1 Consider a random variable X distributed according to the Gaussian mixture

$$p_X(x) = \sum_{k=1}^{n} \alpha_k f_k(x)$$

where the density functions f_k are Gaussian as

$$f_k(x) = \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\{(x - m_k)^2/2\sigma^2\}$$

and the coefficients α_k satisfy

$$\alpha_k > 0, \qquad \sum_{k=1}^{n} \alpha_k = 1$$

We can use the more compact notation

$$X \sim \sum_k \alpha_k \mathbb{N}\left(m_k, \sigma_k^2\right)$$

Find expressions for the mean and variance of X

2 Suppose we have random variables X, Y distributed according to the Gaussian mixture

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim 0.5\mathbb{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right) + 0.5\mathbb{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

(a) Find the best estimate $\hat{X}_{NL}$ of X given the observation $Y = y$ is the minimum variance sense.

(b) Find the best linear estimate $\hat{X}_L$ of X given $Y = y$.

(c) Plot these estimates versus y.

1. X is distributed according to a Gaussian mixture

$$p_X(x) = \sum_{h=1}^{n} \alpha_h f_h(x) \qquad f_h(x) = \frac{1}{\sqrt{2\pi\sigma_h^2}} \exp\{(x-m_h)^2/2\sigma^2\}$$

$$\alpha_h > 0 \qquad \sum_{h=1}^{n} \alpha_h = 1$$

Find the mean and variance

$$m_X = E\{X\} = \int_{-\infty}^{+\infty} x p_X(x)\, dx = \int_{-\infty}^{+\infty} x \sum_{h=1}^{n} \alpha_h f_h(x)\, dx$$

$$= \sum_{h=1}^{n} \alpha_h \int_{-\infty}^{+\infty} x f_h(x)\, dx = \sum_{h=1}^{n} \alpha_h m_h$$

$$m_X = \sum_{h=1}^{n} \alpha_h m_h$$

$$E\{X^2\} = \sigma_X^2 + m_X^2 \qquad \sigma_X^2 = E\{X^2\} - m_X^2$$

$$E\{X^2\} = \int_{-\infty}^{+\infty} x^2 \sum_{h=1}^{n} \alpha_h f_h(x)\, dx = \int_{-\infty}^{+\infty} x^2 (\alpha_1 f_1(x) + \dots + \alpha_n f_n(x))\, dx$$

$$= \alpha_1 \int_{-\infty}^{+\infty} x^2 f_1(x)\, dx + \dots + \alpha_n \int_{-\infty}^{+\infty} x^2 f_n(x)\, dx = \sum_{h=1}^{n} \alpha_h \int_{-\infty}^{+\infty} x^2 f_h(x)\, dx$$

$$= \alpha_1 (\sigma_1^2 + m_1^2) + \dots + \alpha_n (\sigma_n^2 + m_n^2) = \sum_{h=1}^{n} \alpha_h (\sigma_h^2 + m_h^2)$$

$$\sigma_X^2 = \sum_{h=1}^{n} \alpha_h (\sigma_h^2 + m_h^2) - \left[\sum_{h=1}^{n} \alpha_h m_h\right]^2$$

$$E\{(X-m_X)^2\} = E\{(X - \sum_{h=1}^{n} \alpha_h m_h)^2\} = E\{X^2 - 2X\sum \alpha_h m_h + [\sum \alpha_h m_h]^2\}$$

$$E\{X^2\} = \int_{-\infty}^{+\infty} x^2 (\alpha_1 f_1 + \dots + \alpha_n f_n)\, dx = \alpha_1 (\sigma_1^2 + m_1^2) + \dots + \alpha_n (\sigma_n^2 + m_n^2)$$

$$E\{X^2\} = \sum \alpha_h (\sigma_h^2 + m_h^2)$$

$$2E\{X \sum \alpha_h m_h\} = (2\sum \alpha_h m_h) E\{X\} = 2(\sum \alpha_h m_h)(\sum \alpha_h m_h) = 2[\sum \alpha_h m_h]^2$$

$$\Rightarrow E\{(X-m_X)^2\} = [\sum \alpha_h (\sigma_h^2 \cdot m_h^2)] - [\sum \alpha_h m_h]^2 = \sigma_X^2$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim 0.5N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right) + 0.5N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right)$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \qquad\qquad \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$p_{XY}(x,y) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}[x \;\; y]\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) + \frac{1}{2}\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}[x \;\; y]\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$p_{XY}(x,y) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x^2 - 2xy + 2y^2)\right) + \frac{1}{2}\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(2x^2 - 2xy + y^2)\right)$$

$$p_Y(y) = \int_{x=-\infty}^{+\infty} p_{XY}(x,y)\,dx$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{+\infty} \exp\left(-\frac{1}{2}(x^2 - 2xy + 2y^2)\right) + \exp\left(-\frac{1}{2}(2x^2 - 2xy + y^2)\right) dx$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}} \left[\sqrt{\frac{\pi}{2}}\, e^{-y^2/2} \operatorname{erf}\left(\frac{x-y}{\sqrt{2}}\right) + \frac{1}{2}\sqrt{\pi}\, e^{-y^2/4} \operatorname{erf}\left(x - \frac{y}{2}\right) \right] \Bigg|_{x=-\infty}^{+\infty}$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) e^{-y^2/2} (1 - -1) + \frac{1}{4}\frac{1}{\sqrt{2}} e^{-y^2/4} (1 - -1)$$

$$= \frac{1}{4} e^{-y^2/2} + \frac{\sqrt{2}}{4} e^{-y^2/4}$$

$$p_{X|Y=y} = \frac{\frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x^2 - 2xy + 2y^2)\right) + \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(2x^2 - 2xy + y^2)\right)\right)}{\frac{1}{4} e^{-y^2/2} + \frac{\sqrt{2}}{4} e^{-y^2/4}}$$

$$E\{X|Y=y\} = [p_Y(y)]^{-1} \int_{x=-\infty}^{+\infty} x\, p_{XY}(x,y)\,dx$$

$$= [p_Y(y)]^{-1} \left[\int_{x=-\infty}^{+\infty} x\, c \exp\left(-\frac{1}{2}(x^2 - 2xy + 2y^2)\right) dx + \int_{x=-\infty}^{+\infty} x\, c \exp\left(-\frac{1}{2}(2x^2 - 2xy + y^2)\right) dx \right.$$

$$= [p_Y(y)^{-1}] \left\{ \frac{1}{2}\frac{1}{\sqrt{2\pi}} \left[\sqrt{\frac{\pi}{2}}\, e^{-y^2/2}\, y \operatorname{erf}\left(\frac{x-y}{\sqrt{2}}\right) - e^{-\frac{x^2}{2} + xy - y^2} \right] \right.$$

$$\left. + \frac{1}{2}\frac{1}{\sqrt{2\pi}} \left[\frac{1}{4}\sqrt{\pi}\, e^{-y^2/4}\, y \operatorname{erf}\left(x - \frac{y}{2}\right) - \frac{1}{2} e^{-\frac{x^2}{2} + xy - y^2/2} \right] \right\}$$

$$= [p_Y(y)]^{-1}\Big[\frac{1}{2}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}}\, y e^{-y^2/2}(1 - -1) - (0 - 0)$$

$$+ \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{4}\sqrt{\pi} y e^{-y^2/4}(2)\Big]$$

$$= \frac{\frac{1}{4} y e^{-y^2/2} + \frac{1}{8\sqrt{2}} y e^{-y^2/4}}{\frac{1}{4} e^{-y^2/2} + \frac{\sqrt{2}}{4} e^{-y^2/4}}$$

b)

$$\hat{X} = aY + b$$

$$E\{(\hat{X} - X)\}^2 = E\{(aY + b - X)^2\}$$

$$= E\{aY^2 + 2abY - 2aYX - 2bX + b^2 + X^2\}$$

$$= a^2 E\{Y^2\} + 2abE\{Y\} - 2aE\{XY\} - 2bE\{X\} + E\{X^2\} + E\{b^2\}$$

$$= a^2\left(\frac{1}{2}(2+0^2) + \frac{1}{2}(1+0^2)\right) + 2ab(0) - 2a\left(\frac{1}{2}(-1) + \frac{1}{2}(-1)\right) - 2b(0) + \frac{1}{2}(2) + \frac{1}{2}(1) + b^2$$

$$= \frac{3}{2}a^2 - 2a + b^2$$

$$\frac{\partial \hat{X}}{\partial a} = 3a - 2 = 0 \qquad a = 2/3 \qquad \frac{\partial \hat{X}}{\partial b} = 2b = 0 \Rightarrow b = 0$$

$$\hat{X}_{MVL} = \frac{2}{3} Y$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim 0.5N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right) + 0.5N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

$$p_{XY}(x,y) = \frac{1}{2}\frac{1}{2\pi}\exp\left(-\frac{1}{2}[x \; y]\overset{\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix}\right) + \frac{1}{2}\frac{1}{2\pi}\exp\left(-\frac{1}{2}[x \; y]\overset{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$p_{XY}(x,y) = \frac{1}{2}\frac{1}{2\pi}\exp\left(-\frac{1}{2}(x^2 - 2xy + 2y^2)\right) + \frac{1}{2}\frac{1}{2\pi}\exp\left(-\frac{1}{2}(2x^2 - 2xy + y^2)\right)$$

$$p_Y = \int p_{XY}(x,y)\,dx$$

$$p_Y = \frac{1}{2}\frac{1}{2\pi}\int_{x=-\infty}^{+\infty} \exp\left(-\frac{1}{2}(x^2 - 2xy + 2y^2)\right) + \exp\left(-\frac{1}{2}(2x^2 - 2xy + y^2)\right)dx$$

$$p_Y = \left(\frac{1}{2}\right)\left(\frac{1}{2\pi}\right)\int_{x=-\infty}^{+\infty} \exp\left(-\frac{1}{2}(x^2 - 2xy + y^2) - \frac{1}{2}y^2\right)dx$$

$$+ \frac{1}{2}\frac{1}{2\pi}\int_{x=-\infty}^{+\infty} \exp\left(-(x^2 - xy + y^2/4) - y^2/4\right)dx$$

$$= \frac{1}{2}\frac{1}{2\pi}\exp\left(-\frac{1}{2}y^2\right)\frac{1}{2}\sqrt{\pi}\frac{1}{\sqrt{2}}\operatorname{erf}\left(\frac{x-y}{\sqrt{2}}\right)\Bigg|_{-\infty}^{+\infty}$$

$$+ \frac{1}{2}\frac{1}{2\pi}\exp\left(-\frac{1}{4}y^2\right)\frac{1}{2}\sqrt{\pi}\operatorname{erf}\left(x - \frac{y}{2}\right)\Bigg|_{-\infty}^{+\infty}$$

$$= \frac{1}{2}\frac{1}{2\pi}\exp\left(-\frac{1}{2}y^2\right)\cancel{\frac{1}{2}}\sqrt{\pi}\frac{1}{\sqrt{2}}(\cancel{1 - -1}) + \frac{1}{2}\frac{1}{2\pi}\exp\left(-\frac{1}{4}y^2\right)\cancel{\frac{1}{2}}\sqrt{\pi}(\cancel{1 - -1})$$

$$= \frac{1}{2}\frac{1}{2\pi}\sqrt{\frac{\pi}{2}}\exp\left(-\frac{1}{2}y^2\right) + \frac{1}{2}\frac{1}{2\pi}\sqrt{\pi}\exp\left(-\frac{1}{4}y^2\right)$$