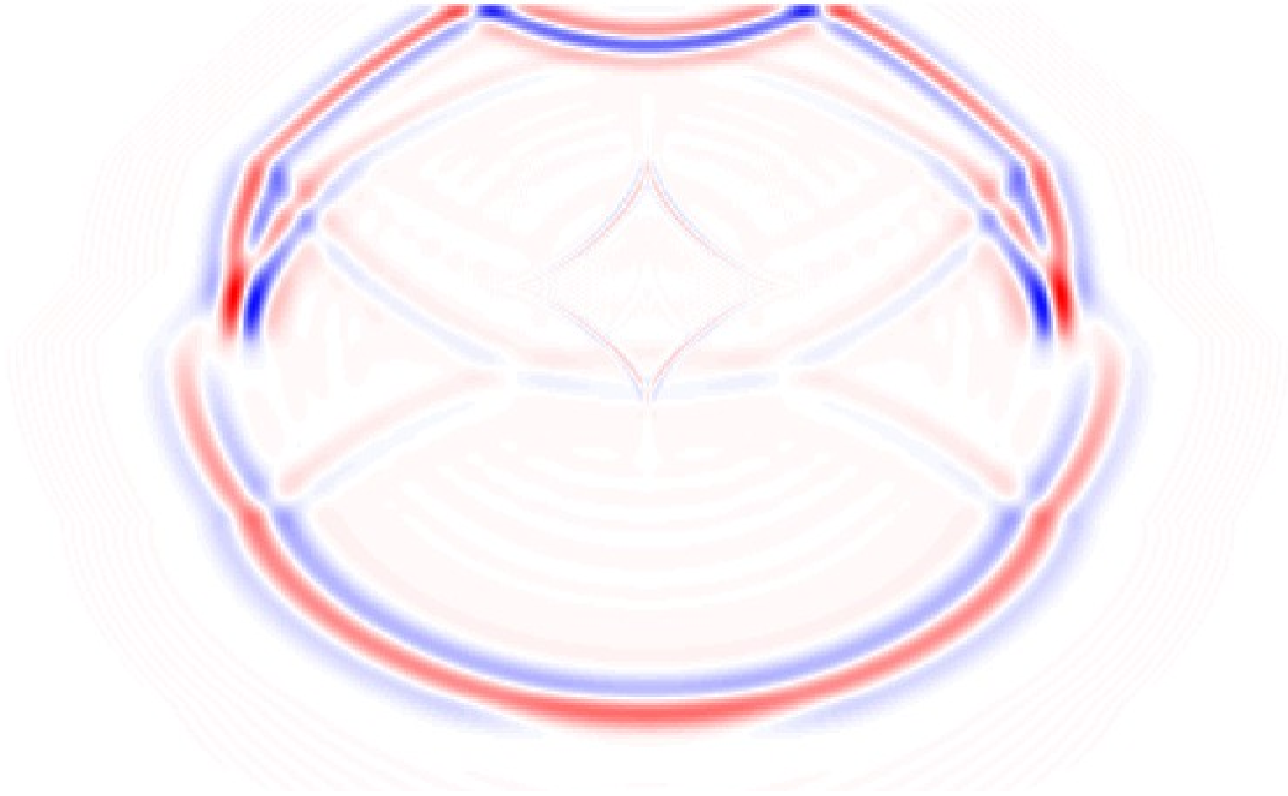
Anisotropic Acoustic Wave Equation



Framing the Problem

- The elastic wave equation is both information rich and elaborate, containing travel time and amplitude characteristics.
- However, numerically solving the elastic wave equation is computationally expensive:
    - This wavefield is described by a three-component displacement vector containing both P and S waves.
    - Simulating wave propagation using finite-difference techniques requires simultaneously computing all three components.
    - The time step and grid spacing must accommodate the slowest velocity shear wave in the model.
- Accounting for anisotropy complicates the picture. On the other hand, conventional isotropic data processing methods lead to lower resolution and incorrect subsurface modeling.

More Problems

- Several methods have been devised to simplify the anisotropic equation:
  - Weak-anisotropy approximation
  - Elliptical approximations
  - Small dip angle approximations

- These approximations have their perspective limitations which sometimes make them ill-suited for practical applications.

The Solution

- Following Alkhalifah (2000) and H. Zhou's (2010) work, implement a coupled system of second-order differential equations subject to the acoustic assumption.
- As Alkhalifah illustrates, the acoustic assumption yields far more kinematically accurate results compared to the aforementioned anisotropic approximations.
- As this system of anisotropic acoustic equations is described by a scalar field rather than a vector field, it is less computationally intensive than the elastic wave equation. The number of equations are reduced while still yielding kinematic results which are comparable to the full elastic equation.
- Moreover, if we can suppress shear wave “artifacts” then we can easily model P-waves without using filters to separate out the S-waves.

But isn't an anisotropic acoustic media physically impossible?

- Most researchers have viewed an anisotropic acoustic media as a physically impossible mathematical trick. Moreover, the diamond shaped wavefronts are widely regarded as artifacts. This view isn't entirely correct.
- As Grechka (2010) shows, simply setting s-waves to zero in the direction of the symmetry axis does not mean s-wave velocity is zero everywhere. The diamond shaped wavefront is a testament to this simple fact.
- Such bizarre s-wave behavior may be representative of finely layered media containing highly compliant solid or viscous fluid layers.
- While it is interesting to know that our “artifacts” have some basis in reality, in practice the diamond shaped s-waves may cause instabilities and should be suppressed for modeling. This s-waves are generally not representative of our actually earth model.

Outline of Derivation

1) Begin with the phase velocity relations.
2) Acoustic approximation: set the s-wave velocity along the symmetry axis to be zero.
3) Derive the dispersion relations for TTI media and multiple by the wavefield function in the Fourier domain.
4) Apply an inverse Fourier transform and use the correspondent relations to obtain the final anisotropic acoustic equation.
5) For an indepth derivation refer to Zhou(2012) and Alkhalifah (2000).

Anisotropic Acoustic Wave Equation in 3D TTI

- Equation (1) governs the propagation of the wavefront while equation (2) acts as an auxiliary function which can be thought to control the anellipticity of the wavefront.

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = (1+2\delta)\mathrm{H}q + (1+2\delta)\mathrm{H}p + \mathrm{H}_0 p \qquad (1)$$

$$\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} = 2(\varepsilon-\delta)\mathrm{H}q + 2(\varepsilon-\delta)\mathrm{H}p \qquad (2)$$

- ε and δ represent Thomsen's dimensionless anisotropic parameters. The differential operators H and H0 equal:

$$\mathrm{H} = A\frac{\partial^2}{\partial x^2} + (BE+D)\frac{\partial^2}{\partial y^2} + (BD+E)\frac{\partial^2}{\partial y^2} - CI\frac{\partial^2}{\partial x \partial y} - CG\frac{\partial^2}{\partial x \partial z} - AF\frac{\partial^2}{\partial y \partial z} \qquad (3)$$

$$\mathrm{H}_0 = B\frac{\partial^2}{\partial x^2} + (AE)\frac{\partial^2}{\partial y^2} + (AD)\frac{\partial^2}{\partial y^2} + CI\frac{\partial^2}{\partial x \partial y} + CG\frac{\partial^2}{\partial x \partial z} + AF\frac{\partial^2}{\partial y \partial z} \qquad (4)$$

- Cross derivatives to account for tilt which is accounted for by combined to rotations: a rotation of angle θ along the y-axis and rotation of φ along the x-axis.

$$A=\cos^2(\theta), B=\sin^2(\theta), C=\sin(2\theta), D=\cos^2(\varphi),$$
$$E=\sin^2(\varphi), F=\sin(2\varphi), G=\cos(\varphi), I=\sin(\varphi)$$

Anisotropic Acoustic Wave Equation in 3D VTI

- Equations for VTI media are readily obtained from the TTI equations (1) and (2) by simply setting θ = φ = 0:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}=(1+2\delta)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(p+q)+\frac{\partial^2 p}{\partial z^2} \qquad (5)$$

$$\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2}=2(\varepsilon-\delta)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(p+q) \qquad (6)$$

- Similarly, equations (1) and (2) degenerate into equations for HTI by setting θ = φ = 90 degrees.

Isotropic Acoustic Wave Equation and Elliptical Anisotropy

- While not physically obvious, the auxiliary q-field can be thought to control the anellipticity of the propagating wave field. For example, let's consider two more degenerate cases.

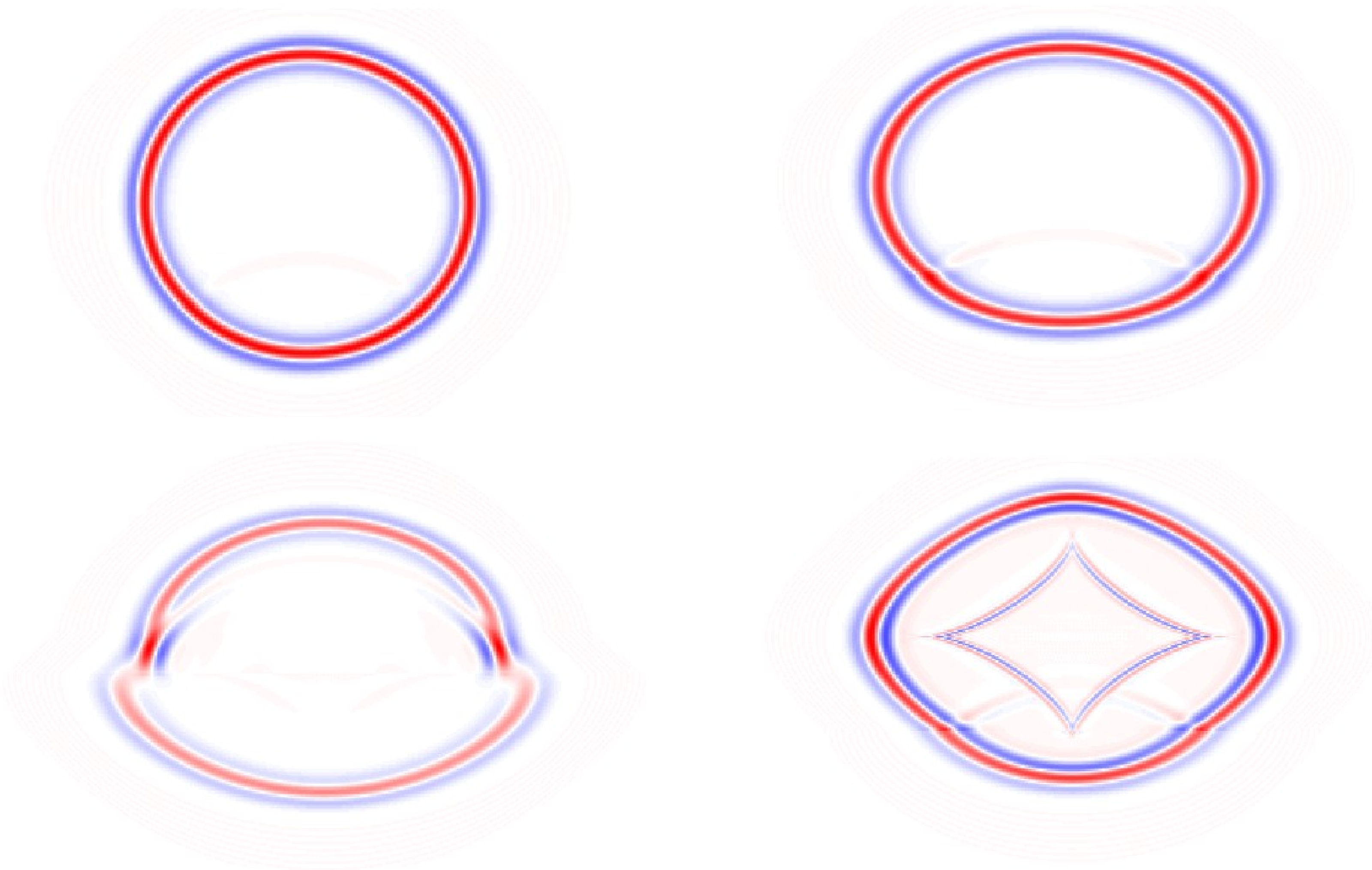
- For isotropic media, ε = δ = 0 causing the q-field (equation (6)) to vanish thereby yielding the familiar acoustic wave equation for a propagating pressure wave:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right)$$

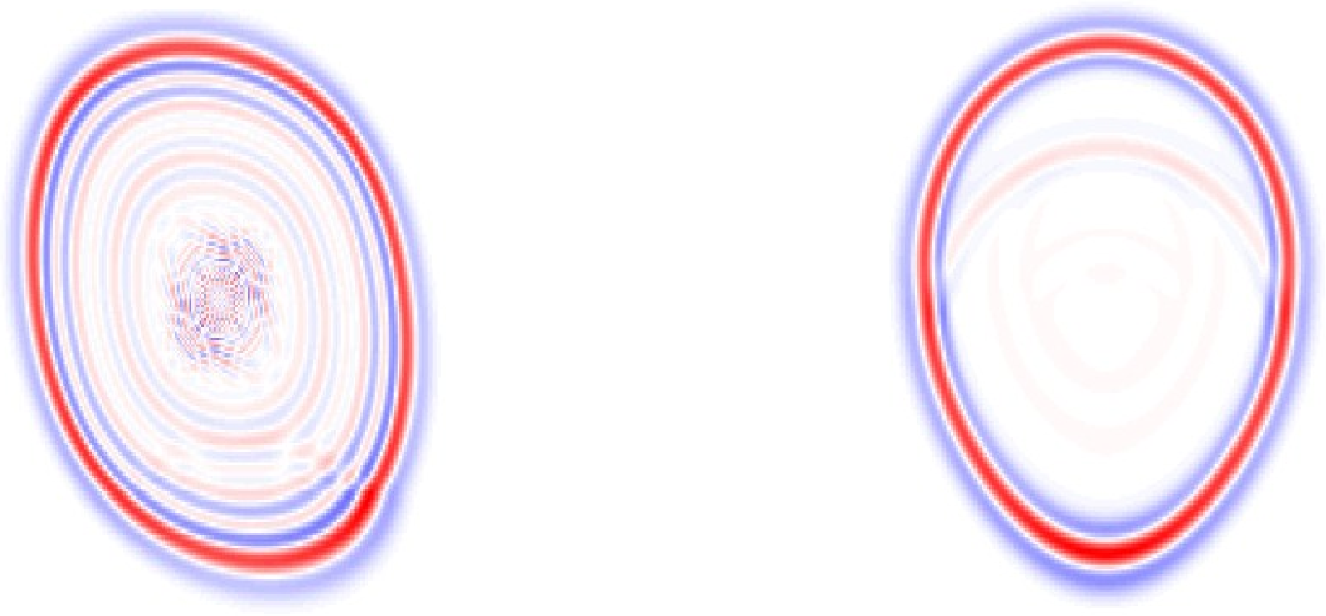
- For elliptical anisotropy, ε = δ ≠ 0 also causing the q field to vanish. However, the p field maintains a δ value which controls the ellipticity of the propagating p-field:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = (1+2\delta)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2 p}{\partial y^2 p}\right) + \frac{\partial^2 p}{\partial z^2}$$

VTI Media and Degenerate Cases



HTI and TTI Media



Time and Spatial Discretization

- Using a second order finite-difference approach, we can approximate the partial differential derivatives in the equations for VTI media. For example:

$$\frac{\partial^2 p}{\partial t^2} \rightarrow \frac{p_y^{n+1} - 2 p_y^n + p_y^{n-1}}{\Delta t^2}$$

- The second-order FD stencil of the system of hyperbolic anisotropic acoustic equations in the time direction can be expressed as:

$$p^{n+1} = 2 p^n - p^{n-1} + c^2 \Delta t^2 [(1+2\delta)(\frac{p_{x+1}^n - 2 p_x^n + p_{x-1}^n}{x^2} + \frac{p_{y+1}^n - 2 p_y^n + p_{y-1}^n}{y^2} + \frac{q_{x+1}^n - 2 q_x^n + q_{x-1}^n}{x^2} + \frac{q_{y+1}^n - 2 q_y^n + q_{y-1}^n}{y^2}) + \frac{p_{z+1}^n - 2 p_z^n + p_{z-1}^n}{z^2}] \qquad (7)$$

$$q^{n+1} = 2 q^n - q^{n-1} + 2 c^2 (\varepsilon - \delta)(\frac{p_{x+1}^n - 2 p_x^n + p_{x-1}^n}{x^2} + \frac{p_{y+1}^n - 2 p_y^n + p_{y-1}^n}{y^2} + \frac{q_{x+1}^n - 2 q_x^n + q_{x-1}^n}{x^2} + \frac{q_{y+1}^n - 2 q_y^n + q_{y-1}^n}{y^2}) \qquad (8)$$

- A second order in time, time stepping algorithm is used. Spatial discretized p and q fields are evaluated at each time step. A conventional grid is used for the FD scheme.

- To reduce numerical dispersions, the spacial derivatives in equations (7) and (8) are approximated by higher orders.

Basic Memory Allocation

- For VTI and TTI models anisotropic parameters need to be accounted for (epsilon, delta, dip, azimuth) in addition to the auxiliary q-field.
- Assuming a constant density media, basic memory allocation can be summarized as follows:

Isotropic Modeling	VTI Modeling	TTI Modeling
p_pastTime	p_pastTime	p_pastTime
p_currentTime	p_currentTime	p_currentTime
Velocity	q_pastTime	q_pastTime
	q_currentTime	q_currentTime
	Velocity	Velocity
	Epsilon	Epsilon
	Delta	Delta
		Dip
		Azimuth

- It requires a huge amount of memory to store full 3D wavefields and models.
- Using high order FD schemes allows for larger grid spacing and time steps resulting in faster computations and less memory usage.

Stability Condition

- The stability condition for the isotropic wave equation can be defined as follows:

$$\Delta t \leq \frac{\Delta d_{min}}{v_{max}} * \left(\frac{2}{\sqrt{a}}\right) \quad (9)$$

$$a = \sum \left(|W_x| + |W_y| + |W_z|\right) \quad (10)$$

- In equation (9), d represents the minimum grid spacing which is governed by minimum velocity, maximum frequency, and the FD scheme order:

$$\Delta d_{min} = min\left(\Delta x, \Delta y, \Delta z\right) \quad (11)$$

$$\Delta x = \frac{v_{min}}{f_{max} * S} \quad (12)$$

- W is the FD coefficient. Equation (13) gives the stability condition for VTI media and equation (14) gives the stability condition for TTI media.

$$a = \frac{1}{3} \sum \left(|W_x| + |W_y| + |W_z|\right)\left(2 + 4\varepsilon + \sqrt{\left(1 + 2\delta\right)}\right) \quad (13)$$

$$\begin{aligned} a &= W\left(a_1 + a_2\right) \\ a_1 &= \left(1 + 2\varepsilon_{max}\right)\left(2 - b\right) \\ b &= \cos^2\theta \sin 2\varphi + \sin 2\theta\left(\sin\varphi + \cos\varphi\right) \end{aligned} \quad (14)$$

Future Work

- Implement the more general system of equations to account for 3D TTI media using a hybrid FD-pseudospectral approach. This method simplifies the calculation of cross terms and allows for a courser, more accurate spatial discretization.
- Parallelize the code so that compute nodes operate under the control of a single master node. Ideally, parallelization will be across sources and the model will not be distributed as subdomains across nodes.
- Move beyond anisotropic wave equation modeling and towards wave equation migration and maybe even inversion.

Beautiful Stuff to Read

- Alkhalifah, T. , 2000, An acoustic wave equation.

  fpr anisotropic media: Geophysics, 65, 1239-1250.
- Grechka, Zhang, Rector, 2004, Shear waves in acoustic anisotropic media:Geophysics, 51, 54-66.
- Zhou, Zhang, Bloor, 2006, An anisotropic acoustic wave equation for VTI media: EAGE.
- Zhou, Zhang, Bloor, 2006, An anisotropic acoustic wave equation for modeling and migration in 2D TTI media.