = x + \tilde{\varepsilon}$, $G(x)$ has more n^{th}-degree risk than $F(x)$ if and only if every n^{th}-degree risk averter prefers F to G.

An expected utility (EU) maximizer has k^{th}-degree risk aversion if $(-1)^{k+1} u^{(k)}(x) > 0$ for all x. In this definition, $u^{(k)}$ is utility function u's k^{th} derivative defined on $[a,b] \in \mathbb{R}_0^+$. Unless stated differently, I assume decision makers to be $(n+j)^{th}$- and $(m+j)^{th}$-degree risk averse, with $n > m \geq 1$ and $j \geq 0$.

To compare two decision makers u and v's attitudes toward the impact of a risk increase on their j^{th} utility derivative, Heinzel (2021) provides the following definitions and results.

Definition 2 *For $j \geq 0$, $u(x)$ is $((n+j)/(m+j))^{th}$-degree Ross more risk averse than $v(x)$ if there exists a scalar $\lambda > 0$ such that*

$$\frac{u^{(n+j)}(x_a)}{v^{(n+j)}(x_a)} \geq \lambda \geq \frac{u^{(m+j)}(x_b)}{v^{(m+j)}(x_b)} \quad \textit{for all} \;\; x_a, x_b \in [a,b] \subseteq \mathbb{R}_0^+ \tag{1}$$

This definition covers Liu and Meyer's $(n/m)^{th}$-degree Ross more risk aversion for $j = 0$.

Representing $((n+j)/(m+j))^{th}$-degree Ross more risk aversion requires to define risk substitution rates at the level of the j^{th} utility derivative.

Definition 3 *The $((n+j)/(m+j))^{th}$-degree risk substitution rate is*

$$T_{j,u}^{(n/m)}(\tilde{x}_l, \tilde{x}_h, \tilde{x}_m) = \frac{Eu^{(j)}(\tilde{x}_l) - Eu^{(j)}(\tilde{x}_h)}{Eu^{(j)}(\tilde{x}_l) - Eu^{(j)}(\tilde{x}_m)} \tag{2a}$$

Alternatively, $T_{j,u}^{(n/m)}$ is the scalar solving

$$Eu^{(j)}(\tilde{x}_h) = \left(1 - T_{j,u}^{(n/m)}\right) Eu^{(j)}(\tilde{x}_l) + T_{j,u}^{(n/m)} Eu^{(j)}(\tilde{x}_m) \tag{2b}$$

Liu and Meyer's rate is covered as $T_0^{(n/m)}$. For any j, $T_j^{(n/m)}$ is positive and unit-free, like $T_0^{(n/m)}$. Equivalently, $T_{j,u}^{(n/m)}$ gauges u's willingness to substitute an m^{th}- for an n^{th}-degree risk increase at the level of the j^{th} utility derivative and u's willingness to substitute an $(m+j)^{th}$- for an $(n+j)^{th}$-degree risk increase.[5]

[5] Heinzel (2021: Lemma 1) notes the equivalence of $(-1)^j u^{(j)}$ being $(n/m)^{th}$-degree Ross more risk averse than $(-1)^j v^{(j)}$ and $((n+j)/(m+j))^{th}$-degree Ross more risk aversion between u and v.

Rates $T_j^{(n/m)}$ can represent $((n+j)/(m+j))^{th}$-degree Ross more risk aversion.

Theorem 1 (Heinzel 2021) *For any $j \geq 0$, the $((n+j)/(m+j))^{th}$-degree Ross more risk averse order between u and v in* (1) *is equivalent to $T_{j,u}^{(n/m)} \geq T_{j,v}^{(n/m)}$.*

For $j = 0$, this theorem covers Liu and Meyer (2013: Theorem 1).

3 Comparative Aversion to Additive Risk

This section extends Liu and Neilson's analysis to the risk impact on utility derivatives. Moreover, I establish the quantitative coincidence of path-dependent risk premia and probability premia with risk substitution rate $T_j^{(n/m)}$ from (2) and clarify $T_j^{(n/m)}$'s role in relation to PRR for any j. Liu and Neilson's original concepts are covered for $j = 0$.

3.1 Path-Dependent Risk Premia

Let $\tilde{x}_h$ have more n^{th}-degree risk than initial wealth $\tilde{x}_l$, and let $\{\tilde{x}(\pi_j)\}_{\pi_j \in [0,B]}$ be the continuous path of m^{th}-degree increasing risk from $\tilde{x}_l$ with $[0,B] \in [0,1)$, $\tilde{x}(0) = \tilde{x}_l$, and so that $\tilde{x}(\pi_j')$ has more m^{th}-degree risk than $\tilde{x}(\pi_j)$ for every $\pi_j' > \pi_j \geq 0$. The *path-dependent* $(m/j)^{th}$*-degree risk premium* is the scalar π_j satisfying

$$Eu^{(j)}(\tilde{x}_h) = Eu^{(j)}(\tilde{x}(\pi_j)) \tag{3}$$

This definition generalizes Machina and Neilson's (1987) random risk premium approach from $Eu(\tilde{x}_h) = Eu(\tilde{x}_l - \pi_0 \tilde{\eta})$, where π_0 is a positive (non-random) "scale-of-premium" parameter and $\tilde{\eta}$ is a nonnegative random variable. Using Machina and Neilson's terms, (3) involves the tradeoff between the impacts on the j^{th} utility derivative of an n^{th}- and an m^{th}-degree increase on $\tilde{x}_l$, the latter in the form of $-\tilde{\eta}$ scaled by π_j. For example, as applied to a saving model with risky future income $\tilde{y}_2$, saving amount s, and safe gross return R, Liu's (2014) precautionary premium θ^{y_2}, from

$$Eu'(\tilde{y}_{2,h} + sR) = Eu'(\tilde{y}_{2,l} + sR - \theta^{y_2})$$

is the special case with $j = m = 1$ and $\tilde{\eta}$ collapsed to unity. Premium θ^{y_2} measures the precautionary motive in terms of the safe reduction in $\tilde{y}_{2,l}$ having the same effect on saving as the n^{th}-degree increase in y_2 risk.

To derive the identity $\pi_j = T_j^{(n/m)}$, relate the random variables in (3) to CDFs F, G, H from Section 2, so that $\tilde{x}\left(\pi_j\right)$ has CDF $\left(1 - \pi_j\right) F + \pi_j H$ for all π_j. Then, rewrite (3) as

$$Eu^{(j)}\left(\tilde{x}_h\right) = \left(1 - \pi_j\right) Eu^{(j)}\left(\tilde{x}_l\right) + \pi_j Eu^{(j)}\left(\tilde{x}_m\right)$$

where π_j is identical to $T_j^{(n/m)}$ in (2b) for all j. For example, Liu's θ^{y_2} is thus equal to $T_1^{(n/m)}$ expressing the willingness to substitute a second- for an $(n+1)^{th}$-degree risk increase.

3.2 Probability Premia

To determine probability premium $p_j^{(n/m)}$, set the j^{th} utility derivative's expectation at initial wealth $\tilde{w}$ equal to a compound lottery that combines the j^{th} utility derivative's expectations at n^{th}-degree increase $\tilde{y}_{+n}$ and m^{th}-degree *decrease* $\tilde{y}_{-m}$ from $\tilde{w}$. A decision maker has thus the preference $\tilde{y}_{-m} \succ_{m+j} \tilde{w} \succ_{n+j} \tilde{y}_{+n}$, where "$\succ_{k+j}$" denotes the strict preference relation regarding a $(k + j)^{th}$-degree risk increase, for $k = m, n$. The $(m/j)^{th}$*-degree probability premium* is the scalar $p_j^{(n/m)}$ solving

$$Eu^{(j)}\left(\tilde{w}\right) = p_j^{(n/m)} Eu^{(j)}\left(\tilde{y}_{-m}\right) + \left(1 - p_j^{(n/m)}\right) Eu^{(j)}\left(\tilde{y}_{+n}\right) \tag{4}$$

Premium $p_j^{(n/m)}$ indicates the weight the decision maker requires for the m^{th}-degree decrease at the expense of the n^{th}-degree increase from initial wealth $\tilde{w}$ in order that the affine combination of the j^{th} utility derivative's expectations at those two lotteries is equal to the j^{th}

utility derivative's expectation at $\tilde{w}$. Liu and Neilson's case is covered for $j = 0$.[6]

To see the identity $p_j^{(n/m)} = T_j^{(n/m)}$, express the distributions of $\tilde{w}$, $\tilde{y}_{-m}$, and $\tilde{y}_{+n}$ using CDFs F, G, and H. For all F, G, H with $G \sim_{m+j} F \succ_{m+j} H$ and $F \succ_{n+j} G$, we have that

$$\frac{F}{2} + \frac{G}{2} \succ_{m+j} \frac{F}{2} + \frac{H}{2} \succ_{n+j} \frac{G}{2} + \frac{H}{2}$$

Replace $\tilde{w}$ by $(F + H)/2$, $\tilde{y}_{-m}$ by $(F + G)/2$, and $\tilde{y}_{+n}$ by $(G + H)/2$, and rewrite (4) as

$$\frac{Eu^{(j)}(\tilde{x}_l) + Eu^{(j)}(\tilde{x}_m)}{2} = p_j^{(n/m)} \frac{Eu^{(j)}(\tilde{x}_l) + Eu^{(j)}(\tilde{x}_h)}{2} + \left(1 - p_j^{(n/m)}\right) \frac{Eu^{(j)}(\tilde{x}_h) + Eu^{(j)}(\tilde{x}_m)}{2}$$
$$\Leftrightarrow \quad Eu^{(j)}(\tilde{x}_h) = \left(1 - p_j^{(n/m)}\right) Eu^{(j)}(\tilde{x}_l) + p_j^{(n/m)} Eu^{(j)}(\tilde{x}_m)$$

where $p_j^{(n/m)}$ is identical to $T_j^{(n/m)}$ in (2b) for all j.

In (4), $p_j^{(n/m)}$ gives a weighting to the m^{th}-degree decrease versus the n^{th}-degree increase from initial wealth $\tilde{w}$ at the level of the j^{th} utility derivative. Premium $p_j^{(n/m)}$'s equality to $T_j^{(n/m)}$ shows that this weighting not only directly depends on, but is indeed quantitatively identical to the willingness to substitute an $(m + j)^{th}$- for an $(n + j)^{th}$-degree risk increase.

3.3 Paying for Risk Reductions

Central to the PRR approach are the choice problem

$$\max_{\alpha_j} (-1)^j Eu^{(j)} [\tilde{x}(\alpha_j)] \tag{5a}$$

[6]Liu and Neilson's $p_0^{(n/m)}$, in turn, covers Pratt's probability premium q from $u(w) = Eu\left(w + \tilde{h}\right) = \frac{1}{2}(1 + q) u(w + h) + \frac{1}{2}(1 - q) u(w - h)$ for $(n, m) = (2, 1)$, $\tilde{w} = w$, $\tilde{z} = w + h$, $\tilde{y} = (w + h, 1/2; w - h, 1/2)$, and relabeling $p/2 = q$.

Close in spirit to Pratt's notion, Jindapon et al. (2021) develop an $(n/m)^{th}$-degree probability premium $p_{n/m}$ that measures the strength of n^{th}-degree risk apportionment in the Eeckhoudt et al. (2009) sense. Contrary to $p_0^{(n/m)}$, $p_{n/m}$ does not coincide with T_0, and $(n/m)^{th}$-degree Ross more risk aversion is only sufficient for $p^u_{n/m} \geq p^v_{n/m}$.

with unique solution $\alpha_j^* \geq 0$ by assumption, and wealth path $\tilde{x}(\alpha_j)$ with $d\tilde{x}(\alpha_j)/d\alpha_j > 0$. As α increases, $\tilde{x}(\alpha_j)$ simultaneously becomes less n^{th}- and more m^{th}-degree risky.

To link α_j to $T_j^{(n/m)}$, let $G(x)$ be the distribution of initial wealth. Then, the decision maker has the opportunity to decrease G's n^{th}-degree risk to $(1-\alpha_j)G + \alpha_j F = G + \alpha_j (F - G)$, at the cost of incurring an m^{th}-degree risk increase $t_j(\alpha_j)(H-F)$, whose unit cost $t_j(0) = 0$ and $t_j', t_j'' > 0$ for all α_j. That tradeoff hinges on the preference $G \prec_{n+j} (1-\alpha_j)G + \alpha_j F \prec_{n+j} F$ on the one hand, and on $G \sim_{m+j} F \succ_{m+j} t_j(\alpha_j)(H-F) \succ_{m+j} H$ on the other. Overall, the decision maker compares, hence, G to $(1-\alpha_j)G + \alpha_j F + t_j(\alpha_j)(H-F)$, where

$$G \begin{cases} \prec_{n+j} (1-\alpha_j)G + \alpha_j F + t_j(\alpha_j)(H-F) \\ \succ_{m+j} (1-\alpha_j)G + \alpha_j F + t_j(\alpha_j)(H-F) \end{cases}$$

Based on this tradeoff, problem (5a) can be reformulated as choosing α_j such as to maximize

$$\begin{aligned} & (-1)^j \int_a^b u^{(j)}(x)\, d\left[G(x) + \alpha_j (F(x) - G(x)) + t_j(\alpha_j)(H(x) - F(x))\right] \\ = \; & (-1)^j \left\{ Eu^{(j)}(\tilde{x}_h) + \alpha_j \left[Eu^{(j)}(\tilde{x}_l) - Eu^{(j)}(\tilde{x}_h)\right] - t_j(\alpha_j)\left[Eu^{(j)}(\tilde{x}_l) - Eu^{(j)}(\tilde{x}_m)\right] \right\} \end{aligned}$$

The first-order condition is equivalent to

$$Eu^{(j)}(\tilde{x}_h) = \left(1 - t_j'(\alpha_j)\right) Eu^{(j)}(\tilde{x}_l) + t_j'(\alpha_j)\, Eu^{(j)}(\tilde{x}_m) \tag{5b}$$

where $t_j'(\alpha_j)$ is equal to $T_j^{(n/m)}$ in (2b) for all j. In words, the decision maker chooses payment α_j such that the marginal unit cost $t_j'(\alpha_j)$ of the m^{th}-degree risk increase at the level of the j^{th} utility derivative, incurred in exchange for the n^{th}-degree risk decrease, is equal to the willingness to substitute an $(m+j)^{th}$- for an $(n+j)^{th}$-degree increase, measured by $T_j^{(n/m)}$. Here, the risk substitution rate measures the willingness to commit to that risk tradeoff.

3.4 The Risk Substitution Rate's Central Role

Proposition 1 summarizes the results for the three preference measures.

Proposition 1 *For any $j \geq 0$, each of* (3)*'s path-dependent $(m/j)^{th}$-degree risk premium π_j,* (4)*'s $(m/j)^{th}$-degree probability premium $p_j^{(n/m)}$, and* (5b)*'s marginal unit cost $t'_j(\alpha_j)$, evaluated at the optimal PPR amount α_j^*, quantitatively coincides with* (2)*'s $T_j^{(n/m)}$.*

Jointly with Theorem 1, Proposition 1 implies that each of the preference measures from (3)–(5) can represent $((n+j)/(m+j))^{th}$-degree Ross more risk aversion.

Theorem 2 *For any $j \geq 0$, u being $((n+j)/(m+j))^{th}$-degree Ross more risk averse than v can equivalently be expressed by* (2)*'s risk substitution rates $T_{j,u}^{(n/m)} \geq T_{j,v}^{(n/m)}$,* (3)*'s path-dependent $(m/j)^{th}$-degree risk premia, $\pi_{j,u} \geq \pi_{j,v}$;* (4)*'s $(m/j)^{th}$-degree probability premia, $p_{j,u}^{(n/m)} \geq p_{j,v}^{(n/m)}$; and* (5)*'s optimal PPR amounts, $\alpha_{j,u}^* \geq \alpha_{j,v}^*$.*

The representation in terms of u's and v's optimal PRR amounts, $\alpha_{j,u}^* \geq \alpha_{j,u}^*$, is due to $t'_j > 0$. For $j = 0$, Theorem 2 covers Liu and Neilson's Theorems 1–3.

Together, Theorems 1 and 2 and Proposition 1 describe the risk substitution rate $T_j^{(n/m)}$'s fundamental role as a risk aversion measure: it lends not only a homogenous interpretation to the preference measures from (3)–(5), namely, as expressing the willingness to substitute an $(m+j)^{th}$- for an $(n+j)^{th}$-degree risk increase. But, $T_j^{(n/m)}$ even *quantitatively predicts* the path-dependent $(m/j)^{th}$-degree risk premium π_j, the $(m/j)^{th}$-degree probability premium $p_j^{(n/m)}$, and the marginal unit cost $t'_j(\alpha_j)$ of the m^{th}-degree risk increase at the level of the j^{th} utility derivative for the equilibrium PRR amount.

The first insight implies that, in each case, the decision maker with the higher preference measure from (3)–(5) also has the higher willingness to substitute an $(m+j)^{th}$- for and $(n+j)^{th}$-degree risk increase. The main importance of the second insight may rather arises in the converse perspective: any quantification of the measures from (3)–(5) simultaneously gauges $T_j^{(n/m)}$. Indeed, the amounts of π_j, $p_j^{(n/m)}$, and $t'_j(\alpha_j^*)$ are interchangeable with $T_j^{(n/m)}$.

4 Comparative Aversion to Multiplicative Risk

This section extends the analysis in Section 3 to decisions under multiplicative risk. Whereas Ekern's definitions and results similarly apply to a risky return $\tilde{\rho}$ on choice x, the definition of $((n+j)/(m+j))^{th}$-degree Ross more risk aversion now needs to be adapted to involve the j^{th} x derivative of utility $h_{[j],u}(\rho) = \rho^j u^{(j)}(x\rho)$ (Heinzel 2021).

The direction in which $h_{[j],u}(\rho)$ moves due to a risk increase depends on countervailing effects: $(-1)^j\, h_{[j],u}(\rho)$ moves positively [negatively] if and only if $(-1)^{n+j-1}\, h_{[j],u}^{(n)}(\rho) > [\, < \,]\ 0$.[7]

Definition 4 *Suppose $j \geq 0$, and let u and v be such that $(-1)^{k+j-1}\, h_{[j],f}^{(k)}(\rho) > [\, < \,]\ 0$ for $k = m, n$ and $f \in \{u, v\}$ and have identical choices of x under the reference return $\tilde{\rho}_l$. Then, u is $((n+j)/(m+j))^{th}$-degree Ross more risk averse than v with respect to a return risk increase from $\tilde{\rho}_l$ to $\tilde{\rho}_h$, if there exists a scalar $\lambda > 0$ such that*

$$\frac{h_{[j],u}^{(n)}(\rho_a)}{h_{[j],v}^{(n)}(\rho_a)} \geq \lambda \geq \frac{h_{[j],u}^{(m)}(\rho_b)}{h_{[j],u}^{(m)}(\rho_b)} \quad \text{for all } \rho_a, \rho_b \text{ such that } x\rho_a, x\rho_b \in [a, b] \tag{6}$$

Note that Definition 4 does not presuppose u's $(n+j)^{th}$- and $(m+1)^{th}$-degree risk aversion. However, those attitudes permit to identify the countervailing effects in $h_{[j],u}^{(n)}(\rho)$.[8]

Risk substitution rate $\widehat{T}_j^{(n/m)}$ now measures the willingness to substitute an m^{th}- for the n^{th}-degree return-risk increase at the level of the j^{th} utility derivative.

Definition 5 *Suppose $j \geq 0$, and u such that $(-1)^{k+j-1}\, h_{[j],u}^{(k)}(\rho) > [\, < \,]\ 0$ for $k = n, m$. The $(n/m)^{th}$-degree risk substitution rate for return risk $\tilde{\rho}$ and the j^{th} utility derivative is given by*

$$\widehat{T}_{j,u}^{(n/m)}(\tilde{\rho}_l, \tilde{\rho}_h, \tilde{\rho}_m) = \frac{E\left[h_{[j],u}(\tilde{\rho}_l)\right] - E\left[h_{[j],u}(\tilde{\rho}_h)\right]}{E\left[h_{[j],u}(\tilde{\rho}_l)\right] - E\left[h_{[j],u}(\tilde{\rho}_m)\right]} \tag{7a}$$

[7] Heinzel (2021) provides an explicit formula for $h_{[j],u}^{(n)}(\rho)$.

[8] For example, in a saving model with risky interest rate, $u\left(s\widetilde{R}\right)$, the first term in $(-1)^n\, h_{[1],u}^{(n)}(R) = (-1)^n \left[u^{(n+1)}\left(s\widetilde{R}\right) sR + nu^{(n)}\left(s\widetilde{R}\right)\right]$ represents the positive precautionary effect and the second term the negative substitution effect if u is $(n+1)^{th}$- and n^{th}-degree risk averse (Bostian and Heinzel 2020).

or, alternatively, $\widehat{T}_{j,u}^{(n/m)}$ *is the scalar solving*

$$E\left[h_{[j],u}\left(\tilde{\rho}_h\right)\right]=\left(1-\widehat{T}_{j,u}^{(n/m)}\right)E\left[h_{[j],u}\left(\tilde{\rho}_l\right)\right]+\widehat{T}_{j,u}^{(n/m)}\,E\left[h_{[j],u}\left(\tilde{\rho}_m\right)\right] \tag{7b}$$

Like $T_j^{(n/m)}$, $\widehat{T}_j^{(n/m)}$ is positive and unit-free. Definition 5 rules out the cases with split signs between numerator and denominator, although empirically conceivable, because a negative risk substitution rate would be hard to interpret. Rate $\widehat{T}_j^{(n/m)}$ measures a decision maker's willingness to substitute an $(m+j)^{th}$- for an $(n+j)^{th}$-degree return-risk increase.

Rates $\widehat{T}_j^{(n/m)}$ can represent Ross more risk aversion as in Definition 4.

Theorem 3 (Heinzel 2021) *For any* $j\geq 0$, *the* $((n+j)/(m+j))^{th}$*-degree Ross more risk averse order between* u *and* v *as in* (6) *is equivalent to* $\widehat{T}_{j,u}^{(n/m)}\geq\widehat{T}_{j,v}^{(n/m)}$.

In the following, I adapt the risk aversion measures from Section 3 to return risk and detail how those adapted measures relate to risk substitution rate $\widehat{T}_j^{(n/m)}$.

4.1 Path-Dependent Multiplicative Risk Premia

Let $\tilde{\rho}_h$ have more n^{th}-degree risk than the initial return $\tilde{\rho}_l$, and let $\{\tilde{\rho}\left(\widehat{\pi}_j\right)\}_{\widehat{\pi}_j\in[0,\widehat{B}]}$ be the continuous path of m^{th}-degree increasing risk from $\tilde{\rho}_l$ with $\left[0,\widehat{B}\right]\in[0,1)$, $\tilde{\rho}\left(0\right)=\tilde{\rho}_l$, and so that $\tilde{\rho}\left(\widehat{\pi}_j'\right)$ has more m^{th}-degree risk than $\tilde{\rho}\left(\widehat{\pi}_j\right)$ for every $\widehat{\pi}_j'>\widehat{\pi}_j\geq 0$. The *path-dependent* $(m/j)^{th}$*-degree multiplicative risk premium* is the scalar $\widehat{\pi}_j$ satisfying:

$$Eh_{[j],u}\left(\tilde{\rho}_h\right)=Eh_{[j],u}\left(\tilde{\rho}\left(\widehat{\pi}_j\right)\right) \tag{8}$$

As adapted to multiplicative, the Machina and Neilson (1987) random risk premium approach risk would read $Eu\left(x\tilde{\rho}_h\right)=Eu\left(x\left(\tilde{\rho}_l-\widehat{\pi}_0\,\tilde{\rho}_\eta\right)\right)$, where $\widehat{\pi}_0$ is a positive (non-random) "scale-of-premium" parameter and $\tilde{\rho}_\eta$ is a nonnegative random return. In those terms, the tradeoff in (8) is between the impacts on the j^{th} utility derivative of an n^{th}-degree and an m^{th}-degree increase from $\tilde{\rho}_l$, the latter in the form of $-\tilde{\rho}_\eta$ scaled by $\widehat{\pi}_j$. Pratt's (1964) multiplicative risk premium, derived from $u\left(x\left(E\tilde{\rho}-\widehat{\pi}_0\right)\right) = Eu\left(x\tilde{\rho}\right)$, is a special case for $(n,m)=(2,1)$ and $\tilde{\rho}_\eta$ collapsed to unity.

Interestingly, when applied at the level of utility derivatives, the parallelism with conventional multiplicative risk premia breaks down, as opposed to additive risk (cf. Section 3.1). To wit, consider now a saving model with risky gross return $\tilde{R}$. In this framework, Bostian and Heinzel's (2018) multiplicative precautionary premium θ^R, from

$$E\left[u'\left(s\left(\tilde{R}_l-\theta^R\right)\right)\tilde{R}_l\right]=E\left[u'\left(s\tilde{R}_h\right)\tilde{R}_h\right] \tag{9}$$

gauges the *proportion* of saving such that the safe change $s\theta^R$ in second-period consumption has the same effect on saving as the risk increase. Premium θ^R is unit-free and trades off the n^{th}-degree return risk increase against a first-degree risk increase – by subtracting $s\theta^R$ – in *future consumption* at the level of marginal utility. Importantly, θ^R may be negative, because the saving reaction to return risk simultaneously depends on a positive precautionary and a negative substitution effect.

When applying the random risk premium approach to the saving model, the path-dependent m^{th}-degree multiplicative precautionary premium $\widehat{\theta}$, from

$$E\left[u'\left(s\left(\tilde{R}_l-\widehat{\theta}\cdot\tilde{R}_\eta\right)\right)\left(\tilde{R}_l-\widehat{\theta}\cdot\tilde{R}_\eta\right)\right]=E\left[u'\left(s\tilde{R}_h\right)\tilde{R}_h\right]$$

captures the decision maker's willingness to substitute an m^{th}- for the n^{th}-degree return risk increase at the level of marginal utility (Heinzel 2021). Like $\widehat{\pi}_j$ from (8), scale-of-premium parameter $\widehat{\theta}$ is positive by definition. However, random variable $\tilde{R}_\eta$ is only positive when the return risk increase induces more saving; when saving decreases, $\tilde{R}_\eta$ is negative. Hence, the signs of precautionary motives $s\,\widehat{\theta}\tilde{R}_\eta$ and $s\,\theta^R$ are the same. But, in general, the two amounts will not coincide due to their different definitions. Because the sign ambiguity concerns any $h_{[j]}\left(\rho\right)$ derivatives for $j\geq 1$, the present remarks apply to any application of the random risk premium approach under return risk in those cases.

The interpretation of $\widehat{\pi}_j$ for generic $j \geq 0$ derives from the fact that $\widehat{\pi}_j = \widehat{T}_j^{(n/m)}$. To establish this identity, relate (8)'s random variables in to Section 2's CDFs F, G, H applied to $\tilde{\rho}$, so that $\tilde{\rho}(\widehat{\pi}_j)$ has CDF $(1-\widehat{\pi}_j)F + \widehat{\pi}_j H$ for all $\widehat{\pi}_j$. Then, rewrite (8) as

$$Eh_{[j],u}(\tilde{\rho}_h) = (1-\widehat{\pi}_j)\, Eh_{[j],u}(\tilde{\rho}_l) + \widehat{\pi}_j Eh_{[j],u}(\tilde{\rho}_m)$$

where $\widehat{\pi}_j$ is identical to $\widehat{T}_j^{(n/m)}$ in (7b) for all j.

4.2 Probability Premia

To determine probability premium $\widehat{p}_j^{(n/m)}$, set the j^{th} utility derivative's expectation at initial wealth $x\tilde{\rho}_w$ equal to a compound lottery combining the j^{th} utility derivative's expectations for the n^{th}-degree increase $\tilde{\rho}_{+n}$ and the m^{th}-degree *decrease* $\tilde{\rho}_{-m}$ from $\tilde{\rho}_w$. A decision maker with $(-1)^{k+j-1}\, h_{[j],u}^{(k)}(\rho) > [<]\ 0$ for $k = n, m$ has the preference $\tilde{\rho}_{-m} \succ_{m+j} \tilde{\rho}_w \succ_{n+j} \tilde{\rho}_{+n}$ $[\,\tilde{\rho}_{-m} \prec_{m+j} \tilde{\rho}_w \prec_{n+j} \tilde{\rho}_{+n}\,]$.[9] The $(m/j)^{th}$*-degree probability premium for return risk increases* is the scalar $\widehat{p}_j^{(n/m)}$ solving

$$Eh_{[j],u}(\tilde{\rho}_w) = \widehat{p}_j^{(n/m)} Eh_{[j],u}(\tilde{\rho}_{-m}) + \left(1-\widehat{p}_j^{(n/m)}\right) Eh_{[j],u}(\tilde{\rho}_{+n}) \tag{10}$$

Premium $\widehat{p}_j^{(n/m)}$ indicates the weight the decision maker requires to incur the m^{th}-degree decrease at the expense of the n^{th}-degree increase from $\tilde{\rho}_w$ in order that the affine combination of the j^{th} utility derivative's expectations at those two lotteries is equal to the j^{th} utility derivative's expectation at initial wealth $x\tilde{\rho}_w$.[10]

To see the identity $\widehat{p}_j^{(n/m)} = \widehat{T}_j^{(n/m)}$, express the distributions of $\tilde{\rho}_w$, $\tilde{\rho}_{-m}$, $\tilde{\rho}_{+n}$ using CDFs

[9] This preference representation is based on Heinzel (2021: Lemma 2), which notes the equivalence of $(-1)^j\, h_{[j],u}$ being $(n/m)^{th}$-degree Ross more risk averse than $(-1)^j\, h_{[j],v}$ and $((n+j)/(m+j))^{th}$-degree Ross more risk aversion as in Definition 5.

[10] Solving (10) shows that $\widehat{p}_j^{(n/m)}$ is equally positive for a decision maker with $(-1)^{k+j-1}\, h_{[j],u}^{(k)}(\rho) > [<]\ 0$ for $k = n, m$, irrespective of whether the given preference or the one in square brackets holds.

F, G, H. For all F, G, H with $G \sim_{m+j} F \succ_{m+j} [\prec_{m+j}] H$ and $F \succ_{n+j} [\prec_{n+j}] G$,

$$\frac{F}{2} + \frac{G}{2} \succ_{m+j} [\prec_{m+j}] \frac{F}{2} + \frac{H}{2} \succ_{n+j} [\prec_{n+j}] \frac{G}{2} + \frac{H}{2}$$

Replace $\tilde{\rho}_w$ by $(F+H)/2$, $\tilde{\rho}_{-m}$ by $(F+G)/2$, and $\tilde{\rho}_{+n}$ by $(G+H)/2$, and rewrite (10) as

$$\frac{Eh_{[j],u}(\tilde{\rho}_l) + Eh_{[j],u}(\tilde{\rho}_m)}{2} = \widehat{p}_j^{(n/m)} \frac{Eh_{[j],u}(\tilde{\rho}_l) + Eh_{[j],u}(\tilde{\rho}_h)}{2} + \left(1 - \widehat{p}_j^{(n/m)}\right) \frac{Eh_{[j],u}(\tilde{\rho}_h) + Eh_{[j],u}(\tilde{\rho}_m)}{2}$$
$$\Leftrightarrow \quad Eh_{[j],u}(\tilde{\rho}_h) = \left(1 - \widehat{p}_j^{(n/m)}\right) Eh_{[j],u}(\tilde{\rho}_l) + \widehat{p}_j^{(n/m)} Eh_{[j],u}(\tilde{\rho}_m)$$

where $\widehat{p}_j^{(n/m)}$ is identical to $\widehat{T}_j^{(n/m)}$ in (7b) for all j. This identity shows that the weighting by $\widehat{p}_j^{(n/m)}$ in (10) not only directly depends on the willingness to substitute an $(m+j)^{th}$- for an $(n+j)^{th}$-degree return risk increase, but quantitatively coincides with it.

4.3 Paying for Return Risk Reductions

Central to the PRR approach applied to return risk are the choice problem

$$\max_{\widehat{\alpha}_j} (-1)^j Eh_{[j],u}[\tilde{\rho}_l(\widehat{\alpha}_j)] \tag{11a}$$

with unique solution $\widehat{\alpha}_j^* \geq 0$ by assumption, and wealth path $x\tilde{\rho}(\widehat{\alpha}_j)$ with $d\tilde{\rho}(\widehat{\alpha}_j)/d\widehat{\alpha}_j > 0$. As $\widehat{\alpha}$ increases, $\tilde{\rho}(\widehat{\alpha}_j)$ simultaneously becomes less n^{th}- and more m^{th}-degree risky.

To link $\widehat{\alpha}_j$ to $\widehat{T}_j$, let $G(\rho)$ be the distribution of initial wealth. Then, the decision maker has the opportunity to decrease G's n^{th}-degree risk to $(1-\widehat{\alpha}_j)G + \widehat{\alpha}_j F = G + \widehat{\alpha}_j(F-G)$, at the cost of incurring an m^{th}-degree risk increase $\widehat{t}_j(\widehat{\alpha}_j)(H-F)$, with unit cost $\widehat{t}_j(0) = 0$ and $\widehat{t}'_j, \widehat{t}''_j > 0$ for all α_j. That tradeoff is, on the one hand, based on the preference $G \prec_{n+j} (1-\widehat{\alpha}_j)G + \widehat{\alpha}_j F \prec_{n+j} F$ and, on the other, on $G \sim_{m+j} F \succ_{m+j} \widehat{t}_j(\widehat{\alpha}_j)(H-F) \succ_{m+j} H$.

Overall, the decision maker compares, hence, G to $(1-\widehat{\alpha}_j)\,G+\widehat{\alpha}_j F+\widehat{t}_j\,(\widehat{\alpha}_j)\,(H-F)$, where

$$G\begin{cases}\prec_{n+j}\,(1-\widehat{\alpha}_j)\,G+\widehat{\alpha}_j F+\widehat{t}_j\,(\widehat{\alpha}_j)\,(H-F)\\ \succ_{m+j}\,(1-\widehat{\alpha}_j)\,G+\widehat{\alpha}_j F+\widehat{t}_j\,(\widehat{\alpha}_j)\,(H-F)\end{cases}$$

Based on this tradeoff, problem (11a) can be reformulated as choosing $\widehat{\alpha}_j$ such as to maximize

$$\begin{aligned}&(-1)^j\int_{\rho_a}^{\rho_b} h_{[j],u}\,(\rho)\,d\left[G\,(\rho)+\widehat{\alpha}_j\,(F\,(\rho)-G\,(\rho))+\widehat{t}_j\,(\widehat{\alpha}_j)\,(H\,(\rho)-F\,(\rho))\right]\\ =\;&(-1)^j\left\{Eh_{[j],u}\,(\tilde{\rho}_h)+\widehat{\alpha}_j\left[Eh_{[j],u}\,(\tilde{\rho}_l)-Eh_{[j],u}\,(\tilde{\rho}_h)\right]-\widehat{t}_j\,(\widehat{\alpha}_j)\left[Eh_{[j],u}\,(\tilde{\rho}_l)-Eh_{[j],u}\,(\tilde{\rho}_m)\right]\right\}\end{aligned}$$

The first-order condition is equivalent to

$$Eh_{[j],u}\,(\tilde{\rho}_h)=\left(1-\widehat{t}'_j\,(\widehat{\alpha}_j)\right)Eh_{[j],u}\,(\tilde{\rho}_l)+\widehat{t}'_j\,(\widehat{\alpha}_j)\,Eh_{[j],u}\,(\tilde{\rho}_m)\tag{11b}$$

where $\widehat{t}'_j\,(\widehat{\alpha}_j)$ is equal to $\widehat{T}_j^{(n/m)}$ in (7b) for all j. In analogy to the case with additive risk, the decision maker chooses payment $\widehat{\alpha}_j$ such that the marginal unit cost $\widehat{t}'_j\,(\widehat{\alpha}_j)$ of the m^{th}-degree increase at the level of the j^{th} utility derivative, incurred in exchange for the n^{th}-degree decrease, is equal to the willingness to substitute an $(m+j)^{th}$- for an $(n+j)^{th}$-degree increase in return risk. Rate $\widehat{T}_j^{(n/m)}$ measures the willingness to commit to that risk tradeoff.

4.4 The Risk Substitution Rate's Central Role

Proposition 2 summarizes the results for the three preference measures.

Proposition 2 *For any decision maker u with $(-1)^{k+j-1}\,h_{[j],u}^{(k)}\,(\rho)>[\,<\,]\;0$ for $k=n,m$, each of (8)'s path-dependent $(m/j)^{th}$-degree multiplicative risk premium $\widehat{\pi}_j$, (10)'s m^{th}-degree probability premium $\widehat{p}_j^{(n/m)}$, and (11b)'s marginal unit cost $\widehat{t}'_j\,(\widehat{\alpha}_j)$, evaluated at the optimal PPR amount $\widehat{\alpha}^*_j$, quantitatively coincides with (7)'s $\widehat{T}_j^{(n/m)}$.*

Proposition 2 provides with Theorem 3 the conditions under which the measures from (8)–(11) can represent $((n+j)/(m+j))^{th}$-degree Ross more risk aversion as in Definition 5.

Theorem 4 *For any two decision makers u, v with $(-1)^{k+j-1}\, h_{[j],f}^{(k)}(\rho) > [<]\ 0$ for $k = n, m$ and $f \in \{u, v\}$, equal initial wealth $x_u = x_v$, and any $j \geq 0$, u being $((n+j)/(m+j))^{th}$-degree Ross more risk averse than v to a return risk increase as in* (6) *can equivalently be expressed by* (7)*'s risk substitution rate $\widehat{T}_j^{(n/m)}$,* (8)*'s path-dependent multiplicative risk premia, $\widehat{\pi}_{j,u} \geq \widehat{\pi}_{j,v}$;* (10)*'s $(m/j)^{th}$-degree probability premia, $\widehat{p}_{j,u}^{(n/m)} \geq \widehat{p}_{j,v}^{(n/m)}$; and* (11a)*'s optimal PPR amounts, $\widehat{\alpha}_{j,u}^{*} \geq \widehat{\alpha}_{j,v}^{*}$.*

Similar to the case with additive risk, Theorems 3 and 4 and Proposition 2 jointly describe risk substitution rate $\widehat{T}_j^{(n/m)}$'s fundamental role as a risk aversion measure in relation to return risk increases. A restriction to the results under return risk comes from the fact that the representations only apply to decision makers with uniform evaluations of n^{th}- and m^{th}-degree risk increases, $(-1)^{k+j-1}\, h_{[j],u}^{(k)}(\rho) > [<]\ 0$ for $k = n, m$, excluding mixed cases.

5 Conclusion

This note extends the risk aversion measures in Liu and Neilson (2019) to apply in the risk comparative statics of utility derivatives and under multiplicative risk and confirms that those measures' close relation to Liu and Meyer's (2013) risk substitution rate holds across all those contexts. The closeness is remarkable. The Liu and Neilson measures and their extensions offer, hence, throughout complementary approaches to explore the decision makers' willingness to substitute risks. The risk substitution rate is the fundamental measure that links them and provides a homogenous interpretation.

References

Bostian, AJ A. and Heinzel, Christoph. Comparative precautionary saving under higher-order risk and recursive utility. *Geneva Risk and Insurance Review*, 43(1): 95–114, 2018.

Bostian, AJ A. and Heinzel, Christoph. Precautionary Saving under Recursive Preferences. 2020. Working paper.

Eeckhoudt, Louis and Schlesinger, Harris. Changes in risk and the demand for saving. *Journal of Monetary Economics*, 55(7): 1329–1336, 2008.

Eeckhoudt, Louis, Schlesinger, Harris, and Tsetlin, Ilia. Apportioning of risks via stochastic dominance. *Journal of Economic Theory*, 144(3): 994–1003, 2009.

Ekern, Steinar. Increasing Nth Degree Risk. *Economics Letters*, 6(4): 329–333, 1980.

Heinzel, Christoph. Comparing Utility Derivative Premia under Additive and Multiplicative Risks. 2021. Working paper.

Jindapon, Paan and Neilson, William S. Higher-order generalizations of Arrow-Pratt and Ross risk aversion: A comparative statics approach. *Journal of Economic Theory*, 136(1): 719–728, 2007.

Jindapon, Paan, Liu, Liqun, and Neilson, William S. Comparative risk apportionment. *Economic Theory Bulletin*, 9: 91–112, 2021.

Liu, Liqun. Precautionary saving in the large: nth degree deteriorations in future income. *Journal of Mathematical Economics*, 52: 169–172, 2014.

Liu, Liqun and Meyer, Jack. Substituting one risk increase for another: A method for measuring risk aversion. *Journal of Economic Theory*, 148(6): 2706–2718, 2013.

Liu, Liqun and Neilson, William S. Alternative Approaches to Comparative nth-Degree Risk Aversion. *Management Science*, 65(8): 3824–3834, 2019.

Machina, Mark J. and Neilson, William S. The Ross Characterization of Risk Aversion: Strengthening and Extension. *Econometrica*, 55(5): 1139–1149, 1987.

Netzer, Nick and Scheuer, Florian. Taxation, insurance, and precautionary labor. *Journal of Public Economics*, 91(7–8): 1519–1531, 2007.

Pratt, John W. Risk Aversion in the Small and in the Large. *Econometrica*, 32(1–2): 122–136, 1964.

Wang, Jianli and Li, Jingyuan. Decreasing Ross risk aversion: Higher-order generalizations and implications. *Journal of Mathematical Economics*, 55: 136–142, 2014.