

Queen's University Belfast

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BSc (Hons) Final Year Project Report

GPU-Accelerated Solution for the Black-Scholes equation

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Abstract

abstract

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Chapter 1

Problem statement

Chapter 2

Derivatives and Option Pricing: Literature Review

2.1 Introduction to Derivatives and Option Pricing

Over the years, derivatives trading has emerged as a fundamental component of modern finance, evolving into a widespread practice across global financial markets. This shift in market dynamics introduces a challenge in determining the fair value (price) of derivative products, and in turn, ensure a balanced outcome for both the buyer and the seller.

A *financial derivative* is primarily known as a financial instrument whose value depends on the values of more basic underlying variables [1]. These derivatives come in many forms, with examples such as *futures contracts*, *options*, and *swaps*, each typically derived from the prices of the traded assets. An *option* is a derivative security which gives the holder the right (but not the obligation) to purchase or sell an underlying asset at a certain price by a predetermined time. The holder may decide to buy or sell the purchased option based on whether the conditions are favourable to them; if not, they may choose to let the option expire worthless. Knowing this definition of an option, it can be divided into two different types: *call* or *put* options. A simple *call* option is a contract that gives the holder the right to buy an asset at an agreed price by a specified time. A simple *put* option, on the other hand, is a contract that gives the holder the right to sell an asset at an agreed price by a specified time. The price at which the option can be exercised is known as the *exercise* or *strike* price, and the date on which the option needs to be exercised by is the *expiration* date or *maturity*. Typically, an option is exercised only at its expiration time—these are known as *European* options. Otherwise, it is considered an *American* option which allows the holder to exercise the option prior to its expiration.

Example of a Call Option

Suppose a European call option for 100 shares of NVIDIA stock is created with a strike price of \$100 and an expiration date in one month. The price of an option for each individual share of the stock is \$8, making the initial investment a total of \$800. If, in one month's time, the stock falls below the strike price (\$100), it would be unfavourable for the holder to exercise the option, as it would then be worthless. The holder then loses their initial investment of \$800. In the favourable case where the stock rises to a price of \$120 at expiry, the option will

be exercised. Here, the holder buys 100 shares of the stock at the agreed strike price (\$100). Selling their stocks immediately would lead to a gain of \$2000, and a net profit of \$1200 (taking into account the initial investment).

Example of a Put Option

A put option has the reverse effect of a call option. Consider a situation where a European put option is purchased with a strike price of \$30 to sell 100 shares of Intel stock. The option is set to expire in 2 months and the price of an option per share is \$5. This makes an initial investment of \$500. Suppose that the price of the stock falls to a value of \$18 at expiry, the holder chooses to exercise the option and purchases 100 shares of Intel stock at \$18. The holder then sells the shares at the agreed strike price, realizing a profit gain of \$1200 and a net profit of \$700. However, if the current market price rises above the strike price at expiry, the holder generally does not exercise the option, suffering a loss of \$500.

2.1.1 The Black-Scholes model

Chapter 3

From the heat equation to Black-Scholes: Theory and Algorithms

3.1 Partial Differential Equations (PDE)

A partial differential equation (PDE) is defined as a differential equation that relates the derivatives of a function such that it depends on more than one independent variable.^[2] This project will look at two significant classes of PDEs, namely the heat equation and the Black-Scholes equation.

3.1.1 Heat Equation

The heat equation describes how temperature diffuses through a medium over a period of time. Let $u(x, \tau)$ be the temperature of a medium at a specific point x and time τ . We denote the derivatives of u with respect to $\tau \geq 0$ and $x \in \mathbb{R}$ by u_τ and u_x , respectively. The heat equation ^[3] states that

$$u_\tau = \kappa u_{xx}, \quad (3.1)$$

where κ is the thermal diffusivity constant. The initial and boundary conditions must be defined to obtain appropriate solutions to the heat equation.

Initial and Boundary conditions over a finite interval

For the heat equation, the initial condition is specified by

$$u(x, 0) = u_0(x), \quad (3.2)$$

which describes the initial temperature across the length of the medium at time $t = 0$.

Assuming that the temperature distribution falls within a finite domain, the *Dirichlet* boundary conditions are applied at the extreme ends of the domain ($0 \leq x \leq L$), that is,

$$u(0, t) = \alpha(t), \quad (3.3a)$$

$$u(L, t) = \beta(t). \quad (3.3b)$$

For the initial and boundary conditions to be well posed, they need to agree at $t = 0$, $x = 0$, and $x = L$

$$\begin{aligned}\alpha(0) &= u_0(0), \\ \beta(0) &= u_0(L).\end{aligned}$$

3.1.2 Black-Scholes Equation

The Black-Scholes (BS) equation is the PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad (3.4)$$

where $V(S, t)$ is the price of the call/put option at time t given the price of the underlying asset, S , t is the time elapsed since the creation of the option, r is the risk-free interest rate, and σ is the volatility of the underlying. Based on the type of option, the BS equation is accompanied by certain terminal and boundary conditions.

European Call Option

For European call options, we use the lower boundary condition (at $S = 0$)

$$V(0, t) = 0, \quad (3.5)$$

which means that the value of option for a zero-value underlying is zero. An “upper” boundary condition is also imposed (at $S \rightarrow \infty$). Practically, this condition can be imposed at a sufficiently large value S_{\max} . A typical choice [4] is

$$V(S_{\max}, t) = S_{\max} - Ke^{-rT}. \quad (3.6)$$

The term Ke^{-rT} represents the present value of the strike price discounted by the risk-free rate r over the time to expiry T . Lastly, we impose the terminal condition (at $t = T$), which is

$$V(S_T, T) = \max(S - K, 0) \quad (3.7)$$

for all S . The option payoff $V(S, T)$ is specified at expiration T , where S represents the current price of the asset and K denotes the strike price.

European Put Option

The case of European put options follows a different set of terminal and boundary conditions. Given the underlying asset value at zero, the payoff becomes approximately K when the option is exercised. Hence, we have the lower boundary condition

$$V(0, t) = Ke^{-rT}. \quad (3.8)$$

On one hand, as the underlying asset S becomes very large, the put option becomes inherently worthless which gives the upper boundary condition ($S \rightarrow \infty$)

$$V(S, t) \rightarrow 0. \quad (3.9)$$

The terminal condition for the put option is set to

$$V(S, T) = \max(K - S, 0) \quad (3.10)$$

3.1.3 Equivalence between the heat equation and the Black-Scholes equation

(add ref to wilmott mathematics of fin derivatives)

Following a change in variables, the Black-Scholes equation can be transformed into the one-dimensional heat equation. Recall that the Black-Scholes equation takes the form of Equation (3.4). Here, the following substitutions can be applied to remove the S term from the equation.

$$t = T - \frac{2}{\sigma^2} \tau \quad (3.11a)$$

$$S = e^x \quad (3.11b)$$

The derivatives can now be rewritten as follows

$$V_t = \frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau} \frac{d\tau}{dt} = -\frac{\sigma^2}{2} \frac{d\tau}{dt} = -\frac{1}{2} \sigma^2 V_\tau \quad (3.12)$$

$$V_S = \frac{\partial V}{\partial S} = \frac{\partial V}{\partial x} \frac{dx}{dS} = \frac{1}{S} \frac{\partial V}{\partial x} = \frac{1}{S} V_x \quad (3.13)$$

$$\begin{aligned} V_{SS} &= \frac{\partial^2 V}{\partial S^2} = \frac{\partial V_S}{\partial S} \\ &= \frac{\partial}{\partial S} \left(\frac{1}{S} \frac{\partial V}{\partial x} \right) \\ &= -\frac{1}{S^2} \frac{\partial V}{\partial x} + \frac{1}{S} \frac{\partial V}{\partial x} \frac{\partial V}{\partial x} \frac{dx}{dS} \\ &= -\frac{1}{S^2} V_x + \frac{1}{S} V_{xx} \frac{1}{S} \\ &= -\frac{1}{S^2} V_x + \frac{1}{S^2} V_{xx}, \end{aligned} \quad (3.14)$$

which leads to the following form of the equation

$$\begin{aligned} V_t + \frac{1}{2} \sigma^2 S^2 V_{SS} + r S V_S - r V &= 0 \\ \Rightarrow -\frac{1}{2} \sigma^2 V_\tau + \frac{1}{2} \sigma^2 (V_{xx} - V_x) + r V_x - r V &= 0 \\ \Rightarrow -\frac{1}{2} \sigma^2 V_\tau + \frac{1}{2} \sigma^2 V_{xx} - \frac{1}{2} \sigma^2 V_x + r V_x - r V &= 0 \\ \Rightarrow -\frac{1}{2} \sigma^2 V_\tau + \frac{1}{2} \sigma^2 V_{xx} + (r - \frac{1}{2} \sigma^2) V_x - r V &= 0, \end{aligned} \quad (3.15)$$

where it can be further simplified by dividing by $-\frac{1}{2} \sigma^2$ to yield

$$V_\tau - V_{xx} + (1 - \lambda) V_x + \lambda V = 0, \quad (3.16)$$

where $\lambda = \frac{2r}{\sigma^2}$. Now, the transformation

$$V(x, \tau) = e^{ax+b\tau} u(x, \tau) \quad (3.17)$$

is introduced to eliminate the V term and its corresponding derivatives and thus convert the equation to the heat equation. The partial derivatives of V in terms of u are

$$V_x = a e^{ax+b\tau} u(x, \tau) + e^{ax+b\tau} u_x(x, \tau) = e^{ax+b\tau} (au + u_x) \quad (3.18)$$

$$V_\tau = b e^{ax+b\tau} u(x, \tau) + e^{ax+b\tau} u_\tau = e^{ax+b\tau} (bu + u_\tau) \quad (3.19)$$

$$V_{xx} = a e^{ax+b\tau} (au + u_x) + e^{ax+b\tau} (au_x + u_{xx}) = e^{ax+b\tau} (u_{xx} + 2au_x + a^2 u) \quad (3.20)$$

Substituting equations (3.18), (3.20), (3.19) into Equation (3.16), using $k = e^{ax+b\tau}$, yields the following:

$$\begin{aligned} k(bu + u_x) - k(u_{xx} + 2au_x + a^2u) + (1 - \lambda)k(au + u_x) + \lambda k u &= 0 \\ bu + u_x - u_{xx} - 2au_x - a^2u + (1 - \lambda)au + (1 - \lambda)u_x + \lambda u &= 0 \end{aligned} \quad (3.21)$$

which gives

$$(b - a^2 + a(1 - \lambda) + \lambda)u + u_\tau - u_{xx} - (2a - (1 - \lambda))u_x = 0 \quad (3.22)$$

Note that the heat equation in (3.1) consists only of u_τ and u_{xx} . To fully transform Equation (3.22) to the heat equation, the terms u_x and u need to be eliminated by setting their coefficients to 0, which can be achieved by the substitutions

$$a = \frac{1 - \lambda}{2}, \quad (3.23)$$

$$b = a^2 - a(1 - \lambda) - \lambda = -\frac{(\lambda + 1)^2}{4} \quad (3.24)$$

Ultimately, leading to the heat equation,

$$u_\tau = u_{xx}. \quad (3.25)$$

3.2 Numerical Methods

The following numerical methods have been used to obtain a solution for the Black-Scholes equation.

3.2.1 Explicit Scheme

Heat equation

The explicit scheme begins with a discretization of the time and space domains of the PDE. For the heat equation, the domain is defined by time $t \in [0, T]$ and space $x \in [0, L]$. The space domain $x \in [0, L]$ is divided into M equal-sized space steps, each of size $\Delta x = \frac{L}{M}$. Similarly, the time interval $t \in [0, T]$ is divided into N equal-sized time steps, each of size $\Delta t = \frac{T}{N}$.

On a discrete grid, the grid points are denoted by

$$x_i = i\Delta x, \quad t_j = j\Delta t,$$

where

$$0 \leq i \leq M \quad \text{and} \quad 0 \leq j \leq N.$$

The explicit scheme follows a forward stepping approach where the solution at the next time step is calculated directly from the known values of the current time step. Knowing the temperature u_i^j at a particular time step, the grid can be leveraged to compute an approximation of the temperature u_i^{j+1} at the subsequent time step. For simplicity, the diffusivity constant κ will be set to 1.

The derivatives of the heat equation represented in (3.1) can then be approximated by the following finite difference approximations:

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{j+1} - u_i^j}{\Delta t} \quad (\text{Time Derivative})$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{1}{\Delta x^2} (u_{i+1}^j - 2u_i^j + u_{i-1}^j) \quad (\text{Spatial Second Derivative})$$

Using these approximations, we arrive at the expression

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} = \frac{1}{\Delta x^2} (u_{i+1}^j - 2u_i^j + u_{i-1}^j), \quad (3.26)$$

which can be rearranged for u_i^{j+1} to give

$$u_i^{j+1} = u_i^j + \frac{\Delta t}{\Delta x^2} (u_{i+1}^j - 2u_i^j + u_{i-1}^j). \quad (3.27)$$

The grid below shows how the temperature at the next time step is computed using the known values at the current time step.

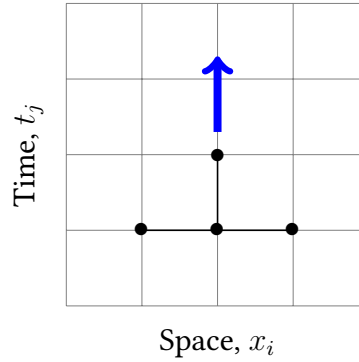


Figure 3.1: Explicit method for the heat equation

This is what is known as the explicit scheme.

Black-Scholes equation

Similar to the heat equation, the Black Scholes-equation can be discretized to a finite grid of points. For this instance, the domain is defined by time $t \in [0, T]$ (where T represents the expiration time of an option) and asset price $S \in [0, S_{max}]$ (where S_{max} is a sufficiently large upper limit for the underlying asset price). This can be represented by a mesh grid of points. The time interval $t \in [0, T]$ is divided into M equal-sized time steps, each of size $\Delta t = \frac{T}{M}$. Similarly, the asset price range $S \in [0, S_{max}]$ is divided into N equal-sized asset steps, each of size $\Delta S = \frac{S_{max}}{N}$.

The price of the option at the current time step will be denoted by V_i^k , where i refers to the index representing the asset price grid point and k refers to the index representing the time step.

From this, it can be said that the grid points are made up of:

$$S_i = i\Delta S, t_k = T - k\Delta t,$$

where

$$0 \leq i \leq N \quad \text{and} \quad 0 \leq k \leq M.$$

Like, in the previous section, the grid can be used to compute an approximation of the option price at the subsequent time step using known values from the current time step. The Black-Scholes equation represents a *backwards* parabolic equation which implies that the subsequent option prices are computed starting from the time at expiration ($t = T$).

For simplicity, the Black-Scholes equation can be rewritten as such:

$$\frac{\partial V}{\partial t} + a(S, t) \frac{\partial^2 V}{\partial S^2} + b(S, t) \frac{\partial V}{\partial S} - c(S, t)V = 0$$

Finite difference approximations will be used to substitute the derivatives of the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} \approx \frac{V_i^k - V_i^{k+1}}{\Delta t} \quad (\text{Theta})$$

$$\frac{\partial V}{\partial S} \approx \frac{V_{i+1}^k - V_{i-1}^k}{2\Delta S} \quad (\text{Delta})$$

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\Delta S^2} \quad (\text{Gamma})$$

Substituting these three approximations, (**Theta**), (**Delta**), and (**Gamma**) into the equation gives us the following:

$$\frac{V_i^k - V_i^{k+1}}{\Delta t} + a(S, t) \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{(\Delta S)^2} + b(S, t) \frac{V_{i+1}^k - V_{i-1}^k}{2\Delta S} - c(S, t)V_i^k = 0$$

The equation is then rearranged such that the $k + 1$ terms and k terms are separated:

$$V_i^{k+1} = A_i^k V_{i-1}^k + (1 - B_i^k) V_i^k + C_i^k V_{i+1}^k$$

What is O? where

$$A_i^k = a_i^k v_1 - \frac{1}{2} b_i^k v_2$$

$$B_i^k = -2a_i^k v_1 + c_i^k \Delta t$$

$$C_i^k = a_i^k v_1 + \frac{1}{2} b_i^k v_2$$

$$v_1 = \frac{\Delta t}{(\Delta S)^2}, \quad v_2 = \frac{\Delta t}{\Delta S}$$

The $O(\Delta t^2, \Delta t(\Delta S)^2)$ notation represents the local truncation error. **SO WHAT????!!!! What is the meaning of local?**

The grid below illustrates how the option price at the subsequent time step can be calculated using the neighbouring grid points.

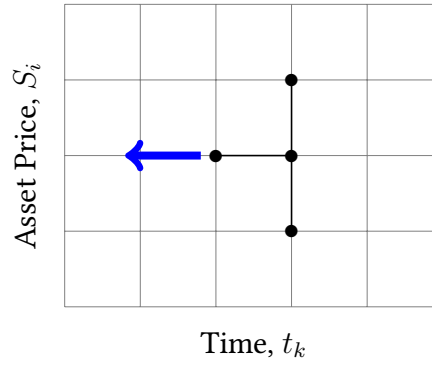


Figure 3.2: Explicit method for the Black-Scholes equation

3.2.2 Implicit Scheme: Crank-Nicolson Method

Despite its straightforward implementation, the explicit method tends to be less stable, as it only takes the current time steps into account for computation. The Crank-Nicolson method addresses this problem by using a combination of the current and subsequent time steps, making it unconditionally stable. It is not confined to the same stability constraints that the explicit method faces.

Heat equation

The Crank-Nicolson method involves the approximation of the derivatives at the midpoint of the current and subsequent time steps. By averaging the spatial second derivatives of both time steps, the following equation is obtained:

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} = \frac{1}{2(\Delta t)^2} (u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1} + u_{i+1}^j - 2u_i^j + u_{i-1}^j), \quad (3.28)$$

which can be rearranged to give

$$u_i^{j+1} = u_i^j + \frac{\Delta t}{2(\Delta x)^2} (u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1} + u_{i+1}^j - 2u_i^j + u_{i-1}^j). \quad (3.29)$$

The equation can be further rewritten to separate the $j + 1$ and j terms:

$$-ru_{i-1}^{j+1} + (1 + 2r)u_i^{j+1} - ru_{i+1}^{j+1} = ru_{i-1}^j + (1 - 2r)u_i^j + ru_{i+1}^j, \quad (3.30)$$

where $r = \frac{\Delta t}{2(\Delta x)^2}$. This leaves us with a tridiagonal system of equations that can be solved to obtain the temperature at the next time step u_i^{j+1} .

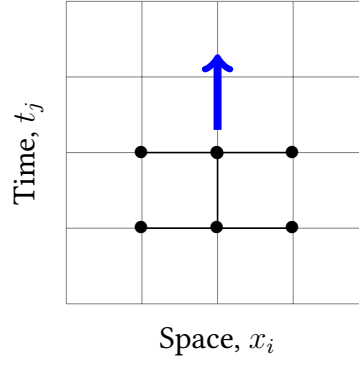


Figure 3.3: Crank-Nicolson method for the heat equation

Black-Scholes equation

Similarly, the Crank-Nicolson method can be applied to the Black-Scholes equation. The equation is rewritten as

$$\frac{V_i^{k+1} - V_i^k}{\Delta t} = \frac{1}{2} (\mathcal{L}(V_i^{k+1}) + \mathcal{L}(V_i^k)) \quad (3.31)$$

such that \mathcal{L} refers to discretized form of the operator that describes how the option value changes. Recall that the following approximations are used for discretization:

$$\frac{\partial V}{\partial t} \approx \frac{V_i^k - V_i^{k+1}}{\Delta t} \quad (\text{Theta})$$

$$\frac{\partial V}{\partial S} \approx \frac{V_{i+1}^k - V_{i-1}^k}{2\Delta S} \quad (\text{Delta})$$

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{(\Delta S)^2} \quad (\text{Gamma})$$

Combine the Explicit (forward) and Implicit (backward) schemes:

$$\begin{aligned} & \frac{V_i^n - V_i^{n+1}}{\Delta t} + \frac{a_i^{k+1}}{2} \left(\frac{V_{i+1}^{k+1} - 2V_i^{k+1} + V_{i-1}^{k+1}}{(\Delta S)^2} \right) + \frac{b_i^{k+1}}{2} \left(\frac{V_{i+1}^{k+1} - V_{i-1}^{k+1}}{2\Delta S} \right) + \frac{1}{2} c_i^{k+1} V_i^{k+1} \\ & + \frac{a_i^k}{2} \left(\frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{(\Delta S)^2} \right) + \frac{b_i^k}{2} \left(\frac{V_{i+1}^k - V_{i-1}^k}{2\Delta S} \right) + \frac{1}{2} c_i^k V_i^k = 0 \end{aligned}$$

Remove the denominators and rearrange the equation such that the $k+1$ and k terms are separated:

$$A_i^{k+1} V_{i-1}^{k+1} + (1 + B_i^{k+1}) V_i^{k+1} + C_i^{k+1} V_{i+1}^{k+1} = -A_i^k V_{i-1}^k + (1 - B_i^k) V_i^k + C_i^k V_{i+1}^k$$

where

$$\begin{aligned} A_i^k &= \frac{b_i^k v_2}{4} - \frac{a_i^k v_1}{2} \\ B_i^k &= a_i^k v_1 - \frac{c_i^k}{2} \Delta t \\ C_i^k &= -\frac{a_i^k v_1}{2} - \frac{b_i^k v_2}{4} \\ v_1 &= \frac{\Delta t}{(\Delta S)^2}, \quad v_2 = \frac{\Delta t}{\Delta S} \end{aligned}$$

This is a linear system of equations written in the matrix form of:

$$AV^{k+1} = BV^k$$

where A and B represent tridiagonal matrices, and V^{k+1} and V^k are vectors containing the option values at the next and current time steps. It is important to note that this is used to solve the interior grid points and only applies to $1 \leq i \leq N_s - 1$, as the boundary conditions address the remaining equations. N_s here refers to the total number of spatial grid points.

This system of equations can be solved to obtain the option values at the next time step V^{k+1} .

a few notes about the errors (local/global)?

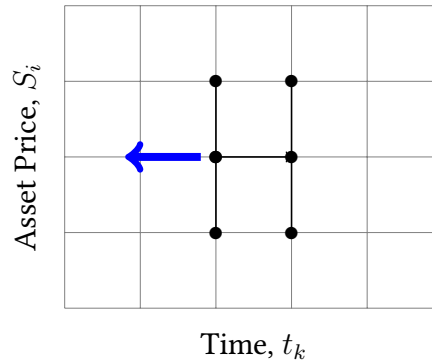


Figure 3.4: Crank-Nicolson method for the Black-Scholes equation

3.3 Stochastic differential equations

3.3.1 Brownian motion

3.3.2 Ito processes

3.3.3 Geometric Brownian motion

3.3.4 GBM for option pricing

Chapter 4

Software

Chapter 5

Simulations

Chapter 6

Conclusions

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