E 231 FALL 2009 | HOMEWORK 5

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1. Let $A \in \mathbb{C}^{n \times n}$ $AA = A$. Find $\mathrm{Spec}\{A\}$

$Av = \lambda v$ $\quad AAv = \lambda v$ $\quad Av = \lambda v$ $\quad A(Av = \lambda v)$

$AAv = \lambda Av \Rightarrow AAv = \lambda(\lambda v) \Rightarrow AAv = \lambda^2 v \Rightarrow Av = \lambda^2 v$

$Av = \lambda v = \lambda^2 v \Rightarrow \lambda = 0, 1$

$\mathrm{Spec}\, A = \{0, 1\}$

2. Let $A \in \mathbb{C}^{n \times n}$, $X \in \mathbb{C}^{n \times n}$

a) When is $(I + \exp(A))$ invertible?

$f(A) = [I + \exp(A)]^{-1}$ $\quad f(s) = (1 + e^s)^{-1}$

$1 + e^s > 1 \quad \forall\, s \in \mathbb{R}$

$\Rightarrow [I + \exp(A)]$ exists $\forall\, A \in \mathbb{C}^{n \times n}$

b) When is $\begin{bmatrix} I & X \\ -X^* & I \end{bmatrix}$ invertible

$\det(A) = \det(I - XI^{-1}(-X^*))\det(I) = \det(I + XX^*)$

$\det(I + XX^*) \neq 0$ for invertibility

$\lambda_i(XX^*) \neq -1 \quad \forall i \Rightarrow \sigma_i(X) =$

$\lambda_i(X^*) = \bar{\lambda_i}(X) \Rightarrow \lambda_i(XX^*) > 0 \quad \forall i$

this is always invertible

c) When is $(I - A)$ invertible

$\frac{1}{1-s} \Rightarrow s \neq 1$ $\quad \det(I - A) \neq 0 \Rightarrow \prod_i \lambda_i(I - A) \neq 0$

$\Rightarrow \lambda_i(A) \neq 1 \quad \forall i$

3 Let $A \in \mathbb{C}^{n \times n}$, Suppose $A: \mathcal{R}(A) \rightarrow \mathcal{N}(A)$. What is spec$\{A\}$

$v \in \mathcal{R}(A) \Rightarrow v = Ax \quad \forall\, x \in \mathbb{C}^n$

$Av = 0 \Rightarrow AAx = 0 \quad \forall\, x \in \mathbb{C}^n$

$Aw = \lambda w \qquad A(Aw = \lambda w) = AAw = \lambda A w$

$AAw = \lambda(\lambda w) = \lambda^2 w \Rightarrow 0 = \lambda^2 w \Rightarrow \lambda = 0$

Spec $\{A\} = \{0, 0, 0 \ldots\}$

5. $A : \mathbb{R}^3 \to \mathbb{R}^3$. $B = \{ b_1, b_2, b_3 \}$ $C = \{ c_1, c_2, c_3 \}$

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad C = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$A(b_1) = 2b_1 - b_2$ $\quad A(b_2) = b_2$ $\quad A(b_3) = 4b_2 + 2b_3$

$A(b_1) = [2 \ -1 \ 0]^T$ $\quad A(b_2) = [0 \ 1 \ 0]$ $\quad A(b_3) = [0 \ 4 \ 2]^T$

$$A(B) = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$c_1 = b_1 + b_2$ $\quad c_2 = b_2 + b_3$ $\quad c_3 = b_1 + b_3$

6 Let S and T be subspaces of $\mathbb{C}^n$

a) Show $(S^\perp)^\perp = S$

$v \in S^\perp \quad w \in (S^\perp)^\perp \quad y \in S$

$w \in (S^\perp)^\perp \Rightarrow w \perp S^\perp \Rightarrow w \in S \Rightarrow (S^\perp)^\perp \subseteq S$

$y \in S \Rightarrow y \perp S^\perp \Rightarrow y \in (S^\perp)^\perp \Rightarrow (S^\perp)^\perp \supseteq S$

$S \subseteq (S^\perp)^\perp, \quad (S^\perp)^\perp \subseteq S \Rightarrow S = (S^\perp)^\perp$

b $S \subseteq T$, show $S^\perp \supseteq T^\perp$

$v \in S^\perp, \ t \in T^\perp \qquad v \perp S, \ t \perp T$

$t \in T^\perp \Rightarrow t \perp T \Rightarrow t \perp S \Rightarrow t \in S^\perp$

$\Rightarrow T^\perp \subseteq S^\perp$

7 Find the Jordan Forms $\qquad Av = \lambda v \quad (A - \lambda I)v = 0$

a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad |sI - A| = (s-1)(s-4) - 4 = s^2 - 5s = 0$

$\lambda_i = 0, 5$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow \quad v_1 + 2v_2 = 0 \quad v_1 = -2v_2 \quad v = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow \quad 2v_1 - v_2 = 0 \quad 2v_1 = v_2 \quad v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$T = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \quad T^{-1} = -\left[\frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}\right] = \frac{1}{5}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$

$A = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} -2/5 & 1/5 \\ 1/5 & 2/5 \end{bmatrix}$

b) $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ $\quad |\lambda I - A| = \lambda(\lambda - 3) + 2 \quad \lambda^2 - 3\lambda + 2$

$(\lambda = 2)(\lambda - 1) = 0 \quad \lambda_i = 1, 2$

$\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow v_1 + v_2 = 0 \quad v_1 = -v_2 \quad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow v_1 + 2v_2 = 0 \quad v_1 = -2v_2 \quad v = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$T = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \quad T^{-1} = \frac{1}{1-2}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$

c) $A = \begin{bmatrix} 2 & 11 \\ 0 & 2 \end{bmatrix}$ $\quad |\lambda I - A| = (\lambda - 2)(\lambda - 2) - 0 = 0 \quad \lambda_i = 2$

$\begin{bmatrix} 0 & 11 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow 0v_1 + 11v_2 = 0 \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 11 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \quad \begin{bmatrix} 11v_2 - 1 \\ 0 \end{bmatrix} = 0 \Rightarrow s = \begin{bmatrix} 0 \\ +\frac{1}{11} \end{bmatrix}$

$T = \begin{bmatrix} 11 & 0 \\ 0 & 1 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} \frac{1}{11} & 0 \\ 0 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$A = \begin{bmatrix} 11 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{11} & 0 \\ 0 & 1 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \quad T = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

$J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} & 0 & 0 \\ \frac{1}{5} & \frac{2}{5} & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -1 \end{bmatrix}$

8 $A \in \mathbb{C}^{n\times n}$ A idempotent, show A is semi-simple

$AA = A \Rightarrow \operatorname{Spec} A = \{0, 1\}$ $\qquad A = T\Lambda T^{-1}$

$(A - \lambda I)v = 0.$

$n=2$

$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \lambda_1 = 1 \quad \lambda_2 = 0$

$(A - \lambda_1 I)v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad (A - \lambda_2 I)v = 0 \Rightarrow v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda_1 = 0 \quad \lambda_2 = 1$

$(A_2 - \lambda_1 I)v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad (A_2 - \lambda_2)v = 0 \Rightarrow v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda_1 = 1 \quad \lambda_2 = 1$

$(A_3 - \lambda_1 I)v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad (A_3 - \lambda_2 I)v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A_4 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad \lambda_1 = 1 \quad \lambda_2 = 0$

$(A_4 - \lambda_1 I)v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad (A_4 - \lambda_2 I)v = 0 \Rightarrow v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \lambda_1 = 1 \quad \lambda_2 = 0$

$(A_5 - \lambda_1 I)v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad (A_5 - \lambda_2 I)v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A_6 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda_1 = 0 \quad \lambda_2 = 1$

$(A_6 - \lambda_1 I)v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad (A_6 - \lambda_2 I)v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A_7 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad \lambda_1 = 0 \quad \lambda_2 = 1$

$(A_7 - \lambda_1 I)v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad (A_7 - \lambda_2 I)v = 0 \Rightarrow v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$A_8 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \lambda_1 = 0 \quad \lambda_2 = 0$$

$$(A_8 - \lambda_1 I)v = 0 \quad \Rightarrow \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (A_8 - \lambda_2 I)v = 0 \quad \Rightarrow \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For $n=2$, all possible matrices $A : AA = A$ have two distinct eigenvectors regardless of the multiplicity of the eigenvalues. Extending this from $n=2$ to $n=3, \ldots$. one can see all idempotent matrices are semi-simple.