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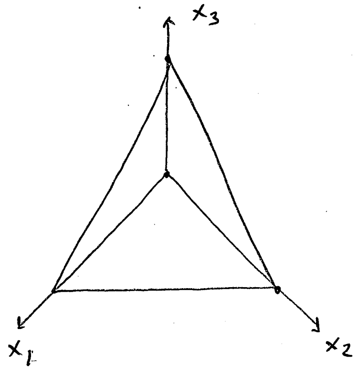
E 231 FALL 2009 HOMEWORK 2

E 231
9-14-09

1. Consider the pyramid in $\mathbb{R}^3$ with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Write as $Ax \le b$



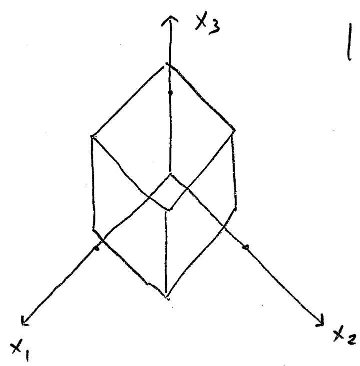
$x_1 + x_2 + x_3 \le 1$

$x_1 \ge 0$
$x_2 \ge 0$
$x_3 \ge 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 \ldots 1 \\ -I_3 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

2. Consider the unit cube in $\mathbb{R}^3$ defined by $\|x\|_\infty \le 1$. Descibe as $Ax \le b$



$\|x\|_\infty \le 1 \Rightarrow$
$-1 \le x_1 \le 1$
$-1 \le x_2 \le 1$
$-1 \le x_3 \le 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} I_3 \\ -I_3 \end{bmatrix}$$

$$b = [1 \ldots\ldots 1]^T$$

3. Consider the unit cube in $\mathbb{R}^n$. $C = \{x \in \mathbb{R}^n : 0 \le x_i \le 1 \quad i = 1 \ldots n\}$

$x_i \le 1 \quad \forall i \in 1 \ldots n$

$x_i \ge 0 \quad \forall i \in 1 \ldots n$

$$\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & & & 1 \\ -1 & 0 & & 0 \\ 0 & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

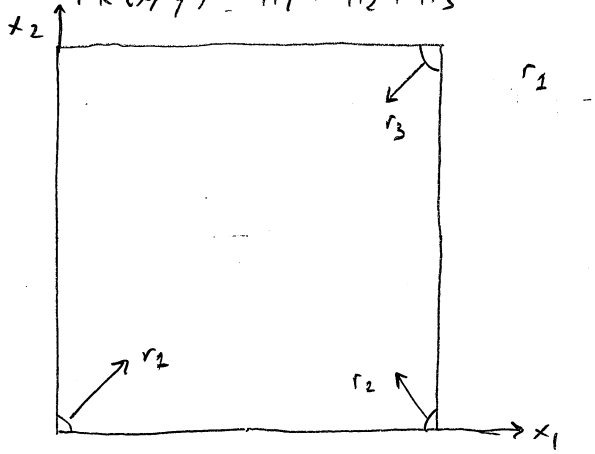
$$A = \left[\frac{I_n}{-I_n} \right]$$

$$b = [\, 1^{1 \times n} \mid 0^{1 \times n} \,]^T$$

4. A square room has 3 heaters. Each heater has independent controlled settings.

$$H_i = (T_i - T_o)e^{-r/R} \qquad T_o = 50°F \qquad R = 3$$

$$TR(x,y) \doteq H_1 + H_2 + H_3$$



$$TR(x_1, x_2) = (T_1 - T_o)e^{-r_1/R} + (T_2 - T_o)e^{-r_2/R} + (T_3 - T_o)e^{-r_3/R}$$

Want $\quad TR(x_1, x_2) \triangleq 70°F$

$$\| TR(x_1, x_2) - 70 \|_\infty \leq \gamma \quad \text{some error bound}$$

$$\Rightarrow \quad TR(x_1, x_2) - 70 \leq \gamma \qquad 70 - TR(x_1, x_2) \leq \gamma \quad \forall x_1, x_2$$

$$(T_1 - T_o)e^{-r_1/R} + (T_2 - T_o)e^{-r_2/R} + (T_3 - T_o)e^{-r_3/R} - 70 \leq \gamma$$

$$(T_o - T_1)e^{-r_1/R} + (T_o - T_2)e^{-r_2/R} + (T_o - T_3)e^{-r_3/R} + 70 \leq \gamma$$

Want to minimize γ over x_1, x_2

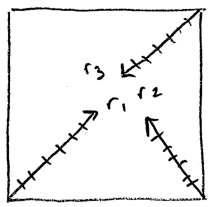
$$\min_{c^*x} : Ax \leq b \qquad x = [\gamma \; T_1 \; T_2 \; T_3 \; r_1 \; r_2 \; r_3]$$

$$\min_x [1\; 0\; 0\; 0\; 0\; 0\; 0][\gamma \; T_1 \; T_2 \; T_3 \; r_1 \; r_2 \; r_3]^T$$

$$\begin{bmatrix} -1 & e^{-r_1/R} & e^{-r_2/R} & e^{-r_3/R} & 0 & 0 & 0 \\ -1 & -e^{-r_1/R} & -e^{-r_2/R} & -e^{-r_3/R} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & 0 & & 0 & 1 & 0 \\ & & & & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma \\ T_1 \\ T_2 \\ T_3 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} \leq \begin{bmatrix} 70 + T_o(e^{-r_1/R} + e^{-r_2/R} + e^{-r_3/R}) \\ -70 - T_o(e^{-r_1/R} + e^{-r_2/R} + e^{-r_3/R}) \\ 5 \\ 5 \\ 5 \end{bmatrix}$$

Cannot do this for all combination of r_1, r_2, r_3

so discritize r_1, r_2, r_3



$$\Rightarrow (T_1 - T_o)e^{-r_{1i}/R} + (T_2 - T_o)e^{-r_{2j}/R} + (T_3 - T_o)e^{-r_3 k/R} - 70 \le \gamma$$

$$(T_o - T_1)e^{-r_{1i}/R} + (T_o - T_2)e^{-r_{2j}/R} + (T_o - T_3)e^{-r_3 k/R} + 70 \le \gamma$$

$$r_{1i} \le \sqrt{5^2 + 5^2}, \quad r_{2j} \le \sqrt{5^2 + 5^2}, \quad r_{3j} \le \sqrt{5^2 + 5^2}$$

$$x = [\gamma \;\; T_1 \;\; T_2 \;\; T_3 \;\; r_{11} \; \cdots \; r_{1n} \;\; r_{21} \; \cdots \; r_{2n} \;\; r_{31} \; \cdots \; r_{3n}]$$

Discretize r_1, r_2, r_3 into n points

$$\min \, [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0^{1\times n} \;\; 0^{1\times n} \;\; 0^{1\times n}][\gamma \;\; T_1 \;\; T_2 \;\; T_3 \;\; r_{11} \cdots r_{1n} \;\; r_{21} \cdots r_{2n} \;\; r_{31} \cdots r_{3n}]^T$$

$$\begin{bmatrix} -1 & e^{-r_{1i}/R} & e^{-r_{2j}/R} & e^{-r_3 k/R} & 0 & \cdots & 0 \\ & \vdots & \text{Over all combinations/ values of } i, j, k & & & & \\ -1 & -e^{-r_{1i}/R} & -e^{-r_{2j}/R} & -e^{-r_3 k/R} & 0 & \cdots & 0 \\ & \vdots & \text{Over all combinations of } i, j, k & & & & \\ 1 & 0 & 0 & \cdots & & & 0 \\ 0 & 1 & 0 & \cdots & & & 0 \\ & & \vdots & & & & \\ 0 & & & & & & 1 \end{bmatrix} \begin{bmatrix} \gamma \\ T_1 \\ T_2 \\ T_3 \\ r_{11} \\ \vdots \\ r_{3n} \end{bmatrix} \le \begin{bmatrix} 70 + T_o(e^{-r_{1i}/R} + e^{-r_{2j}/R} + e^{-r_3 k/R}) \\ \vdots \\ -70 - T_o(e^{-r_{1i}/R} + e^{-r_{2j}/R} + e^{-r_3 k/R}) \\ \sqrt{50} \\ \sqrt{50} \\ \vdots \\ \sqrt{50} \end{bmatrix}$$

5. Consider the set $C = \{x \in \mathbb{R}^n : |x_i| \geq 2 \text{ for } i = 1, \dots, n\}$
Is this a polytope?

$$\begin{array}{cccc} x_1 \geq 2 & x_1 \leq -2 & & -x_1 \leq -2 & x_1 \leq -2 \\ x_2 \geq 2 & x_2 \leq -2 & \Rightarrow & -x_2 \leq -2 & x_2 \leq -2 \\ \vdots & & & \vdots & \vdots \\ x_n \geq 2 & x_n \leq -2 & & -x_n \leq -2 & x_n \leq -2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ -1 & 0 & & 0 \\ 0 & -1 & & 0 \\ & & \ddots & \\ 0 & 0 & & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \leq \begin{bmatrix} -2 \\ -2 \\ \vdots \\ -2 \\ -2 \\ -2 \\ \vdots \\ -2 \end{bmatrix}$$

$$A = \left[\begin{array}{c} I_n \\ \hline -I_n \end{array}\right] \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} -2 \\ \vdots \\ -2 \end{bmatrix} = [-2^{2n \times 1}]$$

This set can be written as $Ax \leq b$ so it is a polytope

6. Consider the optimization problem:

$$\max_{c^* x} \quad \text{subject to} \quad 1 \le \|x\|_\infty \le 2$$

Is the a LP problem?

$$\max_i |x_i| \le 2 \qquad \max_i |x_i| \ge 1$$

7. Let GL_n = the set of all $n \times n$ non singular matrices
Is this set a field

$$GL_n = \{ A^{n\times n} : |A^{n\times n}| \neq 0 \} \qquad A^{n\times n} \in GL_n$$

Check for properties

a) $A^{n\times n} \in \mathbb{F}$ $B^{n\times n} \in \mathbb{F}$ hard to tell
Suppose $B = -A^{n\times n}$
$A + B = 0^{n\times n}$ $\quad |0^{n\times n}| = |A + -A| = |A+B| = 0$
$\Rightarrow |A + B \notin \mathbb{F} \ \forall A, B \in \mathbb{F}$
Not a field

b) $A + B = B + A$ - definition of matrix addition

c) $A + (B + C) = (A + B) + C$ - definition of matrix addition

d) $A \in \mathbb{F}, B \in \mathbb{F}, C \in \mathbb{F}$

$A \cdot (B + C) = AB + AC$ - definition of matrix mult

e $\exists\, 0^{n\times n}$: $A + 0^{n\times n} = A^{n\times n}$ $\forall A \in \mathbb{F}$
$\exists\, I^{n\times n}$: $AI = IA = A \in \mathbb{F}$ $\forall A \in \mathbb{F}$
f $\exists -A^{n\times n}$: $A + (-A) = 0$
thus, this property does not hold

This set is not a field as $A + (-A) = 0$, I

$|A + -A| = |0| = 0 \Rightarrow 0^{n\times n} \notin \mathbb{F}$ and the set does

not satisfy closure

8. Let $\mathbb{R}^{2\times 2}$ be all the set of all 2×2 real matrices

a) Briefly verify $\mathbb{R}^{2\times 2}$ is a vector space

$A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 2}$ $\qquad A_{ij} \in \mathbb{R} \ \forall \ i,j$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad B_{ij} \in \mathbb{R} \ \forall \ i,j$

$A + B \Rightarrow C \qquad C_{ij} = A_{ij} + B_{ij} \in \mathbb{R} \ \forall \ i,j$

$\Rightarrow \quad C = A + B \in \mathbb{R}^{2\times 2}$

$C = \alpha A$ for some $\alpha \in \mathbb{R}$ $\mathbb{F}$

$C_{ij} = \alpha A_{ij} \in \mathbb{R} \ \forall \ i,j$ iff $\alpha \in \mathbb{R}$

$C = \alpha A \in \mathbb{R}^{2\times 2}$

$\Rightarrow \mathbb{R}^{2\times 2}$ is a vector space

b) $$A \in \mathbb{R}^{2\times 2} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} \\ A_{12} \\ A_{21} \\ A_{22} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{11} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{a_1} + \underbrace{\begin{bmatrix} 0 \\ A_{12} \\ 0 \\ 0 \end{bmatrix}}_{a_2} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ A_{21} \\ 0 \end{bmatrix}}_{a_3} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ A_{22} \end{bmatrix}}_{a_4}$$

$A_{ij} \neq 0 \ \forall \ i,j$

$0 = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \alpha_4 a_4 \Rightarrow \alpha_i = 0 \ \forall \ i$

$\Rightarrow \dim(\mathbb{R}^{2\times 2}) = 4$

c) Find a basis for $\mathbb{R}^{2\times 2}$

$$A = \begin{bmatrix} A_{11} \\ A_{12} \\ A_{21} \\ A_{22} \end{bmatrix} = A_{11}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + A_{12}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + A_{21}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + A_{22}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$B = \{ v \in \mathbb{R}^4 : v_i = [\, 0^{n-1} \mid 1 \mid 0^{4-n} \,]^T \}$$

Thus A is a linear combination of v_i and every $A \in \mathbb{R}^{2\times 2}$ must be expressed!

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + A_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + A_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + A_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

d) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ Is the set $\{I, A, A^2\}$ LI?

$A^2 = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$ $\{I, A, A^2\} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 4 \end{bmatrix}$

$\alpha \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 3 \\ 4 \end{bmatrix} = 0 \Rightarrow \alpha = \beta = \gamma = 0$

The set I, A, A^2 is LI in $\mathbb{R}^{2 \times 2}$

9. Which of the following sets is LI in $\mathbb{R}^3$

a) $\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\5\\1\end{bmatrix}$ $\quad \alpha\begin{bmatrix}1\\0\\2\end{bmatrix} + \beta\begin{bmatrix}2\\0\\1\end{bmatrix} + \gamma\begin{bmatrix}0\\5\\1\end{bmatrix} = 0 \quad \Rightarrow \gamma = 0$

$\alpha + 2\beta = 0$
$2\alpha + \beta = 0$ $\quad \alpha + 2\beta = 2\alpha + \beta \quad \alpha = \beta \quad 3\beta = 0$
$3\alpha = 0$

$\Rightarrow \alpha = \beta = \gamma = 0$
This set is LI in $\mathbb{R}^3$

b) $\begin{bmatrix}4\\5\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}$ $\quad \alpha_1\begin{bmatrix}4\\5\\1\end{bmatrix} + \alpha_2\begin{bmatrix}1\\2\\-1\end{bmatrix} + \alpha_3\begin{bmatrix}2\\1\\3\end{bmatrix} = 0$

$\underbrace{\begin{bmatrix}4 & 1 & 2\\5 & 2 & 1\\1 & -1 & 3\end{bmatrix}}_{A}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = 0^{3\times 1}$ $\quad \det A = 4[6+1] - [15-1] + 2[-5-2]$
$= 28 - 14 - 14 = 0$

$|A| = 0 \Rightarrow$ this set is not LI

also $\quad \sum \alpha_h s_h = 0 \Rightarrow \alpha_1 = n \quad \alpha_2 = -2n \quad \alpha_3 = -n$
$\forall\ n \in \mathbb{R}$

c) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}, \begin{bmatrix}7912\\-314\\0.098\end{bmatrix}$ $\quad \dim(\mathbb{R}^3) = 3$, which by definition, is the maximum number of LI vectors that can be found

Therefore, these 4 vectors are not LI since the number of vectors exceeds the dimension of the vector space

10 Let $\mathbb{V}$ be a vector space. Let $B = \{ b_i : i \in I \}$ be a basis for this vector space. Prove B is LI

$v \in V \Rightarrow \quad v = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n$

$v = \beta_1 b_1 + \beta_2 b_2 + \dots + \beta_n b_n \Rightarrow \alpha_i = \beta_i \quad \forall i$

Suppose $v = 0^n$, B is not LI, but LD

$0 = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n$

$\Rightarrow \alpha_i \neq 0$ for some i

However $0 = \alpha_1 b_1 + \alpha_2 b_2 + \dots \alpha_n b_n$

$m0 = m\alpha_1 b_1 + m\alpha_2 b_2 + \dots m\alpha_n b_n = v = 0$

Thus v cannot be expressed uniquely by a linear combination of b_i, so B must be LI

If B is LI $\quad \sum \alpha_i b_i = 0 \Rightarrow \alpha_i = 0 \quad \forall i$

Suppose B is not LI $\quad 0 = \sum \alpha_i b_i = 0 \quad \alpha_i \neq 0$ for some i

Suppose $b_1 = 0 \quad 0 = \alpha_1 (0) + \alpha_2 b_2 + \dots \alpha_n b_n \Rightarrow \alpha_1 = \mathbb{R}$

and this cannot be expressed uniquely, therefore the set B must be LI to satisfy the definition of a basis that any $v \in \mathbb{V}$ can be ! expressed as $\sum \alpha_i b_i$