Dynamics Assignment Report

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Contents

1 Part 1 - Accuracy and Stability Analysis of Newmark's method

1.1 Parameter Selection

The SDOF system equation mentioned in assignment description is

$$\ddot{u} + \omega^2 u = 0 \tag{1}$$

As Eq. (1) describes, the system frequency was calculated as

$$\omega = \sqrt{\frac{k}{m}}$$

The system period was $T = \frac{5}{3}$.

1.2 Stability Analysis of Newmark Method

As the system is a dynamic system and has energy, the Liapunov energy method can be applied to analyze the stability of system.[1]. We will state the basic theorem which point the stability of invariant sets in terms of Lyapunov functions[2]. First consider an autonomous system on an m -dimensional state space, and suppose that Ω is a p -dimensional invariant set. Furthermore, suppose that Ω can be characterized in the following way. There exists a continuous function $g : R^m \to R^{m-p}$ such that:

$$\Omega = \{y \in R^m \mid g(y) = 0\}$$

Clearly, this is the situation of interest in dynamic system stability, in which case $m = 2n, p = 1$, and the map g is linear.
Definition: A scalar function $V(y)$ is said to be positive definite with respect to Ω in an open region $U \supset \Omega$ if:

- $V(y)$ and its first partial derivatives are continuous in U.
- $V(y) = 0$ for $y \in \Omega$
- $V(y)W(g(y))$, where $W(g(y))$ is an ordinary positive definite function on the Image of $U \subset R^p$ under g.

If, in addition, $\dot{V} \leq 0$ on U, V is called a Lyapunov function (with respect to Ω).

1.2.1 Newmarks Method

Consider the following second order differential equation in time t:

$$\ddot{u}(t) + c\dot{u}(t) + ku(t) = p(t) \tag{2}$$

where c and k are non-negative linear values. Let τ be a time step, $U(t)$ be a difference approximation of $u(t), V(t)$ be an approximation of $\dot{u}(t), A(t)$ is an approximation of $\ddot{u}(t)$, β and γ be the fixed real numbers. Then we can represent the Newmark method as follows:

$$A(t) \approx \ddot{u}(t)$$
$$V(t) \approx \dot{u}(t)$$
$$U(t) \approx u(t)$$

$$f(t) = A(t) + cV(t) + kU(t) \tag{3}$$
$$U(t+\tau) = U(t) + \tau V(t) + \frac{1}{2}\tau^2 A(t) + \beta\tau^2(A(t+\tau) - A(t)) \tag{4}$$
$$V(t+\tau) = V(t) + \tau A(t) + \gamma\tau(A(t+\tau) - A(t)) \tag{5}$$

The iteration of Newmarks method Eqs. (3),(4),(5) for Eq. (2) is as follows:
1. Convert the expression of A(t) to the combination of $U(t)$ and $V(t)$ by using Eq. (4) :

$$A(t) = f(t) - (cV(t) + kU(t))$$

2. Compute $A(t+\tau)$ from $f(t+\tau), U(t)$ and $A(t)$:

$$\begin{aligned} A(t+\tau) = &\left(1 + \gamma\tau c + \beta\tau^2 k\right)^{-1} \\ &\times \{-kU(t) - (c + \tau k)V(t) \\ &+ \left(-\tau c + \gamma\tau c - \frac{1}{2}\tau^2 k + \beta\tau^2 k\right) A(t) + f(t+\tau)\Big\} \end{aligned}$$

3. Compute $V(t+\tau)$ from:

$$V(t+\tau) = V(t) + \tau A(t) + \gamma\tau(A(t+\tau) - A(t))$$

4. Compute $U(t+\tau)$ from:

$$U(t+\tau) = U(t) + \tau V(t) + \frac{1}{2}\tau^2 A(t) + \beta\tau^2(A(t+\tau) - A(t))$$

5. Replace t for $t+\tau$ and return to 2

1.2.2 Stability Analysis

Now, some operators was defined with time step τ as follows:

$$D_\tau U(t) \equiv \frac{1}{\tau}(U(t+\tau) - U(t)) \sim \dot{u}\left(t + \frac{\tau}{2}\right)$$
$$D_\tau U(t) \equiv \frac{1}{\tau}(U(t) - U(t-\tau)) \sim \dot{u}\left(t - \frac{\tau}{2}\right)$$
$$D_{\tau\tau} U(t) \equiv \frac{1}{\tau^2}(U(t+\tau) - 2U(t) + U(t-\tau)) \sim \ddot{u}(t)$$
$$\frac{1}{2}\left(D_\tau + D_{\bar{\tau}}\right) U(t) \equiv \frac{1}{2\tau}(U(t+\tau) - U(t-\tau)) \sim \dot{u}(t)$$

Through the Liapunov function(Energy method) there were two equations could be obtained:

$$V_1 =< \frac{1}{2}c\left(D_\tau + D_{\bar{\tau}}\right)U(t), \frac{1}{2}\left(D_\tau + D_{\bar{\tau}}\right)U(t) > \quad (6)$$

$$V_2 =< \left(1+\beta\tau^2 k\right)D_{\tau\bar{\tau}}U(t), \frac{1}{2}\left(D_\tau + D_{\bar{\tau}}\right)U(t) > + < kU(t), \frac{1}{2}\left(D_\tau + D_{\bar{\tau}}\right)U(t) > \quad (7)$$

The $< a, b >$ means inner product between a and b in Eq. (6) and. (7). The definition of inner product is:

$$a = [a_1, a_2, a_3 ... a_n]^T, b = [b_1, b_2, b_3 ... b_n]^T$$
$$< a, b >= a_1 b_1 + a_2 b_2 + a_3 b_3 + ... + a_n b_n$$

The V_1 from Eq. (6) is increasing by time that indicates the energy from system is always increasing beyond its initial positive value (note that $c > 0$), and it is convincing to define V_1 part is unstable in this system.
Nonetheless, after the V_1 combined with Eq. (7), a new energy inequality could be derived:

$$< \left(1+\beta\tau^2 k\right)D_{\tau\bar{\tau}}U(t), \frac{1}{2}\left(D_\tau + D_{\bar{\tau}}\right)U(t) > + < kU(t), \frac{1}{2}\left(D_\tau + D_{\bar{\tau}}\right)U(t) >$$

$$=-< \tfrac{1}{2}c\left(D_\tau + D_{\bar{\tau}}\right)U(t), \tfrac{1}{2}\left(D_\tau + D_{\bar{\tau}}\right)U(t) >\leq 0 \quad (8)$$

From above inequality, it is clear to find the energy is always decreasing beyond its initial value. Thus, $\dot{V}_2 < 0$. Hence, V_2 is a Liapunov function. Eq. (7) could be rewrote as:

$$< \left(1+\beta\tau^2 k\right)D_{\tau\bar{\tau}}U(t), \frac{1}{2}\left(D_\tau + D_{\bar{\tau}}\right)U(t) > + < kU(t), \frac{1}{2}\left(D_\tau + D_{\bar{\tau}}\right)U(t) > \leq 0 \quad (9)$$

By using this inequality and the lemma(note that $k \geq 0$):

$$< (kU(t+\tau), U(t) >=< (kU(t), U(t) > +\tau < (kD_\tau U(t), U(t) >$$

we get:

$$\|D_\tau U(t)\|^2 + \beta\tau^2 \left\|k^{1/2} D_\tau U(t)\right\|^2 + \left\|k^{1/2} U(t)\right\|^2 + \tau < \left(k^{1/2} D_\tau U(t), k^{1/2} U(t) > \leq C_0 \quad (10)$$

where:
$C_0 =< \left(1+\beta\tau^2 k\right)D_\tau U(0), D_\tau U(0) > + < kU(\tau), U(0) >$
$=< \left(1+\beta\tau^2 k\right)D_\tau U(0), D_\tau U(0) > + < kU(0), U(0) > + < (\tau k D_\tau U(0), U(0) >)$
$= \|D_\tau U(0)\|^2 + \beta\tau^2 \left\|k^{1/2} D_\tau U(0)\right\|^2 + \left\|k^{1/2} U(0)\right\|^2$
$+ \tau < k^{1/2} D_\tau U(0), k^{1/2} U(0) >$
The same lemma was used again. We employed Schwarz's inequality on
$+\tau < k^{1/2} D_\tau U(0), k^{1/2} U(0) >$:

$$\left|\tau\left(k^{1/2}D_\tau U(t), k^{1/2}U(t)\right)\right| \leq \|\tau\left(k^{1/2}D_\tau U(t)\right)\| \|k^{1/2}U(t)\|$$
$$= \alpha \left\|\tau\left(k^{1/2}D_\tau U(t)\right)\right\| \times \tfrac{1}{\alpha}\left\|k^{1/2}U(t)\right\| \leq \tfrac{1}{2}\alpha^2\tau^2\left\|k^{1/2}D_\tau U(t)\right\|^2 + \tfrac{1}{2\alpha^2}\left\|k^{1/2}U(t)\right\|^2$$

After the fourth term of the left hand side of Eq. (10) has been moved to the right hand side and applied the inequality above, we derived:

$$\begin{aligned}\|D_\tau U(t)\|^2 + \beta\tau^2\left\|k^{1/2}D_\tau U(t)\right\|^2 + \left\|k^{1/2}U(t)\right\|^2 & \\ \leq C_0 - \tau\left(k^{1/2}D_\tau U(t), k^{1/2}U(t)\right) & \\ \leq C_0 + \left|\tau\left(k^{1/2}D_\tau U(t), k^{1/2}U(t)\right)\right| & \\ \leq C_0 + \frac{1}{2}\alpha^2\tau^2\left\|k^{1/2}D_\tau U(t)\right\|^2 & \\ + \frac{1}{2\alpha^2}\left\|k^{1/2}U(t)\right\|^2 &\end{aligned}$$

Finally moving the second and third terms in the above formula to the left hand side, the energy inequality could be found:

$$\|D_\tau U(t)\|^2 + \tau^2\left(\beta - \frac{\alpha^2}{2}\right)\left\|k^{1/2}D_\tau U(t)\right\|^2 + |\left(1 - \frac{1}{2\alpha^2}\right)\left\|k^{1/2}U(t)\right\|^2 \leq C_0 \tag{11}$$

1.2.3 Stability Condition

In Eq. (11), we set $\alpha^2 = \frac{1}{2}$ [3], then we can get :

$$\|D_\tau U(t)\|^2 \leq C_0 + \tau^2\left(\frac{1}{4} - \beta\right)\left\|k^{1/2}D_\tau U(t)\right\|^2 \tag{12}$$

We considered $\left\|k^{1/2}\right\|$ as operator norm of $k^{1/2}$,
then we have $\left\|k^{1/2}y\right\| \leq \left\|k^{1/2}\right\| \|y\|$
By this fact and Eq. (12), we get:

$$\left(1 - \tau^2\left(\frac{1}{4} - \beta\right)\right)\left\|k^{1/2}\right\|^2 \|D_\tau U(t)\|^2 \leq C_0 \tag{13}$$

When $\tau > 0$:

$$0 < 1 = \tau^2\left(\frac{1}{4} - \beta\right)\left\|k^{1/2}\right\|^2 \Leftrightarrow \tau < \sqrt{\frac{1}{\left(\frac{1}{4} - \beta\right)\|k^1/2\|^2}} \tag{14}$$

Obviously, when $\beta \geq \frac{1}{4}$, the limit of τ would be infinity. Therefore, the conclusion on stability is:

1. If $\beta \geq \frac{1}{4}$ we can find there is no limit for time step τ. That means that is no condition on Δt, and thus the Newmark's Method is unconditional stable.

2. If $0 \leq \beta < \frac{1}{4}$ we can also find an upper limit on Δt that leads to Newmark's Method conditional stable. The condition is:

$$\Delta t < \sqrt{\frac{1}{\left(\frac{1}{4} - \beta\right) \left\|k^{1/2}\right\|^2}}$$

1.3 Derivation of Accuracy Analysis

In this part, we defined an initial value problem $\dot{y} = f(t, y)$ with $y(0) = y_0$ and numerical approximations $u_i \approx y\left(t_i\right)$. The difference:

$$\varepsilon_{i+1} = u_{i+1} - \hat{y}\left(t_{i+1}\right)$$

is called the local error, where $\hat{y}$ is the solution to $\dot{\hat{y}} = f(t, \hat{y})$ with the initial condition $\hat{y}\left(t_i\right) = u_i$

The consistency definition is: When the local error of an algorithm tends to zero as $\mathcal{O}\left(\tau^2\right)$ (τ is a time step), the algorithm is called consistent.

The Taylor expansion was applied to analyze the local errors. Newmark's Method (with $\gamma = \frac{1}{2}$) can be written as:

$$\begin{aligned} u_{i+1} &= u_i + \tau \dot{u}_i + (0.5 - \beta)\tau^2 \ddot{u}_i + \beta \tau^2 \ddot{u}_{i+1} \\ \mathrm{u}_{i+1} &= u_i + \tau \dot{u}_i + (0.5 - \beta)\tau^2 f\left(t_i, u_i, \dot{u}_i\right) + \beta \tau^2 f\left(t_{i+1}, u_{i+1}, \dot{u}_{i+1}\right) \end{aligned} \tag{15}$$

After the initial condition of $\hat{y}$ employed into Eq. (14):

$$u_{i+1} = \hat{y}_i + \tau \dot{\hat{y}}_i + (0.5 - \beta)\tau^2 f\left(t_i, \hat{y}_i, \dot{\hat{y}}_i\right) + \beta \tau^2 f\left(t_{i+1}, \hat{y}_{i+1}, \dot{\hat{y}}_{i+1}\right) \tag{16}$$

The Taylor expansion of $\hat{y}$ is:

$$\hat{y}_{i+1} = \hat{y}_i + \tau \dot{\hat{y}}_i + \frac{\tau^2}{2} \ddot{\hat{y}}_i + \frac{\tau^3}{3!} \hat{y}_i^{(3)} + \mathcal{O}\left(\tau^4\right) \tag{17}$$

Applying Eq. (2) for $\hat{y}$:

$$\hat{y}_{i+1} = \hat{y}_i + \tau \dot{\hat{y}}_i + \frac{\tau^2}{2} f\left(t_i, \hat{y}_i, \dot{\hat{y}}_i\right) + \frac{\tau^3}{3!} \hat{y}_i^{(3)} + \mathcal{O}\left(\tau^4\right) \tag{18}$$

Subtracting Eq. (17) by Eq. (18) we get the expression of local the error of Newmark's Method:

$$\varepsilon_{i+1} = \beta \tau^2 \ddot{\hat{y}}_i - \beta \tau^2 \ddot{\hat{y}}_{i+1} - \frac{\tau^3}{3!} \hat{y}_i^{(3)} - \mathcal{O}\left(\tau^4\right) \tag{19}$$

Follow the definition in this part start, Newmark's method is obviously consistent in general situation.

To further explore the analysis on error, we selected two values of β:

1. $\beta = \frac{1}{2}$

$$\begin{gathered}\varepsilon_{i+1} = \frac{\tau^2}{2}\left(\ddot{y}_i - \ddot{\hat{y}}_{i+1}\right) + \frac{\tau^3}{3!}\hat{y}_i^{(3)} + \mathcal{O}\left(\tau^4\right) \\ \varepsilon_{i+1} = -\frac{\tau^3}{3}\hat{y}_i^{(3)} + \mathcal{O}\left(\tau^4\right) \\ \varepsilon_{i+1} = \mathcal{O}\left(\tau^3\right)\end{gathered}$$

The order of error when $\beta = \frac{1}{2}$ is clearly second order on time due to $\mathcal{O}\left(\tau^2\right)$,

2. $\beta = \frac{1}{6}$

$$\begin{gathered}\varepsilon_{i+1} = -\frac{\tau^3\left(\ddot{\hat{y}}_{i+1} - \ddot{y}_i\right)}{6}\frac{\tau^3}{6}\hat{y}_i^{(3)} + \frac{\tau^4}{4!}\hat{y}_i^{(4)} + \mathcal{O}\left(\tau^5\right) \\ \varepsilon_{i+1} = \frac{\tau^4}{4!}\hat{y}_i^{(4)} + \mathcal{O}\left(\tau^5\right) \\ \varepsilon_{i+1} = \mathcal{O}\left(\tau^4\right)\end{gathered}$$

Thus the accuracy order is third order due to $\mathcal{O}\left(\tau^3\right)$,

1.4 Maximum Kinetic Energy Visualization from Numerical Solution

Through Eq. (14), the limit of time step is: $\Delta t = 0.9188$
The maximum kinetic energy can be calculated by:

$$Kinetic\,energy^{max} = max(\frac{1}{2}mv^2) \tag{20}$$

m and v are mass and velocity of system respectively.

1. When $\beta = \frac{1}{2}$, the method is unconditional stable.

From Fig. (1), it is crystally to express periodic energy in SDOF system. Even the Δt was greater than the limitΔt in the fourth sub-figure, the Energy still has a stable trend in the 100-fold period.

2. When $\beta = \frac{1}{6}$, the method is conditional stable.

From Fig. (2), the system energy was stable with $\Delta t < \Delta t_{limit}$, however, in the third and fourth panels, it showed incredible increasing trend on energy during 100-fold period. From the 3^{rd} and 4^{rd} panels, the unstable energy trend was depicted. Fig. (2) proved the highly correlation between stability and time step when $\beta = \frac{1}{6}$.

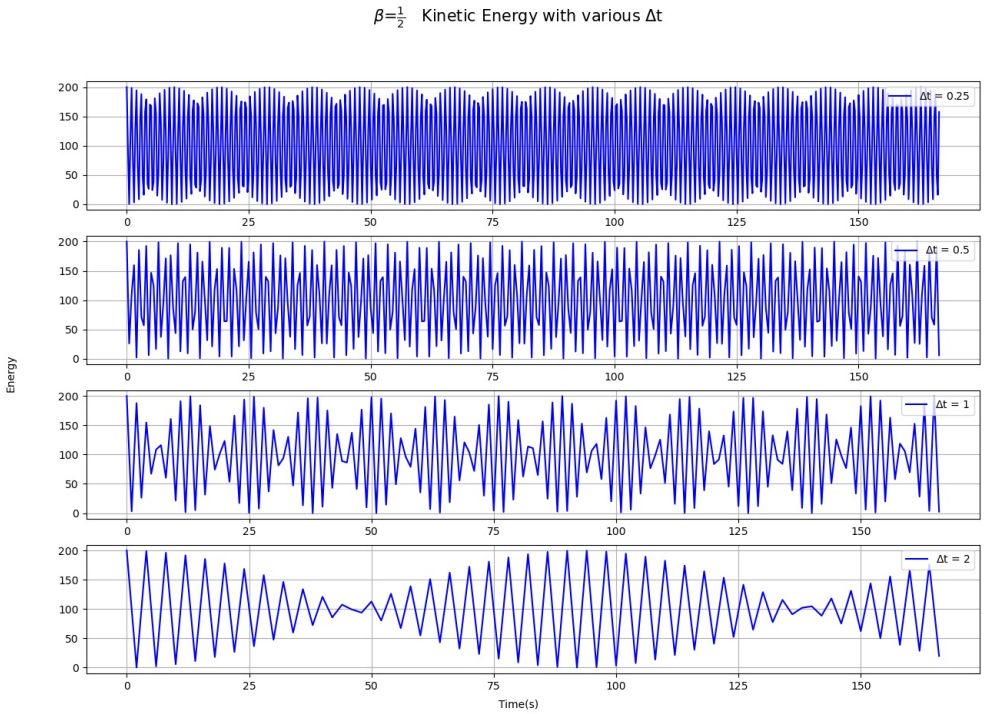

Figure 1: Energy with $\beta = \frac{1}{2}$

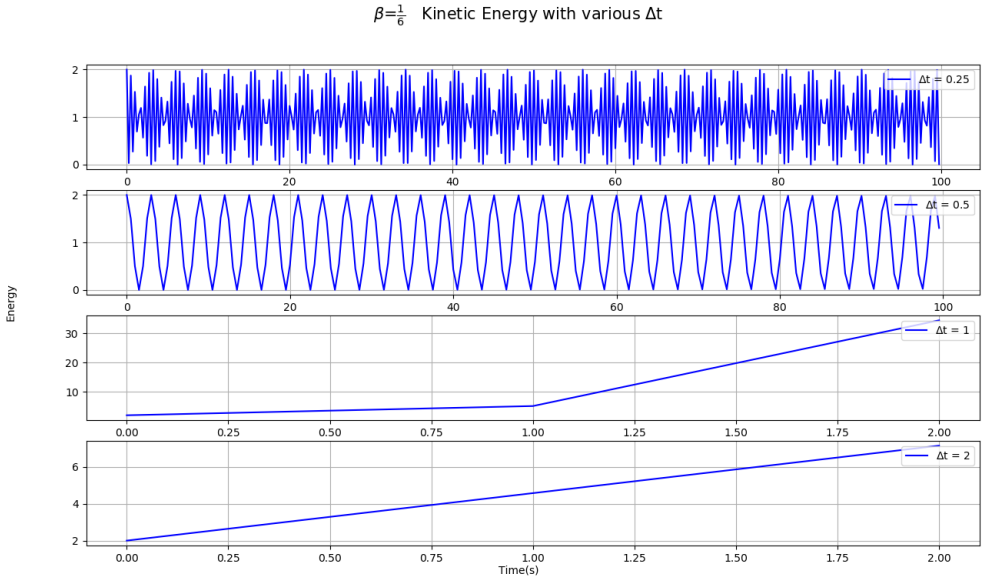

Figure 2: Energy with $\beta = \frac{1}{6}$

1.5 Energy Error Visualization

The error between numerical method with $\beta = \frac{1}{2}$ was depicted on Fig. (5). Obviously, the error showed periodicity and stability due to unconditional stable numerical method.

$$error = Kinetic\,energy_{Numerical\ Solution} - Kinetic\,energy_{Analytical\ Solution} \tag{21}$$

The error between numerical method with $\beta = \frac{1}{6}$ was depicted on Fig. (6). As the method is conditional stable, the error was going to extremely large when the time step was greater than the limit.

1.6 Period Shift

Between the numerical solution and analytical solution, there is a shift due to accuracy of numerical method[4].

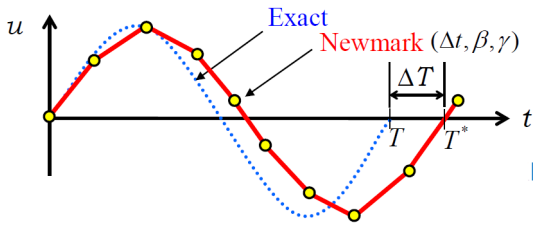

Figure 3: Period Shift

The period of the analytical solution can be calculated directly. While, the period of the numerical solution is hard point. Hence, $Fast\ Fourier\ Transformation(FFT)$[5] has been deployed in period finding. The FFT converts the $x - axis$ of Fig. (3) from $time$ to $frequency$.Thus, the frequency is used as calculating period.

To display the relation between the scale of shift and Δt, $\frac{\Delta T}{T}$ has been plotted with various β and increasing with $\frac{\Delta t}{T}$.

As depicted in Figure. (4):

1. $\beta = 0$ and $\beta = \frac{1}{12}$: ΔT is always negative, thus, the period of the numerical solution is shorter than the analytical one. Furthermore, ΔT decreases with increment of time-step size, that indicates the gap among the numerical solution and analytical solution would be shrunk with bigger Δt.

2. $\beta = \frac{1}{6}$, $\beta = \frac{1}{3}$ and $\beta = \frac{1}{2}$: ΔT is always positive that means the period of numerical solution is longer than analytical one, ΔT raises by increasing Δt. Moreover, the bigger β leads the raising gradient of the tendency of ΔT.

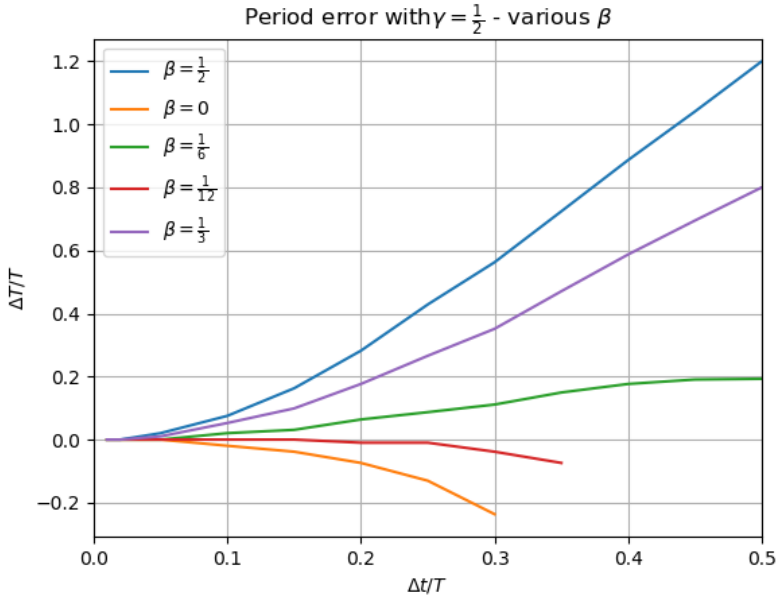

Figure 4: Period Shift with Δt of various β

1.7 Conclusion

In part one, the stability condition of Newmark's method was derived in general form, and range of β of unconditional stable and conditional stable was defined respectively. Meanwhile, the order of accuracy on Newmark's method was found. The visualization of maximum kinetic energy with various time step and two β was depicted, the trend showed unstable when the time step was over the limit which was calculated in **1.2.2**. In unconditional stable situation, the error displayed periodicityhowever, the stability on error collapsed when the time step was greater than limit condition in method with $\beta = \frac{1}{6}$. The crucial part of improvement of accuracy is choosing proper β, various β would generate opposite tendency of period error.

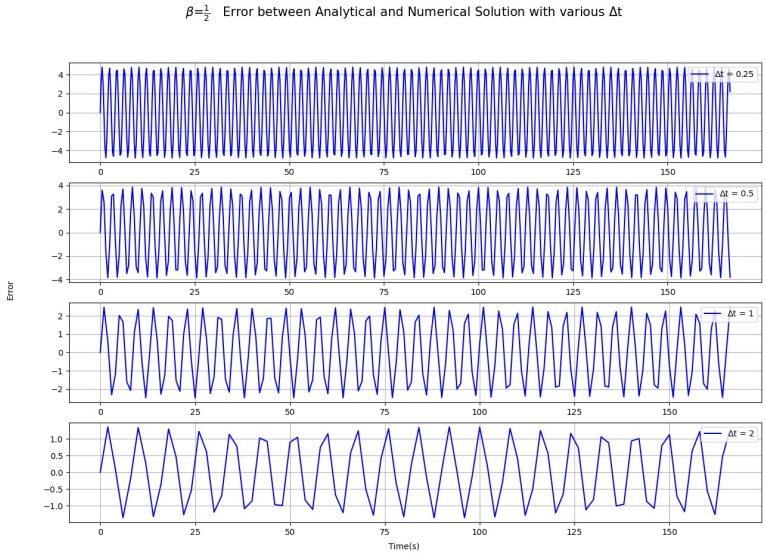

Figure 5: Energy with $\beta = \frac{1}{2}$

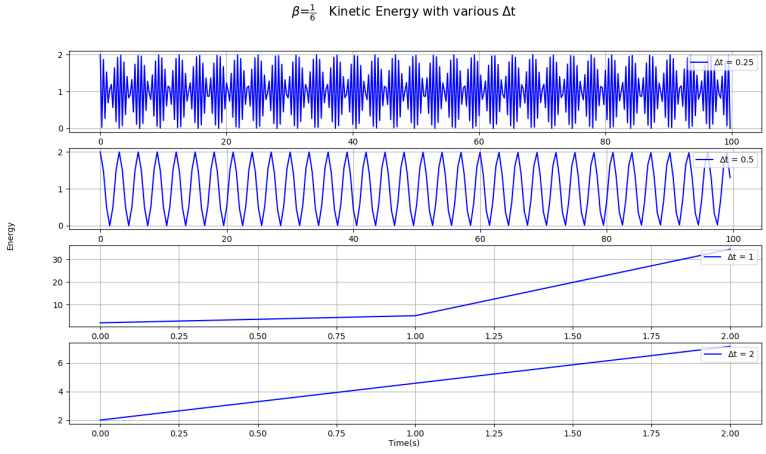

Figure 6: Error with $\beta = \frac{1}{6}$

2 Part 2 - Response Spectra of SDOF Systems under a Ground Motion

2.1 Parameter Selection

In part 2, the dynamic system is expressed as:

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = -a_g(t) \tag{22}$$

$a_g(t)$ is the ground acceleration taken from the El Centro earthquake and visualize in Fig.(7) From the EI Centro data file, the Δt is fixed as $0.02s$. According to the assignment descrip-

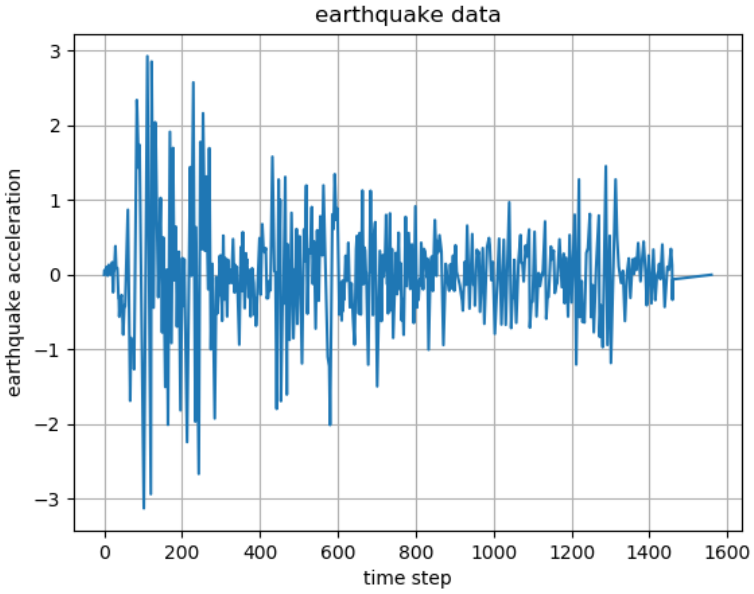

Figure 7: Earthquake Acceleration Visualization

tion, the initial conditions and parameters are selected as:

- $\Delta t = 0.02$
- $\xi = 0.05$
- $u_{initial} = 0$
- $\dot{u}_{initial} = 0$
- Newmarks method parameters: $\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$

2.2 Displacement and Total Acceleration with various Period

For comparing the discrepancy resulted from various period, we set three period, $T = 0.5$, $T = 5$, $T = 10$, and solve Eq. (22) by Newmarks method with defined parameters in **2.1**.

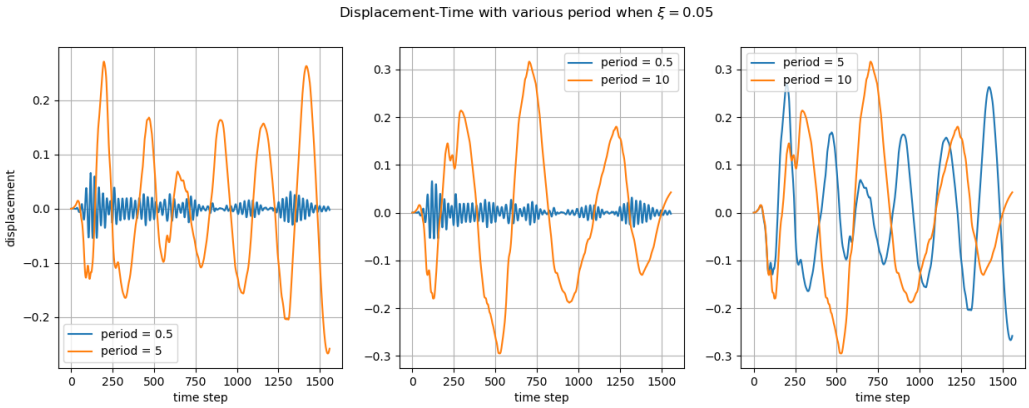

Figure 8: System Displacement with different Period over whole Earthquake

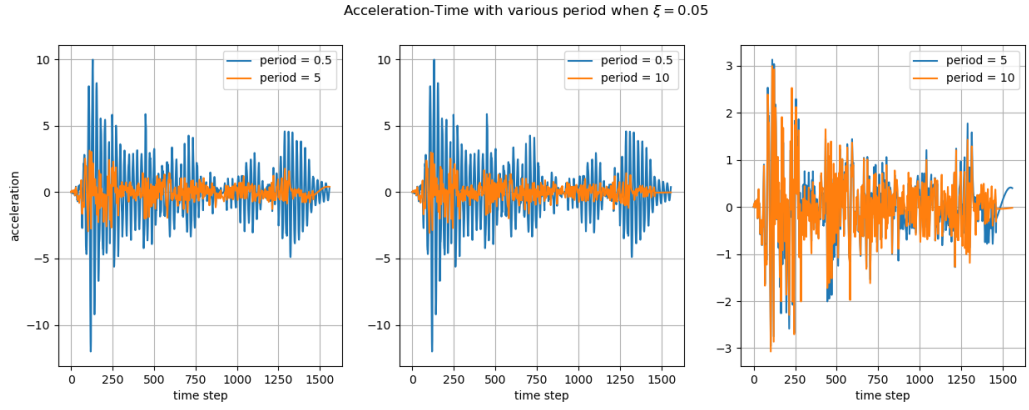

Figure 9: System Acceleration with different Period over whole Earthquake

The numerical results are depicted as Fig. (8) and Fig. (9).
In Fig. (8), the peak displacement becomes larger by the increasing period. However, the peak acceleration decreases with period getting larger.

2.3 Response Spectras of Displacement, Velocity and Acceleration

To find the peak value of the the displacement, velocity and total acceleration for different periods, three figures are plotted in Fig. (10), (11) and (12). The range of period is $[0.01, 10]$ and the period step is 0.05.

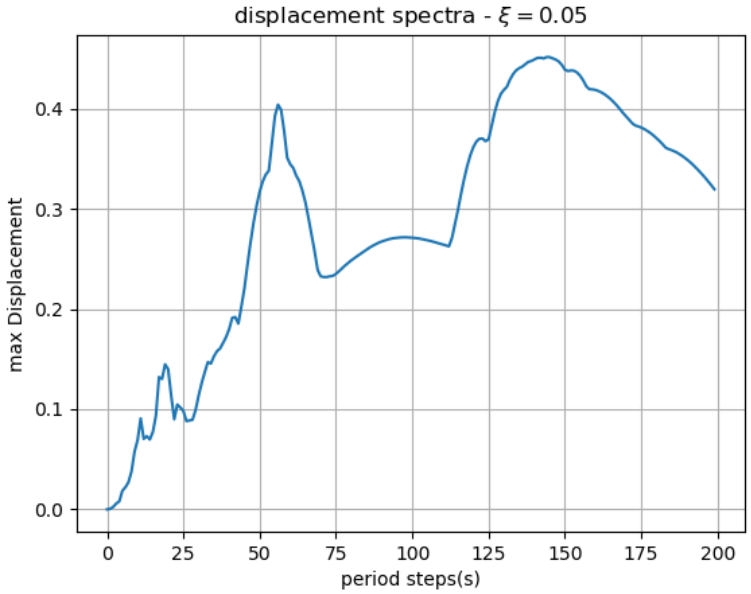

Figure 10: Displacement Response Spectra

There are two ways to find the peak values of acceleration and velocity. The first way is that extracting peak value from each computing of Newmarks method which iterates by period step (called as approximate pseudo spectra). Another way is called as pseudo spectra, it can be calculated by displacement spectra:

$$S_v(\omega, \xi) = wS_u(\omega, \xi) \approx \max_t\{\dot{u}(t)\} \tag{23}$$

$$S_a(\omega, \xi) = w^2 S_u(\omega, \xi) \approx \max_t\{\ddot{u}(t) + a_g(t)\} \tag{24}$$

To compare the difference between two methods. The pseudo spectra and approximate pseudo spectra are displayed in Fig. (11) and (12). As the figures showed, the gap between pseudo results and approximate results increase with period.

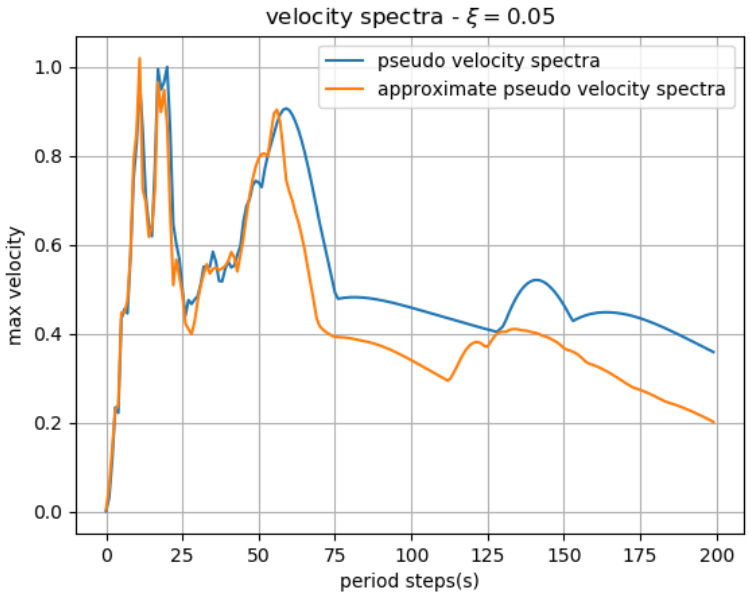

Figure 11: Velocity Response Spectra

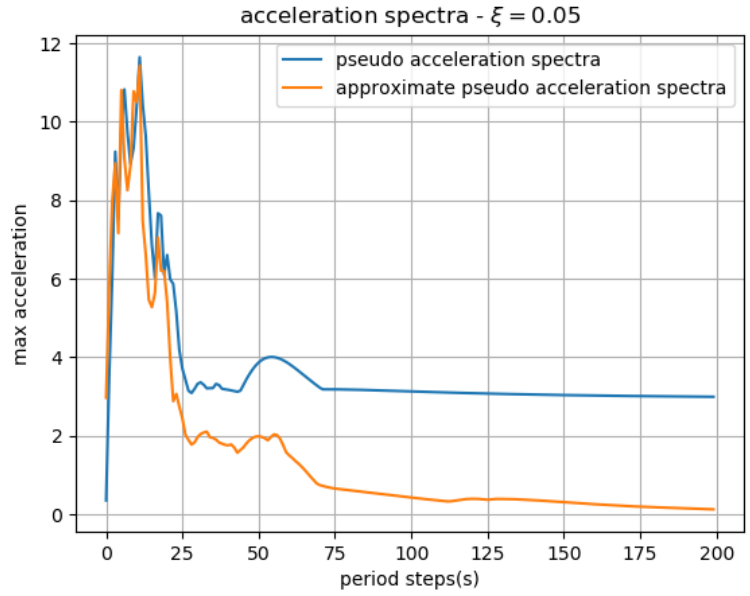

Figure 12: Acceleration Response Spectra

3 Part 3 - Earthquake Analysis of a Simple Frame Structure

3.1 Stiffness and Mass Matrix Calculation

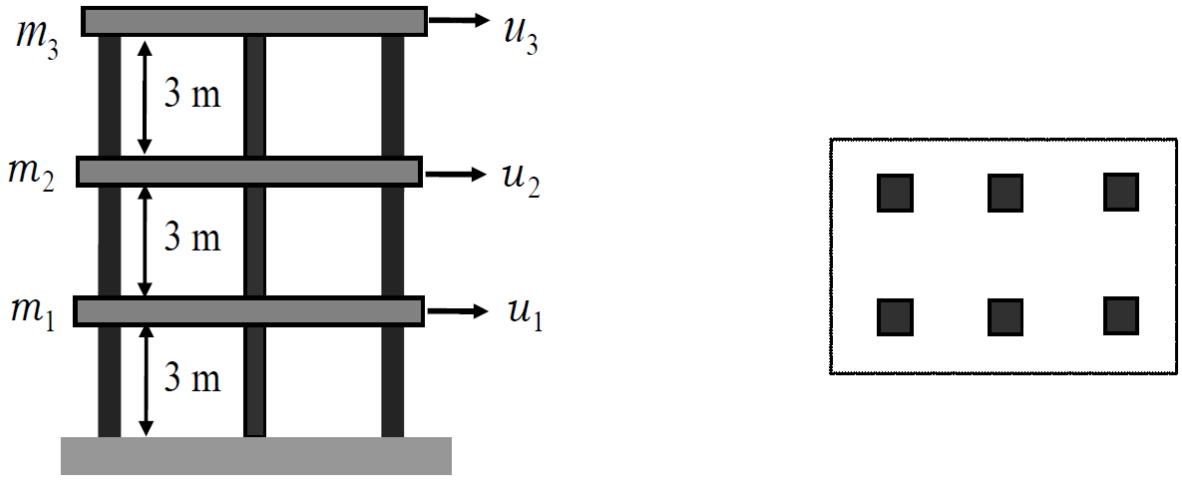

Figure 13: Three storey Build

From Fig. (13), there is a three floors building, and each floor has six concrete columns of dimensions $0.35 * 0.35m^2$. Young's modulus of concrete is assumed to be $E = 25GN/m^2$. The mass of each floor is:

$$\begin{aligned} M_1 &= 400000kg \quad h_1 = 3m \\ M_2 &= 300000kg \quad h_2 = 3m \\ M_3 &= 200000kg \quad h_3 = 3m \end{aligned}$$

Hence, the mas matrix M is:

$$\mathrm{M} = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \tag{25}$$

The stiffness of each column is calculated as:

$$\mathrm{I} = \frac{bh^3}{12} = \frac{a^4}{12} = \frac{0.35^4}{12} \qquad k_{column} = \frac{12EI}{h^3} \tag{26}$$

As each floor has six columns, thus, the stiffness of each floor is expressed as:

$$K_1 = K_2 = K_3 = 8.34 \times 10^7 N/M$$

The stiffness matrix K is assembled as:

$$\mathbf{K} = \begin{bmatrix} K_1 + K_2 & -K_2 & 0 \\ -K_2 & K_2 + K_3 & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix} \tag{27}$$

3.2 First Mode Result

For finding the first mode using one Ritz vector: $r_1 = [1\ 2\ 3]^T$
The reduced M and K are:

$$\begin{aligned} \hat{\mathrm{m}} &= \mathbf{r}_1^T \mathbf{M} \mathbf{r}_1 = 3.4 \times 10^6 (\ \mathrm{kg}) \\ \hat{\mathrm{k}} &= \mathbf{r}_1^T \mathbf{K} \mathbf{r}_1 = 2.5 \times 10^8 (N/m) \end{aligned} \tag{28}$$

The approximate frequency and shape vector are:

$$\omega_1 \approx \hat{\omega}_1 = \sqrt{\hat{k}/\hat{m}} = 8.576(1/s) \tag{29}$$

$$\mathbf{v}_1 \approx \mathbf{r}_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T \tag{30}$$

Once the frequency is calculated, the period $T_1 = \frac{2\pi}{\omega_1}$ can be get. Follow the displacement spectra in *Part* 2, the linear interpolation[6] is applied to find the peak displacement of specific period.
For the peak displacement corresponding to T_1 cannot be found directly in the spectra, the interpolation procedure is:

Algorithm 1: Pseudo Code of Linear Interpolation

Result: The peak value corresponding to input period T
set spectra as an array
find $T_a < T_{input}$ and $T_b > T_{input}$, a and b are indices in spectra array, b=a+1
find the peak value with indices a and b, set them as $V_a = spectra[a]$ and $V_b = spectra[b]$
$Peak\ Value_{T_{input}} = \frac{V_a(T_b - T_{input}) + V_b(T_{input} - T_a)}{T_b - T_a}$

Follow the algorithm. (1), the peak displacement $S_u(T)$ and acceleration $S_a(T)$ are found. After that, the other parameters are calculated as:

$$\begin{gathered} \text{Modal Mass:} \bar{m}_1 = \mathbf{v}_1^T \mathbf{M} \mathbf{v}_1 \\ \text{Load participation factor:} l_1 = \mathbf{v}_1^T \mathbf{M} \mathbf{d} / \bar{m}_1 \\ \text{Earthquake input direction vector:} \mathbf{d} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \\ \text{Modal Force:} \mathrm{V_b} = \mathrm{mS_a\,(T_1)}\ (\text{note that:} m = \text{total mass} = M_1 + M_2 + M_3) \end{gathered} \tag{31}$$

The maximum displacement $U_{1,max}$ and the shear force vector F_i are:

$$\begin{aligned} \mathbf{U}_{1,\max} &= l_1 S_a\,(\ \mathrm{T_1})\, \mathbf{v}_1 \\ \mathrm{F_i} &= \mathrm{V_b} \frac{\mathrm{m_i h_i}}{\sum_{i=1}^{3}\ \mathrm{m_i h_i}} \end{aligned} \tag{32}$$

From Eq. (32),

$$\begin{aligned} U_{1,max} &= [3.44,\ 6.88,\ 10.3] \times 10^{-2} \\ F_1 &= [1.70,\ 2.56,\ 2.56] \times 10^6 \end{aligned}$$

3.3 First two Modes

In this section, the Ritz vectors are:

$$\mathbf{r}_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$$
$$\mathbf{r}_2 = \begin{bmatrix} 1 & 4 & 9 \end{bmatrix}^T$$

The decomposed M and K are:

$$\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 \end{bmatrix}^T \mathbf{M} \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 \end{bmatrix}$$
$$\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 \end{bmatrix}^T \mathbf{K} \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 \end{bmatrix}$$

Follow the similar procedures in **3.2**, the eigenvalue problem needs to be solved:

$$\hat{\mathbf{K}}\mathbf{x}_i = \hat{\omega}_i^2 \hat{\mathbf{M}}\mathbf{x}_i \quad (i = 1, 2) \tag{33}$$

The v is expressed as:

$$\begin{aligned} \mathbf{v}_1 &\approx \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 \end{bmatrix} \mathbf{x}_1 \\ \mathbf{v}_2 &\approx \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 \end{bmatrix} \mathbf{x}_2 \end{aligned} \tag{34}$$

From Eq. (31) and (32), the peak displacement and shear force vector is:

$$\begin{aligned} U_{1,max} &= [1.73,\ 5.88,\ 12.44] \times 10^{-2} \\ U_{2,max} &= [1.50,\ 5.62,\ 12.33] \times 10^{-2} \\ F_1 &= [1.44,\ 2.16,\ 2.16] \times 10^{6} \\ F_2 &= [2.34,\ 3.511,\ 3.511] \times 10^{6} \end{aligned}$$

Using SRSS on $U_{i,max}$ and F_i, the results are:

$$\mathbf{U}_{\max} = \sqrt{\mathbf{U}_{1,\max}^2 + \mathbf{U}_{2,\max}^2 + \mathbf{U}_{3,\max}^2} = \{0.023, 0.081, 0.175\}(\mathrm{m})$$
$$\mathbf{F}_{\max} = \sqrt{\mathbf{F}_{1,\max}^2 + \mathbf{F}_{2,\max}^2 + \mathbf{F}_{3,\max}^2} = \{2.75, 4.12, 4.12\} \times 10^6 \text{ N}$$

3.4 Exact Solution

For finding the exact solution, the $\mathbf{v}_i$ can be calculated from the equation:

$$\left(\mathbf{K} - \omega_i^2 \mathbf{M}\right) \mathbf{v}_i = 0 \tag{35}$$

Follow Eq. (31) and (32), the exact solution of peak displacement and shear force are:

Mode No.	Max displacement Vector (m) $\mathrm{U_{i,max}} = l_i \ \mathrm{S_u}\,(\mathrm{T}_i)\,\mathbf{v}_i$	Total max displacement vector $\mathrm{U_{max}}$ (using SRSS)
1	$[7.8,\ 9.9,\ 7.8]\times10^{-2}$	$[7.9,\ 9.9,\ 7.8]\times10^{-2}$
2	$[1.459\times10^{-2}, 3.106\times10^{-18}, -7.294\times10^{-3}]$	
3	$[3.542\times10^{-4}, -6.2714\times10^{-4}, 3.541\times10^{-4}]$	

Mode No.	Shear force Vector $\mathrm{F}_{i,\max} = l_i \ \mathrm{S}_a\,(\ \mathrm{T}_i)\,\mathbf{M}\mathbf{v}_i$	Total shear force $\mathrm{F_{max}}$ (Using SRSS)
1	$[1.441,\ 2.161,\ 2.161]\times10^{6}$	$[3.298,\ 4.947,\ 4.947]\times10^{6}$
2	$[2.426,\ 3.638,\ 3.638]\times10^{6}$	
3	$[1.708,\ 2.562,\ 2.562]\times10^{6}$	

3.5 Conclusion

In this part, the solution of decomposed modes and exact mode are performed. The results of 1^{st} mode, 2^{nd} mode and exact solution showed different values, while the discrepancy is not substantial. Especially, the shear force vector of 2^{nd} mode that is reduced by two Ritz vectors is almost the same as the result from exact solution table. However, the shear force vector with SRSS is smaller than exact one. To sum up, one Ritz vector is unable to approximate the original system, but two Ritz vectors provide acceptable solution.

4 Reference

References

[1] Cheng-Wu Chen, Wei-Ling Chiang, Chung-Hung Tsai, Chen-Yuan Chen and Morris H. L. Wang. Fuzzy LYAPUNOV Method for Stability Conditions of Nonlinear Systems. *International Journal on Artificial Intelligence Tools. Vol. 15, No. 02, pp. 163-171*, 2006.

[2] Harry G Kwatny. Energy-like Lyapunov functions for power system stability analysis. *IEEE Transactions on Circuits and Systems*, 1985.

[3] Dara Maghdid. Stability and accuracy of Newmark's method. *Lund University* ,2002

[4] J. M. B. Brown. The discretization error of newmark's method for numerical integration in structural dynamics. *The Journal of the International Association for Earthquake Engineering* , January/February 1985.

[5] P.Duhamel, M.Vetterli. Fast fourier transforms: A tutorial review and a state of the art. *Signal Processing. Vol 19, Issue 4, pp. 259-299*, April 1990.

[6] C.Caruso, F.Quarta.Interpolation Methods Comparison. *Computers & Mathematics with Applications, Vol 35, Issue 12, pp 109-126*, June 1998.