$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu \qquad \begin{cases} j^\mu = (c\rho, \vec{j}) \\ F^{\mu\nu} = \dfrac{\partial A^\nu}{\partial x_\mu} - \dfrac{\partial A^\mu}{\partial x_\nu} \end{cases}$$

$$\Rightarrow \partial_\nu F^{\mu\nu} = \partial_\nu \{\partial^\mu A^\nu - \partial^\nu A^\mu\} = \underbrace{\partial^\mu \partial_\nu}_{\text{Schwartz}} A^\nu - \partial_\nu \partial^\nu A^\mu = -\Box A^\mu$$

$$\partial_\nu A^\nu \to 0 \ \text{(gauge)}$$

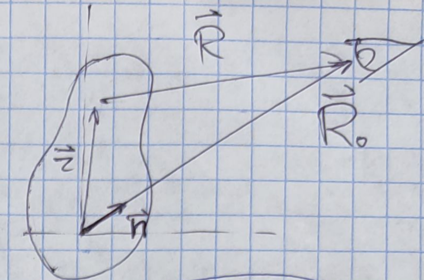
i.e. $\boxed{\Box A^\mu = \frac{4\pi}{c} j^\mu}$ (Lorentz gauge: $\partial_\nu A^\nu = 0$)

$$\Rightarrow A^\mu = -\int d^4 z' \, D_R(z'-z) \, \frac{1}{c} j^\mu(z') \ , \quad D_R(z) = -\frac{1}{4\pi |\vec{z}|} \delta(z^0 - |\vec{z}|)$$

$$\vec{A}(t, \vec{r}) = \int d^3 r' \int d(ct') \frac{1}{c 4\pi R} \delta(ct - ct' - R) \, \vec{j}(t', \vec{r}')$$

$$\boxed{\vec{A}(t, \vec{r}) = \int d^3 r' \, \frac{j^\mu\left(t - \frac{R}{c}, \vec{r}'\right)}{R}} \ , \quad \text{where } R = |\vec{r} - \vec{r}'|$$

At a distance: $R = |\vec{R}_0 - \vec{r}| \simeq R_0 - \vec{n}\vec{r}$



$n = \frac{\vec{R}_0}{R_0}$

$$\vec{A}(t, \vec{R}_0) \simeq \frac{1}{R_0}\int d^3 r\, \vec{j}\left(t - \frac{R_0}{c} - \frac{\vec{n}\vec{r}}{c}, \vec{r}\right)$$

$$\vec{H} = [\nabla \vec{A}] = \vec{e}^i \varepsilon_{ijk} \frac{\partial}{\partial x_j} A^k \simeq \vec{e}^i \varepsilon_{ijk} \frac{\partial A^k}{\partial\left(t - \frac{R_0}{c} - \frac{\vec{n}\vec{r}}{c}\right)} \cdot \frac{\partial\left(t - \frac{\sqrt{x^2+y^2+z^2}}{c} - \frac{\vec{n}\vec{r}}{c}\right)}{\partial x_j} =$$

$$\parallel$$

$$-\frac{1}{c}\frac{2x_j}{2R_0} = -\frac{1}{c} n_j$$

$$= \vec{e}^i \varepsilon_{ijk} \frac{\partial A^k}{\partial t} \cdot \left(-\frac{1}{c}\right) n_j = -\frac{1}{c}[\vec{n} \times \dot{\vec{A}}] = +\frac{1}{c}[\dot{\vec{A}}\, \vec{n}]$$

Planar wave:

$$\Rightarrow \vec{E} = [\vec{H} \times \vec{n}] = [n[n\dot{\vec{A}}]]$$

Fourier:

$$\vec{A}(t,\vec{R}_0) = \frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\vec{A}_\omega(\vec{R}_0)$$

$$\Rightarrow \vec{A}_\omega(\vec{R}_0) = \frac{e^{i\omega R_0}}{R_0}\int d^3r\, \vec{j}_\omega(\vec{r})\, e^{-i\vec{k}\vec{r}}, \quad \vec{k} = \omega\vec{n}$$

$$\vec{j}_\omega(\vec{r}) = \int dt\, e^{i\omega t}\vec{j}(t,\vec{r})$$

Point-part.:

$$\vec{j}(t,\vec{r}) = e\,\vec{v}(t)\,\delta(\vec{r}-\vec{r}_0(t))$$

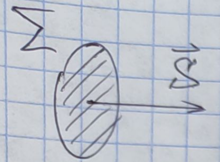
$$\Rightarrow \vec{j}_\omega(\vec{r}) = e\int dt\, \vec{v}(t)e^{i\omega t}\,\delta(\vec{r}-\vec{r}_0(t))$$

$$\Rightarrow \boxed{\vec{A}_\omega(\vec{R}_0) = e\,\frac{e^{i\omega R_0}}{R_0}\int dt\, \vec{v}(t)\, e^{i\omega(t-\vec{n}\vec{r}_0(t))}}$$

Since $[\dot{\vec{A}}]_\omega = \frac{\partial}{\partial t}(\vec{A}_\omega)$ we have: $\dot{\vec{A}}_\omega = i\omega\vec{A}_\omega$

$$\Rightarrow \vec{B}_\omega(\vec{R}_0) = -i\omega\left[\vec{n}\left[\vec{n}\,\vec{A}_\omega(\vec{R}_0)\right]\right] = \boxed{-i\omega\frac{e^{i\omega R_0}}{R_0}\int dt\left[\vec{n}\left[\vec{n}\,\vec{v}(t)\right]\right]e^{i\omega(t-\vec{n}\vec{r}_0(t))}}$$

$$\vec{S} = \frac{1}{4\pi}[\vec{E}\vec{H}] \overset{pl.\,wave}{=} \frac{1}{4\pi}E^2\vec{n}$$ – Poynting's vector



$$S = |\vec{S}| = \frac{d\mathcal{E}}{d\Sigma\, dt}$$

$$\Rightarrow dI = \frac{d\mathcal{E}}{dt} = S\,d\Sigma = R_0^2 S\, d\Omega$$

i.e. $$dI = \frac{E^2}{4\pi}R_0^2\, d\Omega$$

Parseval's theorem

$$\Rightarrow d\mathcal{E} = R_0^2 d\Omega \int_{-\infty}^{+\infty} dt\, \frac{E^2}{4\pi} = 2\cdot R_0^2 d\Omega \int_0^{+\infty} \frac{d\omega}{2\pi}\frac{|\vec{E}_\omega|^2}{4\pi} \overset{def.}{=} \int_0^{+\infty} d\mathcal{E}_{\vec{n}\omega}$$

i.e. $$d\mathcal{E}_{\vec{n}\omega} = \frac{|\vec{E}_\omega|^2}{2\pi}R_0^2\, d\Omega\, \frac{d\omega}{2\pi}$$ – radiation spectrum

Substituting $\vec{E}_\omega$:

$$\boxed{\frac{d\mathcal{E}_{\vec{n}\omega}}{d\Omega\, d\omega} = \frac{e^2\omega^2}{4\pi^2}\left|\left[\vec{n}\left[\vec{n}\vec{J}\right]\right]\right|^2},$$ where $\vec{J} = \int dt\, \vec{v}(t)\, e^{i\omega[t - \vec{n}\vec{r}_0(t)]}$

$$\frac{d\mathcal{E}_{\vec{n}\omega}}{d\Omega\, d\omega} \sim \left|\left[\vec{n}\times\left[\vec{n}\times\vec{J}\right]\right]\right|^2 = |\vec{J}|^2 - |\langle\vec{n}\vec{J}\rangle|^2 \qquad \left(\text{coef. } \frac{e^2\omega^2}{4\pi^2}\right)$$

$$\left[\vec{n}\times\left[\vec{n}\times\vec{J}\right]\right] = \vec{n}\langle\vec{n}\vec{J}\rangle - \vec{J}\langle\vec{n}\vec{n}\rangle = \vec{n}\langle\vec{n}\vec{J}\rangle - \vec{J}$$

$$\left|\left[\vec{n}\left[\vec{n}\vec{J}\right]\right]\right|^2 = \left(\vec{n}\langle\vec{n}\vec{J}\rangle - \vec{J}\right)\left(\vec{n}\langle\vec{n}\vec{J}\rangle - \vec{J}\right)^* = \left(\vec{n}\langle\vec{n}\vec{J}\rangle - \vec{J}\right)\left(\vec{n}\langle\vec{n}\vec{J}^*\rangle - \vec{J}^*\right) =$$

$$= \vec{n}^2\langle\vec{n}\vec{J}\rangle\langle\vec{n}\vec{J}^*\rangle - \langle\vec{n}\vec{J}\rangle\langle\vec{n}\vec{J}^*\rangle - \langle\vec{n}\vec{J}^*\rangle\langle\vec{n}\vec{J}\rangle + |\vec{J}|^2 =$$

$$= |\vec{J}|^2 - \langle\vec{n}\vec{J}\rangle\langle\vec{n}\vec{J}\rangle^* = \boxed{|\vec{J}|^2 - |\langle\vec{n}\vec{J}\rangle|^2}$$

$$|\vec{J}|^2 = \left(\mathrm{Re}\,\vec{J}\right)^2 + \left(\mathrm{Im}\,\vec{J}\right)^2 = \left(\sum_i \mathrm{Re}\,\vec{J}_i\right)^2 + \left(\sum_i \mathrm{Im}\,\vec{J}_i\right)^2$$

$$|\vec{n}\vec{J}|^2 = \left(\mathrm{Re}\langle\vec{n}\vec{J}\rangle\right)^3 + \left(\mathrm{Im}\langle\vec{n}\vec{J}\rangle\right)^2 = \left(\sum_i \mathrm{Re}\langle\vec{n}\vec{J}_i\rangle\right)^2 + \left(\sum_i \mathrm{Im}\langle\vec{n}\vec{J}_i\rangle\right)^2$$

$$\vec{J} = \sum_{i=0}^{nt-1} \vec{J}_i = \vec{J}(\vec{n}, \omega)$$

$$\vec{J}_i(\vec{n}, \omega) = \frac{\Delta t}{\Delta \xi_i} e^{j\omega \xi_i} \Big\{ \underbrace{2\vec{v}_i \sin\left(\frac{\omega \Delta \xi_i}{2}\right)}_{\vec{x}_i} + j \underbrace{\Delta \vec{v}_i \left[\frac{2}{\omega \Delta \xi_i} \sin\left(\frac{\omega \Delta \xi_i}{2}\right) - \cos\left(\frac{\omega \Delta \xi_i}{2}\right) \right]}_{\vec{y}_i} \Big\}$$

$$:= C_i e^{j\varphi_i} \{ \vec{x}_i + j \vec{y}_i \}$$

$$C_i (\cos\varphi_i + j \sin\varphi_i) \{ \vec{x}_i + j \vec{y}_i \}$$

$$C_i \{ \vec{x}_i \cos\varphi_i - \vec{y}_i \sin\varphi_i + j \vec{x}_i \sin\varphi_i + j \vec{y}_i \cos\varphi_i \}$$

$$\vec{J}_i(\vec{n}, \omega) = \underbrace{C_i \{ \vec{x}_i \cos\varphi_i - \vec{y}_i \sin\varphi_i \}}_{\mathrm{Re}\, \vec{J}_i} + j \underbrace{C_i \{ \vec{x}_i \sin\varphi_i + \vec{y}_i \cos\varphi_i \}}_{\mathrm{Im}\, \vec{J}_i}$$