1. Let $A \in \mathbb{R}^{m \times n}$ and consider the subset

$S = \{x : Ax = 0\}$ Is S a subspace?

$x_1 \in S$, $x_2 \in S$, $\alpha \in \mathbb{F}$

$A(x_1 + x_2) = Ax_1 + Ax_2 = 0 + 0 \Rightarrow x_1 + x_2 \in S$

$A(\alpha x_1) = \alpha A x_1 = 0 \Rightarrow \alpha x_1 \in S \ \forall \alpha$

This is a subspace

2. Let $\mathbb{V}$ be an inner product space

a) $x, y \in \mathbb{V}$ $x \perp y$ Prove $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

$\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$

$\langle x, y \rangle = \langle y, x \rangle = 0 \Leftrightarrow x \perp y$

$\langle x+y, x+y \rangle = \langle x, x \rangle + \langle y, y \rangle \Leftrightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$

b) Prove $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

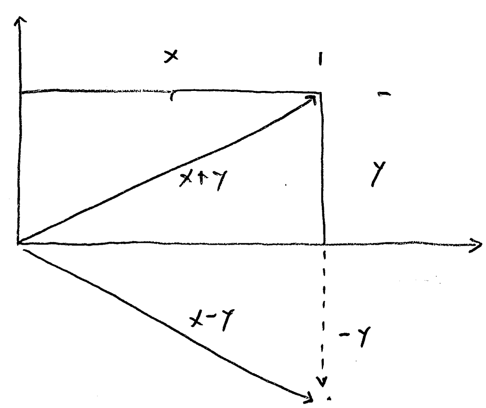
$\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + \langle y, y \rangle$

$\|x-y\|^2 = \langle x-y, x-y \rangle = \langle x, x \rangle + \langle -y, -y \rangle = \langle x, x \rangle + \langle y, y \rangle$

$\|x+y\|^2 + \|x-y\|^2 = \langle x, x \rangle + \langle x, x \rangle + \langle y, y \rangle + \langle y, y \rangle$

$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

$\mathbb{R}^2$



3. Let $v \in \mathbb{C}^n$

a) Show $||v||_1 \leq \sqrt{n}||v||_2 \leq n||v||_\infty$ $\quad ||v||_\infty = [\Sigma |v_i|^\infty]^{1/\infty}$

$$||v||_1 = |\langle v, 1\rangle| \leq \sqrt{\langle v,v\rangle\langle 1,1\rangle} = \sqrt{\langle v,v\rangle n} = \sqrt{n}||v||_2$$

from Cauchy Schwartz

$$\sqrt{n}||v||_2 = \sqrt{\langle v,v\rangle\langle 1,1\rangle} \leq \sqrt[1/\infty]{\langle v,v\rangle^\infty\langle 1,1\rangle^\infty} = \sqrt[1/\infty]{\langle v,v\rangle n^\infty}$$

$$= n\max(v) = n||v||_\infty$$

$$\Rightarrow ||v||_1 \leq \sqrt{n}||v||_2 \leq n||v||_\infty$$

b) Show $||v||_\infty \leq ||v||_2 \leq ||v||_1$

$$||v||_2 = \sqrt{\langle v, v\rangle}$$

$$||v||_1 = |\langle v, 1\rangle|$$

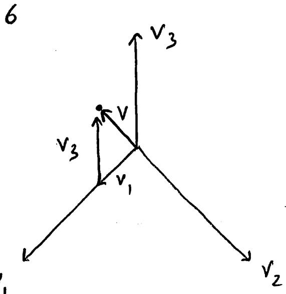
c) Show these inequalities are right

$$\|v\|_1 \leq \sqrt{n}\,\|v\|_2 \leq n\|v\|_\infty$$

$v \in \mathbb{C}^2$ $\qquad v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$\|v\|_1 = 3 \qquad \sqrt{n}\,\|v\|_2 = \sqrt{3}\sqrt{5} = \sqrt{15} \qquad n\|v\|_\infty = 3(2) = 6$

$$3 \leq \sqrt{15} \leq 6$$



$\|v\|_\infty = 2 \qquad \|v\|_2 = \sqrt{5} \qquad \|v\|_1 = 3$

$$2 \leq \sqrt{5} \leq 3$$

$\|v\|_\infty = \max\limits_i |v_i|$

$\|v\|_2 = \sqrt{\sum\limits_i |v_i|^2}$

$\|v\|_1 = \sum\limits_i |v_i|$

$$|v_{max}| \leq \left[|v_1|^2 + |v_2|^2 + \ldots |v_{max}|^2 + \ldots + |v_n|^2 \right]^{\frac{1}{2}}$$

$$\leq |v_1| + |v_2| + \ldots + |v_{max}| + \ldots + |v_n|$$

even if $v_{max} \neq 0$, $v_i = 0$ $\forall\, i \neq max$ the equality holds

4 Consider the vector space $L_2[-1, 1]$ and let $S = \text{span}\{1, t, t^2\}$

a) Find an orthonormal basis using GS

$b_1 = \frac{1}{\|1\|}$ $\quad \|1\|^2 = \langle 1, 1\rangle = \int_{-1}^{1} \bar{1}\, 1\, dt = 2 \quad \|1\| = \sqrt{2}$

$b_1 = 1/\sqrt{2}$

$w_2 = t - \langle \frac{1}{\sqrt{2}}, t\rangle \frac{1}{\sqrt{2}} \qquad \langle \frac{1}{\sqrt{2}}, t\rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} t\, dt = \frac{1}{\sqrt{2}} \frac{1}{2} t^2 \Big|_{-1}^{1}$

$= \frac{1}{\sqrt{2}} \frac{1}{2} (1 - 1) = 0$

$w_2 = t$

$\|t\|^2 = \langle t, t\rangle = \int_{-1}^{1} t^2 dt = \frac{1}{3} t^3 \Big|_{-1}^{2} = \frac{1}{3}(1 - -1) = \frac{2}{3}$

$\|t\| = \sqrt{\frac{2}{3}}$

$b_2 = \sqrt{\frac{3}{2}}\, t$

$w_3 = t^2 - \langle \frac{1}{\sqrt{2}}, t^2\rangle \frac{1}{\sqrt{2}} - \langle \frac{\sqrt{3}}{\sqrt{2}} t, t^2\rangle \frac{\sqrt{3}}{\sqrt{2}} t$

$\langle \frac{1}{\sqrt{2}}, t^2\rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} t^2 = \frac{1}{3\sqrt{2}} t^3 \Big|_{-1}^{1} = \frac{1}{3\sqrt{2}} (1 - -1) = \frac{2}{3\sqrt{2}}$

$\langle \frac{\sqrt{3}}{\sqrt{2}} t, t^2\rangle = \frac{\sqrt{3}}{\sqrt{2}} \int_{-1}^{1} t^3 dt = \frac{\sqrt{3}}{\sqrt{2}} \frac{1}{4} t^4 \Big|_{-1}^{1} = 0$

$w_3 = t^2 - \frac{2}{3\sqrt{2}} \left(\frac{1}{\sqrt{2}}\right) = t^2 - 2/3$

$\|w_3\|^2 = \langle t^2 - \frac{2}{3}, t^2 - \frac{2}{3}\rangle = \int_{-1}^{1} (t^2 - \frac{2}{3})^2 dt = \int_{-1}^{1} t^4 - \frac{4}{3} t^2 + \frac{4}{9}\, dt$

$= \frac{1}{5} t^5 - \frac{4}{9} t^3 + \frac{4}{9} t \Big|_{-1}^{1} = \left(\frac{1}{5} - \frac{4}{9} + \frac{4}{9}\right) - \left(-\frac{1}{5} + \frac{4}{9} - \frac{4}{9}\right) = \frac{2}{5}$

$\|w_3\| = \sqrt{\frac{2}{5}}$

$b_3 = \frac{\sqrt{5}}{\sqrt{2}} (t^2 - 2/3)$

$B = \text{span}\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, t, \frac{\sqrt{5}}{\sqrt{2}} (t^2 - 2/3) \right\}$

b) Find the best approximation for t^3 in S

$$y^{opt} = \sum_{k=1}^{3} \alpha_k b_k \qquad \alpha_k = \langle b_k, t^3 \rangle$$

$$\left\langle \frac{1}{\sqrt{2}}, t^3 \right\rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} t^3 \, dt = \frac{1}{4\sqrt{2}} t^4 \Big|_{-1}^{1} = 0$$

$$\left\langle \frac{\sqrt{3}}{\sqrt{2}} t, t^3 \right\rangle = \int_{-1}^{1} \frac{\sqrt{3}}{\sqrt{2}} t^4 \, dt = \frac{1}{5} \frac{\sqrt{3}}{\sqrt{2}} t^5 \Big|_{-1}^{1} = \frac{\sqrt{3}}{5\sqrt{2}} (1 - -1) = \frac{2\sqrt{3}}{5\sqrt{2}}$$

$$\left\langle \frac{\sqrt{5}}{\sqrt{2}} (t^2 - 2/3), t^3 \right\rangle = \int_{-1}^{1} \frac{\sqrt{5}}{\sqrt{2}} \left(t^2 - \frac{2}{3}\right) t^3 \, dt = \frac{\sqrt{5}}{\sqrt{2}} \left(\frac{1}{6} t^6 - \frac{2}{12} t^4 \right) \Big|_{-1}^{1}$$

$$= \frac{\sqrt{5}}{\sqrt{2}} \left[\left(\frac{1}{6} - \frac{2}{12} \right) - \left(\frac{1}{6} - \frac{2}{12} \right) \right] = 0$$

$$y^{opt} = \alpha_2 b_2 = \frac{2\sqrt{3}}{5\sqrt{2}} \left(\frac{\sqrt{3}}{\sqrt{2}} t \right) = \frac{\cancel{2}(3)}{5(\cancel{2})} t = \frac{3}{5} t$$

Error $\quad e = t^3 - \frac{3}{5} t$

$$\| t^3 - \tfrac{3}{5} t \|^2 = \langle t^3 - \tfrac{3}{5} t, t^3 - 3/5 \, t \rangle = \int_{-1}^{1} t^6 - \frac{6}{5} t^4 + \frac{9}{25} t^2 \, dt$$

$$= \frac{1}{7} t^7 - \frac{6}{25} t^5 + \frac{3}{25} t^3 \Big|_{-1}^{1} = \left(\frac{1}{7} - \frac{6}{25} + \frac{3}{25} \right) - \left(-\frac{1}{7} + \frac{6}{25} - \frac{3}{25} \right)$$

$$= \frac{2}{7} - \frac{12}{25} + \frac{6}{25} = \frac{2}{7} - \frac{6}{25} = \frac{50}{175} - \frac{42}{175} = \frac{8}{175}$$

$$\| e \| = (8/175)^{1/2} = \frac{2\sqrt{2}}{5\sqrt{7}}$$

5. Consider $\mathbb{V} = \mathbb{R}^4$

$S = \text{Span}\{[1\ \ 0\ \ 2\ \ 0]^T, [0\ \ 1\ \ 0\ \ -1]^T, [1\ \ 0\ \ 2\ \ 1]^T\}$

a) $b_1 = \frac{1}{\sqrt{5}}[1\ \ 0\ \ 2\ \ 0]^T = [\,1/\sqrt{5}\ \ 0\ \ 2/\sqrt{5}\ \ 0\,]^T$

$$w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \left\langle \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$b_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}^T = [\,0\ \ 1/\sqrt{2}\ \ 0\ \ -1/\sqrt{2}\,]^T$$

$$w_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} - \left\langle \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\rangle \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \\ 0 \end{bmatrix} - \left\langle \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\rangle \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} - \frac{5}{\sqrt{5}} \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ +1/2 \end{bmatrix}$$

$$b_3 = \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ 0 \\ +\sqrt{2}/2 \end{bmatrix}$$

$$B = \text{Span}\left\{ \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ 0 \\ +\sqrt{2}/2 \end{bmatrix} \right\}$$

b) Find the best approximation for $[-2 \quad 0 \quad 10 \quad 0]^T$

$$y^{opt} = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3$$

$$\alpha_1 = \frac{\langle y, b_1 \rangle}{\|b_1\|^2} = [-2 \quad 0 \quad 10 \quad 0]\begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \\ 0 \end{bmatrix} = \frac{18}{\sqrt{5}}$$

$$\alpha_2 = \frac{\langle y, b_2 \rangle}{\|b_2\|^2} = [-2 \quad 0 \quad 10 \quad 0]\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = 0$$

$$\alpha_3 = \frac{\langle y, b_3 \rangle}{\|b_3\|^2} = [-2 \quad 0 \quad 10 \quad 0]\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = 0$$

$$y^{opt} = \frac{18}{\sqrt{5}}\left[\tfrac{1}{\sqrt{5}} \quad 0 \quad \tfrac{2}{\sqrt{5}} \quad 0\right]^T = \left[\tfrac{18}{5} \quad 0 \quad \tfrac{36}{5} \quad 0\right]$$

$$E = y - y^{opt} = \left[-\tfrac{28}{5} \quad 0 \quad \tfrac{14}{5} \quad 0\right]^T$$

$$\|E\| = \left[\left(\tfrac{28}{5}\right)^2 + \left(\tfrac{14}{5}\right)^2\right]^{\frac{1}{2}} = \frac{1}{5}\left(28^2 + 14^2\right)^{\frac{1}{2}}$$