

课堂练习

练习1：证明概率流通量具有如下性质： $\int d^3x \mathbf{j}(\mathbf{x}, t) = \frac{\langle \hat{\mathbf{p}} \rangle(t)}{m}$

证明：根据埃伦费斯特定理，有 $\frac{d}{dt} \langle \hat{\mathbf{x}} \rangle(t) = \frac{\langle \hat{\mathbf{p}} \rangle(t)}{m}$ ，又知

$$\langle \hat{\mathbf{x}} \rangle(t) = \int \psi(\mathbf{x}, t)^* \hat{\mathbf{x}} \psi(\mathbf{x}, t) d^3x = \int \mathbf{x} \psi(\mathbf{x}, t)^* \psi(\mathbf{x}, t) d^3x$$

因此对时间求导得

$$\frac{d}{dt} \langle \hat{\mathbf{x}} \rangle(t) = \int \mathbf{x} \left[\frac{\partial \psi(\mathbf{x}, t)^*}{\partial t} \psi(\mathbf{x}, t) + \psi(\mathbf{x}, t)^* \frac{\partial \psi(\mathbf{x}, t)}{\partial t} \right] d^3x$$

又知含时薛定谔方程为 $i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = \hat{H} \psi(\mathbf{x}, t)$ ，取复共轭得 $-i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t)^* = \hat{H} \psi(\mathbf{x}, t)^*$ ，而哈密顿算符可写成 $\hat{H} = -\frac{\hbar^2}{2m} \nabla_{\mathbf{x}}^2 + V(\hat{\mathbf{x}})$ ，其中 $V(\hat{\mathbf{x}})$ 为关于算符 $\hat{\mathbf{x}}$ 的实函数，因此有

$$\begin{cases} i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = -\frac{\hbar^2}{2m} \nabla_{\mathbf{x}}^2 \psi(\mathbf{x}, t) + V(\hat{\mathbf{x}}) \psi(\mathbf{x}, t) \\ -i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t)^* = -\frac{\hbar^2}{2m} \nabla_{\mathbf{x}}^2 \psi(\mathbf{x}, t)^* + V(\hat{\mathbf{x}}) \psi(\mathbf{x}, t)^* \end{cases}, \text{ 从而}$$

$$\begin{aligned} & \frac{\partial \psi(\mathbf{x}, t)^*}{\partial t} \psi(\mathbf{x}, t) + \psi(\mathbf{x}, t)^* \frac{\partial \psi(\mathbf{x}, t)}{\partial t} \\ &= -\frac{1}{i\hbar} \left[-\frac{\hbar^2}{2m} \nabla_{\mathbf{x}}^2 \psi(\mathbf{x}, t)^* + V(\hat{\mathbf{x}}) \psi(\mathbf{x}, t)^* \right] \cdot \psi(\mathbf{x}, t) + \psi(\mathbf{x}, t)^* \cdot \frac{1}{i\hbar} \left[-\frac{\hbar^2}{2m} \nabla_{\mathbf{x}}^2 \psi(\mathbf{x}, t) + V(\hat{\mathbf{x}}) \psi(\mathbf{x}, t) \right] \\ &= -\frac{1}{i\hbar} \left[-\frac{\hbar^2}{2m} \nabla_{\mathbf{x}}^2 \psi(\mathbf{x}, t)^* + V(\mathbf{x}) \psi(\mathbf{x}, t)^* \right] \cdot \psi(\mathbf{x}, t) + \psi(\mathbf{x}, t)^* \cdot \frac{1}{i\hbar} \left[-\frac{\hbar^2}{2m} \nabla_{\mathbf{x}}^2 \psi(\mathbf{x}, t) + V(\mathbf{x}) \psi(\mathbf{x}, t) \right] \\ &= \frac{i\hbar}{2m} [-\psi(\mathbf{x}, t) \nabla_{\mathbf{x}}^2 \psi(\mathbf{x}, t)^* + \psi(\mathbf{x}, t)^* \nabla_{\mathbf{x}}^2 \psi(\mathbf{x}, t)] \\ &= \frac{i\hbar}{2m} [-\psi(\mathbf{x}, t) \nabla_{\mathbf{x}}^2 \psi(\mathbf{x}, t)^* - \nabla_{\mathbf{x}} \psi(\mathbf{x}, t) \nabla_{\mathbf{x}} \psi(\mathbf{x}, t)^* + \psi(\mathbf{x}, t)^* \nabla_{\mathbf{x}}^2 \psi(\mathbf{x}, t) + \nabla_{\mathbf{x}} \psi(\mathbf{x}, t) \nabla_{\mathbf{x}} \psi(\mathbf{x}, t)^*] \\ &= \frac{i\hbar}{2m} \nabla_{\mathbf{x}} \cdot [-\psi(\mathbf{x}, t) \nabla_{\mathbf{x}} \psi(\mathbf{x}, t)^* + \psi(\mathbf{x}, t)^* \nabla_{\mathbf{x}} \psi(\mathbf{x}, t)] \end{aligned}$$

记概率通量为

$$\mathbf{j}(\mathbf{x}, t) = -\frac{i\hbar}{2m} [\psi(\mathbf{x}, t)^* \nabla_{\mathbf{x}} \psi(\mathbf{x}, t) - \psi(\mathbf{x}, t) \nabla_{\mathbf{x}} \psi(\mathbf{x}, t)^*]$$

则（此处用到边界条件）

$$\frac{d}{dt} \langle \hat{\mathbf{x}} \rangle(t) = \int_V \mathbf{x} [-\nabla_{\mathbf{x}} \cdot \mathbf{j}(\mathbf{x}, t)] d^3x = [-\mathbf{x} \mathbf{j}(\mathbf{x}, t)]_V - \int_V (\nabla_{\mathbf{x}} \mathbf{x}) \cdot [-\mathbf{j}(\mathbf{x}, t)] d^3x = \int_V \mathbf{j}(\mathbf{x}, t) d^3x$$

故最终 $\int d^3x \mathbf{j}(\mathbf{x}, t) = \frac{\langle \hat{\mathbf{p}} \rangle(t)}{m}$

练习2：推导 $\frac{d\hat{O}_I(t)}{dt} = \frac{1}{i\hbar} [\hat{O}_I(t), \hat{H}_0]$

解：由于 $\hat{O}_I(t) = e^{\frac{i}{\hbar} \hat{H}_0 t} \hat{O}_I(0) e^{-\frac{i}{\hbar} \hat{H}_0 t}$ ，因此对时间求导得

$$\begin{aligned} \frac{d\hat{O}_I(t)}{dt} &= \frac{d[e^{\frac{i}{\hbar} \hat{H}_0 t} \hat{O}_I(0) e^{-\frac{i}{\hbar} \hat{H}_0 t}]}{dt} = \frac{i}{\hbar} [\hat{H}_0 e^{\frac{i}{\hbar} \hat{H}_0 t} \hat{O}_I(0) e^{-\frac{i}{\hbar} \hat{H}_0 t} - e^{\frac{i}{\hbar} \hat{H}_0 t} \hat{O}_I(0) e^{-\frac{i}{\hbar} \hat{H}_0 t} \hat{H}_0] \\ &= \frac{i}{\hbar} [\hat{H}_0 \hat{O}_I(t) - \hat{O}_I(t) \hat{H}_0] = \frac{1}{i\hbar} [\hat{O}_I(t), \hat{H}_0] \end{aligned}$$

练习3：写出相互作用表象和薛定谔表象下时间演化算符之间的关系

解：设 t_0 时刻两种表象的态矢重合，即 $|\alpha(t_0)\rangle_I = |\alpha(t_0)\rangle_S$ ，则 t 时刻，两种表象的态矢满足 $|\alpha(t)\rangle_I = e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}|\alpha(t)\rangle_S$ ，记 $\hat{U}_{\hat{H}_0}^\dagger(t, t_0) = e^{\frac{i}{\hbar}\hat{H}_0(t-t_0)}$ ，则 $|\alpha(t)\rangle_I = \hat{U}_{\hat{H}_0}^\dagger(t, t_0)|\alpha(t)\rangle_S = \hat{U}_{\hat{H}_0}^\dagger(t, t_0)\hat{U}(t, t_0)|\alpha(t_0)\rangle_S$ ，又 $|\alpha(t)\rangle_I = \hat{U}_I(t, t_0)|\alpha(t_0)\rangle_I$ ，因此 $\hat{U}_I(t, t_0) = \hat{U}_{\hat{H}_0}^\dagger(t, t_0)\hat{U}(t, t_0)$

练习4：精确求解含时两能级问题，其中零级哈密顿算符

$\hat{H}_0 = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2|$ ，微扰项 $\hat{H}' = \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1|$

第五章习题

5.1 设 $t = 0$ 时，电子处于 \hat{S}_x 的本征态 $|s_x+\rangle$ ，用海森堡表象求解电子在恒定 z 方向磁场 B 中的进动 $\hat{H} = -(\frac{eB}{mc})\hat{S}_z = \omega\hat{S}_z$ ，获得 $\langle\hat{S}_x\rangle$ ， $\langle\hat{S}_y\rangle$ ， $\langle\hat{S}_z\rangle$ 随时间的变化

解：海森堡表象下，态矢为 $|u\rangle = |s_x+\rangle = \frac{1}{\sqrt{2}}(|s_z+\rangle + |s_z-\rangle)$ ，而算符随时间演化变为：

$$\begin{aligned}\hat{S}_x(t) &= \hat{U}^\dagger(t)\hat{S}_x(0)\hat{U}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{S}_x(0)e^{-\frac{i}{\hbar}\hat{H}t} = e^{\frac{i\omega t}{\hbar}\hat{S}_z(0)}\hat{S}_x(0)e^{-\frac{i\omega t}{\hbar}\hat{S}_z(0)} \\ \hat{S}_y(t) &= \hat{U}^\dagger(t)\hat{S}_y(0)\hat{U}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{S}_y(0)e^{-\frac{i}{\hbar}\hat{H}t} = e^{\frac{i\omega t}{\hbar}\hat{S}_z(0)}\hat{S}_y(0)e^{-\frac{i\omega t}{\hbar}\hat{S}_z(0)} \\ \hat{S}_z(t) &= \hat{U}^\dagger(t)\hat{S}_z(0)\hat{U}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{S}_z(0)e^{-\frac{i}{\hbar}\hat{H}t} = e^{\frac{i\omega t}{\hbar}\hat{S}_z(0)}\hat{S}_z(0)e^{-\frac{i\omega t}{\hbar}\hat{S}_z(0)} = \hat{S}_z(0)\end{aligned}$$

因此 t 时刻各个自旋算符的期望值为

$$\begin{aligned}\langle\hat{S}_x\rangle(t) &= \langle u|\hat{S}_x(t)|u\rangle = [\frac{1}{\sqrt{2}}(\langle s_z+| + \langle s_z-|)]e^{\frac{i\omega t}{\hbar}\hat{S}_z(0)}\hat{S}_x(0)e^{-\frac{i\omega t}{\hbar}\hat{S}_z(0)}[\frac{1}{\sqrt{2}}(|s_z+\rangle + |s_z-\rangle)] \\ &= \frac{1}{2}(\langle s_z+|e^{\frac{i\omega t}{2}} + \langle s_z-|e^{-\frac{i\omega t}{2}})\hat{S}_x(0)(e^{-\frac{i\omega t}{2}}|s_z+\rangle + e^{\frac{i\omega t}{2}}|s_z-\rangle) \\ &= \frac{1}{2}(\langle s_z+|e^{\frac{i\omega t}{2}} + \langle s_z-|e^{-\frac{i\omega t}{2}})\frac{1}{2}(\hat{S}_+(0) + \hat{S}_-(0))(e^{-\frac{i\omega t}{2}}|s_z+\rangle + e^{\frac{i\omega t}{2}}|s_z-\rangle) \\ &= \frac{1}{4}(\langle s_z+|e^{\frac{i\omega t}{2}} + \langle s_z-|e^{-\frac{i\omega t}{2}})(e^{-\frac{i\omega t}{2}}\hbar|s_z-\rangle + e^{\frac{i\omega t}{2}}\hbar|s_z+\rangle) = \frac{\hbar}{4}(e^{i\omega t} + e^{-i\omega t}) \\ &= \frac{\hbar}{4}(\cos\omega t + i\sin\omega t + \cos\omega t - i\sin\omega t) = \frac{\hbar}{2}\cos\omega t\end{aligned}$$

$$\begin{aligned}\langle\hat{S}_y\rangle(t) &= \langle u|\hat{S}_y(t)|u\rangle = [\frac{1}{\sqrt{2}}(\langle s_z+| + \langle s_z-|)]e^{\frac{i\omega t}{\hbar}\hat{S}_z(0)}\hat{S}_y(0)e^{-\frac{i\omega t}{\hbar}\hat{S}_z(0)}[\frac{1}{\sqrt{2}}(|s_z+\rangle + |s_z-\rangle)] \\ &= \frac{1}{2}(\langle s_z+|e^{\frac{i\omega t}{2}} + \langle s_z-|e^{-\frac{i\omega t}{2}})\hat{S}_y(0)(e^{-\frac{i\omega t}{2}}|s_z+\rangle + e^{\frac{i\omega t}{2}}|s_z-\rangle) \\ &= \frac{1}{2}(\langle s_z+|e^{\frac{i\omega t}{2}} + \langle s_z-|e^{-\frac{i\omega t}{2}})\frac{1}{2i}(\hat{S}_+(0) - \hat{S}_-(0))(e^{-\frac{i\omega t}{2}}|s_z+\rangle + e^{\frac{i\omega t}{2}}|s_z-\rangle) \\ &= \frac{1}{4i}(\langle s_z+|e^{\frac{i\omega t}{2}} + \langle s_z-|e^{-\frac{i\omega t}{2}})(-e^{-\frac{i\omega t}{2}}\hbar|s_z-\rangle + e^{\frac{i\omega t}{2}}\hbar|s_z+\rangle) = \frac{\hbar}{4i}(e^{i\omega t} - e^{-i\omega t}) \\ &= \frac{\hbar}{4i}(\cos\omega t + i\sin\omega t - \cos\omega t + i\sin\omega t) = \frac{\hbar}{2}\sin\omega t\end{aligned}$$

$$\langle\hat{S}_z\rangle(t) = \langle u|\hat{S}_z(t)|u\rangle = [\frac{1}{\sqrt{2}}(\langle s_z+| + \langle s_z-|)]\hat{S}_z(0)[\frac{1}{\sqrt{2}}(|s_z+\rangle + |s_z-\rangle)] = \frac{1}{2}(\langle s_z+| + \langle s_z-|)(\frac{\hbar}{2}|s_z+\rangle - \frac{\hbar}{2}|s_z-\rangle) = 0$$

5.2 一个粒子的三维运动对应于哈密顿算符 $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ ，试通过计算

$[\hat{x} \cdot \hat{p}, \hat{H}]$ 获得 $\frac{d\langle\hat{x} \cdot \hat{p}\rangle}{dt} = \langle\frac{\hat{p}^2}{m}\rangle - \langle\hat{x} \cdot \nabla V\rangle$ 。如果方程左侧为零，得到维里定理的量子力学形式。在什么情况下是这样的结果？

解：用矢量的形式，我们可以得到 $\hat{x} = \hat{x}_i\mathbf{i} + \hat{x}_j\mathbf{j} + \hat{x}_k\mathbf{k}$ ， $\hat{p} = \hat{p}_i\mathbf{i} + \hat{p}_j\mathbf{j} + \hat{p}_k\mathbf{k}$ ，因此 $\hat{x} \cdot \hat{p} = \hat{x}_i\hat{p}_i + \hat{x}_j\hat{p}_j + \hat{x}_k\hat{p}_k$ ， $\hat{p}^2 = \hat{p}_i^2 + \hat{p}_j^2 + \hat{p}_k^2$ ，从而代入到 $[\hat{x} \cdot \hat{p}, \hat{H}]$ ，得：

$$[\hat{\mathbf{x}} \cdot \hat{\mathbf{p}}, \hat{H}] = [\hat{x}_i \hat{p}_i + \hat{x}_j \hat{p}_j + \hat{x}_k \hat{p}_k, \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}})] = [\hat{x}_i \hat{p}_i + \hat{x}_j \hat{p}_j + \hat{x}_k \hat{p}_k, \frac{\hat{p}_i^2 + \hat{p}_j^2 + \hat{p}_k^2}{2m} + V(\hat{\mathbf{x}})]$$

$$= \frac{1}{2m} [\hat{x}_i \hat{p}_i + \hat{x}_j \hat{p}_j + \hat{x}_k \hat{p}_k, \hat{p}_i^2 + \hat{p}_j^2 + \hat{p}_k^2] + [\hat{x}_i \hat{p}_i + \hat{x}_j \hat{p}_j + \hat{x}_k \hat{p}_k, V(\hat{\mathbf{x}})]$$

首先我们讨论第一项的结果，对于 $u \in \{i, j, k\}$, $v \in \{i, j, k\}$, 我们有

$$[\hat{x}_u \hat{p}_u, \hat{p}_v^2] = \hat{x}_u [\hat{p}_u, \hat{p}_v^2] + [\hat{x}_u, \hat{p}_v^2] \hat{p}_u = \hat{x}_u \cdot 0 + ([\hat{x}_u, \hat{p}_v] \hat{p}_v + \hat{p}_v [\hat{x}_u, \hat{p}_v]) \hat{p}_u = 2i\hbar \delta_{uv} \hat{p}_v \hat{p}_u$$

因此第一项可以化简为

$$\frac{1}{2m} [\hat{x}_i \hat{p}_i + \hat{x}_j \hat{p}_j + \hat{x}_k \hat{p}_k, \hat{p}_i^2 + \hat{p}_j^2 + \hat{p}_k^2] = \frac{1}{2m} \sum_{\substack{u \in \{i, j, k\} \\ v \in \{i, j, k\}}} [\hat{x}_u \hat{p}_u, \hat{p}_v^2] = \frac{1}{2m} \sum_{\substack{u \in \{i, j, k\} \\ v \in \{i, j, k\}}} 2i\hbar \delta_{uv} \hat{p}_v \hat{p}_u = \frac{i\hbar}{m} \sum_{u \in \{i, j, k\}} \hat{p}_u^2 = \frac{i\hbar}{m} \hat{\mathbf{p}}^2$$

接下来讨论第二项的结果，对于 $u \in \{i, j, k\}$, 我们有

$$[\hat{x}_u \hat{p}_u, V(\hat{\mathbf{x}})] = \hat{x}_u \hat{p}_u V(\hat{\mathbf{x}}) - V(\hat{\mathbf{x}}) \hat{x}_u \hat{p}_u = \hat{x}_u \hat{p}_u V(\hat{\mathbf{x}}) - V(\hat{\mathbf{x}}) \hat{x}_u \hat{p}_u = \hat{x}_u V(\hat{\mathbf{x}}) \hat{p}_u + \hat{x}_u [-i\hbar \nabla_{\hat{x}_u} V(\hat{\mathbf{x}})] - V(\hat{\mathbf{x}}) \hat{x}_u \hat{p}_u$$

$$= V(\hat{\mathbf{x}}) \hat{x}_u \hat{p}_u - i\hbar \hat{x}_u \nabla_{\hat{x}_u} V(\hat{\mathbf{x}}) - V(\hat{\mathbf{x}}) \hat{x}_u \hat{p}_u = -i\hbar \hat{x}_u \nabla_{\hat{x}_u} V(\hat{\mathbf{x}})$$

因此第二项可以化简为

$$[\hat{x}_i \hat{p}_i + \hat{x}_j \hat{p}_j + \hat{x}_k \hat{p}_k, V(\hat{\mathbf{x}})] = \sum_{u \in \{i, j, k\}} [\hat{x}_u \hat{p}_u, V(\hat{\mathbf{x}})] = -i\hbar \sum_{u \in \{i, j, k\}} \hat{x}_u \nabla_{\hat{x}_u} V(\hat{\mathbf{x}}) = -i\hbar \hat{\mathbf{x}} \cdot \nabla V(\hat{\mathbf{x}})$$

最终我们可以得到 $[\hat{\mathbf{x}} \cdot \hat{\mathbf{p}}, \hat{H}] = \frac{i\hbar}{m} \hat{\mathbf{p}}^2 - i\hbar \hat{\mathbf{x}} \cdot \nabla V(\hat{\mathbf{x}})$

回到本题，对 $\langle \hat{\mathbf{x}} \cdot \hat{\mathbf{p}} \rangle$ 求导，得

$$\frac{d\langle \hat{\mathbf{x}} \cdot \hat{\mathbf{p}} \rangle}{dt} = \frac{1}{i\hbar} \langle [\hat{\mathbf{x}} \cdot \hat{\mathbf{p}}, \hat{H}] \rangle = \frac{1}{i\hbar} \langle \frac{i\hbar}{m} \hat{\mathbf{p}}^2 - i\hbar \hat{\mathbf{x}} \cdot \nabla V(\hat{\mathbf{x}}) \rangle = \langle \frac{\hat{\mathbf{p}}^2}{m} \rangle - \langle \hat{\mathbf{x}} \cdot \nabla V \rangle$$

当 $[\hat{\mathbf{x}} \cdot \hat{\mathbf{p}}, \hat{H}] = 0$ 时，即粒子处于定态时，方程左侧为零，从而得到维里定理的量子力学形式。

5.3 $t = 0$ 时，一维自由粒子的波函数为一个高斯波包 $\psi(x) = (\frac{1}{\sigma\sqrt{\pi}})^{\frac{1}{2}} e^{-\frac{1}{2}(\frac{x}{\sigma})^2}$ ，在薛定谔表象中求解 t 时刻的波函数，与 $\langle (\Delta x)^2 \rangle_t \langle (\Delta x)^2 \rangle_0 \geq \frac{\hbar^2 t^2}{4m^2}$ 比较，说明波包随时间越来越弥散

解：对于一维自由粒子，其哈密顿算符为 $\hat{H} = \frac{\hat{p}^2}{2m}$ ，因此时间演化算符可写作 $\hat{U} = e^{-\frac{i}{\hbar} \hat{H} t} = e^{-\frac{it\hat{p}^2}{2m\hbar}}$ ，又根据傅里叶变换，得 $\tilde{\psi}(p, 0) = (2\pi\hbar)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar} p \cdot x} \psi(x, 0) dx$ （其中 $\psi(x, 0)$ 即 $\psi(x)$ ），因此 $\tilde{\psi}(p, t) = \hat{U} \tilde{\psi}(p, 0) = e^{-\frac{itp^2}{2m\hbar}} \tilde{\psi}(p, 0)$ ，再经傅里叶变换得 $\psi(x, t) = (2\pi\hbar)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{\frac{i}{\hbar} p \cdot x} \tilde{\psi}(p, t) dp$ 。现在我们来求解这些表达式：

$$\begin{aligned} \tilde{\psi}(p, 0) &= (2\pi\hbar)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar} p \cdot x} \psi(x, 0) dx = (2\pi\hbar)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar} p \cdot x} \left(\frac{1}{\sigma\sqrt{\pi}}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(\frac{x}{\sigma})^2} dx \\ &= (2\pi\hbar)^{-\frac{1}{2}} \left(\frac{1}{\sigma\sqrt{\pi}}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2} - \frac{i}{\hbar} p x} dx = (2\pi\sqrt{\pi}\hbar\sigma)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2\sigma^2}(x + \frac{i\sigma^2}{\hbar}p)^2 - \frac{\sigma^2 p^2}{2\hbar^2}} dx \\ &= (2\pi\sqrt{\pi}\hbar\sigma)^{-\frac{1}{2}} e^{-\frac{\sigma^2 p^2}{2\hbar^2}} \int_{-\infty}^{+\infty} e^{-\left(\frac{x + \frac{i\sigma^2}{\hbar}p}{\sqrt{2}\sigma}\right)^2} \cdot \sqrt{2}\sigma d\left(\frac{x + \frac{i\sigma^2}{\hbar}p}{\sqrt{2}\sigma}\right) \\ &= (2\pi\sqrt{\pi}\hbar\sigma)^{-\frac{1}{2}} e^{-\frac{\sigma^2 p^2}{2\hbar^2}} \sqrt{2}\sigma \cdot \sqrt{\pi} = \left(\frac{\sigma}{\pi^{\frac{1}{2}}\hbar}\right)^{\frac{1}{2}} e^{-\frac{\sigma^2 p^2}{2\hbar^2}} \\ \tilde{\psi}(p, t) &= e^{-\frac{itp^2}{2m\hbar}} \tilde{\psi}(p, 0) = e^{-\frac{itp^2}{2m\hbar}} \cdot \left(\frac{\sigma}{\pi^{\frac{1}{2}}\hbar}\right)^{\frac{1}{2}} e^{-\frac{\sigma^2 p^2}{2\hbar^2}} = \left(\frac{\sigma}{\pi^{\frac{1}{2}}\hbar}\right)^{\frac{1}{2}} e^{-\left(\frac{it}{2m\hbar} + \frac{\sigma^2}{2\hbar^2}\right)p^2} \end{aligned}$$

$$\begin{aligned}
\psi(x, t) &= (2\pi\hbar)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{\frac{i}{\hbar} p \cdot x} \tilde{\psi}(p, t) dp = (2\pi\hbar)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{\frac{i}{\hbar} p \cdot x} \left(\frac{\sigma}{\pi^{\frac{1}{2}} \hbar}\right)^{\frac{1}{2}} e^{-\left(\frac{it}{2m\hbar} + \frac{\sigma^2}{2\hbar^2}\right)p^2} dp \\
&= (2\pi\hbar)^{-\frac{1}{2}} \left(\frac{\sigma}{\pi^{\frac{1}{2}} \hbar}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\left(\frac{it}{2m\hbar} + \frac{\sigma^2}{2\hbar^2}\right)p^2 + \frac{i}{\hbar} p \cdot x} dp = (2\pi\hbar)^{-\frac{1}{2}} \left(\frac{\sigma}{\pi^{\frac{1}{2}} \hbar}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\left(\frac{it}{2m\hbar} + \frac{\sigma^2}{2\hbar^2}\right)\left(p - \frac{\frac{i}{\hbar} x}{\frac{it}{m\hbar} + \frac{\sigma^2}{\hbar^2}}\right)^2 - \frac{x^2}{2\hbar^2\left(\frac{it}{m\hbar} + \frac{\sigma^2}{\hbar^2}\right)}} dp \\
&= \left(\frac{\sigma}{2\pi^{\frac{3}{2}} \hbar^2}\right)^{\frac{1}{2}} e^{-\frac{x^2}{2\hbar^2\left(\frac{it}{m\hbar} + \frac{\sigma^2}{\hbar^2}\right)}} \int_{-\infty}^{+\infty} e^{-\left[\sqrt{\frac{it}{2m\hbar} + \frac{\sigma^2}{2\hbar^2}}\left(p - \frac{\frac{i}{\hbar} x}{\frac{it}{m\hbar} + \frac{\sigma^2}{\hbar^2}}\right)\right]^2} \cdot \frac{1}{\sqrt{\frac{it}{2m\hbar} + \frac{\sigma^2}{2\hbar^2}}} d\left[\sqrt{\frac{it}{2m\hbar} + \frac{\sigma^2}{2\hbar^2}}\left(p - \frac{\frac{i}{\hbar} x}{\frac{it}{m\hbar} + \frac{\sigma^2}{\hbar^2}}\right)\right] \\
&= \left(\frac{\sigma}{2\pi^{\frac{3}{2}} \hbar^2}\right)^{\frac{1}{2}} e^{-\frac{x^2}{2\hbar^2\left(\frac{it}{m\hbar} + \frac{\sigma^2}{\hbar^2}\right)}} \frac{1}{\sqrt{\frac{it}{2m\hbar} + \frac{\sigma^2}{2\hbar^2}}} \cdot \sqrt{\pi} = \left[\frac{1}{\sigma\pi^{\frac{1}{2}}\left(1 + \frac{i\hbar t}{m\sigma^2}\right)}\right]^{\frac{1}{2}} e^{-\frac{x^2}{2\sigma^2\left(1 + \frac{i\hbar t}{m\sigma^2}\right)}}
\end{aligned}$$

现在我们来求解 $\langle(\Delta x)^2\rangle_0$ 和 $\langle(\Delta x)^2\rangle_t$ ，显然

$$\begin{aligned}
\langle x \rangle_0 &= \int_{-\infty}^{+\infty} x |\psi(x, 0)|^2 dx = \int_{-\infty}^{+\infty} \frac{x}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{\sigma^2}} dx = 0 \\
\langle x^2 \rangle_0 &= \int_{-\infty}^{+\infty} x^2 |\psi(x, 0)|^2 dx = \int_{-\infty}^{+\infty} \frac{x^2}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{\sigma^2}} dx = \frac{\sigma^2}{2}
\end{aligned}$$

$$\begin{aligned}
\langle x \rangle_t &= \int_{-\infty}^{+\infty} x |\psi(x, t)|^2 dx = \int_{-\infty}^{+\infty} \frac{x}{\sigma\pi^{\frac{1}{2}} \sqrt{1 - \frac{\hbar^2 t^2}{m^2 \sigma^4}}} e^{-\frac{x^2}{\sigma^2\left(1 + \frac{\hbar^2 t^2}{m^2 \sigma^4}\right)}} dx = 0 \\
\langle x^2 \rangle_t &= \int_{-\infty}^{+\infty} x^2 |\psi(x, t)|^2 dx = \int_{-\infty}^{+\infty} \frac{x^2}{\sigma\pi^{\frac{1}{2}} \sqrt{1 + \frac{\hbar^2 t^2}{m^2 \sigma^4}}} e^{-\frac{x^2}{\sigma^2\left(1 + \frac{\hbar^2 t^2}{m^2 \sigma^4}\right)}} dx = \frac{\sigma^2\left(1 + \frac{\hbar^2 t^2}{m^2 \sigma^4}\right)}{2}
\end{aligned}$$

因此有

$$\begin{aligned}
\langle(\Delta x)^2\rangle_0 &= \langle(x - \langle x \rangle_0)^2\rangle_0 = \langle x^2 \rangle_0 - \langle x \rangle_0^2 = \frac{\sigma^2}{2} \\
\langle(\Delta x)^2\rangle_t &= \langle(x - \langle x \rangle_t)^2\rangle_t = \langle x^2 \rangle_t - \langle x \rangle_t^2 = \frac{\sigma^2\left(1 + \frac{\hbar^2 t^2}{m^2 \sigma^4}\right)}{2} \\
\langle(\Delta x)^2\rangle_t \langle(\Delta x)^2\rangle_0 &= \frac{\sigma^2\left(1 + \frac{\hbar^2 t^2}{m^2 \sigma^4}\right)}{2} \frac{\sigma^2}{2} = \frac{\sigma^4}{4} + \frac{\hbar^2 t^2}{4m^2} \geq \frac{\hbar^2 t^2}{4m^2}
\end{aligned}$$

5.4 请用海森堡表象求解一维谐振子体系坐标与动量算符随时间演化的问题。如果初始状态是基态 $\langle x|0\rangle$ 平移一段距离 s ，坐标与动量的平均值随时间的变化有什么特征？

解：在海森堡表象下，算符微分为 $\frac{d\hat{B}_H(t)}{dt} = \frac{1}{i\hbar}[\hat{B}_H(t), \hat{H}]$ ，而一维谐振子体系的哈密顿算符为 $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}$ ，因此有（记 $\hat{x}_H(0) \equiv \hat{x}$ ， $\hat{p}_H(0) \equiv \hat{p}$ ）：

$$\begin{aligned}
\frac{d\hat{x}_H(t)}{dt} &= \frac{1}{i\hbar}[\hat{x}_H(t), \hat{H}] = \frac{1}{i\hbar} e^{\frac{i}{\hbar} \hat{H} t} [\hat{x}_H(0), \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}] e^{-\frac{i}{\hbar} \hat{H} t} = \frac{1}{i\hbar} e^{\frac{i}{\hbar} \hat{H} t} \frac{i\hbar \hat{p}}{m} e^{-\frac{i}{\hbar} \hat{H} t} = \frac{\hat{p}_H(t)}{m} \\
\frac{d\hat{p}_H(t)}{dt} &= \frac{1}{i\hbar}[\hat{p}_H(t), \hat{H}] = \frac{1}{i\hbar} e^{\frac{i}{\hbar} \hat{H} t} [\hat{p}_H(0), \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}] e^{-\frac{i}{\hbar} \hat{H} t} = \frac{1}{i\hbar} e^{\frac{i}{\hbar} \hat{H} t} (-i\hbar m\omega^2 \hat{x}) e^{-\frac{i}{\hbar} \hat{H} t} = -m\omega^2 \hat{x}_H(t)
\end{aligned}$$

对两边分别求导，得 $\begin{cases} \frac{d^2 \hat{x}_H(t)}{dt^2} = \frac{1}{m} \frac{d\hat{p}_H(t)}{dt} = -\omega^2 \hat{x}_H(t) \\ \frac{d^2 \hat{p}_H(t)}{dt^2} = -m\omega^2 \frac{d\hat{x}_H(t)}{dt} = -\omega^2 \hat{p}_H(t) \end{cases}$ ，相应的，这个方程组的通解为

$$\begin{cases} \hat{x}_H(t) = A \cos \omega t + B \sin \omega t \\ \hat{p}_H(t) = C \cos \omega t + D \sin \omega t \end{cases}, \text{对这个解求导得} \begin{cases} \frac{d\hat{x}_H(t)}{dt} = -\omega A \sin \omega t + \omega B \cos \omega t \\ \frac{d\hat{p}_H(t)}{dt} = -\omega C \sin \omega t + \omega D \cos \omega t \end{cases}, \text{当} t = 0$$

时，根据上面的条件，可得：

$$\begin{cases} \hat{x}_H(0) = A \\ \hat{p}_H(0) = C \\ \frac{d\hat{x}_H(0)}{dt} = \omega B = \frac{\hat{p}_H(0)}{m} \\ \frac{d\hat{p}_H(0)}{dt} = \omega D = -m\omega^2 \hat{x}_H(0) \end{cases} \Rightarrow \begin{cases} A = \hat{x} \\ B = \frac{\hat{p}}{m\omega} \\ C = \hat{p} \\ D = -m\omega \hat{x} \end{cases}$$

因此在海森堡表象下，坐标与动量算符为 $\begin{cases} \hat{x}_H(t) = \hat{x} \cos \omega t + \frac{\hat{p}}{m\omega} \sin \omega t \\ \hat{p}_H(t) = \hat{p} \cos \omega t - m\omega \hat{x} \sin \omega t \end{cases}$

当初始波函数为 $\psi(x) = \langle x|0' \rangle = (\frac{1}{x_0\sqrt{\pi}})^{\frac{1}{2}} e^{-\frac{(x-s)^2}{2x_0^2}}$ (其中 $x_0 \equiv \sqrt{\frac{\hbar}{m\omega}}$) 时，对算符求平均值，得：

$$\begin{aligned} \langle \hat{x} \rangle(t) &= \langle 0' | \hat{x}_H(t) | 0' \rangle = \int_{-\infty}^{+\infty} \langle 0' | x \rangle \langle x | \hat{x}_H(t) | 0' \rangle dx = \int_{-\infty}^{+\infty} \langle 0' | x \rangle \langle x | (\hat{x} \cos \omega t + \frac{\hat{p}}{m\omega} \sin \omega t) | 0' \rangle dx \\ &= \cos \omega t \int_{-\infty}^{+\infty} x |\langle x | 0' \rangle|^2 dx + \frac{\sin \omega t}{m\omega} \int_{-\infty}^{+\infty} \langle 0' | x \rangle (-i\hbar \nabla \langle x | 0' \rangle) dx \\ &= \cos \omega t \int_{-\infty}^{+\infty} \frac{x}{x_0\sqrt{\pi}} e^{-\frac{(x-s)^2}{x_0^2}} dx + \frac{\sin \omega t}{m\omega} \int_{-\infty}^{+\infty} (\frac{1}{x_0\sqrt{\pi}})^{\frac{1}{2}} e^{-\frac{(x-s)^2}{2x_0^2}} \cdot [-i\hbar (\frac{1}{x_0\sqrt{\pi}})^{\frac{1}{2}} e^{-\frac{(x-s)^2}{2x_0^2}} \cdot (-\frac{x-s}{x_0^2})] dx \\ &= s \cos \omega t \end{aligned}$$

$$\begin{aligned} \langle \hat{p} \rangle(t) &= \langle 0' | \hat{p}_H(t) | 0' \rangle = \int_{-\infty}^{+\infty} \langle 0' | x \rangle \langle x | \hat{p}_H(t) | 0' \rangle dx = \int_{-\infty}^{+\infty} \langle 0' | x \rangle \langle x | (\hat{p} \cos \omega t - m\omega \hat{x} \sin \omega t) | 0' \rangle dx \\ &= \cos \omega t \int_{-\infty}^{+\infty} \langle 0' | x \rangle (-i\hbar \nabla \langle x | 0' \rangle) dx - m\omega \sin \omega t \int_{-\infty}^{+\infty} x |\langle x | 0' \rangle|^2 dx \\ &= \cos \omega t \int_{-\infty}^{+\infty} (\frac{1}{x_0\sqrt{\pi}})^{\frac{1}{2}} e^{-\frac{(x-s)^2}{2x_0^2}} [-i\hbar (\frac{1}{x_0\sqrt{\pi}})^{\frac{1}{2}} e^{-\frac{(x-s)^2}{2x_0^2}} \cdot (-\frac{x-s}{x_0^2})] dx - m\omega \sin \omega t \int_{-\infty}^{+\infty} \frac{x}{x_0\sqrt{\pi}} e^{-\frac{(x-s)^2}{x_0^2}} dx \\ &= -m\omega s \sin \omega t \end{aligned}$$

即坐标与动量的平均值随时间变化，分别呈余弦函数和正弦函数曲线

5.5 在海森堡表象中推导艾伦费斯特定理

解：在海森堡表象下，对算符 \hat{x} 在 t 时刻的期望值 $\langle \hat{x} \rangle(t)$ 求关于时间 t 的导数，得（记海森堡表象下的态矢为 $|u\rangle \equiv |u\rangle_H$ ）：

$$\begin{aligned} \frac{d}{dt} \langle \hat{x} \rangle(t) &= \frac{d}{dt} \langle u | \hat{x}_H(t) | u \rangle = \frac{d}{dt} \langle u | \hat{U}^\dagger(t) \hat{x}_H(0) \hat{U}(t) | u \rangle = \frac{d}{dt} \langle u | e^{\frac{i}{\hbar} \hat{H}t} \hat{x}_H(0) e^{-\frac{i}{\hbar} \hat{H}t} | u \rangle \\ &= \langle u | \frac{i}{\hbar} \hat{H} e^{\frac{i}{\hbar} \hat{H}t} \hat{x}_H(0) e^{-\frac{i}{\hbar} \hat{H}t} | u \rangle + \langle u | e^{\frac{i}{\hbar} \hat{H}t} \hat{x}_H(0) e^{-\frac{i}{\hbar} \hat{H}t} (-\frac{i}{\hbar} \hat{H}) | u \rangle \\ &= \frac{i}{\hbar} \langle u | (\hat{H} \hat{x}_H(t) - \hat{x}_H(t) \hat{H}) | u \rangle = \frac{1}{i\hbar} \langle u | [\hat{x}_H(t), \hat{H}] | u \rangle \end{aligned}$$

而哈密顿算符可写作 $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) = \frac{\hat{p}_H(0)^2}{2m} + V(\hat{x}_H(0))$ ，因此代入得

$$\begin{aligned} \frac{d}{dt} \langle \hat{x} \rangle(t) &= \frac{i}{\hbar} \langle u | e^{\frac{i}{\hbar} \hat{H}t} [\hat{H} \hat{x}_H(0) - \hat{x}_H(0) \hat{H}] e^{-\frac{i}{\hbar} \hat{H}t} | u \rangle \\ &= \frac{i}{\hbar} \langle u | e^{\frac{i}{\hbar} \hat{H}t} \{ [\frac{\hat{p}_H(0)^2}{2m} + V(\hat{x}_H(0))] \hat{x}_H(0) - \hat{x}_H(0) [\frac{\hat{p}_H(0)^2}{2m} + V(\hat{x}_H(0))] \} e^{-\frac{i}{\hbar} \hat{H}t} | u \rangle \\ &= \frac{i}{\hbar} \langle u | e^{\frac{i}{\hbar} \hat{H}t} \{ \frac{\hat{p}_H(0)^2}{2m} \hat{x}_H(0) - \hat{x}_H(0) \frac{\hat{p}_H(0)^2}{2m} \} e^{-\frac{i}{\hbar} \hat{H}t} | u \rangle \\ &= \frac{i}{2\hbar m} \cdot (-2i\hbar) \langle u | e^{\frac{i}{\hbar} \hat{H}t} \hat{p}_H(0) e^{-\frac{i}{\hbar} \hat{H}t} | u \rangle = \frac{\langle \hat{p}_H(t) \rangle}{m} = \frac{\langle \hat{p} \rangle(t)}{m} \end{aligned}$$

5.6 证明 $[\hat{x}, F(\hat{p})] = i\hbar \frac{\partial}{\partial \hat{p}} F(\hat{p})$, $[\hat{p}, G(\hat{x})] = -i\hbar \frac{\partial}{\partial \hat{x}} G(\hat{x})$

证明：首先我们证明 $[\hat{x}, \hat{p}^n] = i\hbar n \hat{p}^{n-1}$, $[\hat{p}, \hat{x}^n] = -i\hbar n \hat{x}^{n-1}$ ，显然

$$\begin{aligned}
[\hat{x}, \hat{p}^n] &= \hat{x}\hat{p}^n - \hat{p}^n\hat{x} = ([\hat{x}, \hat{p}] + \hat{p}\hat{x})\hat{p}^{n-1} - \hat{p}^n\hat{x} = (i\hbar + \hat{p}\hat{x})\hat{p}^{n-1} - \hat{p}^n\hat{x} = i\hbar\hat{p}^{n-1} + \hat{p}\hat{x}\hat{p}^{n-1} - \hat{p}^n\hat{x} \\
&= i\hbar\hat{p}^{n-1} + \hat{p}([\hat{x}, \hat{p}] + \hat{p}\hat{x})\hat{p}^{n-2} - \hat{p}^n\hat{x} = i\hbar\hat{p}^{n-1} + \hat{p}(i\hbar + \hat{p}\hat{x})\hat{p}^{n-2} - \hat{p}^n\hat{x} \\
&= 2i\hbar\hat{p}^{n-1} + \hat{p}^2\hat{x}\hat{p}^{n-2} - \hat{p}^n\hat{x} = \dots = i\hbar n\hat{p}^{n-1}
\end{aligned}$$

$$\begin{aligned}
[\hat{p}, \hat{x}^n] &= \hat{p}\hat{x}^n - \hat{x}^n\hat{p} = \hat{p}\hat{x}^n - \hat{x}^{n-1}([\hat{x}, \hat{p}] + \hat{p}\hat{x}) = \hat{p}\hat{x}^n - \hat{x}^{n-1}(i\hbar + \hat{p}\hat{x}) = \hat{p}\hat{x}^n - i\hbar\hat{x}^{n-1} - \hat{x}^{n-1}\hat{p}\hat{x} \\
&= \hat{p}\hat{x}^n - i\hbar\hat{x}^{n-1} - \hat{x}^{n-2}([\hat{x}, \hat{p}] + \hat{p}\hat{x})\hat{x} = \hat{p}\hat{x}^n - i\hbar\hat{x}^{n-1} - \hat{x}^{n-2}(i\hbar + \hat{p}\hat{x})\hat{x} \\
&= \hat{p}\hat{x}^n - 2i\hbar\hat{x}^{n-1} - \hat{x}^{n-2}\hat{p}\hat{x}^2 = \dots = -i\hbar n\hat{x}^{n-1}
\end{aligned}$$

接下来，将关于算符的函数展开，得 $F(\hat{p}) = \sum_{i=0}^{\infty} c_i \hat{p}^i$ ， $G(\hat{x}) = \sum_{i=0}^{\infty} c_i \hat{x}^i$ ，因此

$$[\hat{x}, F(\hat{p})] = [\hat{x}, \sum_{i=0}^{\infty} c_i \hat{p}^i] = \sum_{i=0}^{\infty} c_i [\hat{x}, \hat{p}^i] = \sum_{i=0}^{\infty} c_i i\hbar n \hat{p}^{n-1} = i\hbar \sum_{i=0}^{\infty} c_i \frac{\partial \hat{p}^n}{\partial \hat{p}} = i\hbar \frac{\partial \sum_{i=0}^{\infty} c_i \hat{p}^n}{\partial \hat{p}} = i\hbar \frac{\partial}{\partial \hat{p}} F(\hat{p})$$

$$[\hat{p}, G(\hat{x})] = [\hat{p}, \sum_{i=0}^{\infty} c_i \hat{x}^i] = \sum_{i=0}^{\infty} c_i [\hat{p}, \hat{x}^i] = \sum_{i=0}^{\infty} c_i (-i\hbar n \hat{x}^{n-1}) = -i\hbar \sum_{i=0}^{\infty} c_i \frac{\partial \hat{x}^n}{\partial \hat{x}} = -i\hbar \frac{\partial \sum_{i=0}^{\infty} c_i \hat{x}^n}{\partial \hat{x}} = -i\hbar \frac{\partial}{\partial \hat{x}} G(\hat{x})$$

5.7 对于自旋1/2的体系，设其处在由0.7概率的 $|s_x + \rangle$ 态和0.3概率的 $|s_y - \rangle$ 态所构成的混合态中，请根据 \hat{S}_z 的本征态表示出该混合态对应的密度算符及相应的密度矩阵

解：因为 $|s_x + \rangle = \frac{1}{\sqrt{2}}(|s_z + \rangle + |s_z - \rangle)$ ， $|s_y - \rangle = \frac{1}{\sqrt{2}}(|s_z + \rangle - i|s_z - \rangle)$ ，所以题中混合态的密度算符为：

$$\begin{aligned}
\hat{\rho} &= 0.7|s_x + \rangle\langle s_x + | + 0.3|s_y - \rangle\langle s_y - | \\
&= 0.7 \cdot \frac{1}{\sqrt{2}}(|s_z + \rangle + |s_z - \rangle) \cdot \frac{1}{\sqrt{2}}(\langle s_z + | + \langle s_z - |) \\
&\quad + 0.3 \cdot \frac{1}{\sqrt{2}}(|s_z + \rangle - i|s_z - \rangle) \cdot \frac{1}{\sqrt{2}}(\langle s_z + | + i\langle s_z - |) \\
&= 0.35(|s_z + \rangle\langle s_z + | + |s_z + \rangle\langle s_z - | + |s_z - \rangle\langle s_z + | + |s_z - \rangle\langle s_z - |) \\
&\quad + 0.15(|s_z + \rangle\langle s_z + | + i|s_z + \rangle\langle s_z - | - i|s_z - \rangle\langle s_z + | + |s_z - \rangle\langle s_z - |) \\
&= 0.5|s_z + \rangle\langle s_z + | + (0.35 + 0.15i)|s_z + \rangle\langle s_z - | + (0.35 - 0.15i)|s_z - \rangle\langle s_z + | + 0.5|s_z - \rangle\langle s_z - |
\end{aligned}$$

写成密度矩阵的形式，即为 $\rho = \begin{pmatrix} 0.5 & 0.35 + 0.15i \\ 0.35 - 0.15i & 0.5 \end{pmatrix}$