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IEOR 262 B
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1 a) $f(x,y,z) = 2x^2 + xy + y^2 + yz + z^2 - 6x - 7y - 8z + 9$

$\Omega = \mathbb{R}^3 \Rightarrow$ FONC $\nabla f = 0$

$\nabla f = [4x + y - 6,\ x + 2y + z - 7,\ y + 2z - 8] = 0$

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$$

$x^* = 1.2$
$y^* = 1.2$
$z^* = 3.4$

b) Verify using SOSC

$$\nabla(\nabla f) = \nabla^2 f = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$|M_{11}| = 4$
$|M_{22}| = 7$
$|\nabla^2 f| = 4(3) - 1(2) = 10$

All principle minors have positive det $\Rightarrow \nabla^2 f \geq 0$
$\nabla^2 f \geq 0 \Rightarrow$ relative minimum

c) $\nabla^2 f \neq f(x,y,z) \Rightarrow \nabla^2 f$ constant for all x,y,z

$\Rightarrow f$ is convex $\Rightarrow$ any relative minimum = global min

2) Let $f_i, i \in I$ be a collection of convex functions on Ω. Show $f(x) = \sup_i f_i(x)$ is also convex where it is finite

$$\forall i \in I \quad f_i(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \quad \forall x_1, x_2 \in \Omega, \forall \lambda \in [0 \;\; 1]$$

For f to be convex, must be a function over a convex set, here a subset of Ω where $f_i(x)$ finite

$$\Omega_f = \{x : x \in \Omega, f_i(x) \text{ finite } \forall i\} \Rightarrow f_i(x) \leq \Gamma \quad \forall x \in \Omega_f$$

$$f_i(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f_i(x_1) + (1-\lambda) f_i(x_2) \leq \lambda \sup_i f_i(x_1) + (1-\lambda) \sup_i f_i(x_2)$$
$$\forall x_1, x_2 \in \Omega_f, \forall \lambda \in [0 \;\; 1]$$

$$\lambda \sup_i f_i(x_1) + (1-\lambda) \sup_i f_i(x_2) \leq \Gamma \quad \forall x_1, x_2 \in \Omega_f$$

$$f(\lambda x_1 + (1-\lambda) x_2) = \sup_{i \in I} f_i(\lambda x_1 + (1-\lambda) x_2) \leq \lambda \sup_i f_i(x_1) + (1-\lambda) \sup_i f_i(x_2) \leq \Gamma$$

$$\Rightarrow \lambda x_1 + (1-\lambda) x_2 \in \Omega_f \quad \forall x_1, x_2 \in \Omega_f, \forall \lambda \in [0 \;\; 1]$$

$$\sup_{i \in I} f_i(\lambda x_1 + (1-\lambda) x_2) \leq \sup_{\substack{i \in I \\ j \in I}} \{\lambda f_i(x_1) + (1-\lambda) f_j(x_2)\} \quad i=j$$

$$\leq \sup_{i \in I} \lambda f_i(x_1) + \sup_{i \in I} (1-\lambda) f_i(x_2) = \lambda \sup_{i \in I} f_i(x_1) + (1-\lambda) \sup_{i \in I} f_i(x_2)$$

$$f(\lambda x_1 + (1-\lambda) x_2) = \sup_{i \in I} f_i(\lambda x_1 + (1-\lambda) x_2) \leq \lambda \sup f_i(x_1) + (1-\lambda) \sup f_i(x_2)$$
$$= \lambda f(x_1) + (1-\lambda) f(x_2$$

$$\forall x_1, x_2 \in \Omega_f, \forall \lambda \in [0 \;\; 1]$$

3) $\gamma(r)$ monotone non-dec ($\gamma(r) \le \gamma(r')$ $r' > r$) convex
f convex on Ω. Show $\gamma(f(x))$ is convex on Ω

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2) \quad \forall \ x_1, x_2 \in \Omega \ , \forall \in [0 \ 1]$$

$$\gamma(\lambda r_1 + (1-\lambda) r_2) \le \lambda \gamma(r_1) + (1-\lambda)\gamma(r_2) \quad \forall \ r \in \mathbb{R} \ , \forall \ \lambda \in [0 \ 1]$$

As $\gamma(r)$ monotone non-dec

$$\begin{aligned} \gamma(f(\lambda x_1 + (1-\lambda)x_2)) &\le \gamma(\lambda f(x_1) + (1-\lambda) f(x_2)) \quad \forall \ x_1, x_2 \in \Omega \ , \forall \lambda \in [0 \ 1] \\ &\le \lambda \gamma(f(x_1)) + (1-\lambda)\gamma(f(x_2)) \quad \gamma \text{ (convex)}, \ \lambda \ge 0 \end{aligned}$$

$$\Rightarrow \gamma(f(\lambda x_1 + (1-\lambda) x_2)) \le \lambda \gamma(f(x_1)) + (1-\lambda)\gamma(f(x_2)) \quad \forall \ x_1, x_2 \in \Omega \ , \ \forall \lambda \in [0 \ 1]$$

4). a) Prove that a concave function is psuedo concave

Convex set Ω $\quad \forall x_1, x_2 \in \Omega, \forall \lambda \in [0\ 1]\ \lambda x_1 + (1-\lambda)x_2 \in \Omega$

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2) \quad \forall x_1, x_2 \in \Omega, \forall \lambda \in [0\ 1]$$

$$f(\lambda(x_1 - x_2) + x_2) \geq \lambda(f(x_1) - f(x_2)) + f(x_2)$$

$$[f(x_2 + \lambda(x_1 - x_2)) - f(x_2)]\lambda^{-1} \geq f(x_1) - f(x_2)$$

$$\frac{[f(x_2 + \lambda(x_1 - x_2)) - f(x_2)]}{\lambda(x_1 - x_2)}(x_1 - x_2) \geq f(x_1) - f(x_2)$$

$$\lim_{\lambda \to 0}$$

$$\nabla f(x)(x_1 - x_2) \geq f(x_1) - f(x_2)$$

$$0 \geq f(x)(x_1 - x_2) \geq f(x_1) - f(x_2) \Rightarrow f(x_2) \geq f(x_1) \text{ Psuedo concave}$$

5

$$x \in \mathbb{R}^n = [x_1 \quad x_2 \quad \dots \quad \dots \quad x_n]^T$$

$$f(x) = b_1 x_1 + b_2 x_2 + \dots + b_n x_n$$
$$+ \int_0^{x_1} c_1 h(y)\,dy + \int_{x_1}^{x_1+x_2} c_2 h(y)\,dy + \int_{x_1+x_2}^{x_1+x_2+x_3} c_3 h(y)\,dy + \dots$$
$$+ \int_{x_1+\dots+x_{n-1}}^{x_1+x_2+\dots x_n} c_n h(y)\,dy + \int_{x_1+\dots+x_n}^{1} c_{n+1} h(y)\,dy$$

$$f(x) = \sum_{h=1}^{n} b_h x_h + \sum_{h=1}^{n} \int_{\sum_{j=1}^{h-1} x_{j-1}}^{\sum_{j=1}^{h} x_j} c_h h(y)\,dy + \int_{\sum_{j=1}^{n} x_j}^{1} c_{n+1} h(y)\,dy$$

$$\nabla f(x) = \Big[\, b_1 + c_1 h(x_1) + c_2\big(h(x_1+x_2) - h(x_1)\big) + c_3\big(h(x_1+x_2+x_3) - h(x_1+x_2)\big)$$
$$+ \dots \dots + c_n\big(h(x_1+\dots+x_n) - h(x_1+\dots x_{n-1})\big) - c_{n+1} h(x_1+\dots x_n)$$
$$h(x_1)(c_1 - c_2) + h(x_2)(c_2 - c_3) + h(x_3)(c_3 - c_4) + \dots$$

$$\sum_{j=1}^{h} x_j = z_h \qquad z_h = \gamma(x) \qquad z_h - z_{h-1} = x_h \qquad z_0 = x_0 = 0$$

$$f(x,z) = \sum_{h=1}^{n} b_h x_h + \int_0^{z_1} c_1 h(y)\,dy + \int_{z_1}^{z_2} c_2 h(y)\,dy + \dots + \int_{z_{n-1}}^{z_n} c_n h(y)\,dy + \int_{z_n}^{1} c_{n+1} h(y)\,dy$$

$$f(z) = \sum_{h=1}^{n} b_h(z_h - z_{h-1}) + \int_0^{z_1} c_1 h(y)\,dy + \dots + \int_{z_n}^{1} c_{n+1} h(y)\,dy$$

$$(\nabla f(z))^T = \begin{bmatrix} b_1 - b_2 + c_1 h(z_1) - c_2 h(z_1) \\ b_2 - b_3 + c_2 h(z_2) - c_3 h(z_2) \\ \vdots \\ b_n + c_n h(z_n) - c_{n+1} h(z_n) \end{bmatrix}^T = \begin{bmatrix} b_1 - b_2 + h(z_1)(c_1 - c_2) \\ b_2 - b_3 + h(z_2)(c_2 - c_3) \\ \vdots \\ b_n + h(z_n)(c_n - c_{n+1}) \end{bmatrix}$$

$$\nabla^2 f(z) = \begin{bmatrix} (c_1 - c_2)\frac{dh}{dz}\Big|_{z=z_1} & & & 0 \\ & (c_2 - c_3)\frac{dh}{dz}\Big|_{z=z_2} & & \\ & & \ddots & \\ 0 & & & (c_n + c_{n+1})\frac{dh}{dz}\Big|_{z=z_n} \end{bmatrix}$$

$\nabla^2 f(z) \geq 0 \iff f(z)$ convex

$$(c_1 - c_2) \frac{dh(z)}{dz}\Big|_{z_1} \geq 0 \quad , \quad (c_2 - c_3) \frac{dh(z)}{dz}\Big|_{z_2} \geq 0$$

From figure 7.2 b $h(x)$ is decreasing in $x \Rightarrow h(z)$ also decreasing $\Rightarrow \frac{dh}{dx} \leq 0 \ \forall x \Rightarrow \frac{dh}{dz} \leq 0 \ \forall z_k$

If this is the case $(c_k - c_{k+1}) \leq 0 \quad \forall k = 1, \ldots n$

This is a fair assumption as sources with lower c_k will be used first

$$\Rightarrow (c_k - c_{k+1}) \leq 0 \quad \forall k = 1 \ldots n$$

$$\Rightarrow (c_k - c_{k+1}) \frac{dh}{dz}\Big|_{z_k} \geq 0 \quad \forall k = 1 \ldots n$$

$$\Rightarrow \nabla^2 f(z) \geq 0 \quad \forall z \quad \Rightarrow f(z) \text{ convex}$$