A05812168
MAE 280A
10-22-08

1 Given $\dot{\underline{x}} = \underline{\underline{A}}\underline{x}$ $A = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$

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a) Find the spectral decomposition of $\underline{\underline{A}}$

$$|A - \lambda I| = (3-\lambda)(-1-\lambda) + 4 = 0 \qquad \lambda^2 - 2\lambda + 1 = 0$$

$$\lambda_i = 1, 1$$

$$\begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix} \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} \qquad \begin{matrix} 3s_1 + s_2 = s_1 \\ -4s_1 - s_2 = s_2 \end{matrix} \qquad \begin{matrix} s_2 = -2s_1 \\ -4s_1 = -2s_2 \end{matrix}$$

$$e_1 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} \qquad e_2 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \qquad |E| = 0$$

A is defective $\Rightarrow$ 1 linearly independent eigenvector

$$\Rightarrow \quad \Lambda = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A g_2 = \lambda_1 g_2 + e_1 \qquad \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix} \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix} = \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix} + \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$$

$$3g_1 + g_2 = g_1 + 1 \qquad g_2 = -2g_1 + 1$$

$$g = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$= E = \begin{bmatrix} e_1 & g \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \qquad |E| = -1 - (-2) = 1$$

$$E^{-1} = \frac{1}{-1-(-2)} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$A = E \Lambda E^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}$$

$$E = \begin{bmatrix} e_1 & g \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \checkmark \qquad E^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \checkmark$$

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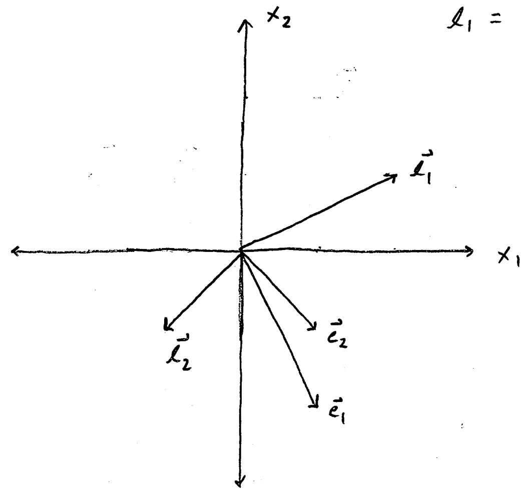
$$\Lambda = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \checkmark$$

b) Plot all the eigenvectors - left and right

$\vec{e}_1 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$ ✓ $\vec{e}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ ✓ $l_1^T = [-2 \quad 1]$ $l_2^T = [-1 \quad -1]$

$l_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ✓ $l_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ ✓



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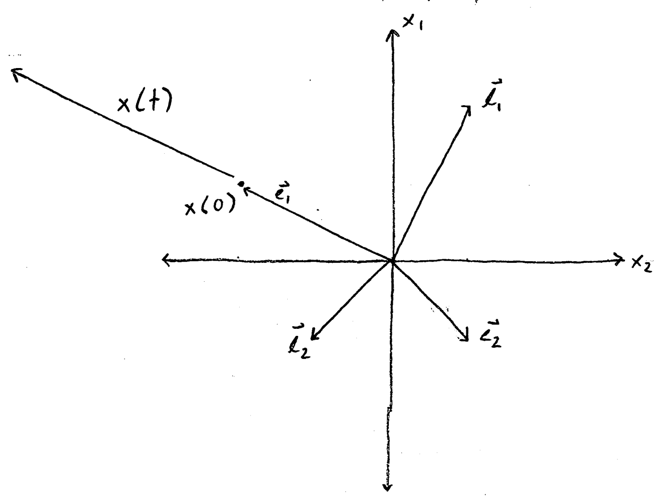
c) Plot in the x_2, x_1 plane $x(t)$ if $x(0) = [1 \quad -2]^T$



As $x(0)$ resides on $\alpha \vec{e}_1$ where α is a scalar, $x(t)$ remains on the $\vec{e}_1$ direction ✓

As $\lambda_1 = 1, \ \lambda_2 = 1,$

$x(t) = E e^{\Lambda t} E^{-1} x(0)$

$= E \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} E^{-1} x(0)$

and this grows to ∞

$x = e^{At} x(0) = e^t \begin{bmatrix} (1+t) & 0 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ close

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d) Compute e^{At}

$$e^{At} = I_2 + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots$$

$$= I_2 + E\Lambda E^{-1}t + \frac{1}{2}E\Lambda \overbrace{E^{-1}E}^{I}\Lambda E^{-1}t^2 + \ldots$$

$$= E(I_2 + \Lambda t + \frac{1}{2}\Lambda^2t^2 + \ldots)E^{-1} = Ee^{\Lambda t}E^{-1}$$

$$\Lambda = J + K \qquad J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad K = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^{At} = Ee^{(J+K)t}E^{-1} = Ee^{Jt}e^{Kt}E^{-1}$$ good

$$e^{Jt} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \qquad e^{Kt} = I_2 + Kt + K^2t^2 = I_2 + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \qquad K^2 = 0$$

$$e^{Jt}e^{Kt} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} = e^t\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$Ee^{Jt}e^{Kt}E^{-1} = e^t\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 2t-1 & t-1 \\ 2 & 1 \end{bmatrix}$$

$$= e^t\begin{bmatrix} 2t+1 & t \\ -4t & 1-2t \end{bmatrix}$$

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2 Find the SVD of A

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 4 \end{bmatrix} \qquad A_{2\times3} = U_{2\times2}\,\Sigma_{2\times3}\,V^T_{3\times3}$$

$$AA^T = \begin{bmatrix} 6 & 10 \\ 10 & 24 \end{bmatrix} \qquad |A-\lambda I| = \lambda^2 - 30\lambda + 44$$

$$= \lambda_i = \tfrac{1}{2}(30 \pm \sqrt{724}) \longrightarrow 15 + \frac{\sqrt{724}}{\cancel{4}2}$$

$$\Sigma = \begin{bmatrix} \Sigma_0 & 0_{2\times1} \end{bmatrix} = \begin{bmatrix} \sqrt{\tfrac{1}{2}(30+\sqrt{724})} & 0 & 0 \\ 0 & \sqrt{\tfrac{1}{2}(30-\sqrt{724})} & 0 \end{bmatrix}$$

$$AA^T u_i = \lambda_i u_i$$

$$\begin{bmatrix} 6 & 10 \\ 10 & 24 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tfrac{1}{2}(30+\sqrt{724}) \qquad 6u_1 + 10u_2 = \tfrac{1}{2}(30+\sqrt{724})u_1$$

$$10u_2 = \tfrac{1}{2}(18+\sqrt{724})u_1 \qquad u_2 = \tfrac{1}{20}(18+\sqrt{724})u_1 = 2.245362405_2$$

$$6u_1 + 10u_2 = \tfrac{1}{2}(30-\sqrt{724})u_1 \qquad 10u_2 = \tfrac{1}{2}(18-\sqrt{724})u_1$$

$$u_2 = \tfrac{1}{20}(18-\sqrt{724})u_1 \qquad u_2 = -0.4453624047$$

$$U = \begin{bmatrix} 1 & 1 \\ \tfrac{1}{20}(18+\sqrt{724}) & \tfrac{1}{20}(18-\sqrt{724}) \end{bmatrix} \Rightarrow \begin{bmatrix} 0.406838585 & 0.91350006 \\ 0.91350006 & -0.406838585 \end{bmatrix} \checkmark$$

$$A^TA = \begin{bmatrix} 5 & 6 & 9 \\ 6 & 8 & 10 \\ 9 & 10 & 17 \end{bmatrix} \qquad |A^TA - \lambda I| = (5-\lambda)[(8-\lambda)(17-\lambda) - 100] + 6(12) + 9(60 - 9(8-\lambda))$$

$$= (5-\lambda)(\lambda^2 - 25\lambda + 36) + 72 + 540 - 72 - 9\lambda$$

$$= 5\lambda^2 - 125\lambda + 180 - \lambda^3 + 25\lambda^2 - 36\lambda + \cancel{72} + 540 - \cancel{72} - 9\lambda$$

$$\lambda^3 + 30\lambda^2 - 44\lambda \Rightarrow \lambda_i = 0,\ \tfrac{1}{2}(30 \pm \sqrt{724})$$

$A^T A V_i = \lambda_i V_i$

$$\begin{bmatrix} 5 & 6 & 9 \\ 6 & 8 & 10 \\ 9 & 10 & 17 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 5 & 6 & 9 \\ 6 & 8 & 10 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \Rightarrow \begin{bmatrix} 5 & 6 & 9 \\ 6 & 0 & 18 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = 0 \qquad \begin{matrix} 6v_1 = -18v_3 \\ v_2 = v_3 \end{matrix}$$

$v_1 = 1 \Rightarrow v_3 = -\frac{1}{3} \qquad v_2 = -\frac{1}{3} \Rightarrow \begin{bmatrix} 0.9045340337 \\ -0.3015113446 \\ -0.3015113446 \end{bmatrix}$ ✓

$\lambda_1 = \frac{1}{2}(30 + \sqrt{724})$

$5v_1 + 6v_2 + 9v_3 = \frac{1}{2}(30 + \sqrt{724})v_1$

$6v_1 + 8v_2 + 10v_3 = \frac{1}{2}(30 + \sqrt{724})v_2$

$9v_1 + 10v_2 + 17v_3 = \frac{1}{2}(30 + \sqrt{724})v_3$

$(5-\lambda)v_1 + 6v_2 + 9v_3 = 0$

$v_3 = -9^{-1}((5-\lambda)v_1 + 6v_2)$

$6v_1 + 8v_2 - \frac{10}{9}((5-\lambda)v_1 + 6v_2) = \lambda v_2$

$6v_1 - \frac{10}{9}(5-\lambda)v_1 = (\lambda + \frac{10}{9}6 - 8)v_2$

set $v_1 = 1 \Rightarrow v_2 = 1.182125318 \Rightarrow v_3 = 1.817874082$

Normalize $\Rightarrow \begin{bmatrix} 0.4187771899 \\ 0.4950471188 \\ 0.7612844511 \end{bmatrix}$ ✓

$\lambda_2 = \frac{1}{2}(30 - \sqrt{724})$ using same relations as above

set $v_1 = 1 \Rightarrow v_2 = 10.15120802 \Rightarrow v_3 = -7.151208019$

Normalize $\Rightarrow \begin{bmatrix} 0.080273576 \\ 0.814873769 \\ -0.5740530408 \end{bmatrix}$ ✓

$$V = \begin{bmatrix} 0.4187771899 & -0.080273576 & 0.9045340337 \\ 0.4950471188 & 0.814873769 & -0.3015113446 \\ 0.7612844511 & -0.5740530408 & -0.3015113446 \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$U\Sigma = \begin{bmatrix} 0.406838585 & 0.91350006 \\ 0.91350006 & -0.406838585 \end{bmatrix} \begin{bmatrix} 5.354193852 & 0 & 0 \\ 0 & 1.24353365 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2.170155879 & 1.135968069 & 0 \\ 4.8727864404 & -0.5059174729 & 0 \end{bmatrix}$$

good work

$$U\Sigma V^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 4 \end{bmatrix}$$ rounding off for precision

sure

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5 Find the value of c such that the constant vectors

$x = [1 \quad 2 \quad 1 \quad 2]^T$, $y = [1 \quad -1 \quad 1 \quad c]^T$ are orthogonal

Def 2.7 x, y are orthogonal if $\langle x, y \rangle = 0$

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [1 \quad 2 \quad 1 \quad 2][1 \quad -1 \quad 1 \quad c]^T dt$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} 1 - 2 + 1 + c \, dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} c \, dt = \frac{ct}{t_2 - t_1} \Big|_{t_1}^{t_2}$$

$= c = 0$ if x, y are orthogonal

Since x, y are constant,

$$\langle x, y \rangle = x \cdot y = x^T y = [1 \quad 2 \quad 1 \quad 2] \begin{bmatrix} 1 \\ -1 \\ 1 \\ c \end{bmatrix} = c = 0$$

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