Project-Perihelion of Mercury

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This project investigates the effect of General Relativity on the orbit of Mercury. Due to its proximity to the Sun, the effect of General Relativity is most significant in case of Mercury. Consider a particle moving under the gravitational influence of a spherical massive object of mass M. The Newtonian expression for acceleration of the particle is

$$\frac{d^2\vec{r}}{dt^2} = -\frac{GM}{r^2}\hat{r}$$

The effect of General relativity effectively modifies this expression for acceleration to

$$\frac{d^2\vec{r}}{dt^2} = -\frac{GM}{r^2}\left(1 + \frac{3l^2}{c^2r^2}\right)\hat{r}$$

where l is the (conserved) angular momentum per unit mass of the particle, $l = |\vec{r} \times \vec{v}|$ and c is the speed of light, $c = 3 \times 10^8$ m/s. The modification is as if the particle, in addition to the standard gravitational inverse square force, experiences a central, attractive force (also proportional to its mass), which goes as the inverse fourth power of distance.

1. To understand the effect of this correction, first simulate the orbit of Mercury in the absence of this correction, based on experimental parameters. First, show that the speed of a planet at perihelion is given by

   $$v_p = \sqrt{\frac{GM_s}{r_p}} \times \sqrt{1+\epsilon}$$

   where M_s is the mass of the Sun, r_p is the perihelion distance from the Sun and ϵ is the eccentricity of the orbit. Next, express lengths in units of the perihelion distnce r_p and time in units of the time period T_0 of an equivalent *circular* orbit of radius r_p around the Sun (the actual orbit is, of course, not circular). Write down the expression for the (dimensionless) acceleration in terms of the dimensionless coordinates. If the orbit of Mercury was circular, what would be the (dimensionless) orbital speed numerically? To set the inital conditions, start with Mercury at perihelion. Given the eccentricity (look up data), what will be the numerical speed at this point? Using the leapfrog algorithm, compute and plot the orbit by running the code for large values of dimensionless time (a few thousand). Check for conservation of both energy and angular momentum by plotting them as functions of time.

2. Next, include the correction term. Write the correction term in terms of the dimensionless distance from the Sun and the ratio $(v_p/c)^2$. Compute the numerical value of this ratio by looking up the value of v_p from data. Run the algorithm again for dimensionless time of about 20,000 (or more) and store the values of coordinates as lists (as usual). However, *do not* plot the entire list. Plot the parts of the orbit for the first few and the last few periods only, on the same plot. In this part of the exercise, do not worry about checking for energy and angular momentum conservation. Given that the correction term shifts the perihelion of Mercury at a constant rate, from the plot, estimate by how many degrees it shifts in a 1000 Earth years.