# 课堂练习

## 练习1：证明旋轨耦合中 $|j=\frac{1}{2},m=-\frac{1}{2}\rangle=\sqrt{\frac{1}{3}}|m_1=0,m_2=-\frac{1}{2}\rangle-\sqrt{\frac{2}{3}}|m_1=-1,m_2=\frac{1}{2}\rangle$

**证明**：首先，根据$m=m_1+m_2$的耦合条件，只有满足$m_1+m_2=-\frac{1}{2}$的未耦合态前面的系数才不为0，此外$m_2=\pm\frac{1}{2}$，因此可设$|j=\frac{1}{2},m=-\frac{1}{2}\rangle=a|m_1=0,m_2=-\frac{1}{2}\rangle+b|m_1=-1,m_2=\frac{1}{2}\rangle$，则由态矢的归一性，以及$|j=\frac{1}{2},m=-\frac{1}{2}\rangle$与$|j=\frac{3}{2},m=-\frac{1}{2}\rangle$的正交性（其中$|j=\frac{3}{2},m=-\frac{1}{2}\rangle=\sqrt{\frac{2}{3}}|m_1=0,m_2=-\frac{1}{2}\rangle+\sqrt{\frac{1}{3}}|m_1=-1,m_2=\frac{1}{2}\rangle$），可得$\begin{cases}|a|^2+|b|^2=1\\ \sqrt{\frac{2}{3}}a+\sqrt{\frac{1}{3}}b=0\end{cases}$，解得$\begin{cases}a=\sqrt{\frac{1}{3}}\\ b=-\sqrt{\frac{2}{3}}\end{cases}$或$\begin{cases}a=-\sqrt{\frac{1}{3}}\\ b=\sqrt{\frac{2}{3}}\end{cases}$，不妨取$a$为正实数，则代入得$|j=\frac{1}{2},m=-\frac{1}{2}\rangle=\sqrt{\frac{1}{3}}|m_1=0,m_2=-\frac{1}{2}\rangle-\sqrt{\frac{2}{3}}|m_1=-1,m_2=\frac{1}{2}\rangle$，证毕

## 练习2：证明$\boldsymbol{R}_x(\varepsilon)\boldsymbol{R}_y(\varepsilon)-\boldsymbol{R}_y(\varepsilon)\boldsymbol{R}_x(\varepsilon)=\boldsymbol{R}_z(\varepsilon^2)-\boldsymbol{I}$

**证明**：根据定义，旋转矩阵展开至二阶项时的形式为：

$$\boldsymbol{R}_x(\varepsilon)=\begin{bmatrix}1&0&0\\0&1-\frac{\varepsilon^2}{2}&-\varepsilon\\0&\varepsilon&1-\frac{\varepsilon^2}{2}\end{bmatrix}\quad \boldsymbol{R}_y(\varepsilon)=\begin{bmatrix}1-\frac{\varepsilon^2}{2}&0&\varepsilon\\0&1&0\\-\varepsilon&0&1-\frac{\varepsilon^2}{2}\end{bmatrix}\quad \boldsymbol{R}_z(\varepsilon)=\begin{bmatrix}1-\frac{\varepsilon^2}{2}&-\varepsilon&0\\\varepsilon&1-\frac{\varepsilon^2}{2}&0\\0&0&1\end{bmatrix}$$

作矩阵乘法得：

$$\boldsymbol{R}_x(\varepsilon)\boldsymbol{R}_y(\varepsilon)=\begin{bmatrix}1&0&0\\0&1-\frac{\varepsilon^2}{2}&-\varepsilon\\0&\varepsilon&1-\frac{\varepsilon^2}{2}\end{bmatrix}\begin{bmatrix}1-\frac{\varepsilon^2}{2}&0&\varepsilon\\0&1&0\\-\varepsilon&0&1-\frac{\varepsilon^2}{2}\end{bmatrix}=\begin{bmatrix}1-\frac{\varepsilon^2}{2}&0&\varepsilon\\\varepsilon^2&1-\frac{\varepsilon^2}{2}&-\varepsilon(1-\frac{\varepsilon^2}{2})\\-\varepsilon(1-\frac{\varepsilon^2}{2})&\varepsilon&(1-\frac{\varepsilon^2}{2})^2\end{bmatrix}$$

$$\boldsymbol{R}_y(\varepsilon)\boldsymbol{R}_x(\varepsilon)=\begin{bmatrix}1-\frac{\varepsilon^2}{2}&0&\varepsilon\\0&1&0\\-\varepsilon&0&1-\frac{\varepsilon^2}{2}\end{bmatrix}\begin{bmatrix}1&0&0\\0&1-\frac{\varepsilon^2}{2}&-\varepsilon\\0&\varepsilon&1-\frac{\varepsilon^2}{2}\end{bmatrix}=\begin{bmatrix}1-\frac{\varepsilon^2}{2}&\varepsilon^2&\varepsilon(1-\frac{\varepsilon^2}{2})\\0&1-\frac{\varepsilon^2}{2}&-\varepsilon\\-\varepsilon&\varepsilon(1-\frac{\varepsilon^2}{2})&(1-\frac{\varepsilon^2}{2})^2\end{bmatrix}$$

两项相减，并忽略三次及更高次项，得：

$$\boldsymbol{R}_x(\varepsilon)\boldsymbol{R}_y(\varepsilon)-\boldsymbol{R}_y(\varepsilon)\boldsymbol{R}_x(\varepsilon)=\begin{bmatrix}0&-\varepsilon^2&\frac{\varepsilon^3}{2}\\\varepsilon^2&0&\frac{\varepsilon^3}{2}\\\frac{\varepsilon^3}{2}&\frac{\varepsilon^3}{2}&0\end{bmatrix}\simeq\begin{bmatrix}0&-\varepsilon^2&0\\\varepsilon^2&0&0\\0&0&0\end{bmatrix}$$

又

$$\boldsymbol{R}_z(\varepsilon^2)-\boldsymbol{I}=\begin{bmatrix}1-\frac{\varepsilon^4}{2}&-\varepsilon^2&0\\\varepsilon^2&1-\frac{\varepsilon^4}{2}&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}-\frac{\varepsilon^4}{2}&-\varepsilon^2&0\\\varepsilon^2&-\frac{\varepsilon^4}{2}&0\\0&0&0\end{bmatrix}\simeq\begin{bmatrix}0&-\varepsilon^2&0\\\varepsilon^2&0&0\\0&0&0\end{bmatrix}$$

因此$\boldsymbol{R}_x(\varepsilon)\boldsymbol{R}_y(\varepsilon)-\boldsymbol{R}_y(\varepsilon)\boldsymbol{R}_x(\varepsilon)=\boldsymbol{R}_z(\varepsilon^2)-\boldsymbol{I}$，原题得证

**练习3：是否有其他方式来推导4.6.7的结论（即$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\boldsymbol{S}_y\theta}=\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$）？**

**解**:

## 第四章习题

### 4.1 用直接计算在$|S_z\pm\rangle$上表示的矩阵元的方法验证4.3.5和4.3.6式，即

$$\boldsymbol{S}_x=\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad \boldsymbol{S}_y=\frac{\hbar}{2}\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix}\quad \boldsymbol{S}_z=\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\quad \boldsymbol{S}_+=\hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}\quad \boldsymbol{S}_-=\hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

### 4.2 在坐标空间表象中，$\hat{L}_z,\hat{L}^2$的本征函数是球谐函数，在动量空间表象中，$\hat{L}_z,\hat{L}^2$的本征函数是什么？

**解**：可以证明，在动量空间表象中，$\hat{L}_z,\hat{L}^2$的本征函数仍然是球谐函数，证明如下：
设在位置空间表象中，任意波函数可表示为$\psi(\boldsymbol{r})=\psi(r,\theta,\varphi)=f(r)Y_L^M(\theta,\varphi)$，则经傅里叶变换后，波函数在动量空间表象的形式为：

$$\widetilde{\psi}(\boldsymbol{p})=(2\pi)^{-\frac{3}{2}}\iiint \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}f(r)Y_L^M(\theta,\varphi)r^2\sin\theta drd\varphi d\theta$$

又平面波按球谐函数展开之后为：

$$\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\mathrm{e}^{-\frac{\mathrm{i}\pi l}{2}}j_l(kr)[Y_l^m(\theta,\varphi)]^*Y_l^m(\theta_p,\varphi_p)$$

因此代入得

$$\begin{aligned}\widetilde{\psi}(\boldsymbol{p})&=(2\pi)^{-\frac{3}{2}}\int r^2dr\iint 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\mathrm{e}^{-\frac{\mathrm{i}\pi l}{2}}j_l(kr)[Y_l^m(\theta,\varphi)]^*Y_l^m(\theta_p,\varphi_p)f(r)Y_L^M(\theta,\varphi)\sin\theta d\varphi d\theta\\&=\sqrt{\frac{2}{\pi}}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\mathrm{e}^{-\frac{\mathrm{i}\pi l}{2}}Y_l^m(\theta_p,\varphi_p)\int r^2j_l(pr)f(r)dr\iint[Y_l^m(\theta,\varphi)]^*Y_L^M(\theta,\varphi)\sin\theta d\varphi d\theta\\&=\sqrt{\frac{2}{\pi}}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\mathrm{e}^{-\frac{\mathrm{i}\pi l}{2}}Y_l^m(\theta_p,\varphi_p)\int r^2j_l(pr)f(r)\delta_{mM}\delta_{lL}dr=\sqrt{\frac{2}{\pi}}\mathrm{e}^{-\frac{\mathrm{i}\pi L}{2}}Y_L^M(\theta_p,\varphi_p)\int r^2j_l(pr)f(r)dr\end{aligned}$$

从而在动量空间表象中，$\hat{L}_z,\hat{L}^2$的本征函数仍然是球谐函数

### 4.3 两个电子自旋耦合相互作用的哈密顿算符为$\hat{H}=A\hat{\boldsymbol{S}}_1\cdot\hat{\boldsymbol{S}}_2$，式中A为常数。求出$\hat{H}=A\hat{\boldsymbol{S}}_1\cdot\hat{\boldsymbol{S}}_2$，$\hat{\boldsymbol{S}}^2=(\hat{\boldsymbol{S}}_1+\hat{\boldsymbol{S}}_2)^2$，$\hat{S}_z=\hat{S}_{z,1}+\hat{S}_{z,2}$用未耦合表象基组表示的共同本征矢和相应的本征值，每一个能级简并度是多少？它们关于两个电子交换的对称性如何？

### 4.4 计算旋转算符在$j=1$的角动量本征态上的表示矩阵，并与4.5.8比较，它们的同异在哪里？

### 4.5 对于轨道角动量算符$\hat{\boldsymbol{L}}$，证明$\hat{\boldsymbol{L}}^2=\hat{\boldsymbol{r}}^2\hat{\boldsymbol{p}}^2-(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})^2+\mathrm{i}\hbar\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}}$

**证明**：由于$\hat{\boldsymbol{L}}=\hat{\boldsymbol{r}}\times\hat{\boldsymbol{p}}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ \hat{r}_x&\hat{r}_y&\hat{r}_z\\ \hat{p}_x&\hat{p}_y&\hat{p}_z\end{vmatrix}$，而$\hat{\boldsymbol{r}}^2=\hat{r}_x^2+\hat{r}_y^2+\hat{r}_z^2$，$\hat{\boldsymbol{p}}^2=\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2$，
$\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}}=\hat{r}_x\hat{p}_x+\hat{r}_y\hat{p}_y+\hat{r}_z\hat{p}_z$，因此有：

$$
\begin{aligned}
\hat{\boldsymbol{L}}^2 &= (\hat{\boldsymbol{r}}\times\hat{\boldsymbol{p}})^2 = [(\hat{r}_y\hat{p}_z-\hat{r}_z\hat{p}_y)\boldsymbol{i}+(\hat{r}_z\hat{p}_x-\hat{r}_x\hat{p}_z)\boldsymbol{j}+(\hat{r}_x\hat{p}_y-\hat{r}_y\hat{p}_x)\boldsymbol{k}]^2 = (\hat{r}_y\hat{p}_z-\hat{r}_z\hat{p}_y)^2+(\hat{r}_z\hat{p}_x-\hat{r}_x\hat{p}_z)^2+(\hat{r}_x\hat{p}_y-\hat{r}_y\hat{p}_x)^2 \\
&= (\hat{r}_y\hat{p}_z\hat{r}_y\hat{p}_z-\hat{r}_y\hat{p}_z\hat{r}_z\hat{p}_y-\hat{r}_z\hat{p}_y\hat{r}_y\hat{p}_z+\hat{r}_z\hat{p}_y\hat{r}_z\hat{p}_y)+(\hat{r}_z\hat{p}_x\hat{r}_z\hat{p}_x-\hat{r}_z\hat{p}_x\hat{r}_x\hat{p}_z-\hat{r}_x\hat{p}_z\hat{r}_z\hat{p}_x+\hat{r}_x\hat{p}_z\hat{r}_x\hat{p}_z) \\
&\quad +(\hat{r}_x\hat{p}_y\hat{r}_x\hat{p}_y-\hat{r}_x\hat{p}_y\hat{r}_y\hat{p}_x-\hat{r}_y\hat{p}_x\hat{r}_x\hat{p}_y+\hat{r}_y\hat{p}_x\hat{r}_y\hat{p}_x) \\
&= [\hat{r}_y\hat{r}_y\hat{p}_z\hat{p}_z+\hat{r}_y([\hat{r}_z,\hat{p}_z]-\hat{r}_z\hat{p}_z)\hat{p}_y+\hat{r}_z([\hat{r}_y,\hat{p}_y]-\hat{r}_y\hat{p}_y)\hat{p}_z+\hat{r}_z\hat{r}_z\hat{p}_y\hat{p}_y] \\
&\quad +[\hat{r}_z\hat{r}_z\hat{p}_x\hat{p}_x+\hat{r}_z([\hat{r}_x,\hat{p}_x]-\hat{r}_x\hat{p}_x)\hat{p}_z+\hat{r}_x([\hat{r}_z,\hat{p}_z]-\hat{r}_z\hat{p}_z)\hat{p}_x+\hat{r}_x\hat{r}_x\hat{p}_z\hat{p}_z] \\
&\quad +[\hat{r}_x\hat{r}_x\hat{p}_y\hat{p}_y+\hat{r}_x([\hat{r}_y,\hat{p}_y]-\hat{r}_y\hat{p}_y)\hat{p}_x+\hat{r}_y([\hat{r}_x,\hat{p}_x]-\hat{r}_x\hat{p}_x)\hat{p}_y+\hat{r}_y\hat{r}_y\hat{p}_x\hat{p}_x] \\
&= [\hat{r}_y^2\hat{p}_z^2+\hat{r}_y(\mathrm{i}\hbar-\hat{r}_z\hat{p}_z)\hat{p}_y+\hat{r}_z(\mathrm{i}\hbar-\hat{r}_y\hat{p}_y)\hat{p}_z+\hat{r}_z^2\hat{p}_y^2]+[\hat{r}_z^2\hat{p}_x^2+\hat{r}_z(\mathrm{i}\hbar-\hat{r}_x\hat{p}_x)\hat{p}_z+\hat{r}_x(\mathrm{i}\hbar-\hat{r}_z\hat{p}_z)\hat{p}_x+\hat{r}_x^2\hat{p}_z^2] \\
&\quad +[\hat{r}_x^2\hat{p}_y^2+\hat{r}_x(\mathrm{i}\hbar-\hat{r}_y\hat{p}_y)\hat{p}_x+\hat{r}_y(\mathrm{i}\hbar-\hat{r}_x\hat{p}_x)\hat{p}_y+\hat{r}_y^2\hat{p}_x^2] \\
&= [\hat{r}_y^2\hat{p}_z^2-\hat{r}_y\hat{r}_z\hat{p}_z\hat{p}_y-\hat{r}_z\hat{r}_y\hat{p}_y\hat{p}_z+\hat{r}_z^2\hat{p}_y^2]+[\hat{r}_z^2\hat{p}_x^2-\hat{r}_z\hat{r}_x\hat{p}_x\hat{p}_z-\hat{r}_x\hat{r}_z\hat{p}_z\hat{p}_x+\hat{r}_x^2\hat{p}_z^2]+[\hat{r}_x^2\hat{p}_y^2-\hat{r}_x\hat{r}_y\hat{p}_y\hat{p}_x-\hat{r}_y\hat{r}_x\hat{p}_x\hat{p}_y+\hat{r}_y^2\hat{p}_x^2] \\
&\quad +2\mathrm{i}\hbar(\hat{r}_x\hat{p}_x+\hat{r}_y\hat{p}_y+\hat{r}_z\hat{p}_z) \\
&= (\hat{r}_x^2+\hat{r}_y^2+\hat{r}_z^2)(\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2)+2\mathrm{i}\hbar(\hat{r}_x\hat{p}_x+\hat{r}_y\hat{p}_y+\hat{r}_z\hat{p}_z) \\
&\quad -(\hat{r}_x^2\hat{p}_x^2+\hat{r}_y^2\hat{p}_y^2+\hat{r}_z^2\hat{p}_z^2)-(\hat{r}_y\hat{r}_z\hat{p}_z\hat{p}_y+\hat{r}_z\hat{r}_y\hat{p}_y\hat{p}_z+\hat{r}_z\hat{r}_x\hat{p}_x\hat{p}_z+\hat{r}_x\hat{r}_z\hat{p}_z\hat{p}_x+\hat{r}_x\hat{r}_y\hat{p}_y\hat{p}_x+\hat{r}_y\hat{r}_x\hat{p}_x\hat{p}_y) \\
&= \hat{\boldsymbol{r}}^2\hat{\boldsymbol{p}}^2+2\mathrm{i}\hbar\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}}-[\hat{r}_x([\hat{r}_x,\hat{p}_x]+\hat{p}_x\hat{r}_x)\hat{p}_x+\hat{r}_y([\hat{r}_y,\hat{p}_y]+\hat{p}_y\hat{r}_y)\hat{p}_y+\hat{r}_z([\hat{r}_z,\hat{p}_z]+\hat{p}_z\hat{r}_z)\hat{p}_z] \\
&\quad -(\hat{r}_y\hat{r}_z\hat{p}_z\hat{p}_y+\hat{r}_z\hat{r}_y\hat{p}_y\hat{p}_z+\hat{r}_z\hat{r}_x\hat{p}_x\hat{p}_z+\hat{r}_x\hat{r}_z\hat{p}_z\hat{p}_x+\hat{r}_x\hat{r}_y\hat{p}_y\hat{p}_x+\hat{r}_y\hat{r}_x\hat{p}_x\hat{p}_y) \\
&= \hat{\boldsymbol{r}}^2\hat{\boldsymbol{p}}^2+2\mathrm{i}\hbar\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}}-\mathrm{i}\hbar(\hat{r}_x\hat{p}_x+\hat{r}_y\hat{p}_y+\hat{r}_z\hat{p}_z)-(\hat{r}_x\hat{p}_x+\hat{r}_y\hat{p}_y+\hat{r}_z\hat{p}_z)^2 \\
&= \hat{\boldsymbol{r}}^2\hat{\boldsymbol{p}}^2-(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}})^2+\mathrm{i}\hbar\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{p}}
\end{aligned}
$$

故原题得证

**另证**：（未完待续）

## 4.6 考虑对应于d 轨道的轨道角动量本征态与电子自旋本征态之间的耦合，写出用未耦合表象的基矢

$$|ls;mm_s\rangle \equiv |lm\rangle\otimes|sm_s\rangle \quad (l=2;m=0,\pm1,\pm2;s=\frac{1}{2};m_s=\pm\frac{1}{2})$$

## 所表示的耦合表象本征态$|jm;ls\rangle$表达式

**解**：首先我们知道$|l-s|\le j\le l+s$，代入得$\frac{3}{2}\le j\le\frac{5}{2}$（实际上$j$只能取$\frac{5}{2}$或$\frac{3}{2}$）。其次，$-\frac{5}{2}=-2-\frac{1}{2}\le m+m_s\le 2+\frac{1}{2}=\frac{5}{2}$，即$-(l+s)\le m+m_s\le l+s$。对于不等式取等号的情形，我们有（以下对耦合表象，只写出$j$和$m_c$，其中$m_c$为耦合后$\hat{J}_z$的本征值，满足$m_c=m+m_s$；对未耦合表象，只写出$m_1$和$m_2$）：

$$|j=\frac{5}{2},m_c=\frac{5}{2}\rangle=|m=2,m_s=\frac{1}{2}\rangle \quad |j=\frac{5}{2},m_c=-\frac{5}{2}\rangle=|m=-2,m_s=-\frac{1}{2}\rangle$$

对第一个式子两边使用总降算符$\hat{J}_-$，得：

$$\hat{J}_-|j=\frac{5}{2},m_c=\frac{5}{2}\rangle=\sqrt{\frac{5}{2}(\frac{5}{2}+1)-\frac{5}{2}(\frac{5}{2}-1)}\hbar|j=\frac{5}{2},m_c=\frac{3}{2}\rangle=\sqrt{5}\hbar|j=\frac{5}{2},m_c=\frac{3}{2}\rangle$$

$$
\begin{aligned}
&\hat{J}_-|m=2,m_s=\frac{1}{2}\rangle=(\hat{L}_-+\hat{S}_-)|m=2,m_s=\frac{1}{2}\rangle=\hat{L}_-|m=2,m_s=\frac{1}{2}\rangle+\hat{S}_-|m=2,m_s=\frac{1}{2}\rangle \\
&=\sqrt{2(2+1)-2(2-1)}\hbar|m=1,m_s=\frac{1}{2}\rangle+\sqrt{\frac{1}{2}(\frac{1}{2}+1)-\frac{1}{2}(\frac{1}{2}-1)}\hbar|m=2,m_s=-\frac{1}{2}\rangle \\
&=2\hbar|m=1,m_s=\frac{1}{2}\rangle+\hbar|m=2,m_s=-\frac{1}{2}\rangle
\end{aligned}
$$

从而有$|j=\frac{5}{2},m_c=\frac{3}{2}\rangle=\sqrt{\frac{4}{5}}|m=1,m_s=\frac{1}{2}\rangle+\sqrt{\frac{1}{5}}|m=2,m_s=-\frac{1}{2}\rangle$，对两边再次使用总降算符$\hat{J}_-$，得：

$$\hat{J}_-|j=\frac{5}{2},m_c=\frac{3}{2}\rangle=\sqrt{\frac{5}{2}(\frac{5}{2}+1)-\frac{3}{2}(\frac{3}{2}-1)}\hbar|j=\frac{5}{2},m_c=\frac{1}{2}\rangle=2\sqrt{2}\hbar|j=\frac{5}{2},m_c=\frac{1}{2}\rangle$$

$$\hat{J}_-|m=1,m_s=\frac{1}{2}\rangle=(\hat{L}_-+\hat{S}_-)|m=1,m_s=\frac{1}{2}\rangle=\hat{L}_-|m=1,m_s=\frac{1}{2}\rangle+\hat{S}_-|m=1,m_s=\frac{1}{2}\rangle$$
$$=\sqrt{2(2+1)-1(1-1)}\hbar|m=0,m_s=\frac{1}{2}\rangle+\sqrt{\frac{1}{2}(\frac{1}{2}+1)-\frac{1}{2}(\frac{1}{2}-1)}\hbar|m=1,m_s=-\frac{1}{2}\rangle$$
$$=\sqrt{6}\hbar|m=0,m_s=\frac{1}{2}\rangle+\hbar|m=1,m_s=-\frac{1}{2}\rangle$$

$$\hat{J}_-|m=2,m_s=-\frac{1}{2}\rangle=(\hat{L}_-+\hat{S}_-)|m=2,m_s=-\frac{1}{2}\rangle=\hat{L}_-|m=2,m_s=-\frac{1}{2}\rangle+\hat{S}_-|m=2,m_s=-\frac{1}{2}\rangle$$
$$=\sqrt{2(2+1)-2(1-1)}\hbar|m=1,m_s=-\frac{1}{2}\rangle+0=2\hbar|m=1,m_s=-\frac{1}{2}\rangle$$

$$\hat{J}_-(\sqrt{\frac{4}{5}}|m=1,m_s=\frac{1}{2}\rangle+\sqrt{\frac{1}{5}}|m=2,m_s=-\frac{1}{2}\rangle)=\sqrt{\frac{4}{5}}\hat{J}_-|m=1,m_s=\frac{1}{2}\rangle+\sqrt{\frac{1}{5}}\hat{J}_-|m=2,m_s=-\frac{1}{2}\rangle$$
$$=\sqrt{\frac{4}{5}}(\sqrt{6}\hbar|m=0,m_s=\frac{1}{2}\rangle+\hbar|m=1,m_s=-\frac{1}{2}\rangle)+\sqrt{\frac{1}{5}}(2\hbar|m=1,m_s=-\frac{1}{2}\rangle)$$
$$=\sqrt{\frac{24}{5}}\hbar|m=0,m_s=\frac{1}{2}\rangle+\sqrt{\frac{16}{5}}|m=1,m_s=-\frac{1}{2}\rangle$$

从而有$|j=\frac{5}{2},m_c=\frac{1}{2}\rangle=\sqrt{\frac{3}{5}}\hbar|m=0,m_s=\frac{1}{2}\rangle+\sqrt{\frac{2}{5}}|m=1,m_s=-\frac{1}{2}\rangle$
对第二个式子两边使用总升算符$\hat{J}_+$，得:

$$\hat{J}_+|j=\frac{5}{2},m_c=-\frac{5}{2}\rangle=\sqrt{\frac{5}{2}(\frac{5}{2}+1)-(-\frac{5}{2})(-\frac{5}{2}+1)}\hbar|j=\frac{5}{2},m_c=-\frac{3}{2}\rangle=\sqrt{5}\hbar|j=\frac{5}{2},m_c=-\frac{3}{2}\rangle$$

$$\hat{J}_+|m=-2,m_s=-\frac{1}{2}\rangle=(\hat{L}_++\hat{S}_+)|m=-2,m_s=-\frac{1}{2}\rangle=\hat{L}_+|m=-2,m_s=-\frac{1}{2}\rangle+\hat{S}_+|m=-2,m_s=-\frac{1}{2}\rangle$$
$$=\sqrt{2(2+1)-(-2)(-2+1)}\hbar|m=-1,m_s=-\frac{1}{2}\rangle+\sqrt{\frac{1}{2}(\frac{1}{2}+1)-(-\frac{1}{2})(-\frac{1}{2}+1)}\hbar|m=-2,m_s=\frac{1}{2}\rangle$$
$$=2\hbar|m=-1,m_s=-\frac{1}{2}\rangle+\hbar|m=-2,m_s=\frac{1}{2}\rangle$$

从而有$|j=\frac{5}{2},m_c=-\frac{3}{2}\rangle=\sqrt{\frac{4}{5}}|m=-1,m_s=-\frac{1}{2}\rangle+\sqrt{\frac{1}{5}}|m=-2,m_s=\frac{1}{2}\rangle$，对两边再次使用总升算符$\hat{J}_+$，得:

$$\hat{J}_+|j=\frac{5}{2},m_c=-\frac{3}{2}\rangle=\sqrt{\frac{5}{2}(\frac{5}{2}+1)-(-\frac{3}{2})(-\frac{3}{2}+1)}\hbar|j=\frac{5}{2},m_c=-\frac{1}{2}\rangle=2\sqrt{2}\hbar|j=\frac{5}{2},m_c=-\frac{1}{2}\rangle$$

$$\hat{J}_+|m=-1,m_s=-\frac{1}{2}\rangle=(\hat{L}_++\hat{S}_+)|m=-1,m_s=-\frac{1}{2}\rangle=\hat{L}_+|m=-1,m_s=-\frac{1}{2}\rangle+\hat{S}_+|m=-1,m_s=-\frac{1}{2}\rangle$$
$$=\sqrt{2(2+1)-(-1)(-1+1)}\hbar|m=0,m_s=-\frac{1}{2}\rangle+\sqrt{\frac{1}{2}(\frac{1}{2}+1)-(-\frac{1}{2})(-\frac{1}{2}+1)}\hbar|m=-1,m_s=\frac{1}{2}\rangle$$
$$=\sqrt{6}\hbar|m=0,m_s=-\frac{1}{2}\rangle+\hbar|m=-1,m_s=\frac{1}{2}\rangle$$

$$\hat{J}_+|m=-2,m_s=\frac{1}{2}\rangle=(\hat{L}_++\hat{S}_+)|m=-2,m_s=\frac{1}{2}\rangle=\hat{L}_+|m=-2,m_s=\frac{1}{2}\rangle+\hat{S}_+|m=-2,m_s=\frac{1}{2}\rangle$$
$$=\sqrt{2(2+1)-(-2)(-2+1)}\hbar|m=-1,m_s=\frac{1}{2}\rangle+0=2\hbar|m=-1,m_s=\frac{1}{2}\rangle$$

$$\hat{J}_+(\sqrt{\frac{4}{5}}|m=-1,m_s=-\frac{1}{2}\rangle+\sqrt{\frac{1}{5}}|m=-2,m_s=\frac{1}{2}\rangle)=\sqrt{\frac{4}{5}}\hat{J}_+|m=-1,m_s=-\frac{1}{2}\rangle+\sqrt{\frac{1}{5}}\hat{J}_+|m=-2,m_s=\frac{1}{2}\rangle$$
$$=\sqrt{\frac{4}{5}}(\sqrt{6}\hbar|m=0,m_s=-\frac{1}{2}\rangle+\hbar|m=-1,m_s=\frac{1}{2}\rangle)+\sqrt{\frac{1}{5}}(2\hbar|m=-1,m_s=\frac{1}{2}\rangle)$$
$$=\sqrt{\frac{24}{5}}\hbar|m=0,m_s=-\frac{1}{2}\rangle+\sqrt{\frac{16}{5}}\hbar|m=-1,m_s=\frac{1}{2}\rangle$$

从而有$|j=\frac{5}{2},m_c=-\frac{1}{2}\rangle=\sqrt{\frac{3}{5}}\hbar|m=0,m_s=-\frac{1}{2}\rangle+\sqrt{\frac{2}{5}}|m=-1,m_s=\frac{1}{2}\rangle$

接下来讨论$j=\frac{3}{2}$的情形，此时$m=\pm\frac{3}{2},\pm\frac{1}{2}$，因此设

$$\begin{cases}|j=\frac{3}{2},m_c=\frac{3}{2}\rangle=c_1|m=1,m_s=\frac{1}{2}\rangle+c_2|m=2,m_s=-\frac{1}{2}\rangle\\|j=\frac{3}{2},m_c=\frac{1}{2}\rangle=c_3|m=0,m_s=\frac{1}{2}\rangle+c_4|m=1,m_s=-\frac{1}{2}\rangle\\|j=\frac{3}{2},m_c=-\frac{1}{2}\rangle=c_5|m=0,m_s=-\frac{1}{2}\rangle+c_6|m=-1,m_s=\frac{1}{2}\rangle\\|j=\frac{3}{2},m_c=-\frac{3}{2}\rangle=c_7|m=-1,m_s=-\frac{1}{2}\rangle+c_8|m=-2,m_s=\frac{1}{2}\rangle\end{cases}$$

其中$c_{1,3,5,7}\in\mathbb{R}^+$，$c_{2,4,6,8}\in\mathbb{R}$，则有：

$$\begin{cases}\langle j=\frac{3}{2},m_c=\frac{3}{2}|j=\frac{3}{2},m_c=\frac{3}{2}\rangle=c_1^2+c_2^2=1\\\langle j=\frac{5}{2},m_c=\frac{3}{2}|j=\frac{3}{2},m_c=\frac{3}{2}\rangle=\sqrt{\frac{4}{5}}c_1+\sqrt{\frac{1}{5}}c_2=0\\\langle j=\frac{3}{2},m_c=\frac{1}{2}|j=\frac{3}{2},m_c=\frac{1}{2}\rangle=c_3^2+c_4^2=1\\\langle j=\frac{5}{2},m_c=\frac{1}{2}|j=\frac{3}{2},m_c=\frac{1}{2}\rangle=\sqrt{\frac{3}{5}}c_3+\sqrt{\frac{2}{5}}c_4=0\\\langle j=\frac{3}{2},m_c=\frac{1}{2}|j=\frac{3}{2},m_c=-\frac{1}{2}\rangle=c_5^2+c_6^2=1\\\langle j=\frac{5}{2},m_c=-\frac{1}{2}|j=\frac{3}{2},m_c=-\frac{1}{2}\rangle=\sqrt{\frac{3}{5}}c_5+\sqrt{\frac{2}{5}}c_6=0\\\langle j=\frac{3}{2},m_c=-\frac{3}{2}|j=\frac{3}{2},m_c=-\frac{3}{2}\rangle=c_7^2+c_8^2=1\\\langle j=\frac{5}{2},m_c=-\frac{3}{2}|j=\frac{3}{2},m_c=-\frac{3}{2}\rangle=\sqrt{\frac{4}{5}}c_7+\sqrt{\frac{1}{5}}c_8=0\end{cases}$$

解得

$$\begin{cases}c_1=\sqrt{\frac{1}{5}},c_2=-\sqrt{\frac{4}{5}}\\c_3=\sqrt{\frac{2}{5}},c_4=-\sqrt{\frac{3}{5}}\\c_5=\sqrt{\frac{2}{5}},c_6=-\sqrt{\frac{3}{5}}\\c_7=\sqrt{\frac{1}{5}},c_8=-\sqrt{\frac{4}{5}}\end{cases}$$

因此有

$$\begin{cases}|j=\frac{3}{2},m_c=\frac{3}{2}\rangle=\sqrt{\frac{1}{5}}|m=1,m_s=\frac{1}{2}\rangle-\sqrt{\frac{4}{5}}|m=2,m_s=-\frac{1}{2}\rangle\\|j=\frac{3}{2},m_c=\frac{1}{2}\rangle=\sqrt{\frac{2}{5}}|m=0,m_s=\frac{1}{2}\rangle-\sqrt{\frac{3}{5}}|m=1,m_s=-\frac{1}{2}\rangle\\|j=\frac{3}{2},m_c=-\frac{1}{2}\rangle=\sqrt{\frac{2}{5}}|m=0,m_s=-\frac{1}{2}\rangle-\sqrt{\frac{3}{5}}|m=-1,m_s=\frac{1}{2}\rangle\\|j=\frac{3}{2},m_c=-\frac{3}{2}\rangle=\sqrt{\frac{1}{5}}|m=-1,m_s=-\frac{1}{2}\rangle-\sqrt{\frac{4}{5}}|m=-2,m_s=\frac{1}{2}\rangle\end{cases}$$