课堂练习

练习1：从 $\begin{cases}\hat{L}_x = -\mathrm{i}\hbar(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}) \\ \hat{L}_y = -\mathrm{i}\hbar(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}) \\ \hat{L}_z = -\mathrm{i}\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})\end{cases}$ **和** $\begin{cases}x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta\end{cases}$ **推导角动量算符在球坐标下的表达式**

解：将球坐标用直角坐标表示，则 $\begin{cases}\tan\varphi = \frac{y}{x} \\ \tan^2\theta = \frac{x^2+y^2}{z^2} \\ r^2 = x^2+y^2+z^2\end{cases}$ ，对x求导并化简得 $\begin{cases}\frac{\partial\varphi}{\partial x} = -\frac{y\cos^2\varphi}{x^2} \\ \frac{\partial\theta}{\partial x} = \frac{x\cos^2\theta}{z^2\tan\theta} \\ \frac{\partial r}{\partial x} = \frac{x}{r}\end{cases}$ ，对y求导并化简得 $\begin{cases}\frac{\partial\varphi}{\partial y} = \frac{\cos^2\varphi}{x} \\ \frac{\partial\theta}{\partial y} = \frac{y\cos^2\theta}{z^2\tan\theta} \\ \frac{\partial r}{\partial y} = \frac{y}{r}\end{cases}$ ，对z求导并化简得 $\begin{cases}\frac{\partial\varphi}{\partial z} = 0 \\ \frac{\partial\theta}{\partial z} = -\frac{(x^2+y^2)\cos^2\theta}{z^3\tan\theta} \\ \frac{\partial r}{\partial z} = \frac{z}{r}\end{cases}$ ，因此有：

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial x} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial}{\partial r}\frac{\partial r}{\partial x} = (-\frac{y\cos^2\varphi}{x^2})\frac{\partial}{\partial\varphi} + (\frac{x\cos^2\theta}{z^2\tan\theta})\frac{\partial}{\partial\theta} + (\frac{x}{r})\frac{\partial}{\partial r}$$
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial y} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial y} + \frac{\partial}{\partial r}\frac{\partial r}{\partial y} = (\frac{\cos^2\varphi}{x})\frac{\partial}{\partial\varphi} + (\frac{y\cos^2\theta}{z^2\tan\theta})\frac{\partial}{\partial\theta} + (\frac{y}{r})\frac{\partial}{\partial r}$$
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial z} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial z} + \frac{\partial}{\partial r}\frac{\partial r}{\partial z} = [-\frac{(x^2+y^2)\cos^2\theta}{z^3\tan\theta}]\frac{\partial}{\partial\theta} + (\frac{z}{r})\frac{\partial}{\partial r}$$

从而得到：

$$\begin{aligned}\hat{L}_x &= -\mathrm{i}\hbar(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}) = -\mathrm{i}\hbar y\{[-\frac{(x^2+y^2)\cos^2\theta}{z^3\tan\theta}]\frac{\partial}{\partial\theta} + (\frac{z}{r})\frac{\partial}{\partial r}\} + \mathrm{i}\hbar z[(\frac{\cos^2\varphi}{x})\frac{\partial}{\partial\varphi} + (\frac{y\cos^2\theta}{z^2\tan\theta})\frac{\partial}{\partial\theta} + (\frac{y}{r})\frac{\partial}{\partial r}] \\ &= \mathrm{i}\hbar\{(\frac{z\cos^2\varphi}{x})\frac{\partial}{\partial\varphi} + [\frac{(x^2+y^2)y\cos^2\theta + z^2y\cos^2\theta}{z^3\tan\theta}]\frac{\partial}{\partial\theta} + (\frac{-yz+zy}{r})\frac{\partial}{\partial r}\} \\ &= \mathrm{i}\hbar\{(\frac{r\cos\theta\cos^2\varphi}{r\sin\theta\cos\varphi})\frac{\partial}{\partial\varphi} + [\frac{r^2\cdot r\sin\theta\sin\varphi\cos^2\theta}{(r\cos\theta)^3\tan\theta}]\frac{\partial}{\partial\theta}\} = \mathrm{i}\hbar(\cot\theta\cos\varphi\frac{\partial}{\partial\varphi} + \sin\varphi\frac{\partial}{\partial\theta})\end{aligned}$$

$$\begin{aligned}\hat{L}_y &= -\mathrm{i}\hbar(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}) = -\mathrm{i}\hbar z[(-\frac{y\cos^2\varphi}{x^2})\frac{\partial}{\partial\varphi} + (\frac{x\cos^2\theta}{z^2\tan\theta})\frac{\partial}{\partial\theta} + (\frac{x}{r})\frac{\partial}{\partial r}] + \mathrm{i}\hbar x\{[-\frac{(x^2+y^2)\cos^2\theta}{z^3\tan\theta}]\frac{\partial}{\partial\theta} + (\frac{z}{r})\frac{\partial}{\partial r}\} \\ &= \mathrm{i}\hbar\{(\frac{zy\cos^2\varphi}{x^2})\frac{\partial}{\partial\varphi} + [\frac{-z^2x - x(x^2+y^2)\cos^2\theta}{z^3\tan\theta}]\frac{\partial}{\partial\theta} + (\frac{-zx+xz}{r})\frac{\partial}{\partial r}\} \\ &= \mathrm{i}\hbar\{[\frac{r\cos\theta\cdot r\sin\theta\sin\varphi\cos^2\varphi}{(r\sin\theta\cos\varphi)^2}]\frac{\partial}{\partial\varphi} + [\frac{-r^2\cdot r\sin\theta\cos\varphi\cos^2\theta}{(r\cos\theta)^3\tan\theta}]\frac{\partial}{\partial\theta}\} = \mathrm{i}\hbar(\cot\theta\sin\varphi\frac{\partial}{\partial\varphi} - \cos\varphi\frac{\partial}{\partial\theta})\end{aligned}$$

$$\begin{aligned}\hat{L}_z &= -\mathrm{i}\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) = -\mathrm{i}\hbar x[(\frac{\cos^2\varphi}{x})\frac{\partial}{\partial\varphi} + (\frac{y\cos^2\theta}{z^2\tan\theta})\frac{\partial}{\partial\theta} + (\frac{y}{r})\frac{\partial}{\partial r}] + \mathrm{i}\hbar y[(-\frac{y\cos^2\varphi}{x^2})\frac{\partial}{\partial\varphi} + (\frac{x\cos^2\theta}{z^2\tan\theta})\frac{\partial}{\partial\theta} + (\frac{x}{r})\frac{\partial}{\partial r}] \\ &= \mathrm{i}\hbar\{(-\cos^2\varphi - \frac{y^2\cos^2\varphi}{x^2})\frac{\partial}{\partial\varphi} + [\frac{(-xy+yx)\cos^2\theta}{z^2\tan\theta}]\frac{\partial}{\partial\theta} + (\frac{-xy+yx}{r})\frac{\partial}{\partial r}\} \\ &= \mathrm{i}\hbar[-\cos^2\varphi - \frac{(r\sin\theta\sin\varphi)^2\cos^2\varphi}{(r\sin\theta\cos\varphi)^2}]\frac{\partial}{\partial\varphi} = -\mathrm{i}\hbar\frac{\partial}{\partial\varphi}\end{aligned}$$

由以上公式还可推出

$$\begin{aligned}\hat{L}_\pm = \hat{L}_x \pm \mathrm{i}\hat{L}_y &= [\mathrm{i}\hbar(\cot\theta\cos\varphi\frac{\partial}{\partial\varphi} + \sin\varphi\frac{\partial}{\partial\theta})] \pm \mathrm{i}[\mathrm{i}\hbar(\cot\theta\sin\varphi\frac{\partial}{\partial\varphi} - \cos\varphi\frac{\partial}{\partial\theta})] \\ &= \mathrm{i}\hbar[(\cos\varphi \pm \mathrm{i}\sin\varphi)\cot\theta\frac{\partial}{\partial\varphi} + (\sin\varphi \mp \mathrm{i}\cos\varphi)\frac{\partial}{\partial\theta}] = \hbar\mathrm{e}^{\pm\mathrm{i}\varphi}(\mathrm{i}\cot\theta\frac{\partial}{\partial\varphi} \pm \frac{\partial}{\partial\theta})\end{aligned}$$

$$
\begin{aligned}
\hat{L}^2 &= \hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2=\hat{L}_+\hat{L}_- -\hbar\hat{L}_z+\hat{L}_z^2\\
&=\hbar e^{i\varphi}(i\cot\theta\frac{\partial}{\partial\varphi}+\frac{\partial}{\partial\theta})\cdot\hbar e^{-i\varphi}(i\cot\theta\frac{\partial}{\partial\varphi}-\frac{\partial}{\partial\theta})-\hbar\cdot(-i\hbar\frac{\partial}{\partial\varphi})+(-i\hbar\frac{\partial}{\partial\varphi})\cdot(-i\hbar\frac{\partial}{\partial\varphi})\\
&=\hbar e^{i\varphi}\{i\cot\theta\frac{\partial}{\partial\varphi}[\hbar e^{-i\varphi}(i\cot\theta\frac{\partial}{\partial\varphi}-\frac{\partial}{\partial\theta})]+\frac{\partial}{\partial\theta}[\hbar e^{-i\varphi}(i\cot\theta\frac{\partial}{\partial\varphi}-\frac{\partial}{\partial\theta})]\}+i\hbar^2\frac{\partial}{\partial\varphi}-\hbar^2\frac{\partial^2}{\partial\varphi^2}\\
&=\hbar e^{i\varphi}\{i\cot\theta[-i\hbar e^{-i\varphi}(i\cot\theta\frac{\partial}{\partial\varphi}-\frac{\partial}{\partial\theta})+\hbar e^{-i\varphi}(i\cot\theta\frac{\partial^2}{\partial\varphi^2}-\frac{\partial^2}{\partial\varphi\partial\theta})]+[\hbar e^{-i\varphi}(-\frac{i}{\sin^2\theta}\frac{\partial}{\partial\varphi}+i\cot\theta\frac{\partial^2}{\partial\theta\partial\varphi}-\frac{\partial^2}{\partial\theta^2})]\}\\
&\quad+i\hbar^2\frac{\partial}{\partial\varphi}-\hbar^2\frac{\partial^2}{\partial\varphi^2}\\
&=\hbar^2\{\cot\theta(i\cot\theta\frac{\partial}{\partial\varphi}-\frac{\partial}{\partial\theta})+i\cot\theta(i\cot\theta\frac{\partial^2}{\partial\varphi^2}-\frac{\partial^2}{\partial\varphi\partial\theta})+(-\frac{i}{\sin^2\theta}\frac{\partial}{\partial\varphi}+i\cot\theta\frac{\partial^2}{\partial\theta\partial\varphi}-\frac{\partial^2}{\partial\theta^2})\}+i\hbar^2\frac{\partial}{\partial\varphi}-\hbar^2\frac{\partial^2}{\partial\varphi^2}\\
&=\hbar^2\{i\cot^2\theta\frac{\partial}{\partial\varphi}-\cot\theta\frac{\partial}{\partial\theta}-\cot^2\theta\frac{\partial^2}{\partial\varphi^2}-i\cot\theta\frac{\partial^2}{\partial\varphi\partial\theta}-\frac{i}{\sin^2\theta}\frac{\partial}{\partial\varphi}+i\cot\theta\frac{\partial^2}{\partial\theta\partial\varphi}-\frac{\partial^2}{\partial\theta^2}+i\frac{\partial}{\partial\varphi}-\frac{\partial^2}{\partial\varphi^2}\}\\
&=-\hbar^2(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta})=-\hbar^2[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})]
\end{aligned}
$$

练习2：写出如下对易关系表达式：$[\hat{L}_z,\hat{x}]=?$，$[\hat{L}_z,\hat{p}_x]=?$，$[\hat{L}_z,\hat{\boldsymbol{r}}^2]=?$

解：因为$\hat{L}_z=\hat{x}\hat{p}_y-\hat{y}\hat{p}_x$，$\hat{\boldsymbol{r}}^2=\hat{x}^2+\hat{y}^2+\hat{z}^2$，所以：

$$[\hat{L}_z,\hat{x}]=[\hat{x}\hat{p}_y-\hat{y}\hat{p}_x,\hat{x}]=[\hat{x}\hat{p}_y,\hat{x}]-[\hat{y}\hat{p}_x,\hat{x}]=\hat{x}[\hat{p}_y,\hat{x}]+[\hat{x},\hat{x}]\hat{p}_y-\hat{y}[\hat{p}_x,\hat{x}]-[\hat{y},\hat{x}]\hat{p}_x=-\hat{y}\cdot(-i\hbar)=i\hbar\hat{y}$$

$$[\hat{L}_z,\hat{p}_x]=[\hat{x}\hat{p}_y-\hat{y}\hat{p}_x,\hat{p}_x]=[\hat{x}\hat{p}_y,\hat{p}_x]-[\hat{y}\hat{p}_x,\hat{p}_x]=\hat{x}[\hat{p}_y,\hat{p}_x]+[\hat{x},\hat{p}_x]\hat{p}_y-\hat{y}[\hat{p}_x,\hat{p}_x]-[\hat{y},\hat{p}_x]\hat{p}_x=i\hbar\hat{p}_y$$

$$
\begin{aligned}
[\hat{L}_z,\hat{\boldsymbol{r}}^2]&=[\hat{x}\hat{p}_y-\hat{y}\hat{p}_x,\hat{x}^2+\hat{y}^2+\hat{z}^2]=[\hat{x}\hat{p}_y,\hat{x}^2]+[\hat{x}\hat{p}_y,\hat{y}^2]+[\hat{x}\hat{p}_y,\hat{z}^2]-[\hat{y}\hat{p}_x,\hat{x}^2]-[\hat{y}\hat{p}_x,\hat{y}^2]-[\hat{y}\hat{p}_x,\hat{z}^2]\\
&=\hat{x}[\hat{p}_y,\hat{x}^2]+[\hat{x},\hat{x}^2]\hat{p}_y+\hat{x}[\hat{p}_y,\hat{y}^2]+[\hat{x},\hat{y}^2]\hat{p}_y-\hat{y}[\hat{p}_x,\hat{x}^2]-[\hat{y},\hat{x}^2]\hat{p}_x-\hat{y}[\hat{p}_x,\hat{y}^2]-[\hat{y},\hat{y}^2]\hat{p}_x\\
&=\hat{x}([\hat{p}_y,\hat{y}]\hat{y}+\hat{y}[\hat{p}_y,\hat{y}])-\hat{y}([\hat{p}_x,\hat{x}]\hat{x}+\hat{x}[\hat{p}_x,\hat{x}])=\hat{x}[-i\hbar\hat{y}+\hat{y}\cdot(-i\hbar)]-\hat{y}[-i\hbar\hat{x}+\hat{x}\cdot(-i\hbar)]\\
&=-2i\hbar[\hat{x},\hat{y}]=0
\end{aligned}
$$

练习3：处在状态$\gamma=\frac{1}{\sqrt{6}}\begin{pmatrix}1+i\\2\end{pmatrix}$的自旋1/2粒子，测量$S_x$得到结果为$\frac{\hbar}{2}$的概率是多少？

解：由于$|s_x\pm\rangle=\frac{1}{\sqrt{2}}(|s_z+\rangle\pm|s_z-\rangle)$，因此$|s_z\pm\rangle=\frac{1}{\sqrt{2}}(|s_x+\rangle\pm|s_x-\rangle)$，从而对题中的自旋1/2粒子，有：

$$|\gamma\rangle=\frac{1+i}{\sqrt{6}}|s_z+\rangle+\frac{2}{\sqrt{6}}|s_z-\rangle=\frac{1+i}{\sqrt{6}}\cdot\frac{1}{\sqrt{2}}(|s_x+\rangle+|s_x-\rangle)+\frac{2}{\sqrt{6}}\cdot\frac{1}{\sqrt{2}}(|s_x+\rangle-|s_x-\rangle)=\frac{3+i}{2\sqrt{3}}|s_x+\rangle+\frac{-1+i}{2\sqrt{3}}|s_x-\rangle$$

因此测量S_x得到结果为$\frac{\hbar}{2}$的概率为$P(S_x=\frac{\hbar}{2})=|\frac{3+i}{2\sqrt{3}}|^2=\frac{5}{6}$

练习4：推导自旋角动量算符$\hat{S}_x$和$\hat{S}_+$的矩阵形式

解：我们知道$\begin{cases}\hat{S}_+|\alpha\rangle=0\\\hat{S}_+|\beta\rangle=\hbar|\alpha\rangle\end{cases}$，$\begin{cases}\hat{S}_-|\alpha\rangle=\hbar|\beta\rangle\\\hat{S}_-|\beta\rangle=0\end{cases}$，且$\hat{S}_x=\frac{1}{2}(\hat{S}_++\hat{S}_-)$，因此有：

$$
\begin{aligned}
\boldsymbol{S}_+&=\begin{pmatrix}\langle\alpha|\hat{S}_+|\alpha\rangle&\langle\alpha|\hat{S}_+|\beta\rangle\\\langle\beta|\hat{S}_+|\alpha\rangle&\langle\beta|\hat{S}_+|\beta\rangle\end{pmatrix}=\begin{pmatrix}\langle\alpha|\cdot0&\langle\alpha|\cdot\hbar|\alpha\rangle\\\langle\beta|\cdot0&\langle\beta|\cdot\hbar|\alpha\rangle\end{pmatrix}=\hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}\\
\boldsymbol{S}_-&=\begin{pmatrix}\langle\alpha|\hat{S}_-|\alpha\rangle&\langle\alpha|\hat{S}_-|\beta\rangle\\\langle\beta|\hat{S}_-|\alpha\rangle&\langle\beta|\hat{S}_-|\beta\rangle\end{pmatrix}=\begin{pmatrix}\langle\alpha|\cdot\hbar|\beta\rangle&\langle\alpha|\cdot0\\\langle\beta|\cdot\hbar|\beta\rangle&\langle\beta|\cdot0\end{pmatrix}=\hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}\\
\boldsymbol{S}_x&=\frac{1}{2}(\boldsymbol{S}_++\boldsymbol{S}_-)=\frac{1}{2}(\hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}+\hbar\begin{pmatrix}0&0\\1&0\end{pmatrix})=\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}
\end{aligned}
$$

练习5：证明对Pauli矩阵构成的向量σ和三维几何空间矢量a,b，有关系式 $(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})=\boldsymbol{a}\cdot\boldsymbol{b}+\mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b})$

证明：根据题意，$\boldsymbol{\sigma}=\sigma_i\mathbf{i}+\sigma_j\mathbf{j}+\sigma_k\mathbf{k}$，$\boldsymbol{a}=a_i\mathbf{i}+a_j\mathbf{j}+a_k\mathbf{k}$，$\boldsymbol{b}=b_i\mathbf{i}+b_j\mathbf{j}+b_k\mathbf{k}$。另一方面，根据Pauli矩阵的对易关系$[\sigma_i,\sigma_j]=2\mathrm{i}\sum_k\varepsilon_{ijk}\sigma_k$和反对易关系$\{\sigma_i,\sigma_j\}=2\delta_{ij}$，得

$\sigma_i\sigma_j=\frac{1}{2}([\sigma_i,\sigma_j]+\{\sigma_i,\sigma_j\})=\mathrm{i}\sum_k\varepsilon_{ijk}\sigma_k+\delta_{ij}$，因此有：

$$
\begin{aligned}
&(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})=(\sigma_ia_i+\sigma_ja_j+\sigma_ka_k)(\sigma_ib_i+\sigma_jb_j+\sigma_kb_k)\\
&=(a_ib_i\sigma_i^2+a_jb_j\sigma_j^2+a_kb_k\sigma_k^2)+(a_ib_j\sigma_i\sigma_j+a_jb_i\sigma_j\sigma_i)+(a_ib_k\sigma_i\sigma_k+a_kb_i\sigma_k\sigma_i)+(a_jb_k\sigma_j\sigma_k+a_kb_j\sigma_k\sigma_j)\\
&=(a_ib_i+a_jb_j+a_kb_k)+(a_ib_j\mathrm{i}\sigma_k-a_jb_i\mathrm{i}\sigma_k)+(-a_ib_k\mathrm{i}\sigma_j+a_kb_i\mathrm{i}\sigma_j)+(a_jb_k\mathrm{i}\sigma_i-a_kb_j\mathrm{i}\sigma_i)\\
&=\boldsymbol{a}\cdot\boldsymbol{b}+\mathrm{i}[\sigma_k\cdot(\boldsymbol{a}\times\boldsymbol{b})_k+\sigma_j\cdot(\boldsymbol{a}\times\boldsymbol{b})_j+\sigma_i\cdot(\boldsymbol{a}\times\boldsymbol{b})_i]==\boldsymbol{a}\cdot\boldsymbol{b}+\mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b})
\end{aligned}
$$

故原题得证

练习6：证明旋轨耦合中 $|j=\frac{3}{2},m=-\frac{1}{2}\rangle=\sqrt{\frac{2}{3}}|m_1=0,m_2=-\frac{1}{2}\rangle+\sqrt{\frac{1}{3}}|m_1=-1,m_2=\frac{1}{2}\rangle$

证明：我们知道$|j=\frac{3}{2},m=-\frac{3}{2}\rangle=|m_1=-1,m_2=-\frac{1}{2}\rangle$，将总上升算符$\hat{J}_+=\hat{L}_++\hat{S}_+$作用在该式左边，得：

$$
\hat{J}_+|j=\frac{3}{2},m=-\frac{3}{2}\rangle=\sqrt{\frac{3}{2}(\frac{3}{2}+1)-(-\frac{3}{2})(-\frac{3}{2}+1)}\hbar|j=\frac{3}{2},m=-\frac{1}{2}\rangle=\sqrt{3}|j=\frac{3}{2},m=-\frac{1}{2}\rangle
$$

$$
\begin{aligned}
&\hat{J}_+|m_1=-1,m_2=-\frac{1}{2}\rangle=(\hat{L}_++\hat{S}_+)|m_1=-1,m_2=-\frac{1}{2}\rangle=\hat{L}_+|m_1=-1,m_2=-\frac{1}{2}\rangle+\hat{S}_+|m_1=-1,m_2=-\frac{1}{2}\rangle\\
&=\sqrt{1(1+1)-(-1)(-1+1)}\hbar|m_1=0,m_2=-\frac{1}{2}\rangle+\sqrt{\frac{1}{2}(\frac{1}{2}+1)-(-\frac{1}{2})(-\frac{1}{2}+1)}\hbar|m_1=-1,m_2=\frac{1}{2}\rangle\\
&=\sqrt{2}|m_1=0,m_2=-\frac{1}{2}\rangle+|m_1=-1,m_2=\frac{1}{2}\rangle
\end{aligned}
$$

联立两式得$\sqrt{3}|j=\frac{3}{2},m=-\frac{1}{2}\rangle=\sqrt{2}|m_1=0,m_2=-\frac{1}{2}\rangle+|m_1=-1,m_2=\frac{1}{2}\rangle$，因此 $|j=\frac{3}{2},m=-\frac{1}{2}\rangle=\sqrt{\frac{2}{3}}|m_1=0,m_2=-\frac{1}{2}\rangle+\sqrt{\frac{1}{3}}|m_1=-1,m_2=\frac{1}{2}\rangle$，证毕