# 课堂练习

## 练习1：证明$\hat{D}^{-1}(ds)\hat{\boldsymbol{x}}\hat{D}(ds)=\hat{\boldsymbol{x}}+ds$

**证明**：（方法1）对任意位置表象的态矢$|\boldsymbol{x}\rangle$，有：

$$\begin{aligned}\hat{D}^{-1}(ds)\hat{\boldsymbol{x}}\hat{D}(ds)|\boldsymbol{x}\rangle&=\hat{D}^{-1}(ds)\hat{\boldsymbol{x}}|\boldsymbol{x}+ds\rangle=\hat{D}^{-1}(ds)[(\boldsymbol{x}+ds)|\boldsymbol{x}+ds\rangle]=\hat{D}^{-1}(ds)[(\boldsymbol{x}+ds)|\boldsymbol{x}+ds\rangle]\\&=(\boldsymbol{x}+ds)\hat{D}(-ds)|\boldsymbol{x}+ds\rangle=(\boldsymbol{x}+ds)|\boldsymbol{x}\rangle\end{aligned}$$

而$(\hat{\boldsymbol{x}}+ds)|\boldsymbol{x}\rangle=\hat{\boldsymbol{x}}|\boldsymbol{x}\rangle+ds|\boldsymbol{x}\rangle=\boldsymbol{x}|\boldsymbol{x}\rangle+ds|\boldsymbol{x}\rangle=(\boldsymbol{x}+ds)|\boldsymbol{x}\rangle$，因此
$\hat{D}^{-1}(ds)\hat{\boldsymbol{x}}\hat{D}(ds)|\boldsymbol{x}\rangle=(\hat{\boldsymbol{x}}+ds)|\boldsymbol{x}\rangle$，即$\hat{D}^{-1}(ds)\hat{\boldsymbol{x}}\hat{D}(ds)=\hat{\boldsymbol{x}}+ds$
（方法2）根据位置算符$\hat{\boldsymbol{x}}$与坐标平移算符$\hat{D}(ds)$的对易关系$[\hat{\boldsymbol{x}},\hat{D}(ds)]=\hat{\boldsymbol{x}}\hat{D}(ds)-\hat{D}(ds)\hat{\boldsymbol{x}}=ds$，有：

$$\begin{aligned}\hat{D}^{-1}(ds)\hat{\boldsymbol{x}}\hat{D}(ds)&=\hat{D}^{-1}(ds)\{[\hat{\boldsymbol{x}},\hat{D}(ds)]+\hat{D}(ds)\hat{\boldsymbol{x}}\}=\hat{D}^{-1}(ds)\{ds+\hat{D}(ds)\hat{\boldsymbol{x}}\}=\hat{D}^{-1}(ds)ds+\hat{D}^{-1}(ds)\hat{D}(ds)\hat{\boldsymbol{x}}\\&=\hat{D}(-ds)ds+[\hat{D}^{-1}(ds)\hat{D}(ds)]\hat{\boldsymbol{x}}=(1-\mathrm{i}\hat{\boldsymbol{K}}\cdot ds)ds+\hat{I}\,\hat{\boldsymbol{x}}=ds-\mathrm{i}\hat{\boldsymbol{K}}\cdot ds^2+\hat{\boldsymbol{x}}\approx\hat{\boldsymbol{x}}+ds\end{aligned}$$

## 练习2：证明$\langle\boldsymbol{x}|\hat{D}(ds)|u\rangle=\langle x-ds|u\rangle$

**证明**：因为
$\langle\boldsymbol{x}|\hat{D}(ds)|u\rangle=\langle u|\hat{D}^{\dagger}(ds)|\boldsymbol{x}\rangle^*=\langle u|\hat{D}^{-1}(ds)|\boldsymbol{x}\rangle^*=\langle u|\hat{D}(-ds)|\boldsymbol{x}\rangle^*=\langle u|\boldsymbol{x}-ds\rangle^*=\langle\boldsymbol{x}-ds|u\rangle$
，故原式得证

## 练习3：在坐标表象中证明，坐标算符和动量算符满足基本对易关系$[\hat{x},\hat{p}]=\mathrm{i}\hbar$

**证明**：对任意的态矢$|\psi\rangle,|\phi\rangle$，有：

$$\begin{aligned}\langle\psi|[\hat{x},\hat{p}]|\phi\rangle&=\langle\psi|(\hat{x}\hat{p}-\hat{p}\hat{x})|\phi\rangle=\langle\psi|\hat{x}\hat{p}|\phi\rangle-\langle\psi|\hat{p}\hat{x}|\phi\rangle=\int\langle\psi|x\rangle\langle x|\hat{x}\hat{p}|\phi\rangle dx-\int\langle\psi|x\rangle\langle x|\hat{p}\hat{x}|\phi\rangle dx\\&=\int\langle\psi|x\rangle(\langle x|\hat{x})\hat{p}|\phi\rangle dx-\int\langle\psi|x\rangle\langle x|\hat{p}(\hat{x}|\phi\rangle)dx=\int\langle\psi|x\rangle(\langle x|x)\hat{p}|\phi\rangle dx-\int\langle\psi|x\rangle\{-\mathrm{i}\hbar\nabla[\langle x|(\hat{x}|\phi\rangle)]\}dx\\&=\int x\langle\psi|x\rangle\langle x|\hat{p}|\phi\rangle dx-\int\langle\psi|x\rangle\{-\mathrm{i}\hbar\nabla\langle x|\hat{x}|\phi\rangle\}dx=\int x\langle\psi|x\rangle[-\mathrm{i}\hbar\nabla\langle x|\phi\rangle]dx-\int\langle\psi|x\rangle\{-\mathrm{i}\hbar\nabla(x\langle x|\phi\rangle)\}dx\\&=-\mathrm{i}\hbar\int x\langle\psi|x\rangle\nabla\langle x|\phi\rangle dx+\mathrm{i}\hbar\int\langle\psi|x\rangle\{\langle x|\phi\rangle+x\nabla\langle x|\phi\rangle\}dx=\mathrm{i}\hbar\int\langle\psi|x\rangle\langle x|\phi\rangle dx=\mathrm{i}\hbar\langle\psi|\phi\rangle=\langle\psi|(\mathrm{i}\hbar\hat{I})|\phi\rangle\end{aligned}$$

因此$[\hat{x},\hat{p}]=\mathrm{i}\hbar\hat{I}=\mathrm{i}\hbar$，证毕

## 练习4：证明角动量的对易关系$[\hat{L}_i,\hat{L}_j]=\mathrm{i}\hbar\sum_k\varepsilon_{ijk}\hat{L}_k$，其中$\varepsilon_{ijk}$为Levi-Civita符号，若$ijk$由1,2,3的偶置换变成则$\varepsilon_{ijk}=1$，若$ijk$由1,2,3的奇置换变成则$\varepsilon_{ijk}=-1$，若$ijk$中任意一对相等则$\varepsilon_{ijk}=0$

**证明**：定义下标中的$i+1$为$[(i+1)\bmod 3]\in\{1,2,3\}$，$i-1$为$[(i-1)\bmod 3]\in\{1,2,3\}$，则：

$$\begin{aligned}[\hat{L}_i,\hat{L}_j]&=[\hat{x}_{i+1}\hat{p}_{i-1}-\hat{x}_{i-1}\hat{p}_{i+1},\hat{x}_{j+1}\hat{p}_{j-1}-\hat{x}_{j-1}\hat{p}_{j+1}]\\&=[\hat{x}_{i+1}\hat{p}_{i-1},\hat{x}_{j+1}\hat{p}_{j-1}]-[\hat{x}_{i+1}\hat{p}_{i-1},\hat{x}_{j-1}\hat{p}_{j+1}]-[\hat{x}_{i-1}\hat{p}_{i+1},\hat{x}_{j+1}\hat{p}_{j-1}]+[\hat{x}_{i-1}\hat{p}_{i+1},\hat{x}_{j-1}\hat{p}_{j+1}]\end{aligned}$$

结合$[\hat{x}_i,\hat{x}_j]=[\hat{p}_i,\hat{p}_j]=0$，$[\hat{x}_i,\hat{p}_j]=-[\hat{p}_i,\hat{x}_j]=\mathrm{i}\hbar\delta_{ij}$，得：

$$\begin{aligned}[\hat{x}_{i+1}\hat{p}_{i-1},\hat{x}_{j+1}\hat{p}_{j-1}]&=\hat{x}_{i+1}[\hat{p}_{i-1},\hat{x}_{j+1}\hat{p}_{j-1}]+[\hat{x}_{i+1},\hat{x}_{j+1}\hat{p}_{j-1}]\hat{p}_{i-1}\\&=\hat{x}_{i+1}([\hat{p}_{i-1},\hat{x}_{j+1}]\hat{p}_{j-1}+\hat{x}_{j+1}[\hat{p}_{i-1},\hat{p}_{j-1}])+([\hat{x}_{i+1},\hat{x}_{j+1}]\hat{p}_{j-1}+\hat{x}_{j+1}[\hat{x}_{i+1},\hat{p}_{j-1}])\hat{p}_{i-1}\\&=-\mathrm{i}\hbar\delta_{i-1,j+1}\hat{x}_{i+1}\hat{p}_{j-1}+\mathrm{i}\hbar\delta_{i+1,j-1}\hat{x}_{j+1}\hat{p}_{i-1}\end{aligned}$$

$$\begin{aligned}[\hat{x}_{i+1}\hat{p}_{i-1},\hat{x}_{j-1}\hat{p}_{j+1}] &= \hat{x}_{i+1}[\hat{p}_{i-1},\hat{x}_{j-1}\hat{p}_{j+1}]+[\hat{x}_{i+1},\hat{x}_{j-1}\hat{p}_{j+1}]\hat{p}_{i-1}\\ &= \hat{x}_{i+1}([\hat{p}_{i-1},\hat{x}_{j-1}]\hat{p}_{j+1}+\hat{x}_{j-1}[\hat{p}_{i-1},\hat{p}_{j+1}])+([\hat{x}_{i+1},\hat{x}_{j-1}]\hat{p}_{j+1}+\hat{x}_{j-1}[\hat{x}_{i+1},\hat{p}_{j+1}])\hat{p}_{i-1}\\ &= -\mathrm{i}\hbar\delta_{i-1,j-1}\hat{x}_{i+1}\hat{p}_{j+1}+\mathrm{i}\hbar\delta_{i+1,j+1}\hat{x}_{j-1}\hat{p}_{i-1}\end{aligned}$$

$$\begin{aligned}[\hat{x}_{i-1}\hat{p}_{i+1},\hat{x}_{j+1}\hat{p}_{j-1}] &= \hat{x}_{i-1}[\hat{p}_{i+1},\hat{x}_{j+1}\hat{p}_{j-1}]+[\hat{x}_{i-1},\hat{x}_{j+1}\hat{p}_{j-1}]\hat{p}_{i+1}\\ &= \hat{x}_{i-1}([\hat{p}_{i+1},\hat{x}_{j+1}]\hat{p}_{j-1}+\hat{x}_{j+1}[\hat{p}_{i+1},\hat{p}_{j-1}])+([\hat{x}_{i-1},\hat{x}_{j+1}]\hat{p}_{j-1}+\hat{x}_{j+1}[\hat{x}_{i-1},\hat{p}_{j-1}])\hat{p}_{i+1}\\ &= -\mathrm{i}\hbar\delta_{i+1,j+1}\hat{x}_{i-1}\hat{p}_{j-1}+\mathrm{i}\hbar\delta_{i-1,j-1}\hat{x}_{j+1}\hat{p}_{i+1}\end{aligned}$$

$$\begin{aligned}[\hat{x}_{i-1}\hat{p}_{i+1},\hat{x}_{j-1}\hat{p}_{j+1}] &= \hat{x}_{i-1}[\hat{p}_{i+1},\hat{x}_{j-1}\hat{p}_{j+1}]+[\hat{x}_{i-1},\hat{x}_{j-1}\hat{p}_{j+1}]\hat{p}_{i+1}\\ &= \hat{x}_{i-1}([\hat{p}_{i+1},\hat{x}_{j-1}]\hat{p}_{j+1}+\hat{x}_{j-1}[\hat{p}_{i+1},\hat{p}_{j+1}])+([\hat{x}_{i-1},\hat{x}_{j-1}]\hat{p}_{j+1}+\hat{x}_{j-1}[\hat{x}_{i-1},\hat{p}_{j+1}])\hat{p}_{i+1}\\ &= -\mathrm{i}\hbar\delta_{i+1,j-1}\hat{x}_{i-1}\hat{p}_{j+1}+\mathrm{i}\hbar\delta_{i-1,j+1}\hat{x}_{j-1}\hat{p}_{i+1}\end{aligned}$$

从而有：

$$\begin{aligned}[\hat{L}_i,\hat{L}_j] &= (-\mathrm{i}\hbar\delta_{i-1,j+1}\hat{x}_{i+1}\hat{p}_{j-1}+\mathrm{i}\hbar\delta_{i+1,j-1}\hat{x}_{j+1}\hat{p}_{i-1})-(-\mathrm{i}\hbar\delta_{i-1,j-1}\hat{x}_{i+1}\hat{p}_{j+1}+\mathrm{i}\hbar\delta_{i+1,j+1}\hat{x}_{j-1}\hat{p}_{i-1})\\ &\quad -(-\mathrm{i}\hbar\delta_{i+1,j+1}\hat{x}_{i-1}\hat{p}_{j-1}+\mathrm{i}\hbar\delta_{i-1,j-1}\hat{x}_{j+1}\hat{p}_{i+1})+(-\mathrm{i}\hbar\delta_{i+1,j-1}\hat{x}_{i-1}\hat{p}_{j+1}+\mathrm{i}\hbar\delta_{i-1,j+1}\hat{x}_{j-1}\hat{p}_{i+1})\\ &= \mathrm{i}\hbar\delta_{i+1,j+1}(\hat{x}_{i-1}\hat{p}_{j-1}-\hat{x}_{j-1}\hat{p}_{i-1})+\mathrm{i}\hbar\delta_{i+1,j-1}(\hat{x}_{j+1}\hat{p}_{i-1}-\hat{x}_{i-1}\hat{p}_{j+1})\\ &\quad +\mathrm{i}\hbar\delta_{i-1,j+1}(\hat{x}_{j-1}\hat{p}_{i+1}-\hat{x}_{i+1}\hat{p}_{j-1})+\mathrm{i}\hbar\delta_{i-1,j-1}(\hat{x}_{i+1}\hat{p}_{j+1}-\hat{x}_{j+1}\hat{p}_{i+1})\end{aligned}$$

当$j=i$时，上式第二、三项为0（因$\delta_{i+1,j-1}=\delta_{i-1,j+1}=0$），此时有：

$$[\hat{L}_i,\hat{L}_i]=\mathrm{i}\hbar\delta_{i+1,i+1}(\hat{x}_{i-1}\hat{p}_{i-1}-\hat{x}_{i-1}\hat{p}_{i-1})+\mathrm{i}\hbar\delta_{i-1,i-1}(\hat{x}_{i+1}\hat{p}_{i+1}-\hat{x}_{i+1}\hat{p}_{i+1})=0=\mathrm{i}\hbar\sum_k\varepsilon_{iik}\hat{L}_k$$

当$j=i+1$时，上式第一、二、四项为0（因$\delta_{i+1,j+1}=\delta_{i+1,j-1}=\delta_{i-1,j-1}=0$），此时有：

$$[\hat{L}_i,\hat{L}_{i+1}]=\mathrm{i}\hbar\delta_{i-1,i-1}(\hat{x}_i\hat{p}_{i+1}-\hat{x}_{i+1}\hat{p}_i)=\mathrm{i}\hbar\hat{L}_{i-1}=\mathrm{i}\hbar\sum_k\varepsilon_{i(i+1)k}\hat{L}_k$$

当$j=i-1$时，上式第一、三、四项为0（因$\delta_{i+1,j+1}=\delta_{i-1,j+1}=\delta_{i-1,j-1}=0$），此时有：

$$[\hat{L}_i,\hat{L}_{i-1}]=\mathrm{i}\hbar\delta_{i+1,i+1}(\hat{x}_i\hat{p}_{i-1}-\hat{x}_{i-1}\hat{p}_i)=-\mathrm{i}\hbar\hat{L}_{i+1}=\mathrm{i}\hbar\sum_k\varepsilon_{i(i-1)k}\hat{L}_k$$

综上，原命题得证

# 第二章习题

### 1.证明$\delta$函数的下列性质：1）$\delta(ax)=\frac{\delta(x)}{|a|}\ (a\neq 0)$；2）$\delta(x)=\delta(-x)$，即$\delta(x)$为偶函数；3）定义$\delta$函数的导数为$\delta'(x-x')\equiv\frac{d}{dx}\delta(x-x')$，则有$\delta'(x-x')=\delta(x-x')\frac{d}{dx'}$；4）$\delta(f(x))=\sum_i\frac{\delta(x_i)}{|f'(x_i)|}$，其中$x_i$是方程的第$i$个根，$f'(x)$表示对$f(x)$的一阶导数，这里要求$f(x)$是个光滑函数，并且$f'(x_i)\neq 0$

**证明**：1）令$t=ax$，两边对$x$求微分，则$dt=adx$。当$a>0$时，有
$\int_{-\infty}^{+\infty}f(x)\delta(ax)dx=\frac{1}{a}\int_{-\infty}^{+\infty}f(t/a)\delta(t)dt=\frac{f(t/a)}{a}=\frac{f(x)}{a}$；当$a<0$时，有
$\int_{-\infty}^{+\infty}f(x)\delta(ax)dx=\frac{1}{a}\int_{+\infty}^{-\infty}f(t/a)\delta(t)dt=-\frac{1}{a}\int_{-\infty}^{+\infty}f(t/a)\delta(t)dt=-\frac{f(t/a)}{a}=-\frac{f(x)}{a}$。对以上两种情形，均可改用$|a|$表示，从而得
$\int_{-\infty}^{+\infty}f(x)\delta(ax)dx=\frac{1}{|a|}\int_{-\infty}^{+\infty}f(t/a)\delta(t)dt=\frac{f(t/a)}{|a|}=\frac{f(x)}{|a|}=\frac{1}{|a|}\int_{-\infty}^{+\infty}f(x)\delta(x)dx$，故
$\delta(ax)=\frac{\delta(x)}{|a|}\quad(a\neq 0)$
2）此题可看作上一题中取$a=-1$的情形，证明见上
3）因为

$$\int_{-\infty}^{+\infty}f(x)\delta^{'}(x-x^{'})dx=\int_{-\infty}^{+\infty}f(x)\frac{d\delta(x-x^{'})}{dx}dx=[f(x)\delta(x-x^{'})]_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}\frac{df(x)}{dx}\delta(x-x^{'})dx$$
$$=-\int_{-\infty}^{+\infty}\delta(x-x^{'})\frac{df(x)}{dx}dx$$

所以$\delta^{'}(x-x^{'})=-\delta(x-x^{'})\frac{d}{dx}$。另一方面，因为

$$\begin{aligned}\int_{-\infty}^{+\infty}f(x)\delta^{'}(x-x^{'})dx^{'}&=\int_{-\infty}^{+\infty}f(x)\frac{d\delta(x-x^{'})}{dx}dx^{'}=-\int_{-\infty}^{+\infty}f(x)\frac{d\delta(x-x^{'})}{dx^{'}}dx^{'}\\&=-[f(x)\delta(x-x^{'})]_{-\infty}^{+\infty}+\int_{-\infty}^{+\infty}\delta(x-x^{'})\frac{df(x)}{dx^{'}}dx^{'}\\&=\int_{-\infty}^{+\infty}\delta(x-x^{'})\frac{df(x)}{dx^{'}}dx^{'}\end{aligned}$$

所以$\delta^{'}(x-x^{'})=\delta(x-x^{'})\frac{d}{dx^{'}}$。
注：此处$\frac{d\delta(x-x^{'})}{dx}=-\frac{d\delta(x-x^{'})}{dx^{'}}$，是因为$\frac{d\delta(x-x^{'})}{dx}=\frac{d\delta(x-x^{'})}{d(x-x^{'})}\cdot\frac{d(x-x^{'})}{dx}=\frac{d\delta(x-x^{'})}{d(x-x^{'})}$，
$\frac{d\delta(x-x^{'})}{dx^{'}}=\frac{d\delta(x-x^{'})}{d(x-x^{'})}\cdot\frac{d(x-x^{'})}{dx^{'}}=-\frac{d\delta(x-x^{'})}{d(x-x^{'})}$，故联立得证。
4）首先考察$x_i$附近的一个邻域$U(x_i,\varepsilon)$，计算在该领域上的积分$\int_{x_i-\varepsilon}^{x_i+\varepsilon}g(x)\delta(f(x))dx$。令$y=f(x)$，两边对$x$求微分，则$dy=f^{'}(x)dx$。若$f^{'}(x)>0$，则$f(x_i+\varepsilon)>f(x_i-\varepsilon)$，因此：

$$\begin{aligned}\int_{x_i-\varepsilon}^{x_i+\varepsilon}g(x)\delta(f(x))dx&=\int_{f(x_i-\varepsilon)}^{f(x_i+\varepsilon)}g(x)\delta(y)\frac{dy}{|f^{'}(x)|}=\int_{f(x_i-\varepsilon)}^{f(x_i+\varepsilon)}\frac{g(x)}{|f^{'}(x)|}\delta(y-0)dy\\&=[\frac{g(x)}{|f^{'}(x)|}]_{y=0}=[\frac{g(x)}{|f^{'}(x)|}]_{g(x)=0}=\frac{g(x_i)}{|f^{'}(x_i)|}\end{aligned}$$

若$f^{'}(x)<0$，则$f(x_i+\varepsilon)<f(x_i-\varepsilon)$，因此：

$$\begin{aligned}\int_{x_i-\varepsilon}^{x_i+\varepsilon}g(x)\delta(f(x))dx&=\int_{f(x_i-\varepsilon)}^{f(x_i+\varepsilon)}g(x)\delta(y)\frac{dy}{-|f^{'}(x)|}=\int_{f(x_i+\varepsilon)}^{f(x_i-\varepsilon)}\frac{g(x)}{|f^{'}(x)|}\delta(y-0)dy\\&=[\frac{g(x)}{|f^{'}(x)|}]_{y=0}=[\frac{g(x)}{|f^{'}(x)|}]_{g(x)=0}=\frac{g(x_i)}{|f^{'}(x_i)|}\end{aligned}$$

将积分扩展至整个实数域，则有$\int_{-\infty}^{+\infty}g(x)\delta(f(x))dx=\sum_{i=1}^{n}\frac{g(x_i)}{|f^{'}(x_i)|}$，
另一方面，$\int_{x_i-\varepsilon}^{x_i+\varepsilon}g(x)\delta(x-x_i)dx=g(x_i)$，两边同时除以$|f^{'}(x_i)|$，得
$\int_{x_i-\varepsilon}^{x_i+\varepsilon}\frac{g(x)}{|f^{'}(x_i)|}\delta(x-x_i)dx=\frac{g(x_i)}{|f^{'}(x_i)|}$，从而对所有的$x_i$求和得
$\sum_{i=1}^{n}\int_{x_i-\varepsilon}^{x_i+\varepsilon}\frac{g(x)}{|f^{'}(x_i)|}\delta(x-x_i)dx=\sum_{i=1}^{n}\frac{g(x_i)}{|f^{'}(x_i)|}$。将积分扩展至整个实数域，得：

$$\sum_{i=1}^{n}\int_{-\infty}^{+\infty}\frac{g(x)}{|f^{'}(x_i)|}\delta(x-x_i)dx=\sum_{i=1}^{n}\int_{x_i-\varepsilon}^{x_i+\varepsilon}\frac{g(x)}{|f^{'}(x_i)|}\delta(x-x_i)dx=\sum_{i=1}^{n}\frac{g(x_i)}{|f^{'}(x_i)|}\quad\text{（利用}\delta(x-x_i)\text{在邻域}U(x_i,\varepsilon)\text{均为}0\text{的性质）}$$

交换积分符号和求和符号得：

$$\sum_{i=1}^{n}\int_{-\infty}^{+\infty}\frac{g(x)}{|f^{'}(x_i)|}\delta(x-x_i)dx=\int_{-\infty}^{+\infty}g(x)\sum_{i=1}^{n}\frac{\delta(x-x_i)}{|f^{'}(x_i)|}dx=\sum_{i=1}^{n}\frac{g(x_i)}{|f^{'}(x_i)|}=\int_{-\infty}^{+\infty}g(x)\delta(f(x))dx$$

从而一一对应得$\delta(f(x))=\sum_{i=1}^{n}\frac{\delta(x-x_i)}{|f^{'}(x_i)|}$

**2.求出波函数$\psi(x)=A\mathrm{e}^{-\frac{x^2}{2\sigma^2}}$的归一化因子，然后求出动量空间的波函数形式。你发现什么特征？**

**解**：因为$\int_{-\infty}^{+\infty}\psi^*(x)\psi(x)dx=\int_{-\infty}^{+\infty}|A|^2\mathrm{e}^{-\frac{x^2}{\sigma^2}}dx=\int_{-\infty}^{+\infty}|A|^2\sigma\mathrm{e}^{-\frac{x^2}{\sigma^2}}d(\frac{x}{\sigma})=|A|^2\sigma\sqrt{\pi}=1$，所以归一化因子为$|A|=\pm\sqrt{\frac{1}{\sigma\sqrt{\pi}}}$，若波函数的系数为正实数，则可以取$A=\sqrt{\frac{1}{\sigma\sqrt{\pi}}}$。而该波函数在动量空间的波函数形式为：

$$
\begin{aligned}
\phi(p)=\langle p|\phi\rangle&=\int_{-\infty}^{+\infty}\langle p|x\rangle\langle x|\phi\rangle dx=(2\pi\hbar)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\cdot x}\psi(x)dx=(2\pi\hbar)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\cdot x}\cdot A\mathrm{e}^{-\frac{x^2}{2\sigma^2}}dx\\
&=(2\pi\hbar)^{-\frac{1}{2}}A\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{x^2}{2\sigma^2}-\frac{\mathrm{i}}{\hbar}p\cdot x}dx=(2\pi\hbar)^{-\frac{1}{2}}A\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{p^2\sigma^2}{2\hbar^2}}\cdot\mathrm{e}^{-\frac{(x+\frac{\mathrm{i}}{\hbar}p\sigma^2)^2}{2\sigma^2}}dx\\
&=(2\pi\hbar)^{-\frac{1}{2}}A\mathrm{e}^{-\frac{p^2\sigma^2}{2\hbar^2}}\int_{-\infty}^{+\infty}\sqrt{2}\sigma\mathrm{e}^{-\frac{(x+\frac{\mathrm{i}}{\hbar}p\sigma^2)^2}{2\sigma^2}}d(\frac{x+\frac{\mathrm{i}}{\hbar}p\sigma^2}{\sqrt{2}\sigma})=(2\pi\hbar)^{-\frac{1}{2}}A\mathrm{e}^{-\frac{p^2\sigma^2}{2\hbar^2}}\cdot\sqrt{2}\sigma\sqrt{\pi}\\
&=(2\pi\hbar)^{-\frac{1}{2}}A\mathrm{e}^{-\frac{p^2\sigma^2}{2\hbar^2}}\cdot\sqrt{2}\sigma\sqrt{\pi}=\frac{1}{\sqrt{2\pi\hbar}}\cdot\sqrt{\frac{1}{\sigma\sqrt{\pi}}}\cdot\mathrm{e}^{-\frac{p^2\sigma^2}{2\hbar^2}}\cdot\sqrt{2}\sigma\sqrt{\pi}=\sqrt{\frac{\sigma}{\hbar\sqrt{\pi}}}\mathrm{e}^{-\frac{p^2\sigma^2}{2\hbar^2}}
\end{aligned}
$$

这两种表象下的波函数具有如下特征：（1）$\psi(x)$和$\phi(p)$互为傅里叶变换的关系；（2）$\psi(x)$和$\phi(p)$均具有类似于高斯函数的形式，即均可表示为$f(q)=\sqrt{\frac{1}{\sigma\sqrt{\pi}}}\mathrm{e}^{-\frac{q^2}{2\sigma^2}}$，其中$q$为广义坐标（无论是位置坐标还是动量坐标）。

### 3.令$|a\rangle=|s_z+\rangle$，$|b\rangle=|s_x+\rangle$，1）写出算符$|a\rangle\langle b|$以$\hat{S}_z$的本征态为基矢的矩阵表示；2）计算态矢$|u\rangle=\alpha(|a\rangle+|b\rangle)$中的归一化因子$\alpha$；3）对状态$|u\rangle$测量得到$s_z=\frac{1}{2}\hbar$和$s_z=-\frac{1}{2}\hbar$的概率分别是多少？

**解**：1）易知$|b\rangle=|s_x+\rangle=\frac{1}{\sqrt{2}}(|s_z+\rangle+|s_z-\rangle)$，因此
$|a\rangle\langle b|=|s_z+\rangle\cdot\frac{1}{\sqrt{2}}(\langle s_z+|+\langle s_z-|)=\frac{1}{\sqrt{2}}(|s_z+\rangle\langle s_z+|+|s_z+\rangle\langle s_z-|)$，而$\hat{S}_z$的本征态即为$|s_z+\rangle$和$|s_z-\rangle$，因此：

$$
\begin{cases}
\langle s_z+|a\rangle\langle b|s_z+\rangle=\langle s_z+|\cdot\frac{1}{\sqrt{2}}(|s_z+\rangle\langle s_z+|+|s_z+\rangle\langle s_z-|)\cdot|s_z+\rangle=\frac{1}{\sqrt{2}}(\langle s_z+|s_z+\rangle\langle s_z+|s_z+\rangle+\langle s_z+|s_z+\rangle\langle s_z-|s_z+\rangle)=\frac{1}{\sqrt{2}}\\
\langle s_z+|a\rangle\langle b|s_z-\rangle=\langle s_z+|\cdot\frac{1}{\sqrt{2}}(|s_z+\rangle\langle s_z+|+|s_z+\rangle\langle s_z-|)\cdot|s_z-\rangle=\frac{1}{\sqrt{2}}(\langle s_z+|s_z+\rangle\langle s_z+|s_z-\rangle+\langle s_z+|s_z+\rangle\langle s_z-|s_z-\rangle)=\frac{1}{\sqrt{2}}\\
\langle s_z-|a\rangle\langle b|s_z+\rangle=\langle s_z-|\cdot\frac{1}{\sqrt{2}}(|s_z+\rangle\langle s_z+|+|s_z+\rangle\langle s_z-|)\cdot|s_z+\rangle=\frac{1}{\sqrt{2}}(\langle s_z-|s_z+\rangle\langle s_z+|s_z+\rangle+\langle s_z-|s_z+\rangle\langle s_z-|s_z+\rangle)=0\\
\langle s_z-|a\rangle\langle b|s_z-\rangle=\langle s_z-|\cdot\frac{1}{\sqrt{2}}(|s_z+\rangle\langle s_z+|+|s_z+\rangle\langle s_z-|)\cdot|s_z-\rangle=\frac{1}{\sqrt{2}}(\langle s_z-|s_z+\rangle\langle s_z+|s_z-\rangle+\langle s_z-|s_z+\rangle\langle s_z-|s_z-\rangle)=0
\end{cases}
$$

从而算符$|a\rangle\langle b|$以$\hat{S}_z$的本征态为基矢的矩阵表示为$\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\0&0\end{bmatrix}$

2）因为$|u\rangle=\alpha(|a\rangle+|b\rangle)=\alpha[|s_z+\rangle+\frac{1}{\sqrt{2}}(|s_z+\rangle+|s_z-\rangle)]=\alpha[\frac{\sqrt{2}+1}{\sqrt{2}}|s_z+\rangle+\frac{1}{\sqrt{2}}|s_z-\rangle]$，所以根据$\langle u|u\rangle=1$，得$\alpha^2[(\frac{\sqrt{2}+1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2]=1$，解得$\alpha=\pm\sqrt{\frac{1}{2+\sqrt{2}}}$。如果要求各个基矢的因子均为正数，则$\alpha$可取$\sqrt{\frac{1}{2+\sqrt{2}}}$

3）我们知道，测量得到$s_z=\frac{1}{2}\hbar$时，对应的态矢为$|s_z+\rangle$；测量得到$s_z=-\frac{1}{2}\hbar$时，对应的态矢为$|s_z-\rangle$。因此对状态$|u\rangle$测量，得到$s_z=\frac{1}{2}\hbar$的概率为
$P(s_z=\frac{1}{2}\hbar)=\alpha^2(\frac{\sqrt{2}+1}{\sqrt{2}})^2=\frac{1}{2+\sqrt{2}}\cdot\frac{3+2\sqrt{2}}{2}=\frac{2+\sqrt{2}}{4}$；得到$s_z=-\frac{1}{2}\hbar$的概率为
$P(s_z=-\frac{1}{2}\hbar)=\alpha^2(\frac{1}{\sqrt{2}})^2=\frac{1}{2+\sqrt{2}}\cdot\frac{1}{2}=\frac{2-\sqrt{2}}{4}$

### 4.证明对应于有限平移$\boldsymbol{s}$，存在如下恒等式$\hat{D}^{-1}(\boldsymbol{s})\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{s})=\hat{\boldsymbol{x}}+\boldsymbol{s}$（看看你能用几种方法证明？）

**证明**：（方法1）对任意位置表象的态矢$|\boldsymbol{x}\rangle$，有：

$$
\hat{D}^{-1}(\boldsymbol{s})\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{s})|\boldsymbol{x}\rangle=\hat{D}^{-1}(\boldsymbol{s})\hat{\boldsymbol{x}}|\boldsymbol{x}+\boldsymbol{s}\rangle=\hat{D}^{-1}(\boldsymbol{s})(\boldsymbol{x}+\boldsymbol{s})|\boldsymbol{x}+\boldsymbol{s}\rangle=(\boldsymbol{x}+\boldsymbol{s})\hat{D}^{-1}(\boldsymbol{s})|\boldsymbol{x}+\boldsymbol{s}\rangle=(\boldsymbol{x}+\boldsymbol{s})\hat{D}(-\boldsymbol{s})|\boldsymbol{x}+\boldsymbol{s}\rangle=(\boldsymbol{x}+\boldsymbol{s})|\boldsymbol{x}\rangle
$$

而$(\hat{\boldsymbol{x}}+\boldsymbol{s})|\boldsymbol{x}\rangle=\hat{\boldsymbol{x}}|\boldsymbol{x}\rangle+\boldsymbol{s}|\boldsymbol{x}\rangle=\boldsymbol{x}|\boldsymbol{x}\rangle+\boldsymbol{s}|\boldsymbol{x}\rangle=(\boldsymbol{x}+\boldsymbol{s})|\boldsymbol{x}\rangle$，因此$\hat{D}^{-1}(\boldsymbol{s})\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{s})|\boldsymbol{x}\rangle=(\hat{\boldsymbol{x}}+\boldsymbol{s})|\boldsymbol{x}\rangle$，从而$\hat{D}^{-1}(\boldsymbol{s})\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{s})=\hat{\boldsymbol{x}}+\boldsymbol{s}$

（方法2）对任意位置表象的态矢$|\boldsymbol{x}\rangle$，有$\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{s})|\boldsymbol{x}\rangle=\hat{\boldsymbol{x}}|\boldsymbol{x}+\boldsymbol{s}\rangle=(\boldsymbol{x}+\boldsymbol{s})|\boldsymbol{x}+\boldsymbol{s}\rangle$，$\hat{D}(\boldsymbol{s})\hat{\boldsymbol{x}}|\boldsymbol{x}\rangle=\hat{D}(\boldsymbol{s})(\boldsymbol{x}|\boldsymbol{x}\rangle)=\boldsymbol{x}\hat{D}(\boldsymbol{s})|\boldsymbol{x}\rangle=\boldsymbol{x}|\boldsymbol{x}+\boldsymbol{s}\rangle$，因此$[\hat{\boldsymbol{x}},\hat{D}(\boldsymbol{s})]|\boldsymbol{x}\rangle=(\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{s})-\hat{D}(\boldsymbol{s})\hat{\boldsymbol{x}})|\boldsymbol{x}\rangle=\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{s})|\boldsymbol{x}\rangle-\hat{D}(\boldsymbol{s})\hat{\boldsymbol{x}}|\boldsymbol{x}\rangle=\boldsymbol{s}|\boldsymbol{x}+\boldsymbol{s}\rangle$，从而有：

$$\begin{aligned}\hat{D}^{-1}(\boldsymbol{s})\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{s})|\boldsymbol{x}\rangle&=\hat{D}^{-1}(\boldsymbol{s})\{[\hat{\boldsymbol{x}},\hat{D}(\boldsymbol{s})]+\hat{D}(\boldsymbol{s})\hat{\boldsymbol{x}}\}|\boldsymbol{x}\rangle=\hat{D}^{-1}(\boldsymbol{s})[\hat{\boldsymbol{x}},\hat{D}(\boldsymbol{s})]|\boldsymbol{x}\rangle+\hat{D}^{-1}(\boldsymbol{s})\hat{D}(\boldsymbol{s})\hat{\boldsymbol{x}}|\boldsymbol{x}\rangle=\hat{D}^{-1}(\boldsymbol{s})\boldsymbol{s}|\boldsymbol{x}+\boldsymbol{s}\rangle+[\hat{D}^{-1}(\boldsymbol{s})\hat{D}(\boldsymbol{s})]\hat{\boldsymbol{x}}|\boldsymbol{x}\rangle\\&=\boldsymbol{s}\hat{D}(-\boldsymbol{s})|\boldsymbol{x}+\boldsymbol{s}\rangle+\hat{I}\,\hat{\boldsymbol{x}}|\boldsymbol{x}\rangle=\boldsymbol{s}|\boldsymbol{x}\rangle+\hat{\boldsymbol{x}}|\boldsymbol{x}\rangle=(\hat{\boldsymbol{x}}+\boldsymbol{s})|\boldsymbol{x}\rangle\end{aligned}$$

故$\hat{D}^{-1}(\boldsymbol{s})\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{s})=\hat{\boldsymbol{x}}+\boldsymbol{s}$

## 5.假定函数$F(x)$和$G(x)$关于$x=0$泰勒展开收敛，证明如下对易关系恒等式：$[\hat{x},F(\hat{p})]=\mathrm{i}\hbar\frac{\partial F}{\partial\hat{p}}$，$[\hat{p},G(\hat{x})]=-\mathrm{i}\hbar\frac{\partial G}{\partial\hat{x}}$

**证明**：我们首先证明$[\hat{x},\hat{p}^n]=\mathrm{i}\hbar n\hat{p}^{n-1},[\hat{p},\hat{x}^n]=-\mathrm{i}\hbar n\hat{x}^{n-1}$，根据对易关系$[\hat{x},\hat{p}]=\hat{x}\hat{p}-\hat{p}\hat{x}=\mathrm{i}\hbar$，得：

$$\begin{aligned}[\hat{x},\hat{p}^n]&=\hat{x}\hat{p}^n-\hat{p}^n\hat{x}=(\hat{x}\hat{p})\hat{p}^{n-1}-\hat{p}^n\hat{x}=([\hat{x},\hat{p}]+\hat{p}\hat{x})\hat{p}^{n-1}-\hat{p}^n\hat{x}=(\mathrm{i}\hbar+\hat{p}\hat{x})\hat{p}^{n-1}-\hat{p}^n\hat{x}=\mathrm{i}\hbar\hat{p}^{n-1}+\hat{p}\hat{x}\hat{p}^{n-1}-\hat{p}^n\hat{x}\\&=\mathrm{i}\hbar\hat{p}^{n-1}+\hat{p}(\hat{x}\hat{p})\hat{p}^{n-2}-\hat{p}^n\hat{x}=\mathrm{i}\hbar\hat{p}^{n-1}+\hat{p}([\hat{x},\hat{p}]+\hat{p}\hat{x})\hat{p}^{n-2}-\hat{p}^n\hat{x}=\mathrm{i}\hbar\hat{p}^{n-1}+\hat{p}(\mathrm{i}\hbar+\hat{p}\hat{x})\hat{p}^{n-2}-\hat{p}^n\hat{x}\\&=\mathrm{i}\hbar\hat{p}^{n-1}+(\mathrm{i}\hbar\hat{p}+\hat{p}^2\hat{x})\hat{p}^{n-2}-\hat{p}^n\hat{x}=\mathrm{i}\hbar\hat{p}^{n-1}+\mathrm{i}\hbar\hat{p}^{n-1}+\hat{p}^2\hat{x}\hat{p}^{n-2}-\hat{p}^n\hat{x}=2\mathrm{i}\hbar\hat{p}^{n-1}+\hat{p}^2\hat{x}\hat{p}^{n-2}-\hat{p}^n\hat{x}\\&=\cdots=\mathrm{i}\hbar n\hat{p}^{n-1}+\hat{p}^n\hat{x}-\hat{p}^n\hat{x}=\mathrm{i}\hbar n\hat{p}^{n-1}\end{aligned}$$

$$\begin{aligned}[\hat{p},\hat{x}^n]&=\hat{p}\hat{x}^n-\hat{x}^n\hat{p}=(\hat{p}\hat{x})\hat{x}^{n-1}-\hat{x}^n\hat{p}=(-[\hat{x},\hat{p}]+\hat{x}\hat{p})\hat{x}^{n-1}-\hat{x}^n\hat{p}=(-\mathrm{i}\hbar+\hat{x}\hat{p})\hat{x}^{n-1}-\hat{x}^n\hat{p}=-\mathrm{i}\hbar\hat{x}^{n-1}+\hat{x}\hat{p}\hat{x}^{n-1}-\hat{x}^n\hat{p}\\&=-\mathrm{i}\hbar\hat{x}^{n-1}+\hat{x}(\hat{p}\hat{x})\hat{x}^{n-2}-\hat{x}^n\hat{p}=-\mathrm{i}\hbar\hat{x}^{n-1}+\hat{x}(-[\hat{x},\hat{p}]+\hat{x}\hat{p})\hat{x}^{n-2}-\hat{x}^n\hat{p}=-\mathrm{i}\hbar\hat{x}^{n-1}+\hat{x}(-\mathrm{i}\hbar+\hat{x}\hat{p})\hat{x}^{n-2}-\hat{x}^n\hat{p}\\&=-\mathrm{i}\hbar\hat{x}^{n-1}+(-\mathrm{i}\hbar\hat{x}+\hat{x}^2\hat{p})\hat{x}^{n-2}-\hat{x}^n\hat{p}=-\mathrm{i}\hbar\hat{x}^{n-1}-\mathrm{i}\hbar\hat{x}^{n-1}+\hat{x}^2\hat{p}\hat{x}^{n-2}-\hat{x}^n\hat{p}=-2\mathrm{i}\hbar\hat{x}^{n-1}+\hat{x}^2\hat{p}\hat{x}^{n-2}-\hat{x}^n\hat{p}\\&=\cdots=-\mathrm{i}\hbar n\hat{x}^{n-1}+\hat{x}^n\hat{p}-\hat{x}^n\hat{p}=-\mathrm{i}\hbar n\hat{x}^{n-1}\end{aligned}$$

接下来，我们让$F(\hat{p})$和$G(\hat{x})$在原点处进行泰勒展开，得$F(\hat{p})=\sum\limits_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}\hat{p}^k,G(\hat{x})=\sum\limits_{k=0}^{\infty}\frac{G^{(k)}(0)}{k!}\hat{x}^k$，因此：

$$[\hat{x},F(\hat{p})]=[\hat{x},\sum_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}\hat{p}^k]=\sum_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}[\hat{x},\hat{p}^k]=\sum_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}(\mathrm{i}\hbar k\hat{p}^{k-1})=\mathrm{i}\hbar\sum_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}\frac{\partial\hat{p}^k}{\partial\hat{p}}=\mathrm{i}\hbar\frac{\partial\sum\limits_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}\hat{p}^k}{\partial\hat{p}}=\mathrm{i}\hbar\frac{\partial F}{\partial\hat{p}}$$

$$[\hat{p},G(\hat{x})]=[\hat{p},\sum_{k=0}^{\infty}\frac{G^{(k)}(0)}{k!}\hat{x}^k]=\sum_{k=0}^{\infty}\frac{G^{(k)}(0)}{k!}[\hat{p},\hat{x}^k]=\sum_{k=0}^{\infty}\frac{G^{(k)}(0)}{k!}(-\mathrm{i}\hbar k\hat{x}^{k-1})=-\mathrm{i}\hbar\sum_{k=0}^{\infty}\frac{G^{(k)}(0)}{k!}\frac{\partial\hat{x}^k}{\partial\hat{x}}=-\mathrm{i}\hbar\frac{\partial\sum\limits_{k=0}^{\infty}\frac{G^{(k)}(0)}{k!}\hat{x}^k}{\partial\hat{x}}=-\mathrm{i}\hbar\frac{\partial G}{\partial\hat{x}}$$