1. $Spec(A) = \{-1, -2\}$ $\quad v_1 = [1 \quad 0]^T \quad v_2 = [2 \quad 1]^T$

a) $A = U\Lambda U^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

b) $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \dots = U\left[I + \Lambda t + \frac{1}{2!}\Lambda^2 t^2 + \dots\right]U^{-1}$

$= U\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}U^{-1} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$

2. Show that

$$A = \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} > 0 \iff R > 0 \text{ and } P - QR^{-1}Q^* > 0$$

$$A = \begin{bmatrix} I & \\ - & I \end{bmatrix}\begin{bmatrix} P - QR^{-1}Q^* & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} I & \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & -QR^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix}\begin{bmatrix} I & 0 \\ -R^{-1}Q^* & I \end{bmatrix} = \begin{bmatrix} P - QR^{-1}Q^* & 0 \\ 0 & R \end{bmatrix}$$

$$\begin{bmatrix} P - QR^{-1}Q^* & 0 \\ Q^* & R \end{bmatrix}\begin{bmatrix} I & 0 \\ -R^{-1}Q^* & I \end{bmatrix} = \begin{bmatrix} P - QR^{-1}Q^* & 0 \\ 0 & R \end{bmatrix}$$

$$\begin{bmatrix} I & -QR^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix}\begin{bmatrix} I & 0 \\ -R^{-1}Q^* & I \end{bmatrix}$$

3. Let $A \in \mathbb{C}^{n \times n}$, $A > 0$ $\quad A = A^* = U \Lambda U^*$

$\lambda_1, \lambda_2, \ldots, \lambda_n = \mathrm{Re}(\lambda_i(A)) > 0$

$$A^{1/2} = U \begin{bmatrix} \lambda_1^{1/2} & & \\ & \ddots & \\ & & \lambda_n^{1/2} \end{bmatrix} U^*$$

a) Show $A^{1/2} A^{1/2} = A$

$$A^{1/2}A^{1/2} = U \begin{bmatrix} \lambda_1^{1/2} & & \\ & \ddots & \\ & & \lambda_n^{1/2} \end{bmatrix} U^* U \begin{bmatrix} \lambda_1^{1/2} & & \\ & \ddots & \\ & & \lambda_n^{1/2} \end{bmatrix} U^* = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^* = A$$

b Show $A^{1/2} \geq 0$

$A > 0 \Rightarrow \mathrm{Spec}(A) \geq 0 \Rightarrow \lambda_i(A) \geq 0 \ \forall i, \ \lambda_i \in \mathbb{R}$

$\mathrm{Spec}(A^{1/2}) = [\mathrm{Spec}(A)]^{1/2} \Rightarrow \lambda_i(A^{1/2}) \geq 0 \ \forall i, \ \lambda_i(A^{1/2}) \in \mathbb{R}$

$\Rightarrow A^{1/2} \geq 0$

c Find $A^{1/2}$, $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$

$|sI - A| = (s-4)(s-4) - 9 = s^2 - 8s + 7 = (s-1)(s-7) = 0$

$\lambda_i = 1, 7 \qquad \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow v_1 = -v_2 \qquad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow v_1 = v_2 \qquad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$U_N = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \qquad U = \begin{bmatrix} \frac{v_1}{\|v_1\|} & \frac{v_2}{\|v_2\|} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$$A^{1/2} = U \Lambda^{1/2} U^* = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{7} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1.8229 & 0.8229 \\ 0.8229 & 1.8229 \end{bmatrix}$$

4 Suppose $A > 0$, $B > 0$ $\quad A = A^* = U \Lambda U^*$ $\quad B = B^* = V \Gamma V^*$

a) Show $\lambda_i(AB) > 0 \quad \forall i$

$Spec\{A\} = \{\lambda_1, \lambda_2 \ldots \lambda_n\} \quad \lambda_i > 0 \ \forall i$

$Spec\{B\} = \{\gamma_1, \gamma_2 \ldots \gamma_n\} \quad \gamma_i > 0 \ \forall i$

$Spec\{f(A)\} = \{f(\lambda_1), \ldots, f(\lambda_n)\} \qquad f(A) = AB$

$= \{\lambda_1 B, \ldots, \lambda_n B\}$

b Is $AB > 0$? $Spec\{AB\} > 0 \qquad AB = (AB)^* = B^* A^*$?

$(AB)^* = B^* A^* = BA \qquad [(AB)^*]^* = (B^* A^*)^*$

$AB = A^* B^* = (BA)^* = (B^* A^*)^* = [(AB)^*]^*$

5 Let $P > 0$ $\quad P = P^* = U \Lambda U^*$

a) Show P^{-1} exists, $P^{-1} > 0$

$$|P| = \prod \lambda_i(P) > 0 \iff \lambda_i(P) > 0$$

$$\Rightarrow P^{-1} \text{ exists} \quad \lambda_i(P^{-1}) = 1/\lambda_i(P) \Rightarrow \lambda_i(P^{-1}) > 0 \quad \forall i$$

$$\Rightarrow P^{-1} > 0 \quad P = U \Lambda U^* \quad P^{-1} = (U^*)^{-1} \Lambda U^{-1} = U \Lambda^{-1} U^* = P^{-1} = (P^{-1})^*$$

$$\Rightarrow P^{-1} > 0$$

b) $$\sup_{x \neq 0} \frac{x^* P x}{x^* x} = \lambda_{max}(P) \qquad \frac{x^* P x}{x^* x} = \frac{x^* U \Lambda U^* x}{x^* x}$$

$$y = U^* x \qquad x = U y$$

$$\sup_{x \neq 0} \frac{x^* P x}{x^* x} = \sup_{y \neq 0} \frac{y^* \Lambda y}{y^* U^* U y} = \sup_{y \neq 0} \frac{y^* \Lambda y}{y^* y}$$

$$= \sup_{y \neq 0} \frac{\lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2}{y_1^2 + y_2^2 + \dots + y_n^2} = \lambda_{max}(P)$$

c) $$\inf_{x \neq 0} \frac{x^* P x}{x^* x} = \lambda_{min}(P) \qquad y = U^* x \quad x = U y$$

$$\inf_{x \neq 0} \frac{x^* P x}{x^* x} = \inf_{y \neq 0} \frac{y^* \Lambda y}{y^* y} = \frac{\lambda_1^2 y_1^2 + \dots + \lambda_n^2 y_n^2}{y_1^2 + \dots + y_n^2} = \lambda_{min}(P)$$

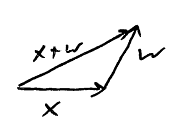
d) $\|x\| = \sqrt{x^* P x}$ qualifies as a norm

$$\|x\| = \sqrt{x^* U \Lambda U^* x} \qquad y = U^* x \qquad \|x\| = \sqrt{y^* \Lambda y}$$

a) $\|x\| = [y^* \Lambda y]^{\frac{1}{2}} = [\lambda_1 y_1^2 + \dots + \lambda_n y_n^2]^{\frac{1}{2}} > 0 \qquad \lambda_i(P) > 0 \ \forall i$

b) $\|x\| = 0 = [\lambda_1 y_1^2 + \dots + \lambda_n y_n^2]^{\frac{1}{2}} \Rightarrow y = 0 \qquad \lambda_i(P) > 0 \ \forall i$

c) $\|x + w\| = [(x+w)^* P (x+w)]^{\frac{1}{2}} \qquad h = U^*(x+w)$



$\|x + w\| = [\lambda_1 h_1^2 + \dots + \lambda_n h_n^2]^{\frac{1}{2}}$

$\|x\| + \|w\| = [\lambda_1 x_1^2 + \dots + \lambda_n x_n^2]^{\frac{1}{2}} + [\lambda_1 w_1^2 + \dots + \lambda_n w_n^2]^{\frac{1}{2}}$

$\Rightarrow \|x + w\| \leq \|x\| + \|w\|$

d) $\|\alpha x\| = [\bar{\alpha} x^* U \Lambda U^* x \alpha]^{\frac{1}{2}} = [(\bar{\alpha}\alpha)(\lambda_1 x_1^2 + \dots + \lambda_n x_n^2)]^{\frac{1}{2}} =$

$= |\alpha| (\lambda_1 x_1^2 + \dots + \lambda_n x_n^2)^{\frac{1}{2}} = |\alpha| \|x\|$

6.

a) Compute $\sin(A)$ $\quad A = \begin{bmatrix} -3\pi & 4\pi \\ -5\pi & 6\pi \end{bmatrix}$ $\quad |sI - A| = (s+3\pi)(s-6\pi) + 20\pi^2$

$s^2 - 3\pi s + 2\pi^2 = 0 \qquad (s-2\pi)(s-\pi) = 0 \qquad \lambda_i = \pi, 2\pi$

$\lambda = \pi \quad \begin{bmatrix} -4\pi & 4\pi \\ -6\pi & 6\pi \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \qquad 4\pi v_1 = 4\pi v_2 \qquad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2\pi \quad \begin{bmatrix} -5\pi & 4\pi \\ -5\pi & 4\pi \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \qquad -5\pi v_1 + 4\pi v_2 = 0 \qquad v_1 = \frac{4}{5} v_2 \qquad v = \begin{bmatrix} 4/5 \\ 1 \end{bmatrix}$

$T = \begin{bmatrix} 1 & 4/5 \\ 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & -4/5 \\ 0 & 1 \end{bmatrix} \qquad \sin(A) = T\sin(\Lambda)T^{-1}$

$$\sin(A) = \begin{bmatrix} 1 & 4/5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sin(\pi) & 0 \\ 0 & \sin(2\pi) \end{bmatrix}\begin{bmatrix} 1 & -4/5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b) Suppose $A \in \mathbb{C}^{n\times n}$, $\mathrm{Spec}(A) \subseteq \{\lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) > 0\}$
show $I+A$ is invertible

$\mathrm{Spec}\{I+A\} = 1 + \mathrm{Spec}\{A\} \Rightarrow \mathrm{Re}\{\mathrm{Spec}\{I+A\}\} > 0$

$\Rightarrow \lambda_i(I+A) \neq 0 \quad \forall i \quad \Rightarrow (I+A)^{-1}$ exists

c) What can you say about the eigenvalues of $(I+A)^{-1}(I-A)$

$\mathrm{Spec}\{I+A\} = 1 + \mathrm{Spec}\{A\} \qquad \mathrm{Spec}\{I+A\}^{-1} = \dfrac{1}{1+\mathrm{Spec}\{A\}}$

$\mathrm{Spec}\{I-A\} = 1 - \mathrm{Spec}\{A\}$

$\mathrm{Spec}\{(I+A)^{-1}(I-A)\} = \underline{1-\mathrm{Spec}\{A\}}$

7 Show that $\|AB\|_{i2} \leq \|A\|_{i2} \|B\|_{i2}$

$$\|AB\|_{i2} = \max_x \frac{\|ABx\|_2}{\|x\|_2} = \max_x \frac{[x^*B^*A^*ABx]^{\frac{1}{2}}}{[x^*x]^{\frac{1}{2}}}$$

$$\|A\|_{i2} = \max_x \frac{\|Ax\|_2}{\|x\|_2} = \max_x \frac{[x^*A^*Ax]^{\frac{1}{2}}}{[x^*x]^{\frac{1}{2}}}$$

8 SVD

a Show that $\rho(A) \leq \bar{\sigma}(A)$