1. X, Y independent random variables $X \sim N(0, 1)$ $Y \sim U[2\ 3]$

a) find joint density $p_{XY}(x, y)$

X, Y independent $\quad p_{XY}(x,y) = p_X(x)\, p_Y(y)$

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \qquad p_Y(y) = \begin{cases} 1 & y \in [2\ 3] \\ 0 & \text{else} \end{cases}$$

$$p_{XY}(x, y) = \begin{cases} \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right) & y \in [2\ 3] \quad , \forall x \\ 0 & \text{else} \end{cases}$$

b) Find the conditional density $p_{X|Y=y}(x|y)$

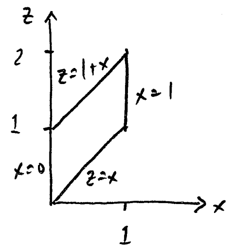
$$p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{p_Y(y)} = \frac{p_X(x)\, p_Y(y)}{p_Y(y)} = p_X(x)$$

$$p_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right)$$

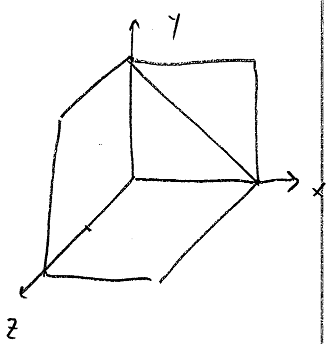
2 Let X, Y be independent $X \sim U[0\ \ 1]$ $Y \sim [0\ \ 1]$

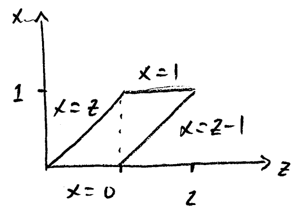
$Z = X + Y$

a) Find $p_{XZ}(x, z)$



$$p_{xz} = \begin{cases} 1 & x \in [0\ \ 1],\ z \in [x\ \ 1+x] \\ 0 & \text{else} \end{cases}$$

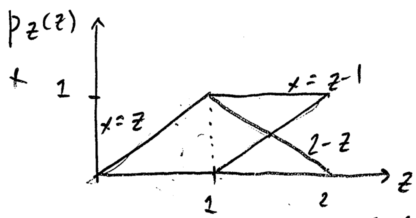




$$p_z(z) = \int_x p_{xz}\, dx$$

$$p_z(z) \int_x 1 \quad x \in [0\ \ 1],\ z \in [x\ \ 1+x]\, dx \Rightarrow \int 1 \quad z \in [0\ \ 2],\ x \in [z\ \ z-1]\, dx$$

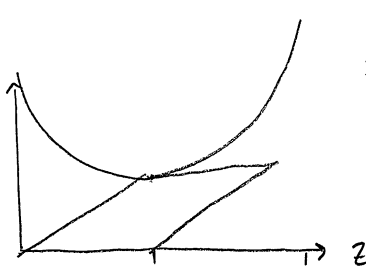
$$= \int_{z-1}^{z} 1\, dx = x\Big|_{z-1}^{z} = 1 \quad z \in [0\ \ 2],\ x \in [z\ \ z-1]$$



$$p_z(z) = \begin{cases} z & z \in [0\ \ 1] \\ 2-z & z \in [1\ \ 2] \\ 0 & \text{else} \end{cases}$$

$$p_{x|z} = \frac{p_{xz}}{p_z} = \frac{\begin{cases} 1 & x \in [0\ \ 1],\ z \in [x\ \ 1+x] \\ 0 & \text{else} \end{cases}}{\begin{cases} z & z \in [0\ \ 1] \\ 2-z & z \in [1\ \ 2] \\ 0 & \text{else} \end{cases}}$$

$$= \begin{cases} \frac{1}{z} & z \in [0\ \ 1],\ x \in [0\ \ z] \\ \frac{1}{2-z} & z \in [1\ \ 2],\ x \in [z-1\ \ 1] \\ 0 & \text{else} \end{cases}$$



c)

$$E\{X|Z\} = \int_X x \, p_{X|Z}(x|z)\, dx =$$

$$\int_0^z x \frac{1}{z}\, dx = \frac{1}{2} x^2 \frac{1}{z} \Big|_0^z = \frac{1}{2} z^2 z^{-1} = \frac{z}{2}$$

$$\int_{z-1}^1 x \frac{1}{2-z}\, dx = \frac{1}{2} x^2 \frac{1}{2-z} \Big|_{z-1}^1 = \frac{1}{2}\left(\frac{1}{2-z}\right)\left(\cancel{1} - z^2 + 2z - \cancel{1}\right)$$

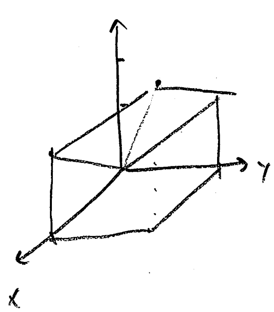
$$= \frac{1}{2}\, \frac{2z - z^2}{2-z} = \frac{z}{2}$$

d)

$$E\{Z|X\} = E\{X + Y|X\} = E\{X|X\} + E\{Y|X\}$$

independent

$$= x + \frac{1}{2}$$



3 Let X, Y be RV

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

a) Find $p_X(x)$

$$X = [1 \quad 0]\begin{bmatrix} X \\ Y \end{bmatrix} \qquad m_X = [1 \quad 0]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1$$

$$\sigma_X^2 = [1 \quad 0]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$

$$X \sim N(1, 1)$$

b) Find $p_R(r)$ $\quad R = X + Y$

$$R = [1 \quad 1]\begin{bmatrix} X \\ Y \end{bmatrix} \qquad m_R = [1 \quad 1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$$\sigma_R^2 = [1 \quad 1]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2$$

$$R \sim N(0, 2)$$

c) Find $p_S(s)$ $\quad S = X - Y$

$$S = [1 \quad -1]\begin{bmatrix} X \\ Y \end{bmatrix} \qquad m_S = [1 \quad -1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2$$

$$\sigma_S^2 = [1 \quad -1]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2$$

$$S \sim N(2, 2)$$

d) Find $E\{R | S = s\}$ $\quad Z = [X \quad Y \quad R \quad S]^T$

$$Z = A\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} \qquad m_X = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$\Lambda_{ZZ} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix}$$

$$\sigma_{RS} = 0$$

$$\begin{bmatrix} R \\ S \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right.$$

$$p_{RS}(r,s) = \frac{1}{(2\pi)\left(\begin{vmatrix}2 & 0\\ 0 & 2\end{vmatrix}\right)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}[(r-0)\ (s-2)]\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}^{-1}\begin{bmatrix}r\\ s-2\end{bmatrix}\right)$$

$$= \frac{1}{2\pi(2)} \exp\left(-\frac{1}{2}[r\ (s-2)]\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}r\\ s-2\end{bmatrix}\right)$$

$$= \frac{1}{4\pi} \exp\left(-\frac{1}{4}\left(r^2+(s-2)^2\right)\right)$$

$$p_S = \frac{1}{\sqrt{2\pi(2)}} \exp\left(-\frac{1}{2}\frac{(s-2)^2}{2}\right) = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{1}{4}(s-2)^2\right)$$

$$p_{R|S=s}(r|s) = \frac{\frac{1}{4\pi} \exp\left(-\frac{1}{4}\left(r^2+(s-2)^2\right)\right)}{\frac{1}{2\sqrt{\pi}} \exp\left(-\frac{1}{4}(s-2)^2\right)}$$

$$= \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{1}{4}r^2\right) = \frac{1}{\sqrt{2\pi 2}} \exp\left(-\frac{1}{4}r^2\right)$$

$$\Rightarrow p_{R|S}(r|s) \sim N(0,2)$$

Also uncorrelated $\Rightarrow$ independence if Gaussian

$$\Rightarrow p_{R|S}(r|s) = p_R(r)$$

$$E(R|S=s) = m_R + \Lambda_{RS}\Lambda_{SS}^{-1}(s-m_S) = m_R = 0$$

4 Can $\Lambda = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ be a covariance matrix?

$\Lambda = \Lambda^*$ $\quad |\Lambda_{11}| = 2 > 0$ $\quad |\Lambda_{22}| = 8 - 9 < 0$ $\Rightarrow \Lambda \not> 0$

5 Let $X_1 \dots X_n$ be IID RV with mean m covariance σ^2

$Z = \frac{1}{n}\sum_{i=1}^{n} X_i$ $\qquad \begin{bmatrix} X_1 \\ \vdots \\ X_i \\ \vdots \\ X_n \end{bmatrix} \sim \begin{bmatrix} m \\ \vdots \\ m \\ \vdots \\ m \end{bmatrix}, \begin{bmatrix} \sigma^2 & & & \\ & \sigma^2 & & \\ & & \ddots & \\ & & & \sigma^2 \end{bmatrix}$

a) Compute $E\{Z\}$

$$E\{Z\} = E\{\tfrac{1}{n}\sum_{i=1}^{n} X_i\} = \tfrac{1}{n} E\{\sum_{i=1}^{n} X_i\} = \tfrac{1}{n}\sum_{i=1}^{n} E\{X_i\}$$

$$= \frac{1}{n}\sum_{i=1}^{n} m = \frac{1}{n} nm = m$$

b) Compute $E\{Z^2\}$

$$E\{Z^2\} = E\{(\tfrac{1}{n}\Sigma X_i)(\tfrac{1}{n}\sum_{i=1} X_i)^*\} = \tfrac{1}{n^2} E\{(\Sigma X_i)(\Sigma X_i)\}$$

$$= \frac{1}{n^2} E\{(X_1 + \dots + X_n)(X_1 + \dots + X_n)\} = \frac{1}{n^2} E\{X_1^2 + \dots + X_n^2 + 2X_1X_2 + \dots$$

$$= \frac{1}{n^2} E\{\sum_{i=1}\sum_{j=1} X_i X_j\} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E\{X_i X_j\}$$

$$= \frac{1}{n^2}\left[n(\sigma^2 + m^2)\right] + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E\{(1-\delta_{ij})X_i X_j\}$$

$$= \frac{1}{n}(\sigma^2 + m^2) + \frac{1}{n^2} 2n m^2 = \frac{1}{n}(\sigma^2 + 3m^2) \;?$$

5 $X_1 \ldots X_n$ IID variables mean m, variance σ^2

$$Z = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \sim \begin{bmatrix} m \\ m \\ \vdots \\ m \end{bmatrix}, \ \sigma^2 I_n$$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

a) Compute $E\{Z\}$

$$Z = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}[1 \ldots 1]\begin{bmatrix} X_1 \\ X_n \end{bmatrix} = \frac{1}{n} a^* X$$

$$E\{Z\} = E\{\tfrac{1}{n} a^* X\} = \frac{a^*}{n} E\{X\} = \frac{a^*}{n}\begin{bmatrix} m \\ \vdots \\ m \end{bmatrix} = \frac{nm}{n} = m$$

b) Compute $E\{Z^2\}$

$$E\{Z^2\} = E\{\tfrac{1}{n} a^* X X^* a \tfrac{1}{n}\} = \frac{1}{n^2} a^* E\{XX^*\} a$$

$$= \frac{1}{n^2} a^* \sigma^2 I a = \frac{\sigma^2 n}{n^2} = \frac{\sigma^2}{n}$$

6. Define $X \sim U[0\ \ 1]$, $Y = X^2$, $Z = [X\ \ Y]^T$

Calculate Λ_{ZZ} $\quad p_X = 1 \quad 0 \le x \le 1$

$m_X = \int_{-\infty}^{+\infty} x\,1\,dx = \frac{1}{2}$

$m_Y = E\{Y\} = \int_{-\infty}^{+\infty} x^2\, 1\,dx = \frac{1}{3}$

$$\Lambda_{ZZ} = E\left\{\begin{bmatrix} X-\frac{1}{2} \\ Y-\frac{1}{3}\end{bmatrix}\begin{bmatrix} X-\frac{1}{2} & Y-\frac{1}{3}\end{bmatrix}\right\} = E\left\{\begin{bmatrix} (X-\frac{1}{2})^2 & (X-\frac{1}{2})(Y-\frac{1}{3}) \\ (Y-\frac{1}{3})(X-\frac{1}{2}) & (Y-\frac{1}{3})^2\end{bmatrix}\right\}$$

$$E\{(X-\tfrac{1}{2})^2\} = \int_0^1 (x-\tfrac{1}{2})^2\,dx = \tfrac{1}{3}(x-\tfrac{1}{2})^3\Big|_0^1 = \tfrac{1}{3}(\tfrac{1}{4}+\tfrac{1}{4}) = \tfrac{1}{12}$$

$$E\{(X-\tfrac{1}{2})(Y-\tfrac{1}{3})\} = \int_0^1 (x-\tfrac{1}{2})(x^2-\tfrac{1}{3})\,dx = \int_0^1 x^3 - \tfrac{1}{2}x^2 - \tfrac{1}{3}x + \tfrac{1}{6}\,dx$$

$$= \tfrac{1}{4}x^4 - \tfrac{1}{6}x^3 - \tfrac{1}{6}x^2 + \tfrac{1}{6}x\Big|_0^1 = \tfrac{1}{4} - \tfrac{1}{6} - \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{12}$$

$$E\{(Y-\tfrac{1}{3})^2\} = \int_0^1 (x^2-\tfrac{1}{3})^2\,dx = \int_0^1 x^4 - \tfrac{2}{3}x^2 + \tfrac{1}{9}\,dx$$

$$= \tfrac{1}{5}x^5 - \tfrac{2}{9}x^3 + \tfrac{1}{9}x\Big|_0^1 = \tfrac{1}{5} - \tfrac{2}{9} + \tfrac{1}{9} = 4/45$$

$$\Lambda_{ZZ} = \begin{bmatrix} 1/12 & 1/12 \\ 1/12 & 4/45\end{bmatrix}$$

7. Let $X \sim U[0\ \ 1]$. Find $f: \mathbb{R} \to \mathbb{R}$ so that $Y = f(X) \sim N(0, \sigma^2)$

$$p_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

$$p_Y(y) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{y^2}{2\sigma^2}\right)$$

$$\boxed{\begin{aligned} p_Y(y) &= p_X(f^{-1}(y)) \left| \frac{df^{-1}(y)}{dy} \right| \\ p_Y(y) &= p_X(g(y)) \left| \frac{dg(y)}{dy} \right| \end{aligned}}$$

$Y = f(X) \qquad X = f^{-1}(Y)$

$X = g(Y) \qquad Y = g^{-1}(X)$

$X = f^{-1}(g^{-1}(X))$

$\Rightarrow g^{-1} = f(X)$

$$\frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(-\frac{y^2}{2\sigma^2}\right) = 1 \left| \frac{d}{dy} g(y) \right|$$

$$g(y) = \int \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-y^2}{2\sigma^2}\right) dy = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{\sqrt{\pi}}{\sqrt{2}} \operatorname{erf}\left(\frac{y}{\sqrt{2}\sigma}\right)$$

$$g(y) = \frac{1}{2\sigma^2} \operatorname{erf}\left(\frac{y}{\sqrt{2}\sigma}\right) = x$$

$$\operatorname{erf}\left(\frac{y}{\sqrt{2}\sigma}\right) = 2\sigma^2 x$$

$$\Rightarrow y = \sqrt{2}\sigma \operatorname{erf}^{-1}(2\sigma^2 x)$$

8 Let $X_1 \dots X_n$ be IID RV

$$\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \sim \begin{bmatrix} m \\ \vdots \\ m \end{bmatrix}, \begin{bmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{bmatrix}$$

a) Compute $E\{X_1 \mid X_1 + X_2 + \dots + X_n = s\}$

$$s = [1 \ 1 \ \dots \ 1] \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \Rightarrow m_s = nm$$

$$\sigma_s^2 = [1 \, 1 \dots 1] \sigma^2 I \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = n\sigma^2$$

$m_s = nm \qquad \sigma_s^2 = n\sigma^2$

$$Z = \begin{bmatrix} X_1 \\ \vdots \\ X_n \\ s \end{bmatrix} = \begin{bmatrix} & I_n & \\ 1 & 1 \dots & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \qquad m_Z = \begin{bmatrix} m \\ m \\ \vdots \\ m \\ nm \end{bmatrix}$$

$$\Lambda_{ZZ} = \begin{bmatrix} \sigma^2 & & & \sigma^2 \\ & \sigma^2 & & \vdots \\ & & \ddots & \sigma^2 \\ & & \sigma^2 & \sigma^2 \\ \sigma^2 & \dots & \sigma^2 & n\sigma^2 \end{bmatrix} \qquad s = \sum_{i=1}^{n} X_i$$

IID $\Rightarrow$ $E\{X_1 \mid S = s\} = E\{X_2 \mid S = s\} = \dots = E\{X_n \mid S = s\} = \gamma$

$$n\gamma = \sum_{i=1}^{n} E\{X_i \mid S = s\} = E\{\sum_{i=1}^{n} X_i \mid S = s\} = E\{X_1 + X_2 + \dots + X_n \mid S = s\} = s$$

$n\gamma = s \quad \Rightarrow \quad \gamma = s/n$

b) Compute for $k \le n$ $E\left\{\dfrac{X_1 + \dots + X_k}{X_1 + \dots + X_n}\right\}$

$$E\left\{\frac{X_1 + \dots + X_k}{S}\right\} = \alpha$$

9 Let X, Y be RV with joint density

$$p_{XY}(x,y) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x^2+y^2)\right)$$

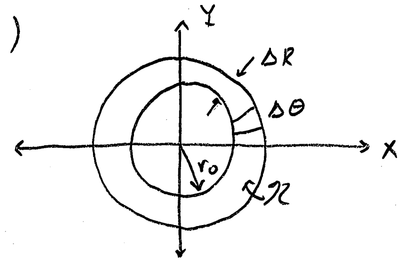
$$R^2 = X^2 + Y^2$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

Define $R > 0$ $\quad \theta \in [0, 2\pi)$

$X = R\cos\theta \quad , \quad Y = R\sin\theta$

Find $p_{R\theta}(r, \theta)$



$$\text{Prob}((X,Y) \in \mathcal{R}) = \int_{\mathcal{R}} p_{XY}\, dx\, dy = \int_{\mathcal{R}} \frac{1}{2\pi} \exp\left(-\frac{x^2 - y^2}{2}\right) dx\, dy$$

$$= \int_{\mathcal{R}} \frac{1}{2\pi} \exp\left(-\frac{r^2}{2}\right) dx\, dy \approx \frac{1}{2\pi} \exp\left(-\frac{r_0^2}{2}\right) \int_{\mathcal{R}} dx\, dy$$

$$= \frac{1}{2\pi} \exp\left(\frac{-r^2}{2}\right) A(\mathcal{R}) = \text{Prob}((R, \theta) \in \mathcal{R}) = \int_{\mathcal{R}} p_{R\theta}(r, \theta)\, dr\, d\theta$$

10 Let $X \sim N(m, \Lambda)$

$$m = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

a) Find Q : $X = QZ$ Z being independent, $m_z = 0$ $\sigma_z^2 = 1$

$$Z \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$m_X = Q m_Z \qquad \underline{0} = Q \, \underline{0} \quad \Rightarrow \quad Q = \text{Anything } 3 \times 3$$

$$\Lambda_{ZZ} = I_{3\times 3} \qquad \Lambda_{XX} = Q \Lambda_{ZZ} Q^* = Q Q^*$$

$$\lambda(\Lambda_{XX}) = [0.5858, \ 2, \ 3.4142] \quad \Rightarrow \quad \Lambda_{XX} > 0$$

$$\Lambda_{XX} = U H U^* \qquad \Lambda_{XX}^{\frac{1}{2}} = U H^{\frac{1}{2}} U^*$$

$$Q = U H^{\frac{1}{2}} U^* \quad \Rightarrow \quad \Lambda_{XX} = U H^{\frac{1}{2}} U^* (U H^{\frac{1}{2}} U^*)^*$$

$$\Lambda_{XX} = U H^{\frac{1}{2}} H^{\frac{1}{2}} U^* = U H U^*$$

$$U = \begin{bmatrix} -\sqrt{2}/2 & 0 & 1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \\ 1/2 & 1/\sqrt{2} & 1/2 \end{bmatrix} \qquad H^{\frac{1}{2}} = \begin{bmatrix} 0.7654 & & \\ & \sqrt{2} & \\ & & 1.8478 \end{bmatrix}$$

$$\Rightarrow Q = \begin{bmatrix} 1.3066 & 0.3827 & 0.3827 \\ 0.3827 & 1.3604 & -0.0538 \\ 0.3827 & -0.0538 & 1.3604 \end{bmatrix}$$

11 Let $e_k \sim N(0, \Lambda_{EE})$, and suppose e_k, e_j uncorrelated for $k \neq j$

$$x_{k+1} = Ax_k + Be_k \qquad \text{with} \quad x_0 = 0$$

a) Find an expression of x_k in terms of e_j

$$x_1 = A x_0 + Be_0 = Be_0$$

$$x_2 = Ax_1 + Be_1 = ABe_0 + Be_1$$

$$x_3 = Ax_2 + Be_2 = AABe_0 + ABe_1 + Be_2$$

$$x_k = A^{k-1} Be_0 + A^{k-2} Be_1 + \dots + ABe_{k-2} + Be_{k-1}$$

$$x_k = \sum_{j=0}^{k-1} A^j Be_{k-1-j} = \sum_{i=0}^{k-1} A^{k-1-i} Be_i$$

b) Find the density function for x_k

x_k Gaussian - Gaussian

$$E\{x_k\} = E\{\sum_{i=0}^{k-1} A^{k-1-i} Be_i\} = \sum_{i=0}^{k-1} A^{k-1-i} B E\{e_i\} = 0$$

$$E\{x_k x_k^*\} = E\{\sum_{i=0}^{k-1} A^{(k-1-i)} Be_i \Sigma e_i^* B^* A^{*(k-1-i)}\}$$

$$= E\{\sum_{i=0}^{k-1} A^{(k-1-i)} Be_i e_i^* B^* A^{*(k-1-i)}\}$$

$$\Lambda_{xx}^k = \sum_{i=0}^{k-1} A^{(k-1-i)} B E\{e_i e_i^*\} B^* A^{*(k-1-i)} = \sum_{i=0}^{k-1} A^{(k-1-i)} B \Lambda_{EE} B^* A^{*(k-1-i)}$$

$$P_k = \sum_{i=0}^{k-1} A^{(k-1-i)} B \Lambda_{EE} B^* A^{*(k-1-i)}$$

c) What happens as $k \to \infty$, Assumptions

$$P_{k+1} = \sum_{i=0}^{k} A^{(k-i)} B\Lambda_{EE} B^* A^{*(k-i)} = \sum_{i=0}^{k-1} AA^{(k-1-i)} B\Lambda_{EE} B^* A^{*(k-1-i)} A^* + B\Lambda_{EE} B^*$$

$$P_{k+1} = AP_k A^* + B\Lambda_{EE} B^*$$