IEOR 262B	HOMEWORK 2		Michael Janki 21204659 IEOR 262B

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1) $g(x) = \begin{bmatrix} -x_1 \\ -x_2 \\ x_2 - (x_1-1)^2 \end{bmatrix} \leq 0$

Show $x^* = [1 \quad 0]^T$ is feasible but not a regular point

$-1 \leq 0$, $0 \leq 0$, $0 - (0)^2 \leq 0 \Rightarrow$ feasible

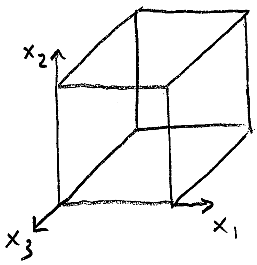
$$\nabla_x g(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -2(x_1-1) & 1 \end{bmatrix}$$

$$\nabla_x g(x^*) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ not LI}$$

$$0\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \Rightarrow \exists \alpha \neq 0 \text{ s.t. } \sum_{i=1}^{3} \alpha_i v_i = 0$$

$x^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ feasible, not regular

2) maximize volume



$$\max_x \; x_1 x_2 x_3 \equiv -\min_x \; -x_1 x_2 x_3$$

$$\begin{aligned} 3x_1x_2 + 4x_1x_3 + 2x_2x_3 - 72 &= 0 \qquad h(x) = 0 \\ -x_1 &\le 0 \\ -x_2 &\le 0 \qquad g(x) \le 0 \\ -x_3 &\le 0 \end{aligned}$$

$$L(x,\lambda,\mu) = -x_1x_2x_3 + \lambda(3x_1x_2 + 4x_1x_3 + 2x_2x_3 - 72) + \mu^T(-x)$$

1) FONC $\nabla_x L(x^*) = 0$, $\mu^T g(x^*) = 0$, $\mu \ge 0$

$$\nabla_x L = \begin{bmatrix} -x_2x_3 + \lambda(3x_2 + 4x_3) - \mu_1 \\ -x_1x_3 + \lambda(3x_1 + 2x_3) - \mu_2 \\ -x_1x_2 + \lambda(4x_1 + 2x_2) - \mu_3 \end{bmatrix}^T = 0$$

$\mu^T g(x) = 0$

$$-\mu_1x_1 - \mu_2x_2 - \mu_3x_3 = 0 \qquad \mu_1, \mu_2, \mu_3 \ge 0$$

$x_1, x_2, x_3 > 0$ to be physically possible

$\Rightarrow -\mu_1x_1 - \mu_2x_2 - \mu_3x_3 \le 0$ if $\mu \ge 0$

$-(\mu_1x_1 + \mu_2x_2 + \mu_3x_3) = 0 \Rightarrow \mu = 0$

$h(x) = 0$

$g(x) \le 0$

2)

$$\begin{aligned} -x_2x_3 + \lambda(3x_2 + 4x_3) &= 0 \\ -x_1x_3 + \lambda(3x_1 + 2x_3) &= 0 \\ -x_1x_2 + \lambda(4x_1 + 2x_2) &= 0 \\ 3x_1x_2 + 4x_1x_3 + 2x_2x_3 &= 72 \end{aligned}$$

$x > 0 \Rightarrow \lambda \ne 0$

$$\lambda(3x_2 + 4x_3)x_1 - \lambda(3x_1 + 2x_3)x_2 = 0$$
$$\lambda(3\cancel{x_1x_2} + 4x_1x_3 - 3\cancel{x_1x_2} - 2x_2x_3) = 0$$
$$\lambda(4x_1 - 2x_2)x_3 = 0$$

$4x_1 = 2x_2 \qquad x_2 = 2x_1$

$$\lambda(3x_1 + 2x_3)x_2 - \lambda(4x_1 + 2x_2)x_3 = 0$$
$$\lambda(3x_1x_2 + 2\cancel{x_2x_3} - 4x_1x_3 - 2\cancel{x_2x_3}) = 0$$
$$\lambda(3x_2 - 4x_3)x_1 = 0$$

$3x_2 = 4x_3 \qquad x_2 = \frac{4}{3}x_3$

$$\lambda(3x_2 + 4x_3)x_1 - \lambda(4x_1 + 2x_2)x_3 = 0$$
$$\lambda(3x_1x_2 + 4\cancel{x_1x_3} - 4\cancel{x_1x_3} - 2x_2x_3) = 0$$
$$\lambda(3x_1 - 2x_3)x_2 = 0$$

$3x_1 = 2x_3 \qquad x_1 = \frac{2}{3}x_3$

$$3\left(\tfrac{2}{3}x_3\right)\left(\tfrac{4}{3}x_3\right) + 4\left(\tfrac{2}{3}x_3\right)x_3 + 2\left(\tfrac{4}{3}x_3\right)x_3 = 72$$

$$x_3^2\left(\tfrac{8}{3}+\tfrac{8}{3}+\tfrac{8}{3}\right) = 72 \qquad 8x_3^2 = 72 \qquad x_3^2 = 9 \qquad x_3 = 3,\ x_2 = 4,\ x_1 = 2$$

$$-4(3) + \lambda(12+12) = 0 \Rightarrow \lambda = \tfrac{1}{2}$$

$$x^* = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \qquad \lambda^* = \tfrac{1}{2} \qquad \mu^* = 0_{3\times 1}$$

3)

$$\nabla_x f(x) = [-x_2x_3 \quad -x_1x_3 \quad -x_1x_2]$$

$$\nabla_x^2 f(x) = \begin{bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{bmatrix} \nabla_x f(x) = \begin{bmatrix} 0 & -x_3 & -x_2 \\ -x_3 & 0 & -x_1 \\ -x_2 & -x_1 & 0 \end{bmatrix}$$

$\mu \geq 0$

$\mu^T g(x^*) = 0$

$\nabla_x f(x^*) + \lambda^T \nabla_x h(x^*) + \mu^T \nabla_x g(x^*) = 0$

$$\nabla_x^2 f(x^*) = \begin{bmatrix} 0 & -3 & -4 \\ -3 & 0 & -2 \\ -4 & -2 & 0 \end{bmatrix}$$

$$\nabla_x h(x) = [3x_2 + 4x_3,\ 3x_1 + 2x_3,\ 4x_1 + 2x_2]$$

$$\nabla_x^2 h(x) = \begin{bmatrix} 0 & 3 & 4 \\ 3 & 0 & 2 \\ 4 & 2 & 0 \end{bmatrix} = \nabla_x^2 h(x^*)$$

$$\nabla_x^2 L(x^*, \lambda^*, \mu^*) = \begin{bmatrix} 0 & -1.5 & -2 \\ -1.5 & 0 & -1 \\ -2 & -1 & 0 \end{bmatrix} = H$$

$$\nabla h(x^*) = [3(4)+4(3),\ 3(2)+2(3),\ 4(2)+2(4)] = [24 \quad 12 \quad 16]$$

$\mu_1 = \mu_2 = \mu_3 = 0 \Rightarrow \nabla g_{1,2,3}(x^*)y = 0$ condition not necessary

$$\nabla h(x^*)y = 0 \qquad [6 \quad 3 \quad 4]y = 0 \Rightarrow y_1 = \begin{bmatrix} 1 \\ 0 \\ -1.5 \end{bmatrix} \quad y_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ -1.5 & 0 \end{bmatrix} \qquad E^T \nabla_x^2 L(x^*, \lambda^*, \mu^*) E = \begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}$$

$|E_1| = 6 > 0$

$|E_2| = 27 > 0$

$\Rightarrow \nabla_x^2 L(x^*, \lambda^*, \mu^*) > 0$ on M SOSC met

3)

$$L = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \qquad h = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad M = \{ y \mid h^T y = 0 \}$$

1) Find L_M

$$M = \{ y \mid h^T y = 0 \} = M = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$e_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \bar{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad e_2 = \bar{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{e}_1^T \bar{e}_1 = \frac{1}{2}(1+1) = 1 \qquad \bar{e}_1^T e_2 = 0 \qquad \bar{e}_2^T e_2 = 1$$

$$E = [\bar{e}_1 \ \ \bar{e}_2] \qquad E^T L E = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 1 \end{bmatrix} = L_M$$

2

$$|L_M - \lambda I| = \left(-\frac{1}{2} - \lambda\right)(1 - \lambda) - \frac{1}{2} = 0$$

$$\lambda^2 - \frac{1}{2}\lambda - 1 = 0 \qquad \lambda = \frac{1}{2}\left[\frac{1}{2} \pm \sqrt{\frac{1}{4} + 4} \right] = \frac{1}{2}\left[\frac{1}{2} \pm \sqrt{\frac{17}{4}} \right]$$

$$\lambda = \frac{1}{4} \pm \frac{\sqrt{17}}{4}$$

3

$$\begin{bmatrix} 0 & h^T \\ -h & L - I\lambda \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 4-\lambda & 3 & 2 \\ -1 & 3 & 1-\lambda & 1 \\ 0 & 2 & 1 & 1-\lambda \end{bmatrix} = B$$

$$|B| = -1 \left\{ 1((1-\lambda)^2 - 1) - 3(1(1-\lambda) - 0) + 2(1) \right\}$$

$$+1 \left\{ 1(3(1-\lambda) - 2(1)) - (4-\lambda)((1-\lambda) - 0) + 2(-2) \right\}$$

$$-(1 - 2\lambda + \lambda^2 - 1 - 3 + 3\lambda + 2) + (3 - 3\lambda - 2 - 4 + 4\lambda + 1\lambda - \lambda^2 - 4)$$

$$-2\lambda^2 + \lambda + 2 = 0 \Rightarrow \lambda^2 - \frac{1}{2}\lambda - 1 = 0 \qquad \lambda = \frac{1}{4} \pm \frac{\sqrt{17}}{4}$$

4

$$B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 4 & 3 & 2 \\ 1 & 3 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

$n = 3$
$m = 1$
$n - m = 2$

$|B| = 2$

$$|B_3| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 3 \\ 1 & 3 & 1 \end{vmatrix} = -1(1-3) + 1(3-4) = 2 - 1 = 1$$

$\text{sgn}(2) \neq \text{sgn}(-1^m) \Rightarrow L \not> 0$ on M

4) $x(k)$ - money at year k

$$x(k+1) = \alpha x(k) - u(k) \qquad \alpha \geq 1$$
$$x(0) = F$$

enjoyment $\Psi(u(k))$ $\quad \Psi$ smooth

$$J = \sum_{k=0}^{N} \Psi(u(k))\beta^k \qquad 0 < \beta^k < 1 \qquad \text{Want } x(N+1) = 0$$

1 $\quad \max\limits_{u(k)} J = \sum_{k=0}^{N} \Psi(u(k))\beta^k$ $\Bigg|$

$x(k+1) = \alpha x(k) - u(k) \quad k = 0 \dots N$

$x(0) = F$

$x(N+1) = 0$

$0 < \beta^k < 1$

$x(k) \geq 0 \quad \forall\ k = 1 \dots N+1$

$u(k) \geq 0 \quad \forall\ k = 1 \dots N$

$$x(0) = F$$
$$x(1) = \alpha x(0) - u(0)$$
$$x(2) = \alpha x(1) - u(1) = \alpha(\alpha x(0) - u(0)) - u(1)$$
$$x(3) = \alpha x(2) - u(2) = \alpha(\alpha^2 x(0) - \alpha u(0) - u(1)) - u(2)$$

$$x(N) = \alpha^N x(0) - \alpha^{N-1} u(0) - \alpha^{N-2} u(1) - \dots - u(N-1)$$
$$x(N+1) = \alpha^{N+1} x(0) - \alpha^N u(0) - \alpha^{N-1} u(1) - \dots - \alpha u(N-1) - u(N) = 0$$

$$x(k) = \alpha^k F - \sum_{j=0}^{N} \alpha^{(N-j)} u(j)$$

$$\equiv$$

$$u(N) = \alpha^{N+1} x(0) - \sum_{j=0}^{N-1} \alpha^{(N-j)} u(j)$$

If $x(k) < 0$ then $u(k) < 0$ impossible

$$\min_{\substack{u(k) \\ k=0\dots N}} J = -\sum_{k=0}^{N} \Psi(u(k))\beta^k \quad \Bigg| \quad u(N) = \alpha^{N+1} x(0) - \sum_{j=0}^{N-1} \alpha^{(N-j)} u(j) = 0, \quad x(0) = F$$

$$L(u, \lambda) = -\sum_{k=0}^{N} \Psi(u(k))\beta^k + \lambda\left(\sum_{j=0}^{N-1} \alpha^{(N-j)} u(j) - \alpha^{N+1} F \right)$$

2) $\Psi(u(k)) = u(k)^{1/2}$

$$L(u,\lambda) = -\sum_{k=0}^{N} + \beta^k u(k)^{\frac{1}{2}} + \lambda\left[\sum_{j=0}^{N-1} \alpha^{(N-j)} u(j) - \alpha^{N+1} F\right]$$

FONC $\quad \nabla_u L = 0 \qquad \nabla_\lambda L = 0$

$$\nabla_u L = \left[-\beta^0 \tfrac{1}{2} u(0)^{-\frac{1}{2}} + \lambda \alpha^N u(0),\ -\beta^1 \tfrac{1}{2} u(1)^{-\frac{1}{2}} + \lambda \alpha^{N-1} u(1), \ldots\right.$$

$$\frac{\partial L}{\partial u(k)} = -\frac{1}{2}\beta^k u(k)^{-\frac{1}{2}} + \lambda \alpha^{(N-k)} u(k) = 0 \quad \forall\, k$$

$$\frac{1}{2}\beta^k u(k)^{-\frac{1}{2}} = \lambda \alpha^{(N-k)} u(k) \qquad u(k)^{3/2} = \frac{\beta^k}{2\lambda\alpha^{(N-k)}}$$

$$u(k) = \left[\frac{\beta^k}{2\lambda\alpha^{(N-k)}}\right]^{2/3}$$

$$L(u,\lambda) = -\sum_{k=0}^{N} + \beta^k \left[\frac{\beta^k}{2\lambda\alpha^{(N-k)}}\right]^{1/3} + \lambda\left[\sum_{j=0}^{N-1} \alpha^{(N-j)} \left[\frac{\beta^j}{2\lambda\alpha^{(N-j)}}\right]^{2/3} - \alpha^{N+1} F\right]$$

$$\nabla_\lambda L = \beta^k \left[\frac{\beta^k}{2\alpha^{(N-k)}}\right]^{1/3}\left(-\frac{1}{3}\lambda^{-\frac{4}{3}}\right) + \left[\sum_{j=0}^{N-1} \alpha^{(N-j)} \left[\frac{\beta^j}{2\lambda\alpha^{(N-j)}}\right]^{2/3}\right] - \alpha^{N+1} F$$

$$+ \lambda\left[\sum_{j=0}^{N-1} \alpha^{(N-j)} \left(\frac{\beta^j}{2\alpha^{(N-j)}}\right)^{2/3}\left(-\frac{2}{3}\right)\left(\lambda^{-5/3}\right)\right] = 0$$

$$\Rightarrow \lambda^*$$

$$u(k)^* = \left(\frac{\beta^k}{2\lambda^*\alpha^{(N-k)}}\right)^{2/3}$$

5) $L \in \mathbb{R}^{n \times n}$ $\quad A \in \mathbb{R}^{m \times n}$

$L_M > 0$ $\quad M = \{x \mid Ax = 0\}$

Show $\begin{bmatrix} L & A^T \\ A & O_{m \times m} \end{bmatrix} > 0$

$AL^{-1}A^T$

$$\begin{bmatrix} L & 0 \\ A & I \end{bmatrix} \begin{bmatrix} I & L^{-1}A^T \\ 0 & -AL^{-1}A^T \end{bmatrix}$$

$$= \underbrace{\det(L\,I)}_{>0} \underbrace{\det(-AL^{-1}A^T)}_{\leq 0}$$

If singular $\begin{bmatrix} L & A^T \\ A & O_{m \times m} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$ $\quad u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$

$Lu + A^T v = 0$

$Au = 0 \Rightarrow u \in M$; $u^T A^T = 0^T$

$u^T(Lu + A^T v) = u^T L u + \underbrace{u^T A^T}_{0} v = 0$

$\Rightarrow u^T L u = 0$ contradicts $L > 0$ on M as $u \in M$

$\Rightarrow \begin{bmatrix} L & A^T \\ A & O_m \end{bmatrix} > 0$