课堂练习

练习1：证明如下等式：（1）$[\hat{A},\hat{B}\hat{C}]=[\hat{A},\hat{B}]\hat{C}+\hat{B}[\hat{A},\hat{C}]$；（2）$[\hat{A}\hat{B},\hat{C}]=\hat{A}[\hat{B},\hat{C}]+[\hat{A},\hat{C}]\hat{B}$

证明：因为：

$$\begin{aligned}[\hat{A},\hat{B}\hat{C}]&=\hat{A}(\hat{B}\hat{C})-(\hat{B}\hat{C})\hat{A}=\hat{A}\hat{B}\hat{C}-\hat{B}\hat{C}\hat{A}+\hat{B}\hat{A}\hat{C}-\hat{B}\hat{A}\hat{C}=(\hat{A}\hat{B}\hat{C}-\hat{B}\hat{A}\hat{C})+(-\hat{B}\hat{C}\hat{A}+\hat{B}\hat{A}\hat{C})\\&=(\hat{A}\hat{B}-\hat{B}\hat{A})\hat{C}+\hat{B}(\hat{A}\hat{C}-\hat{C}\hat{A})=[\hat{A},\hat{B}]\hat{C}+\hat{B}[\hat{A},\hat{C}]\end{aligned}$$

$$\begin{aligned}[\hat{A}\hat{B},\hat{C}]&=(\hat{A}\hat{B})\hat{C}-\hat{C}(\hat{A}\hat{B})=\hat{A}\hat{B}\hat{C}-\hat{C}\hat{A}\hat{B}+\hat{A}\hat{C}\hat{B}-\hat{A}\hat{C}\hat{B}=(\hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B})+(-\hat{C}\hat{A}\hat{B}+\hat{A}\hat{C}\hat{B})\\&=\hat{A}(\hat{B}\hat{C}-\hat{C}\hat{B})+(\hat{A}\hat{C}-\hat{C}\hat{A})\hat{B}=\hat{A}[\hat{B},\hat{C}]+[\hat{A},\hat{C}]\hat{B}\end{aligned}$$

故原题得证

练习2：以$|s_x+\rangle$和$|s_x-\rangle$为基矢来表示$|s_z\pm\rangle$和$|s_y\pm\rangle$

解：我们知道$|s_x+\rangle=\frac{1}{\sqrt{2}}(|s_z+\rangle+|s_z-\rangle)$，$|s_x-\rangle=\frac{1}{\sqrt{2}}(|s_z+\rangle-|s_z-\rangle)$，因此两式相加得$|s_x+\rangle+|s_x-\rangle=\sqrt{2}|s_z+\rangle$，即$|s_z+\rangle=\frac{1}{\sqrt{2}}(|s_x+\rangle+|s_x-\rangle)$；两式相减得$|s_x+\rangle-|s_x-\rangle=\sqrt{2}|s_z-\rangle$，即$|s_z-\rangle=\frac{1}{\sqrt{2}}(|s_x+\rangle-|s_x-\rangle)$。因此，$|s_z\pm\rangle$的表达式为$|s_z\pm\rangle=\frac{1}{\sqrt{2}}(|s_x+\rangle\pm|s_x-\rangle)$。
我们再来看看$|s_y\pm\rangle$，由于$|s_y\pm\rangle=\frac{1}{\sqrt{2}}(|s_z+\rangle\pm\mathrm{i}|s_z-\rangle)$，因此将上述的$|s_z\pm\rangle$的表达式代入，得

$$|s_y+\rangle=\frac{1}{\sqrt{2}}(|s_z+\rangle+\mathrm{i}|s_z-\rangle)=\frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}(|s_x+\rangle+|s_x-\rangle)+\frac{\mathrm{i}}{\sqrt{2}}(|s_x+\rangle-|s_x-\rangle)]=\frac{1+\mathrm{i}}{2}|s_x+\rangle+\frac{1-\mathrm{i}}{2}|s_x-\rangle$$

$$|s_y-\rangle=\frac{1}{\sqrt{2}}(|s_z+\rangle-\mathrm{i}|s_z-\rangle)=\frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}(|s_x+\rangle+|s_x-\rangle)-\frac{\mathrm{i}}{\sqrt{2}}(|s_x+\rangle-|s_x-\rangle)]=\frac{1-\mathrm{i}}{2}|s_x+\rangle+\frac{1+\mathrm{i}}{2}|s_x-\rangle$$

从而$|s_y\pm\rangle=\frac{1\pm\mathrm{i}}{2}|s_x+\rangle+\frac{1\mp\mathrm{i}}{2}|s_x-\rangle$

练习3：根据狄拉克δ函数的定义$\delta(x)=\begin{cases}0 & (x\neq 0)\\ \infty & (x=0)\end{cases}$，以及相应的推论$\int_{-\infty}^{+\infty}\delta(x)dx=\int_{-\varepsilon}^{+\varepsilon}\delta(x)dx=1$，证明$\int_{-\infty}^{+\infty}f(x)\delta(x)dx=f(0)$

证明：利用狄拉克δ函数的定义可得$\int_{-\infty}^{+\infty}f(x)\delta(x)dx=\int_{-\varepsilon}^{+\varepsilon}f(x)\delta(x)dx$，对于连续有界函数$f(x)$，根据积分第一中值定理，有$\int_{-\varepsilon}^{+\varepsilon}f(x)\delta(x)dx=f(x_0)\int_{-\varepsilon}^{+\varepsilon}\delta(x)dx$，其中$-\varepsilon\leq x_0\leq+\varepsilon$，该式对任意$\varepsilon$成立，因此当$\varepsilon\to 0$时，有$f(x_0)\to f(0)$；又根据狄拉克$\delta$函数的推论，有$\int_{-\varepsilon}^{+\varepsilon}\delta(x)dx=1$，因此$\int_{-\infty}^{+\infty}f(x)\delta(x)dx=f(x_0)\int_{-\varepsilon}^{+\varepsilon}\delta(x)dx=f(0)\cdot 1=f(0)$，证毕

练习4：坐标本征态$|x\rangle$在坐标表象中的表达式是什么？

解：将坐标本征态在坐标表象中展开，得$|x\rangle=\int_{-\infty}^{+\infty}|x'\rangle\langle x'|x\rangle dx'=\int_{-\infty}^{+\infty}\delta(x'-x)|x'\rangle dx'$，因此坐标本征态$|x\rangle$在坐标表象中的表达式恰好为$\delta(x'-x)$

练习5：试问$\delta(x)$是否为平方可积的函数，并给出理由

解：$\delta(x)$不是平方可积的函数，因为$\int_{-\infty}^{+\infty}\delta^2(x)dx=\int_{-\infty}^{+\infty}\delta(x)\cdot\delta(x-0)dx=\delta(0)=\infty$，故$\delta(x)$不是平方可积的函数

练习6：验证$\hat{\boldsymbol{x}}|\boldsymbol{x}\rangle = \boldsymbol{x}|\boldsymbol{x}\rangle$，其中$\hat{\boldsymbol{x}} = \hat{x}\boldsymbol{e}_x + \hat{y}\boldsymbol{e}_y + \hat{z}\boldsymbol{e}_z = \sum_{i=1}^{3}\hat{x}_i\boldsymbol{e}_i$，表示一个"矢量"算符，$|\boldsymbol{x}\rangle = |x\rangle \otimes |y\rangle \otimes |z\rangle$，表示线性空间中三个不同方向的矢量的直积

解：$\hat{\boldsymbol{x}}|\boldsymbol{x}\rangle = (\hat{x}\boldsymbol{e}_x + \hat{y}\boldsymbol{e}_y + \hat{z}\boldsymbol{e}_z)|x\rangle \otimes |y\rangle \otimes |z\rangle = (x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z)|x\rangle \otimes |y\rangle \otimes |z\rangle = \boldsymbol{x}|\boldsymbol{x}\rangle$

练习7：证明$\hat{D}^{-1}(d\boldsymbol{s})\hat{\boldsymbol{x}}\hat{D}(d\boldsymbol{s}) = \hat{\boldsymbol{x}} + d\boldsymbol{s}$

证明：取任意位置态矢$|\boldsymbol{x}\rangle$，有：

$$
\begin{aligned}
[\hat{D}^{-1}(d\boldsymbol{s})\hat{\boldsymbol{x}}\hat{D}(d\boldsymbol{s})]|\boldsymbol{x}\rangle &= \hat{D}^{-1}(d\boldsymbol{s})\hat{\boldsymbol{x}}[\hat{D}(d\boldsymbol{s})|\boldsymbol{x}\rangle] = \hat{D}^{-1}(d\boldsymbol{s})\hat{\boldsymbol{x}}|\boldsymbol{x}+d\boldsymbol{s}\rangle = \hat{D}^{-1}(d\boldsymbol{s})(\hat{\boldsymbol{x}}|\boldsymbol{x}+d\boldsymbol{s}\rangle) = \hat{D}^{-1}(d\boldsymbol{s})[(\boldsymbol{x}+d\boldsymbol{s})|\boldsymbol{x}+d\boldsymbol{s}\rangle] \\
&= (\boldsymbol{x}+d\boldsymbol{s})\hat{D}^{-1}(d\boldsymbol{s})|\boldsymbol{x}+d\boldsymbol{s}\rangle = (\boldsymbol{x}+d\boldsymbol{s})\hat{D}(-d\boldsymbol{s})|\boldsymbol{x}+d\boldsymbol{s}\rangle = (\boldsymbol{x}+d\boldsymbol{s})|\boldsymbol{x}\rangle
\end{aligned}
$$

而$(\hat{\boldsymbol{x}}+d\boldsymbol{s})|\boldsymbol{x}\rangle = \hat{\boldsymbol{x}}|\boldsymbol{x}\rangle + d\boldsymbol{s}|\boldsymbol{x}\rangle = \boldsymbol{x}|\boldsymbol{x}\rangle + d\boldsymbol{s}|\boldsymbol{x}\rangle = (\boldsymbol{x}+d\boldsymbol{s})|\boldsymbol{x}\rangle$，因此
$[\hat{D}^{-1}(d\boldsymbol{s})\hat{\boldsymbol{x}}\hat{D}(d\boldsymbol{s})]|\boldsymbol{x}\rangle = (\hat{\boldsymbol{x}}+d\boldsymbol{s})|\boldsymbol{x}\rangle$，从而算符相等，即$\hat{D}^{-1}(d\boldsymbol{s})\hat{\boldsymbol{x}}\hat{D}(d\boldsymbol{s}) = \hat{\boldsymbol{x}}+d\boldsymbol{s}$，证毕