课堂练习

练习1：证明任意矢量用一组基矢的展开是唯一的

证明：反证法，设$|\alpha\rangle$在基矢$\{|u_i\rangle\}$下存在至少两种展开$|\alpha\rangle=\sum\limits_{i=1}^{n}\alpha_i|u_i\rangle=\sum\limits_{i=1}^{n}\alpha_i^{'}|u_i\rangle$，其中$\alpha_i$与$\alpha_i^{'}$不全相等，则移项得$\sum\limits_{i=1}^{n}(\alpha_i-\alpha_i^{'})|u_i\rangle=0$，若第$k_1,k_2,\ldots,k_m$项满足
$\alpha_{k_1}\neq\alpha_{k_1}^{'},\alpha_{k_2}\neq\alpha_{k_2}^{'},\ldots,\alpha_{k_m}\neq\alpha_{k_m}^{'}$，原式可化为$\sum\limits_{i=1}^{n}(\alpha_{k_i}-\alpha_{k_i}^{'})|u_{k_i}\rangle=0$，即
$|u_{k_1}\rangle,|u_{k_2}\rangle,\ldots,|u_{k_m}\rangle$线性相关，但基矢$\{|u_i\rangle\}$之间满足线性无关，矛盾！因此任意矢量用一组基矢的展开是唯一的。

练习2：证明$S_{1/2}$是个二维的复数线性空间

证明：易知S_2是个二维的复数线性空间⇔线性无关的向量（右矢）个数最多有两个，故先设两个右矢：$|a\rangle=\begin{bmatrix}a_1\\a_2\end{bmatrix}$，$|b\rangle=\begin{bmatrix}b_1\\b_2\end{bmatrix}$，满足$n_a|a\rangle+n_b|b\rangle=0$，则有$\begin{cases}n_aa_1+n_bb_1=0\\n_aa_2+n_bb_2=0\end{cases}$，由此得$\begin{cases}n_b(b_1a_2-b_2a_1)=0\\n_b(a_1b_2-a_2b_1)=0\end{cases}$。对以上情形，只要保证$\begin{cases}a_1,a_2,b_1,b_2\neq0\\a_1b_2\neq b_1a_2\end{cases}$，即可得到$n_a=n_b=0$，从而$|a\rangle$与$|b\rangle$线性无关，即$S_2$中线性无关的向量（右矢）个数可以为两个。

接下来，我们还要证明S_2中线性无关的向量（右矢）个数不能为三个或更多个。设三个右矢：$|a\rangle=\begin{bmatrix}a_1\\a_2\end{bmatrix}$，$|b\rangle=\begin{bmatrix}b_1\\b_2\end{bmatrix}$，$|c\rangle=\begin{bmatrix}c_1\\c_2\end{bmatrix}$，满足$n_a|a\rangle+n_b|b\rangle+n_c|c\rangle=0$，则有$\begin{cases}n_aa_1+n_bb_1+n_cc_1=0\\n_aa_2+n_bb_2+n_cc_2=0\end{cases}$，该方程组中齐次线性方程的个数小于变量个数，故有无穷组非零解，从而存在不全为零的n_a,n_b,n_c，使$n_a|a\rangle+n_b|b\rangle+n_c|c\rangle=0$，即$|a\rangle,|b\rangle,|c\rangle$线性相关。对更多右矢的情形，同理可证它们均满足线性相关。

综上，S_2是个二维的复数线性空间。

注：如果不用线性方程组的性质，第二部分亦可按如下说明：由$\begin{cases}n_aa_1+n_bb_1+n_cc_1=0\\n_aa_2+n_bb_2+n_cc_2=0\end{cases}$，得$\begin{cases}n_b(b_1a_2-b_2a_1)+n_c(c_1a_2-c_2a_1)=0\\n_a(a_1b_2-a_2b_1)+n_c(c_1b_2-c_2b_1)=0\\n_a(a_1c_2-a_2c_1)+n_b(b_1c_2-b_2c_1)=0\end{cases}$，若$|a\rangle,|b\rangle,|c\rangle$线性无关，则前提条件为$\begin{cases}a_1b_2\neq a_2b_1\\b_1c_2\neq b_2c_1\\c_1a_2\neq c_2a_1\end{cases}$（否则在$|a\rangle,|b\rangle,|c\rangle$中取任意一对向量，必满足线性相关，矛盾！），此时可得到如下比例关系：$n_a:n_b:n_c=(b_1c_2-b_2c_1):(c_1a_2-a_1c_2):(a_1b_2-a_2b_1)$，从而存在不全为零的$n_a,n_b,n_c$，使$n_a|a\rangle+n_b|b\rangle+n_c|c\rangle=0$，即$|a\rangle,|b\rangle,|c\rangle$线性相关，这与$|a\rangle,|b\rangle,|c\rangle$线性无关矛盾。对更多右矢的情形，同理可证它们均满足线性相关。

练习3：证明基矢$\psi_n(x)=\sqrt{\frac{2}{a}}\sin(\frac{n\pi}{a}x),n=1,2,3,\ldots$是正交归一的

证明：证明过程可分为两部分：

1. 正交
   取两个不相同的基矢$\psi_m(x)$和$\psi_n(x)$，其中$m\neq n$，则其内积为：

$$\begin{aligned}\langle\psi_m|\psi_n\rangle&=\int_0^a\psi_m^*(x)\psi_n(x)dx=\int_0^a\frac{2}{a}\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{a}x)dx=\int_0^a\frac{1}{a}[\cos\frac{(m-n)\pi}{a}x-\cos\frac{(m+n)\pi}{a}x]dx\\&=[\frac{1}{(m-n)\pi}\sin\frac{(m-n)\pi}{a}x-\frac{1}{(m+n)\pi}\sin\frac{(m+n)\pi}{a}x]_0^a=0-0=0\end{aligned}$$

2. 归一

对基矢$\psi_n(x)$，求其与自身的内积，得：

$$\begin{aligned}\langle\psi_n|\psi_n\rangle &= \int_0^a \psi_n^*(x)\psi_n(x)dx = \int_0^a \frac{2}{a}\sin^2(\frac{n\pi}{a}x)dx = \int_0^a \frac{1}{a}(1-\cos\frac{2n\pi}{a}x)dx \\ &= [\frac{x}{a} - \frac{1}{2n\pi}\sin\frac{2n\pi}{a}x]_0^a = 1 - 0 = 1\end{aligned}$$

综上，原命题得证。

练习4：推导出Gram-Schmidt正交归一化过程中α_2的表达式

解：易知$|\phi_2\rangle = \alpha_2(|u_2\rangle - |\phi_1\rangle\langle\phi_1|u_2\rangle)$，该基矢满足正交归一条件，即$\langle\phi_2|\phi_2\rangle = 1$，从而代入得：

$$\begin{aligned}\alpha_2^*(\langle u_2| - \langle u_2|\phi_1\rangle\langle\phi_1|)\cdot\alpha_2(|u_2\rangle - |\phi_1\rangle\langle\phi_1|u_2\rangle) &= |\alpha_2|^2(\langle u_2|u_2\rangle - \langle u_2|\phi_1\rangle\langle\phi_1|u_2\rangle - \langle u_2|\phi_1\rangle\langle\phi_1|u_2\rangle + \langle u_2|\phi_1\rangle\langle\phi_1|\phi_1\rangle\langle\phi_1|u_2\rangle) \\ &= |\alpha_2|^2(\langle u_2|u_2\rangle - \langle u_2|\phi_1\rangle\langle\phi_1|u_2\rangle) = 1\text{（利用}\langle\phi_1|\phi_1\rangle = 1\text{）}\end{aligned}$$

因此$|\alpha_2|^2 = \frac{1}{\langle u_2|u_2\rangle - \langle u_2|\phi_1\rangle\langle\phi_1|u_2\rangle}$，即$\alpha_2$的模为$|\alpha_2| = (\langle u_2|u_2\rangle - \langle u_2|\phi_1\rangle\langle\phi_1|u_2\rangle)^{-1/2}$，当然，由于我们无法得知$\alpha_2$的相位角，故无法写出$\alpha_2$的具体形式

练习5：证明Gram-Schmidt正交归一化过程中α_k的表达式为 $\alpha_k = [\langle u_k|u_k\rangle - \sum_{i=1}^{k-1}\langle u_k|\varphi_i\rangle\langle\varphi_i|u_k\rangle]^{-\frac{1}{2}}$

证明：该式最好与$|\varphi_k\rangle = \alpha_k(|u_k\rangle - \sum_{i=1}^{k-1}|\varphi_i\rangle\langle\varphi_i|u_k\rangle)$一起证明，此处$|\varphi_k\rangle$已归一化。首先，我们可以得知$|\varphi_1\rangle = |u_1\rangle\cdot\langle u_1|u_1\rangle^{-\frac{1}{2}} = \alpha_1|u_1\rangle$，并且$|\varphi_2\rangle = \alpha_2(|u_2\rangle - |\varphi_1\rangle\langle\varphi_1|u_2\rangle)$，其中$|\alpha_2| = (\langle u_2|u_2\rangle - \langle u_2|\phi_1\rangle\langle\phi_1|u_2\rangle)^{-1/2}$，当$\alpha_2$取实数时，有$\alpha_2 = (\langle u_2|u_2\rangle - \langle u_2|\phi_1\rangle\langle\phi_1|u_2\rangle)^{-1/2}$。然后，我们假设$|\varphi_k\rangle$和$\alpha_k$的表达式在$k = l$时成立，则对$|u_{l+1}\rangle$正交归一化时，设$|\varphi_{l+1}\rangle = \alpha_{l+1}|u_{l+1}\rangle + \sum_{i=1}^{l}\beta_i|\varphi_i\rangle$，结合$\langle\varphi_j|\varphi_{l+1}\rangle = 0\ (j = 1,2,\dots,l)$得：

$$\langle\varphi_j|\varphi_{l+1}\rangle = \langle\varphi_j|\cdot(\alpha_{l+1}|u_{l+1}\rangle + \sum_{i=1}^{l}\beta_i|\varphi_i\rangle) = \alpha_{l+1}\langle\varphi_j|u_{l+1}\rangle + \sum_{i=1}^{l}\beta_i\delta_{ji} = 0$$

因此$\beta_j = -\alpha_{l+1}\langle\varphi_j|u_{l+1}\rangle$，代回$|\varphi_{l+1}\rangle = \alpha_{l+1}|u_{l+1}\rangle + \beta|\varphi_l\rangle$得
$|\varphi_{l+1}\rangle = \alpha_{l+1}(|u_{l+1}\rangle - \sum_{i=1}^{l}|\varphi_i\rangle\langle\varphi_i|u_{l+1}\rangle)$，又由归一化条件知$\langle\varphi_{l+1}|\varphi_{l+1}\rangle = 1$，因此有：

$$\begin{aligned}\langle\varphi_{l+1}|\varphi_{l+1}\rangle &= \alpha_{l+1}^*(\langle u_{l+1}| - \sum_{i=1}^{l}\langle u_{l+1}|\varphi_i\rangle\langle\varphi_i|)\cdot\alpha_{l+1}(|u_{l+1}\rangle - \sum_{i=1}^{l}|\varphi_i\rangle\langle\varphi_i|u_{l+1}\rangle) \\ &= |\alpha_{l+1}|^2(\langle u_{l+1}|u_{l+1}\rangle - 2\sum_{i=1}^{l}\langle u_{l+1}|\varphi_i\rangle\langle\varphi_i|u_{l+1}\rangle + \sum_{i=1}^{l}\sum_{j=1}^{l}\langle u_{l+1}|\varphi_i\rangle\langle\varphi_i|\varphi_j\rangle\langle\varphi_j|u_{l+1}\rangle) \\ &= |\alpha_{l+1}|^2(\langle u_{l+1}|u_{l+1}\rangle - 2\sum_{i=1}^{l}\langle u_{l+1}|\varphi_i\rangle\langle\varphi_i|u_{l+1}\rangle + \sum_{i=1}^{l}\langle u_{l+1}|\varphi_i\rangle\langle\varphi_i|u_{l+1}\rangle)\quad\text{（根据克罗内克符号的性质）} \\ &= |\alpha_{l+1}|^2(\langle u_{l+1}|u_{l+1}\rangle - \sum_{i=1}^{l}\langle u_{l+1}|\varphi_i\rangle\langle\varphi_i|u_{l+1}\rangle) = 1\end{aligned}$$

从而$|\alpha_{l+1}| = [\langle u_{l+1}|u_{l+1}\rangle - \sum_{i=1}^{l}\langle u_{l+1}|\varphi_i\rangle\langle\varphi_i|u_{l+1}\rangle]^{-\frac{1}{2}}$，当$\alpha_{l+1}$取实数时，有
$\alpha_{l+1} = [\langle u_{l+1}|u_{l+1}\rangle - \sum_{i=1}^{l}\langle u_{l+1}|\varphi_i\rangle\langle\varphi_i|u_{l+1}\rangle]^{-\frac{1}{2}}$。综上，由数学归纳法，得原式成立，证毕