1. Find the rank and nullities of the following matrices

a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 2 & 4 & 0 \end{bmatrix}$ $R(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$N(A) = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\text{rank}(A) = 2$
$\nu(A) = 1$

b) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 4 & 0 \\ 1 & 0 \end{bmatrix}$ $R(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ $\text{rank}(A) = 2$ $\nu(A) = 0$

c) $\begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 11 & 0 & 0 & 0 \end{bmatrix}$ $R(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \right\}$

$N(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\text{rank}(A) = 3$
$\nu(A) = 1$

2 Prove or supply counterexample

a) $\operatorname{rank}(A) = \operatorname{rank}(AA)$ - False

$A \in \mathbb{C}^{2\times 2}$ $\quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \operatorname{rank}(A) = \underline{1}$

$AA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \operatorname{rank}(AA) = 0$

b) $\operatorname{rank}(A) = \operatorname{rank}(AA^*)$ - True

Let $A \in \mathbb{C}^{n\times n}$, $w \in \mathbb{W}^n$, $v \in \mathbb{V}^n$

$w = Av = \begin{bmatrix} A_1 \end{bmatrix} v_1 + \begin{bmatrix} A_2 \end{bmatrix} v_2 + \dots + \begin{bmatrix} A_n \end{bmatrix} v_n \quad \forall\ v \in \mathbb{V}$

$\Rightarrow w = \operatorname{span}\left\{ \begin{bmatrix} A_1 \end{bmatrix}, \begin{bmatrix} A_2 \end{bmatrix}, \dots \begin{bmatrix} A_n \end{bmatrix} \right\} \forall\ v \in \mathbb{V}$

Let $y \in Y^n$, $x \in X^n$

$y = AA^*x = A\hat{x} = \begin{bmatrix} A_1 \end{bmatrix} \hat{x}_1 + \begin{bmatrix} A_2 \end{bmatrix} \hat{x}_2 + \dots + \begin{bmatrix} A_n \end{bmatrix} \hat{x}_n$

$\Rightarrow y = \operatorname{span}\left\{ [A_1], [A_2], \dots, [A_n] \right\} \quad \forall\ \hat{x}_n = Ax$

$\Rightarrow \mathcal{R}(A) = \mathcal{R}(AA^*)$

c) Let $A \in \mathbb{C}^{p\times m}$, $Q \in \mathbb{C}^{m\times m}$, $\operatorname{rank}(Q) = m$

then $\operatorname{rank}(AQ) = \operatorname{rank}(A)$ $\quad AQ \in \mathbb{C}^{p\times m}$

True - $w \in \mathbb{C}^m$, $v \in \mathbb{C}^m$ $\quad w = Qv$

$\Rightarrow w = \operatorname{span}\left\{ \begin{bmatrix} Q_1 \end{bmatrix}, \dots, \begin{bmatrix} Q_m \end{bmatrix} \right\}$

$y \in \mathbb{C}^p$ $\quad y = AQv = Aw$

$y = \operatorname{span}\left\{ \begin{bmatrix} A_1 \end{bmatrix}, \dots, \begin{bmatrix} A_m \end{bmatrix} \right\}$

3. Find the Range and Null Spaces

a) $A = I \in \mathbb{C}^{n \times n}$

$$\mathcal{R}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\} = \mathbb{C}^n$$

$$\mathcal{N}(A) = \{ \underline{0} \}$$

b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \mathcal{R}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$$\mathcal{N}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

c) $A = \begin{bmatrix} Q & 0 \end{bmatrix} \in \mathbb{C}^{n \times (n+m)}$ $\quad Q \in \mathbb{C}^{n \times n}$

$$\mathcal{R}(A) = \mathcal{R}(Q) = \text{span}\left\{ \begin{bmatrix} Q_1 \end{bmatrix}, \ldots \begin{bmatrix} Q_n \end{bmatrix} \right\} = \mathbb{C}^n$$

$$\mathcal{N}(A) = \text{span}\left\{ \begin{bmatrix} 0_n \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0_n \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ldots \begin{bmatrix} 0_n \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \mathcal{N}(Q) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

d) $A = \begin{bmatrix} I & R \end{bmatrix} \in \mathbb{C}^{n \times (n \times m)}$

$$R(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \right\} \quad \mathbb{C}^n$$

$$\mathcal{N}(A) = \begin{bmatrix} -Rx \\ x \end{bmatrix}$$

4. Which of the following maps are linear?

a) $f(X) = AX + XB$ Linear

$f(X_1 + X_2) = A(X_1 + X_2) + (X_1 + X_2)B = AX_1 + X_1B + AX_2 + X_2B$

$f(X_1) + f(X_2) = AX_1 + X_1B + AX_2 + X_2B$

$f(\alpha X) = A(\alpha X) + \alpha XB = \alpha(AX + XB) = \alpha f(X)$

b) $f(X) = AX + BXC$ Linear

$f(X_1 + X_2) = A(X_1 + X_2) + B(X_1 + X_2)C = AX_1 + AX_2 + (BX_1 + BX_2)C$

$= AX_1 + BX_1C + AX_2 + BX_2C$

$f(X_1) + f(X_2) = AX_1 + BX_1C + AX_2 + BX_2C$

$f(\alpha X) = A(\alpha X) + B(\alpha X)C = \alpha AX + \alpha BXC = \alpha f(X)$

c) $AX + XBX$ - Non Linear

$f(X_1 + X_2) = A(X_1 + X_2) + (X_1 + X_2)B(X_1 + X_2)$

$= AX_1 + AX_2 + (X_1 + X_2)(BX_1 + BX_2) = AX_1 + AX_2 + X_1BX_1 + X_2BX_1 + X_1BX_2 + X_2BX_2$

$f(X_1) + f(X_2) = AX_1 + X_1BX_1 + AX_2 + X_2BX_2$

$f(\alpha X) = A(\alpha X) + (\alpha X)B(\alpha X) = \alpha AX + \alpha^2 XBX \neq \alpha f(X)$

d) $f(X) = A^*XA - X$ Linear

$f(X_1 + X_2) = A^*(X_1 + X_2)A - (X_1 + X_2) = (A^*X_1 + A^*X_2)A - (X_1 + X_2)$

$= A^*X_1A - X_1 + A^*X_2A - X_2$

$f(X_1) + f(X_2) = A^*XA - X_1 + A^*X_2A - X_2$

$f(\alpha X) = A^*(\alpha X)A - \alpha X = \alpha A^*XA - X = \alpha(f(X))$

e) $f(x) = \|x\| = \left[\langle x, x\rangle\right]^{\frac{1}{2}}$ - Non Linear

$$f(x_1+x_2) = \left[\langle x_1+x_2, x_1+x_2\rangle\right]^{\frac{1}{2}} = \left[x_1^*x_1 + x_1^*x_2 + x_2^*x_1 + x_2^*x_2\right]^{\frac{1}{2}}$$

$$f(x_1)+f(x_2) = \left[\langle x_1, x_1\rangle\right]^{\frac{1}{2}} + \left[\langle x_2, x_2\rangle\right]^{\frac{1}{2}} = \left[x_1^*x_1\right]^{\frac{1}{2}} + \left[x_2^*x_2\right]^{\frac{1}{2}}$$

f) $f(x) = \langle v, x\rangle$ - Linear

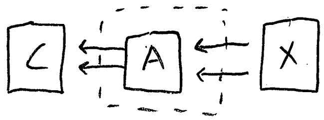
$$f(x_1+x_2) = \langle v, x_1+x_2\rangle = v^*(x_1+x_2) = v^*x_1 + v^*x_2$$

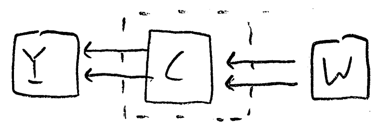
$$f(x_1)+f(x_2) = v^*x_1 + v^*x_2$$

$$f(\alpha x) = \langle v, \alpha x\rangle = \alpha v^* x = \alpha f(x)$$

5 Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{\ell \times p}$, $C \in \mathbb{C}^{m \times p}$ $X \in \mathbb{C}^{n \times p}$

a) Show $AX = C$ only solvable iff $\mathcal{R}(C) \subseteq \mathcal{R}(A)$





$C = AX$ $\qquad$ $\underline{Y} = CW$ $\qquad$ $\underline{Y} = AXW$

$\underline{Y} \in \mathcal{R}(C) \Rightarrow \underline{Y} \in \mathcal{R}(A) \Rightarrow \mathcal{R}(C) \subseteq \mathcal{R}(A)$

$Ax_i = c_i \Rightarrow c_i \in \mathcal{R}(A) \ \forall i \Rightarrow \mathcal{R}(C) \subseteq \mathcal{R}(A)$

b Suppose $AX = C$, Show its unique iff $\mathcal{N}(A) = \{0\}$

Let X_1, X_2 be unique solutions : $AX_1 = AX_2 = C$

$AX_1 - AX_2 = C - C = 0$ $\qquad$ $A(X_1 - X_2) = 0$

$X_1 - X_2 \in \mathcal{N}(A)$ $\qquad$ $X_1 - X_2 \neq 0$

For $\mathcal{N}(A) = 0$ $\qquad$ $X_1 - X_2 = 0 \Rightarrow X_1 = X_2$

X_1 a unique solution

c) $XB = C$ solvable if $\mathcal{N}(B) \subseteq \mathcal{N}(C)$

$B^* X^* = C^*$ solvable if $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$

$\Rightarrow \mathcal{R}^\perp(B^*) \subseteq \mathcal{R}^\perp(C^*) \Rightarrow \mathcal{N}(B) \subseteq \mathcal{N}(C)$

6. Under what conditions does $TA = TB \Rightarrow A = B$

$C = TA \qquad D = TB$

$C_i = TA_i \qquad D_i = TB_i$

$TA = TB \Rightarrow C_i = D_i \Rightarrow TA_i = TB_i \quad \forall \; A_i, B_i$

$T(A_i - B_i) = 0$

if $N(T) = 0 \Rightarrow A_i - B_i = 0 \Rightarrow A_i = B_i$

else for some $A, B \quad A_i - B_i \in N(T)$

$\Rightarrow N(T) = \{0\}$ such that $TA = TB \Rightarrow A = B$

7. Let $A \in \mathbb{C}^{m \times n}$ for $k = 0, \ldots,$ define $N_k = N(A^k)$

a) Show $\forall\ k = 0, 1, \ldots$ $N_k \subseteq N_{k+1}$

Let $x \in N(A^k) \Rightarrow A^k x = 0 \Rightarrow AA^k x = 0$

$\Rightarrow A^{k+1} x = 0 \Rightarrow x \in N(A^{k+1})$

Let $y \in N(A^{k+1}) \Rightarrow A^{k+1} y = 0 \Rightarrow AA^k y = A^k A y = 0$

$A^{-1}[AA^k y = A^k A y = 0] \Rightarrow A^k y = A^{k-1} A y = 0$

$\Rightarrow y \in N(A^k) \Rightarrow y \in N(A^{k+1}) \Rightarrow y \in N(A^k)$

b) If for some integer ℓ $N_\ell = N_{\ell+1}$ then $N_\ell = N_{\ell+1} = N_{\ell+2} \ldots$

$x \in N(A^\ell), \in N(A^{\ell+1})$

$\Rightarrow A^\ell x = A^{\ell+1} x = 0$

Show $N(A^\ell) \subseteq N(A^{\ell+1})$

$x \in A^{\ell+1}$ $A^{\ell+1} x = 0$ $A^\ell A x = AA^\ell x = 0$

$A^{-1}(AA^\ell x) = 0 \Rightarrow A^\ell x = 0 \Rightarrow x \in N(A^\ell)$

$\Rightarrow N(A^\ell) \subseteq N(A^{\ell+1})$

Show $N(A^{\ell+1}) \subseteq N(A^\ell)$

$y \in N(A^\ell)$ $A^\ell y = 0 \Rightarrow AA^\ell y = A^{\ell+1} y = 0$

$\Rightarrow y \in N(A^{\ell+1}) \Rightarrow N(A^{\ell+1}) \subseteq N(A^\ell)$

Thus by assertion

$w \in N(A^{\ell+2})$ $A^{\ell+2} w = A^{\ell+1} A w = A A^{\ell+1} w = 0$

$\Rightarrow A^{\ell+1} w = 0 \Rightarrow w \in N(A^{\ell+1}) \Rightarrow N(A^{\ell+1}) \subseteq N(A^{\ell+2})$

$z \in N(A^{\ell+1})$ $A^{\ell+1} z = AA^{\ell+1} z = A^{\ell+2} z = 0$

$\Rightarrow z \in N(A^{\ell+2}) \Rightarrow N(A^{\ell+2}) \subseteq N(A^{\ell+1})$

$\Rightarrow N_\ell = N_{\ell+1} = N_{\ell+2} \ldots$

8 Prove if $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times r}$

$\operatorname{rank}(B) - \operatorname{nullity}(A) \leq \operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$

$C = AB \quad C_i = AB_i \quad C_i \in R(A) \Rightarrow R(C) \in R(A)$