MAE 280 A | HOMEWORK

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36/45

2.31 Prove that $x^H A x$ is an imaginary number if A is skew-hermitian $A = -A^H$.

$$A = -A^H \Rightarrow \quad Re(A_{ij}) = 0 \ \forall\, i,j \qquad A_{ij} = A_{ji} \qquad A = -\bar{A}$$

$$x^H A x = [\bar{x}_1 \ \bar{x}_2 \ \cdots \ \bar{x}_n] \begin{bmatrix} A_{11} & & \\ & \ddots & \\ & & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Does $x^H A x = -\overline{x^H A x}$?

$$-\overline{x^H A x} = -\bar{x}^H \bar{A} \bar{x} = x^T A \bar{x} = \bar{x}^T A x = x^H A x \quad \checkmark$$

If A is skew-symmetric $A = -A^T$ $\quad Im(A_{ij}) = 0 \ \forall\, i,j$
$A_{ij} = -A_{ji} \qquad A = -A^H$

$$x^H A x = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\bar{x}_1 \ \bar{x}_2] \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

$$= -\bar{x}_1 x_2 + x_1 \bar{x}_2 = -\overline{(a+jb)}(c+jd) + (a+jb)\overline{(c+jd)}$$

$$= -(a-jb)(c+jd) + (a+jb)(c-jd)$$

$$= -(ac + jad - jbc + bd) + (ac - jad + jbc + bd) = 0$$

$x^H A x$ is real by Thereom 2.13

$(x^H A x)^H = x^H A x$ if real (by Thereom 2.13)

$x^H A x = x^H A^H x = x^H A^T x$ (A is real)

$$A = -A^T \Rightarrow x^H A x = x^H A^T x = -x^H A^T x \Rightarrow x^H A x = 0 \quad \checkmark$$

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2.32 Prove that every square matrix $A = A_{SH} + A_H$

where $A_H = \frac{1}{2}(A + A^H)$ $A_{SH} = \frac{1}{2}(A - A^H)$

$(A + A^H)^H = (A + A^H)$ for A_H to be true

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2.33 Show that, for Hermitian A with eigenvalues λ_i $\quad A = A^H$

$$x^H A x = \sum_{i=1}^{n} \lambda_i z_i^2 \qquad z = E^H x \qquad A \in \mathbb{R}^{n \times n}$$

$$x^H A x = x^H E \Lambda E^{-1} x = [x_1 \; \dots \; x_n][e_1 \; e_2 \; \dots \; e_n]\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}[e_1 \; e_2 \; \dots \; e_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

A is nondefective by Thereom 2.16 $\Rightarrow$ Λ is diagonal

$$E^{-1} = E^H \quad \Rightarrow \quad x^H A x = (x^H E) \Lambda E^H x \qquad Im(E_{ij}) = 0 \;\; \forall i,j$$

$$z = E^H x \quad \Rightarrow \quad x^H A x = z^H \Lambda z = \sum_{i=1}^{n} \bar{z}_i \lambda_i z_i = \sum_{i=1}^{n} \lambda_i z_i^2 \;\checkmark$$

if $z_i \in \mathbb{R} \;\; \forall i$

Thereom 2.3

a1 $\quad K = K^H \qquad K = A \qquad x^H K x > 0$

$x^H A x = \sum \lambda_i z_i^2$, $\; z_i^2 > 0 \;\; \forall i$, therefore

for $\sum \lambda_i z_i^2 > 0 \;\; \forall \; z_i^2 \geq 0 \qquad \lambda_i > 0 \;\; \forall i$ ✓

b1 $\quad K \geq 0 \qquad K = A \qquad x^H A x \geq 0$

$x^H A x = \sum \lambda_i z_i^2$, $\; z_i^2 > 0 \;\; \forall i$ therefore ✓

for $\sum \lambda_i z_i^2 \geq 0 \;\; \forall \; z_i^2 \geq 0$, $\; \lambda_i \geq 0 \;\; \forall i$

c1 $\quad K < 0 \qquad K = A \qquad x^H A x < 0$

$x^H A x = \sum \lambda_i z_i^2$, $\; z_i^2 \geq 0 \;\; \forall i$ therefore ✓

for $\sum \lambda_i z_i^2 < 0 \;\; \forall \; z_i^2 \geq 0$, $\; \lambda_i < 0 \;\; \forall i$

d1 $\quad K \leq 0 \qquad K = A \qquad x^H A x \leq 0$

$x^H A x = \sum \lambda_i z_i^2$, $\; z_i^2 \geq 0 \;\; \forall i$ therefore ✓

for $\sum \lambda_i z_i^2 \leq 0 \;\; \forall \; z_i^2 \geq 0 \qquad \lambda_i \leq 0 \;\; \forall i$

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2.34 On the x_1, x_2 plane, plot $x^H A x = 1$

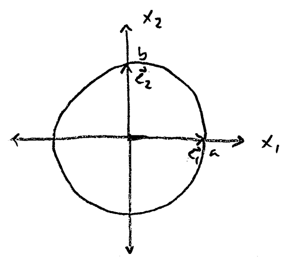
i $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $x^H A x = [\bar{x}_1 \;\; \bar{x}_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\bar{x}_1 + x_2\bar{x}_2 = |x_1|^2 + |x_2|^2 = 1$

$x_1^2 + x_2^2 = 1$ ✓

$|A - \lambda I| = (1-\lambda)^2 \Rightarrow \lambda_1 = \lambda_2 = 1$ $m_1 = 2$

$g_1 = 2$ $\Rightarrow a = 1, \;\; b = 1$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix} = 0 \Rightarrow \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $a' = 1, \;\; b' = 1$



major axis $= a'\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

minor axis $= b'\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

✓

ii $A = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$ $x^H A x = x_1^2 + 10x_2^2 = 1$ $a = 1$ $b = (10)^{-\frac{1}{2}}$

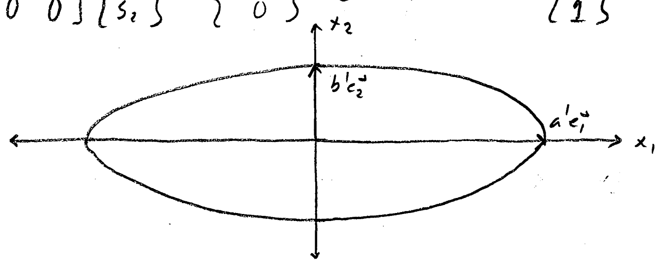
$|A - \lambda I| = (1-\lambda)(10-\lambda) \Rightarrow \lambda_1 = 1$ $\lambda_2 = 10$

$a' = 1$

$b' = 0.316227766 = 10^{-\frac{1}{2}}$

$\begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix} \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = \begin{Bmatrix} s_1 + s_2 \\ 9s_2 \end{Bmatrix} = 0 \Rightarrow \vec{e}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$ major axis $a'\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ✓

$\begin{bmatrix} -9 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = \begin{Bmatrix} -9s_1 \\ 0 \end{Bmatrix} = 0 \Rightarrow \vec{e}_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ minor axis $b'\vec{e}_2 = \begin{bmatrix} 0 \\ 10^{-\frac{1}{2}} \end{bmatrix}$ ✓



✓

iii $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ $x^H A x = [x_1 \;\; x_2] \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 5x_2 \end{bmatrix} = x_1^2 + 4x_1x_2 + 5x_2^2 = 1$

$|A - \lambda I| = (1-\lambda)(5-\lambda) - 4 = 0$ $\lambda^2 - 6\lambda + 1 = 0$ $\lambda = \frac{1}{2}(6 \pm \sqrt{32}) = 3 \pm 2\sqrt{2}$

$\begin{bmatrix} 1-\lambda_i & 2 \\ 2 & 5-\lambda_i \end{bmatrix} \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = 0$ $(1-\lambda_i)s_1 + 2s_2 = 0$ $s_1 = [-2/(1-\lambda_i)]s_2$

$\lambda_1 \Rightarrow s_1 = 0.4142135624 s_2$ $\lambda_2 \Rightarrow s_1 = -2.414213562 s_2$

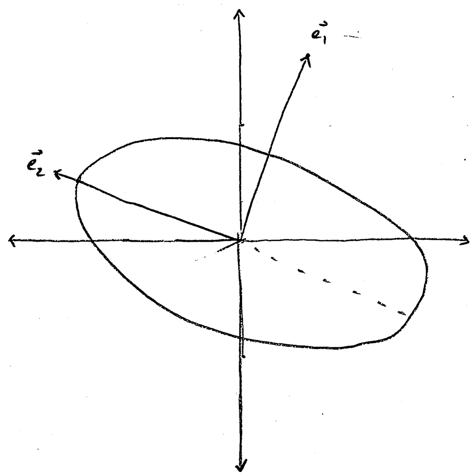
$\vec{e}_1 = \begin{bmatrix} 0.3826834324 \\ 0.9238795325 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} -0.9238795325 \\ 0.38268343324 \end{bmatrix}$

$x_1^2 + 4x_1x_2 + 5x_2^2 - 1 = 0 \quad \Rightarrow \quad \emptyset = 22.5$

$a = 1 \quad b = 2 \quad c = 5 \quad s = -1$

$a' = \left[2(-4 - (1)(5)(-1)) / \left[(4-5) \left\{ (5-1)\sqrt{1 + {}^{16}/_{(-4)^2}} - (6) \right\} \right] \right]^{\frac{1}{2}} \Rightarrow a' = 2.414213562$

$b' = \left[2(-4 - (1)(5)(-1)) / \left[(4-5) \left\{ (1-5)\sqrt{1+1} - (6) \right\} \right] \right]^{\frac{1}{2}} \Rightarrow b' = 0.4142135624$



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2.35 Repeat $x^H A x = 1$

i) $A = \begin{bmatrix} 1 & 5 \\ 5 & 10 \end{bmatrix}$ $\qquad [\bar{x}_1 \quad \bar{x}_2] \begin{bmatrix} 1 & 5 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\bar{x}_1 \quad \bar{x}_2] \begin{bmatrix} x_1 + 5x_2 \\ 5x_1 + 10x_2 \end{bmatrix} =$

$x_1^2 + 10x_1x_2 + 10x_2^2 = 1$

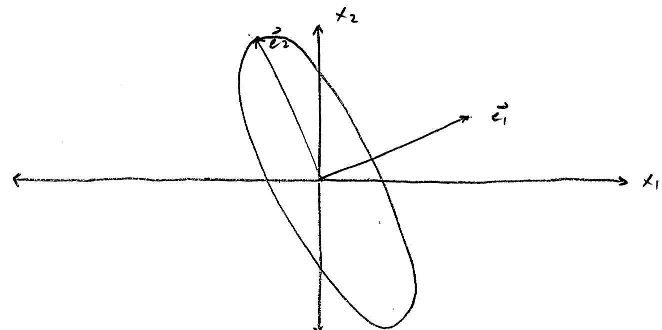
$|A - \lambda I| = (1-\lambda)(10-\lambda) - 25 = 0 \qquad \lambda^2 - 11\lambda - 15 = 0 \qquad \lambda_i = \frac{1}{2}(11 \pm \sqrt{181})$

$\begin{bmatrix} 1-\lambda_1 & 5 \\ 5 & 10-\lambda_1 \end{bmatrix} \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = 0 \qquad (1-\lambda_1)s_1 = 5s_2 \qquad s_1 = \frac{5}{1-\lambda_1} s_2 = -0.4453624047$

$\vec{e}_1 = \begin{bmatrix} -0.4068385849 \\ 0.9135000634 \end{bmatrix}$ ok $\quad s_1 = \frac{5}{1-\lambda_2} s_2 = 2.245362405 s_2 \qquad \vec{e}_2 = \begin{bmatrix} 0.9135000635 \\ 0.4068385849 \end{bmatrix}$ ✓

$x_1^2 + 10x_1x_2 + 10x_2^2 - 1 = 0 \qquad \theta = \frac{1}{2}\cot^{-1}\left(\frac{9}{20}\right)$

$a' =$
$b' =$



ii $A = \begin{bmatrix} 0 & 5 \\ 5 & 10 \end{bmatrix} \qquad x^H A x = [x_1 \quad x_2] \begin{bmatrix} 0 & 5 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \quad x_2] \begin{bmatrix} 5x_2 \\ 5x_1 + 10x_2 \end{bmatrix} = 10x_1x_2 + 10x_2^2 = 1$

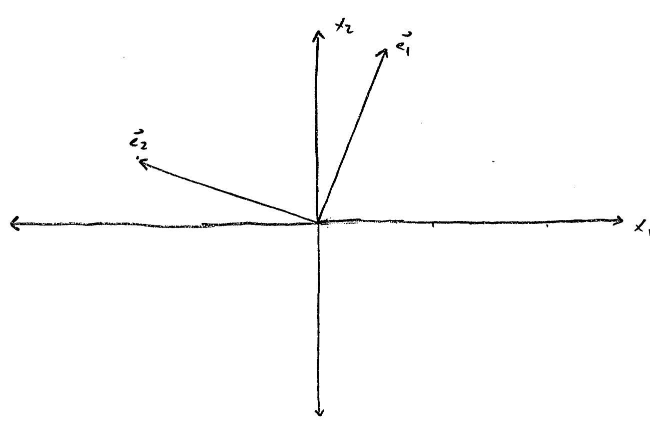
$|A - \lambda I| = (-\lambda)(10-\lambda) - 25 = 0 \qquad \lambda^2 - 10\lambda - 25 = 0 \qquad \lambda = \frac{1}{2}(10 \pm \sqrt{100+100}) = \frac{1}{2}(10 \pm 10\sqrt{2})$

$\begin{bmatrix} -\lambda_i & 5 \\ 5 & 10-\lambda_i \end{bmatrix} \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} \Rightarrow \qquad -\lambda_i s_1 + 5s_2 = 0 \qquad s_1 = \frac{5}{\lambda_1} s_2$

$\lambda_1 \Rightarrow \qquad s_1 = 0.4142135624 \qquad \Rightarrow \quad \vec{e}_1 = \begin{bmatrix} 0.3826834324 \\ 0.9238795325 \end{bmatrix}$ ✓

$\lambda_2 \Rightarrow \qquad s_1 = -2.414213562 s_2 \quad \Rightarrow \quad \vec{e}_2 = \begin{bmatrix} -0.9238795325 \\ 0.3826834324 \end{bmatrix}$ ✓

$\theta = \frac{1}{2}\cot^{-1}\left(\frac{10}{20}\right) = 13.28$



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2.36 A spacecraft can be described by

$\dot{w}_1 + \lambda w_2 = 0$

$\dot{w}_2 - \lambda w_1 = 0$

Show these can be put into the form $[\dot{w}_1 \ \dot{w}_2]^T = A[w_1 \ w_2]^T$ where $A = -A^H$. Compute $\lambda(A)$, $e(A)$. Write $x \triangleq E\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$\begin{aligned} \dot{w}_1 &= -\lambda w_2 \\ \dot{w}_2 &= \lambda w_1 \end{aligned} \Rightarrow \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix} \checkmark$$

$$A = \begin{bmatrix} 0 & -(a+bj) \\ a+jb & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 0 & a+jb \\ -(a+bj) & 0 \end{bmatrix} \qquad b = 0 \Rightarrow A = -A^T$$

eigenvalues denoted by s

$$|A - sI| = s^2 + \lambda^2 = 0 \qquad s = \pm j\lambda$$

$$\begin{bmatrix} -j\lambda & -\lambda \\ \lambda & -j\lambda \end{bmatrix}\begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} \qquad -j\lambda s_1 - \lambda s_2 = 0 \qquad -js_1 = +s_2 \qquad \vec{e}_1 = \begin{bmatrix} 1 \\ -j \end{bmatrix} \checkmark$$

$$\begin{bmatrix} j\lambda & -\lambda \\ \lambda & j\lambda \end{bmatrix}\begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} \qquad j\lambda s_1 - \lambda s_2 = 0 \qquad js_1 = s_2 \qquad \vec{e}_2 = \begin{bmatrix} 1 \\ +j \end{bmatrix} \checkmark$$

$$E = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \qquad \Lambda = \begin{bmatrix} +j\lambda & 0 \\ 0 & -j\lambda \end{bmatrix} \qquad \begin{aligned} A &= E\Lambda E^{-1} \\ AE &= E\Lambda \end{aligned}$$

$$E^{-1} = \frac{1}{2j}\begin{bmatrix} j & -1 \\ j & 1 \end{bmatrix} \qquad EE^{-1} = \frac{1}{2j}\begin{bmatrix} 2j & 0 \\ 0 & 2j \end{bmatrix} = I_2 \qquad EE^H = I$$

$$\dot{w} = Aw = -A^H w = -(E\Lambda E^{-1})^H w = -E^H \Lambda^H E w$$

$$x = Ew \Rightarrow E\dot{w} = -\Lambda^H E w \qquad \dot{x} = -\Lambda^H x. \ \times$$

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2.37 Prove Thereom 2.17, 2.18

2.17 The skew Hermitian matrix A has thes properties

$A = -A^H \Rightarrow Re(A_{ij}) = 0 \ \forall i,j \quad A_{ij} = A_{ji} \quad A = -\bar{A}$

i jA is Hermitian $jA = jA^H$

$$A = \begin{bmatrix} ja_{11} & ja_{12} \\ ja_{21} & ja_{22} \end{bmatrix} \qquad jA = -\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R} \qquad a_{12} = a_{21}$$

$jA \Rightarrow A_{ij} \in \mathbb{R} \ \forall ij, \ A_{ij} = A_{ji} \Rightarrow A = A^H$

ii A is nondefective $\Rightarrow$ n LI e_i $\quad \lambda_i \neq \lambda_j$

$$Ae_i = e_i \lambda_i \qquad e_j^H(Ae_i = e_i\lambda_i) \qquad e_j^H A e_i = -\overline{e_i^H A e_j}$$
$$e_i^H(Ae_j = e_j \lambda_j)$$

$$e_j^H e_i \lambda_i = -\overline{e_i^H e_j \lambda_j} = -e_j^H e_i \bar{\lambda}_j \qquad \underbrace{\lambda_j = -\bar{\lambda}_j}_{\text{part iii}} \qquad e_j^H e_i \lambda_i = e_j^H e_i \lambda_j$$

$\Rightarrow e_j^H e_i = 0$ as $\lambda_i \neq \lambda_j$

iii $Re(\lambda_i(A)) = 0 \quad \forall i$

$$Ae_i = e_i\lambda_i \qquad e_i^H(Ae_i = e_i\lambda_i) = e_i^H A e_i = e_i^H e_i \lambda_i$$

$$\lambda_i = \underbrace{(e_i^H e_i)^{-1}}_{\text{real}} \underbrace{e_i^H A e_i}_{\text{Imaginary by iv}} \Rightarrow \lambda_i \in \mathbb{I}, \ Re(\lambda_i(A)) = 0 \ \forall i$$

iv $x^H A x \in \mathbb{I} \ \forall x \in \mathbb{C}^n \qquad x^H A x \stackrel{?}{=} -\overline{x^H A x} \qquad A = -A^H$

$$-\overline{x^H A x} = -x^H \bar{A} x = x^H A x$$

v Eigenvectors associated w/ distinct eigenvalues are $\perp$

$$\lambda_i \neq \lambda_j \qquad Ae_i = e_i\lambda_i \qquad e_j^H(Ae_i = e_i\lambda_i) \qquad e_j^H A e_i = -e_i^H A^H e_j$$
$$e_i^H(Ae_j = e_j\lambda_j)$$

$$e_j^H e_i \lambda_j = -\overline{e_i^H e_j \lambda_j} = -e_j^H e_i \bar{\lambda}_j = e_j^H e_i \lambda_j \quad \text{by part iii}$$

$\Rightarrow e_j^H e_i = 0, \ e_i^H e_j = 0$

2.18 Unitary matrices A are $A^H A = I$ $\quad A^H = A^{-1}$ $\quad A \in \mathbb{R}^n$

i $||Ax||^2 = ||x||^2$

$$||Ax||^2 = (Ax)^H(Ax) = x^H A^H A x = x^H x = ||x||^2$$ ✓

ii All eigenvalues of A have modulus $|\lambda| = 1$

from 1 $\quad x^H A^H A x = ||x||^2$ $\quad A = U \Sigma V^H$ $\quad V^H V = U^H U = I$

$$x^H V \Sigma^H U^H U \Sigma V^H x = x^H V \Sigma^H \Sigma V^H x = x^H V \Lambda V^H x$$

$$= x^H V \begin{bmatrix} |\lambda_1| & & \\ & \ddots & \\ & & |\lambda_n| \end{bmatrix} V^H x$$

X

iii Eigenvectors associated with distinct eigenvalues are $\perp$

$\lambda_i \neq \lambda_j$ $\quad A e_i = e_i \lambda_i$ $\quad e_j^H (A e_i = e_i \lambda_i)$

$$e_j^H A e_i = \overline{e_i^H A e_j}$$ X

iv X

v X

$\not{b}$

2.38 Prove that there are no negative eigenvalues of $A^H A$ or AA^H

$A = U\Sigma V^H \qquad U^H U = V^H V = I$

$A^H A = V\Sigma^H U^H U \Sigma V^H = V\Sigma^H \Sigma V^H$

$$\Sigma^H \Sigma = \begin{bmatrix} [\Sigma_0]^H & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} [\Sigma_0] & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [\bar{\Sigma}_0] & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} [\Sigma_0] & 0 \\ 0 & 0 \end{bmatrix}$$

$$[\bar{\Sigma}_0][\Sigma_0] = \begin{bmatrix} |\sigma_1|^2 & & \\ & \ddots & \\ & & |\sigma_n|^2 \end{bmatrix}, \quad |\sigma_1|^2 = \lambda_1 \quad \dots \quad |\sigma_n|^2 = \lambda_n$$

$|\sigma_i|^2 \in \mathbb{R}, \Rightarrow \lambda_i \in \mathbb{R} \ \forall i$

$AA^H = U\Sigma V^H V \Sigma^H U^H = U\Sigma\Sigma^H U^H$

$$\Sigma\Sigma^H = \begin{bmatrix} [\Sigma_0] & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} [\bar{\Sigma}_0] & 0 \\ 0 & 0 \end{bmatrix} =$$

$$[\Sigma][\bar{\Sigma}_0] = \begin{bmatrix} |\sigma_1|^2 & & \\ & \ddots & \\ & & |\sigma_n|^2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$|\sigma_i|^2 \in \mathbb{R} \Rightarrow \lambda_i \in \mathbb{R} \ \forall i$

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