ROB 541 Assignment 1: Groups

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1. Create a software structure for handling tangent vectors. Your vector class should have the attributes
   - An ndarray with the components of the vector
   - The configuration at which it is tangent to the configuration space (an ndarray, or, if you have a class for manifold elements, an instance of this class).
   - Methods for handling addition, scalar multiplication, and matrix multiplication of vectors.

2. Make a function for generating the derivative in the direction of a function at a given point. This function should take as input:
   - a function f whose own inputs are
     - a configuration value
     - a δ "scaling" value

     and returns another configuration value (which is the same as the input configuration if δ is zero).
   - a configuration at which to evaluate the derivative

   and return the numerical derivative of f with respect to δ at the specified point, stored as an element of your vector class, located at the configuration at which f is evaluated.

   Deliverables:
   - Replicate the Cartesian-expressed polar vector fields in Fig. 2.7 and Example 2.3.1 using your direction-derivative code.
   - Demonstrate that you can equivalently create the vector fields by using the numerical Jacobian of a function F to map a single generating vector out to points on the field.

3. Make a "group tangent vector" class that stores the location of the vector as a group element instead of a simple list of coordinate values or manifold element.

   Deliverables: Replicate the groupwise vector basis fields in Fig. 2.10 b and c, using your direction-derivative function with the group actions of elements that are δ away from the identity in single components.

4. Extend your group element class so that it has left and right lifted actions that act on vectors. These lifted actions should be generated automatically from the group action using numerical differentiation tools, and their outputs should return group tangent vectors anchored at the new location. Also provide an adjoint action (named `Ad`) on vectors, which combines the left and right lifted actions.

   Deliverables:

   - Replicate the groupwise vector basis fields in Fig. 2.10 b and c, using lifted actions to transfer unit-coefficient vectors at the identity out to each point.
   - Find the scale-shift vector adjoint to $\overset{\circ\rightarrow}{g} = \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix}^T$ at $g = (\frac{1}{2}, -1)$, and make a set of plots similar to Fig. 2.11 demonstrating this adjoint relationship.

5. Make a class for vectors tangent to matrix-represented groups. Its primary data storage for the vector information should be its matrix form. The constructor should accept this matrix directly, but should also accept a list of parameters and use the direction-derivatives of the group **representation function** to construct a representation basis, and then use this basis to construct the vector's representation matrix. Similarly, there should be a "getter" method for de-representing the matrix into a list of coefficients (see equation 2.172 for this algorithm).

6. Extend your representation group element class so that its lifted actions directly multiply the group representation by the vector representation.

   Deliverables: Use the representation versions of your group element and tangent vector classes as drop-in replacements for the groupwise basis field and adjoint element plots you created in the previous exercises.

7. Instantiate the matrix-representation versions of your group, group element, and group tangent vector classes.

   Deliverables:

   - Make a figure similar to Fig. 2.22(b), illustrating the motion of a set of objects that all have body velocity $\overset{\circ\rightarrow}{g} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$ at a similar set of points as in the original figure.
   - Make a figure similar to Fig. 2.23(b), illustrating the motion of a set of objects that all have spatial velocity $\overset{\leftarrow\circ}{g} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ at a similar set of points as in the original figure.
   - For rigidly attached group elements $g_1 = (0, 1, -\frac{\pi}{4})$ and $g_2 = (1, 2, -\frac{\pi}{2})$, find the velocity of g_2 if the velocity of g_1 is $\overset{\circ\rightarrow}{g} = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$. Illustrate the two frames, an object they are both attached to (in the style of Fig. 2.24), and the velocity vectors of the two frames.