课堂练习

练习1：证明无限深方势阱中，波函数满足正交关系$\int_0^a \psi_m^*(x)\psi_n(x)dx=\delta_{mn}$，其中$\psi_n(x)=\sqrt{\frac{2}{a}}\sin(\frac{n\pi}{a}x)$

证明：当$m=n$时，有：

$$\int_0^a \psi_m^*(x)\psi_n(x)dx=\int_0^a \frac{2}{a}\sin^2(\frac{n\pi}{a}x)dx=\int_0^a \frac{2}{a}\frac{1-\cos(\frac{2n\pi}{a}x)}{2}dx=[\frac{x}{a}-\frac{\sin(\frac{2n\pi}{a}x)}{2n\pi}]_0^a=1$$

当$m\neq n$时，有：

$$\int_0^a \psi_m^*(x)\psi_n(x)dx=\int_0^a \frac{2}{a}\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{a}x)dx=\int_0^a \frac{2}{a}\frac{\cos[\frac{(m-n)\pi}{a}x]-\cos[\frac{(m+n)\pi}{a}x]}{2}dx=[\frac{\sin[\frac{(m-n)\pi}{a}x]}{(m-n)\pi}-\frac{\sin[\frac{(m+n)\pi}{a}x]}{(m+n)\pi}]_0^a=0$$

综上可知$\int_0^a \psi_m^*(x)\psi_n(x)dx=\delta_{mn}$

练习2：将箱中粒子的势函数定义为$V(x)=\begin{cases}0 & (|x|<\frac{a}{2})\\ +\infty & (|x|\geq\frac{a}{2})\end{cases}$，写出相应的本征能量和本征波函数

解：由于势能函数$V(x)$为偶函数，因此波函数必满足一定的宇称（即波函数要么为奇函数，要么为偶函数）。又当$|x|<\frac{a}{2}$时，将势能函数代入定态薛定谔方程，得$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}=E\psi$，或$\frac{d^2\psi}{dx^2}=-\frac{2mE}{\hbar^2}\psi$。记$k=\sqrt{\frac{2mE}{\hbar^2}}$，则波函数的解为$\psi(x)=A\mathrm{e}^{\mathrm{i}kx}+B\mathrm{e}^{-\mathrm{i}kx}\ (|x|<\frac{a}{2})$；当$|x|\geq\frac{a}{2}$时，因$V(x)=+\infty$，故波函数为$\psi(x)=0\ (|x|\geq\frac{a}{2})$。结合波函数的连续性，得$\begin{cases}\psi(\frac{a}{2})=A\mathrm{e}^{\frac{\mathrm{i}ka}{2}}+B\mathrm{e}^{-\frac{\mathrm{i}ka}{2}}=0\\ \psi(-\frac{a}{2})=A\mathrm{e}^{-\frac{\mathrm{i}ka}{2}}+B\mathrm{e}^{\frac{\mathrm{i}ka}{2}}=0\end{cases}$。
接下来，我们联立这两个等式，得$\mathrm{e}^{\mathrm{i}ka}=\mathrm{e}^{-\mathrm{i}ka}$，即$\mathrm{e}^{2\mathrm{i}ka}=1$，从而有$2ka=2n\pi\ (n\in\mathbb{Z}^+)$，即$k=\frac{n\pi}{a}\ (n\in\mathbb{Z}^+)$，相应的，本征能量为$E_n=\frac{n^2\pi^2\hbar^2}{2ma^2}$。
将k与n的关系式代回边界条件，得$\begin{cases}\psi(\frac{a}{2})=A\mathrm{e}^{\frac{\mathrm{i}n\pi}{2}}+B\mathrm{e}^{-\frac{\mathrm{i}n\pi}{2}}=0\\ \psi(-\frac{a}{2})=A\mathrm{e}^{-\frac{\mathrm{i}n\pi}{2}}+B\mathrm{e}^{\frac{\mathrm{i}n\pi}{2}}=0\end{cases}$。当$n=2p\ (p\in\mathbb{Z}^+)$时，可得$A+B=0$，即$A=-B$，此时

$$\psi(x)=A(\mathrm{e}^{\mathrm{i}kx}-\mathrm{e}^{-\mathrm{i}kx})=2\mathrm{i}A\sin(kx)=A^{'}\sin(kx)=A^{'}\sin(\frac{n\pi x}{a})=A^{'}\sin(\frac{2p\pi x}{a})\ (|x|<\frac{a}{2})$$

接下来归一化得

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}|\psi(x)|^2dx=\int_{-\frac{a}{2}}^{\frac{a}{2}}|A^{'}|^2\sin^2(kx)dx=[\frac{|A^{'}|^2x}{2}-\frac{|A^{'}|^2\sin(2kx)}{4k}]_{-\frac{a}{2}}^{\frac{a}{2}}=\frac{|A^{'}|^2a}{2}=1$$

即$|A^{'}|=\sqrt{\frac{2}{a}}$，故当$A^{'}$取正实数时，有$\psi_n(x)=\sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})\ (|x|<\frac{a}{2},n\text{ is even})$，或写作$\psi_p(x)=\sqrt{\frac{2}{a}}\sin(\frac{2p\pi x}{a})\ (|x|<\frac{a}{2},p\in\mathbb{Z}^+)$，此时本征能量可改写为$E_p=\frac{2p^2\pi^2\hbar^2}{ma^2}\ (p\in\mathbb{Z}^+)$。
当$n=2p-1\ (p\in\mathbb{Z}^+)$时，可得$A-B=0$，即$A=B$，此时

$$\psi(x)=B(\mathrm{e}^{\mathrm{i}kx}+\mathrm{e}^{-\mathrm{i}kx})=2B\cos(kx)=B^{'}\cos(kx)=B^{'}\cos(\frac{n\pi x}{a})=B^{'}\cos[\frac{(2p-1)\pi x}{a}]\ (|x|<\frac{a}{2})$$

接下来归一化得

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}|\psi(x)|^2dx=\int_{-\frac{a}{2}}^{\frac{a}{2}}|B^{'}|^2\cos^2(kx)dx=[\frac{|B^{'}|^2x}{2}+\frac{|B^{'}|^2\sin(2kx)}{4k}]_{-\frac{a}{2}}^{\frac{a}{2}}=\frac{|B^{'}|^2a}{2}=1$$

即$|B^{'}|=\sqrt{\frac{2}{a}}$，故当$B^{'}$取正实数时，有$\psi_n(x)=\sqrt{\frac{2}{a}}\cos(\frac{n\pi x}{a})\ (|x|<\frac{a}{2},n \text{ is odd})$，或写作$\psi_p(x)=\sqrt{\frac{2}{a}}\cos[\frac{(2p-1)\pi x}{a}]\ (|x|<\frac{a}{2},p\in\mathbb{Z}^+)$，此时本征能量可改写为$E_p=\frac{(2p-1)^2\pi^2\hbar^2}{2ma^2}\ (p\in\mathbb{Z}^+)$。

综上，本征波函数为$\psi(x)=\sqrt{\frac{2}{a}}\cdot\begin{cases}\cos(\frac{n\pi x}{a}) \text{ when } n \text{ is odd}\\ \sin(\frac{n\pi x}{a}) \text{ when } n \text{ is even}\end{cases}\ (|x|<\frac{a}{2},n\in\mathbb{Z}^+)$，相应的本征能量为$E_n=\frac{n^2\pi^2\hbar^2}{2ma^2}\ (n\in\mathbb{Z}^+)$。

练习3：求湮灭（湮没）算符$\hat{a}$和创造（产生）算符$\hat{a}^\dagger$的对易关系$[\hat{a},\hat{a}^\dagger]$

解：我们知道$\hat{a}=\sqrt{\frac{m\omega}{2\hbar}}(\hat{x}+\frac{\mathrm{i}\hat{p}}{m\omega})$，$\hat{a}^\dagger=\sqrt{\frac{m\omega}{2\hbar}}(\hat{x}-\frac{\mathrm{i}\hat{p}}{m\omega})$，因此：

$$\begin{aligned}[\hat{a},\hat{a}^\dagger]=\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}&=\sqrt{\frac{m\omega}{2\hbar}}(\hat{x}+\frac{\mathrm{i}\hat{p}}{m\omega})\cdot\sqrt{\frac{m\omega}{2\hbar}}(\hat{x}-\frac{\mathrm{i}\hat{p}}{m\omega})-\sqrt{\frac{m\omega}{2\hbar}}(\hat{x}-\frac{\mathrm{i}\hat{p}}{m\omega})\cdot\sqrt{\frac{m\omega}{2\hbar}}(\hat{x}+\frac{\mathrm{i}\hat{p}}{m\omega})\\&=\frac{m\omega}{2\hbar}[(\hat{x}+\frac{\mathrm{i}\hat{p}}{m\omega})(\hat{x}-\frac{\mathrm{i}\hat{p}}{m\omega})-(\hat{x}-\frac{\mathrm{i}\hat{p}}{m\omega})(\hat{x}+\frac{\mathrm{i}\hat{p}}{m\omega})]=\frac{m\omega}{2\hbar}[(\hat{x}^2+\frac{\mathrm{i}\hat{p}\hat{x}}{m\omega}-\frac{\mathrm{i}\hat{x}\hat{p}}{m\omega}+\frac{\hat{p}^2}{m^2\omega^2})-(\hat{x}^2-\frac{\mathrm{i}\hat{p}\hat{x}}{m\omega}+\frac{\mathrm{i}\hat{x}\hat{p}}{m\omega}+\frac{\hat{p}^2}{m^2\omega^2})]\\&=\frac{m\omega}{2\hbar}(-\frac{2\mathrm{i}}{m\omega})(\hat{x}\hat{p}-\hat{p}\hat{x})=\frac{m\omega}{2\hbar}(-\frac{2\mathrm{i}}{m\omega})\mathrm{i}\hbar=1\end{aligned}$$

练习4：证明如下等式：$\hat{a}^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle$

证明：我们知道占据数算符定义为$\hat{N}=\hat{a}^\dagger\hat{a}$，它满足$\hat{N}|n\rangle=n|n\rangle$。又根据湮没算符和产生算符满足$[\hat{a},\hat{a}^\dagger]=\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}=1$，因此$\hat{N}^\dagger=\hat{a}\hat{a}^\dagger=\hat{N}+1$。另一方面，由于

$$\begin{aligned}\hat{N}\hat{a}^\dagger|n\rangle=([\hat{N},\hat{a}^\dagger]+\hat{a}^\dagger\hat{N})|n\rangle&=[\hat{N},\hat{a}^\dagger]|n\rangle+\hat{a}^\dagger\hat{N}|n\rangle=[\hat{a}^\dagger\hat{a},\hat{a}^\dagger]|n\rangle+\hat{a}^\dagger\hat{N}|n\rangle=(\hat{a}^\dagger[\hat{a},\hat{a}^\dagger]+[\hat{a}^\dagger,\hat{a}^\dagger]\hat{a})|n\rangle+\hat{a}^\dagger\hat{N}|n\rangle\\&=(\hat{a}^\dagger\cdot 1+0\cdot\hat{a})|n\rangle+\hat{a}^\dagger\hat{N}|n\rangle=\hat{a}^\dagger|n\rangle+n\hat{a}^\dagger|n\rangle=(n+1)\hat{a}|n\rangle\end{aligned}$$

故$\begin{cases}\hat{a}^\dagger|n\rangle=c_\uparrow|n+1\rangle\\ \langle n|\hat{a}=\langle n+1|c_\uparrow^*\end{cases}$，从而$(\langle n|\hat{a})(\hat{a}^\dagger|n\rangle)=(\langle n+1|c_\uparrow^*)(c_\uparrow|n+1\rangle)=|c_\uparrow|^2$，结合$(\langle n|\hat{a})(\hat{a}^\dagger|n\rangle)=\langle n|\hat{a}\hat{a}^\dagger|n\rangle=\langle n|(\hat{N}+1)|n\rangle=n+1$，得$|c_\uparrow|^2=n+1$，即$|c_\uparrow|=\sqrt{n+1}$，当$c_\uparrow$为正实数时，即有$\hat{a}^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle$，证毕

练习5：计算矩阵元$\langle n|\hat{x}^2|n\rangle$

解：易知$\hat{x}=\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)$，而$\hat{a}|n\rangle=\sqrt{n}|n-1\rangle$，$\hat{a}^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle$，因此

$$\hat{x}|n\rangle=\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)|n\rangle=\sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n}|n-1\rangle+\sqrt{n+1}|n+1\rangle)$$

$$\begin{aligned}\hat{x}^2|n\rangle=\hat{x}(\hat{x}|n\rangle)&=\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)\cdot\sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n}|n-1\rangle+\sqrt{n+1}|n+1\rangle)\\&=\frac{\hbar}{2m\omega}[(\hat{a}+\hat{a}^\dagger)(\sqrt{n}|n-1\rangle)+(\hat{a}+\hat{a}^\dagger)(\sqrt{n+1}|n+1\rangle)]\\&=\frac{\hbar}{2m\omega}[\sqrt{n}(\sqrt{n-1}|n-2\rangle+\sqrt{n}|n\rangle)+\sqrt{n+1}(\sqrt{n+1}|n\rangle+\sqrt{n+2}|n+2\rangle)]\\&=\frac{\hbar}{2m\omega}[\sqrt{n(n-1)}|n-2\rangle+(2n+1)|n\rangle+\sqrt{(n+1)(n+2)}|n+2\rangle]\end{aligned}$$

从而左乘$\langle n|$，得

$$\begin{aligned}\langle n|\hat{x}^2|n\rangle&=\langle n|\cdot\frac{\hbar}{2m\omega}[\sqrt{n(n-1)}|n-2\rangle+(2n+1)|n\rangle+\sqrt{(n+1)(n+2)}|n+2\rangle]\\&=\frac{\hbar}{2m\omega}[\sqrt{n(n-1)}\delta_{n,n-2}+(2n+1)\delta_{n,n}+\sqrt{(n+1)(n+2)}\delta_{n,n+2}]=\frac{(2n+1)\hbar}{2m\omega}\end{aligned}$$

另解：矩阵元$\langle n|\hat{x}^2|n\rangle$可看作$\langle n|\hat{x}\hat{I}\,\hat{x}|n\rangle$，将单位算符改写，可得：

$$\begin{aligned}\langle n|\hat{x}^2|n\rangle &= \sum_{i=0}^{\infty}\langle n|\hat{x}|i\rangle\langle i|\hat{x}|n\rangle = \sum_{i=0}^{\infty}\sqrt{\frac{\hbar}{2m\omega}}(\sqrt{i}\delta_{n,i-1}+\sqrt{i+1}\delta_{n,i+1})\cdot\sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n}\delta_{i,n-1}+\sqrt{n+1}\delta_{i,n+1}) \\ &= \frac{\hbar}{2m\omega}(\sqrt{n-1+1}\cdot\sqrt{n}+\sqrt{n+1}\cdot\sqrt{n+1}) = \frac{(2n+1)\hbar}{2m\omega}\end{aligned}$$

练习6：推导谐振子模型中$\langle n|\hat{T}|n\rangle$和$\langle n|\hat{V}|n\rangle$的表达式，并验证该体系满足不含时的维里定理

解：首先，根据练习5的结论可得

$\langle n|\hat{V}|n\rangle = \langle n|\frac{1}{2}m\omega^2\hat{x}^2|n\rangle = \frac{1}{2}m\omega^2\langle n|\hat{x}^2|n\rangle = \frac{1}{2}m\omega^2\cdot\frac{(2n+1)\hbar}{2m\omega} = \frac{(2n+1)\hbar\omega}{4}$。

另一方面，由于$\hat{p} = \mathrm{i}\sqrt{\frac{m\hbar\omega}{2}}(-\hat{a}+\hat{a}^\dagger)$，因此

$\hat{p}^2 = -\frac{m\hbar\omega}{2}(-\hat{a}+\hat{a}^\dagger)^2 = -\frac{m\hbar\omega}{2}(\hat{a}^2+(\hat{a}^\dagger)^2-\{\hat{a},\hat{a}^\dagger\})$，从而

$$\begin{aligned}\langle n|\hat{p}^2|n\rangle &= \langle n|-\frac{m\hbar\omega}{2}(\hat{a}^2+(\hat{a}^\dagger)^2-\{\hat{a},\hat{a}^\dagger\})|n\rangle = -\frac{m\hbar\omega}{2}(\langle n|\hat{a}^2|n\rangle+\langle n|(\hat{a}^\dagger)^2|n\rangle-\langle n|\{\hat{a},\hat{a}^\dagger\}|n\rangle) \\ &= -\frac{m\hbar\omega}{2}(\langle n|\cdot\sqrt{n(n-1)}|n-2\rangle+\langle n|\cdot\sqrt{(n+1)(n+2)}|n+2\rangle-\langle n|\cdot(2\hat{N}+1)|n\rangle) = \frac{(2n+1)m\hbar\omega}{2}\end{aligned}$$

故有$\langle n|\hat{T}|n\rangle = \langle n|\frac{\hat{p}^2}{2m}|n\rangle = \frac{1}{2m}\langle n|\hat{p}^2|n\rangle = \frac{(2n+1)\hbar\omega}{4}$，

维里定律的表述为：对势能服从r^n的体系，其平均势能$\langle V\rangle$与平均动能$\langle T\rangle$的关系为$\langle T\rangle = \frac{n\langle V\rangle}{2}$。在谐振子体系中，$n=2$，因此我们有$\frac{2\cdot\langle n|\hat{V}|n\rangle}{2} = \frac{(2n+1)\hbar\omega}{4} = \langle n|\hat{T}|n\rangle$，谐振子模型恰好满足维里定律。

练习7：推导势垒台阶模型中，当$E>0$时透射流通量的表达式

解：粒子流通量的表达式为$\boldsymbol{j} = \frac{\mathrm{i}\hbar}{2m}(\psi\nabla\psi^*-\psi^*\nabla\psi)$，而势垒台阶模型的散射解为

$$\psi(x) = \begin{cases}\mathrm{e}^{\mathrm{i}k_1x}+\frac{k_1-k_2}{k_1+k_2}\mathrm{e}^{-\mathrm{i}k_1x}\ (x<0)\\ \frac{2k_1}{k_1+k_2}\mathrm{e}^{\mathrm{i}k_2x}\ (x>0)\end{cases}$$

其中$k_1\equiv\sqrt{\frac{2m(E+V_0)}{\hbar^2}}$，$k_2\equiv\sqrt{\frac{2mE}{\hbar^2}}$。因此透射流通量由$x>0$时$\mathrm{e}^{-\mathrm{i}k_1x}$的部分提供，相应的表达式为

$$\boldsymbol{j}_{\mathrm{in}} = \frac{\mathrm{i}\hbar}{2m}[\mathrm{e}^{\mathrm{i}k_1x}\cdot(-\mathrm{e}^{-\mathrm{i}k_1x}\cdot\mathrm{i}k_1)-\mathrm{e}^{-\mathrm{i}k_1x}\cdot(\mathrm{e}^{\mathrm{i}k_1x}\cdot\mathrm{i}k_1)] = \frac{\hbar k_1}{m}$$

$$\boldsymbol{j}_{\mathrm{out}} = \frac{\mathrm{i}\hbar}{2m}[\frac{2k_1}{k_1+k_2}\mathrm{e}^{\mathrm{i}k_2x}\cdot(-\frac{2k_1}{k_1+k_2}\mathrm{e}^{-\mathrm{i}k_2x}\cdot\mathrm{i}k_2)-\frac{2k_1}{k_1+k_2}\mathrm{e}^{-\mathrm{i}k_2x}\cdot(\frac{2k_1}{k_1+k_2}\mathrm{e}^{\mathrm{i}k_2x}\cdot\mathrm{i}k_2)] = \frac{4\hbar k_1^2k_2}{(k_1+k_2)^2m}$$

$$T = \frac{\boldsymbol{j}_{\mathrm{out}}}{\boldsymbol{j}_{\mathrm{in}}} = \frac{\frac{4\hbar k_1^2k_2}{(k_1+k_2)^2m}}{\frac{\hbar k_1}{m}} = \frac{4k_1k_2}{(k_1+k_2)^2}$$

练习8：（1）具体推导矩形势垒钻穿模型中，当$0<E<V_0$时S的表达式；（2）证明：从该表达式出发，在保持$aV_0\equiv\gamma$的条件下，求$a\to0$的极限，可得到δ势垒的透射系数$|S|^2 = \frac{1}{1+\frac{m\gamma^2}{2E\hbar^2}}$

解：（1）矩形势垒钻穿模型满足$V=\begin{cases}0\ (|x|\geq\frac{a}{2})\\V_0\ (|x|<\frac{a}{2})\end{cases}$，代入薛定谔方程，得$\begin{cases}-\frac{\hbar^2}{2m}\frac{d^2\psi}{d^2x}=E\psi\ (|x|\geq\frac{a}{2})\\-\frac{\hbar^2}{2m}\frac{d^2\psi}{d^2x}+V_0\psi=E\psi\ (|x|<\frac{a}{2})\end{cases}$，变形即可得$\begin{cases}\frac{d^2\psi}{d^2x}+\frac{2mE}{\hbar^2}\psi=0\ (|x|\geq\frac{a}{2})\\\frac{d^2\psi}{d^2x}+\frac{2m(V_0-E)}{\hbar^2}\psi=0\ (|x|<\frac{a}{2})\end{cases}$，为讨论问题方便，设$k_1^2=\frac{2mE}{\hbar^2}$，$k_2^2=\frac{2m(V_0-E)}{\hbar^2}$，并假设平面波$\mathrm{e}^{\mathrm{i}k_1x}$从负无穷处向正方向传播，那么对应的解为$\psi(x)=\begin{cases}\mathrm{e}^{\mathrm{i}k_1x}+B\mathrm{e}^{-\mathrm{i}k_1x}\ (x\leq-\frac{a}{2})\\C\mathrm{e}^{k_2x}+D\mathrm{e}^{-k_2x}\ (|x|<\frac{a}{2})\\S\mathrm{e}^{\mathrm{i}k_1x}\ (x\geq\frac{a}{2})\end{cases}$，其导函数为$\psi'(x)=\begin{cases}\mathrm{i}k_1(\mathrm{e}^{\mathrm{i}k_1x}-B\mathrm{e}^{-\mathrm{i}k_1x})\ (x\leq-\frac{a}{2})\\k_2(C\mathrm{e}^{k_2x}-D\mathrm{e}^{-k_2x})\ (|x|<\frac{a}{2})\\\mathrm{i}k_1S\mathrm{e}^{\mathrm{i}k_1x}\ (x\geq\frac{a}{2})\end{cases}$

接下来，考虑到边界连续条件及波函数光滑条件，体系应满足$\begin{cases}\psi_{x\to(-\frac{a}{2})^-}=\psi_{x\to(-\frac{a}{2})^+}\\\psi'_{x\to(-\frac{a}{2})^-}=\psi'_{x\to(-\frac{a}{2})^+}\\\psi_{x\to(\frac{a}{2})^-}=\psi_{x\to(\frac{a}{2})^+}\\\psi'_{x\to(\frac{a}{2})^-}=\psi'_{x\to(\frac{a}{2})^+}\end{cases}$，代入可得

$\begin{cases}\mathrm{e}^{-\frac{\mathrm{i}k_1a}{2}}+B\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}}=C\mathrm{e}^{-\frac{k_2a}{2}}+D\mathrm{e}^{\frac{k_2a}{2}}\\\mathrm{i}k_1(\mathrm{e}^{-\frac{\mathrm{i}k_1a}{2}}-B\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}})=k_2(C\mathrm{e}^{-\frac{k_2a}{2}}-D\mathrm{e}^{\frac{k_2a}{2}})\\C\mathrm{e}^{\frac{k_2a}{2}}+D\mathrm{e}^{-\frac{k_2a}{2}}=S\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}}\\k_2(C\mathrm{e}^{\frac{k_2a}{2}}-D\mathrm{e}^{-\frac{k_2a}{2}})=\mathrm{i}k_1S\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}}\end{cases}$，经化简为

$$\begin{cases}C\mathrm{e}^{-\frac{k_2a}{2}}+D\mathrm{e}^{\frac{k_2a}{2}}=\mathrm{e}^{-\frac{\mathrm{i}k_1a}{2}}+B\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}} & ①\\C\mathrm{e}^{-\frac{k_2a}{2}}-D\mathrm{e}^{\frac{k_2a}{2}}=\frac{\mathrm{i}k_1}{k_2}(\mathrm{e}^{-\frac{\mathrm{i}k_1a}{2}}-B\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}}) & ②\\C\mathrm{e}^{\frac{k_2a}{2}}+D\mathrm{e}^{-\frac{k_2a}{2}}=S\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}} & ③\\C\mathrm{e}^{\frac{k_2a}{2}}-D\mathrm{e}^{-\frac{k_2a}{2}}=\frac{\mathrm{i}k_1}{k_2}S\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}} & ④\end{cases}$$

$\frac{①+②}{2}\cdot\mathrm{e}^{\frac{k_2a}{2}}$，得$C=\frac{1}{2}\mathrm{e}^{\frac{k_2a}{2}}(\frac{\mathrm{i}k_1+k_2}{k_2}\mathrm{e}^{-\frac{\mathrm{i}k_1a}{2}}+\frac{-\mathrm{i}k_1+k_2}{k_2}B\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}})$；$\frac{③+④}{2}\cdot\mathrm{e}^{-\frac{k_2a}{2}}$，得$C=\frac{1}{2}\mathrm{e}^{-\frac{k_2a}{2}}\cdot\frac{\mathrm{i}k_1+k_2}{k_2}S\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}}=\frac{\mathrm{i}k_1+k_2}{2k_2}S\mathrm{e}^{\frac{(\mathrm{i}k_1-k_2)a}{2}}$，代入可得

$$\frac{1}{2}\mathrm{e}^{\frac{k_2a}{2}}(\frac{\mathrm{i}k_1+k_2}{k_2}\mathrm{e}^{-\frac{\mathrm{i}k_1a}{2}}+\frac{-\mathrm{i}k_1+k_2}{k_2}B\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}})=\frac{\mathrm{i}k_1+k_2}{2k_2}S\mathrm{e}^{\frac{(\mathrm{i}k_1-k_2)a}{2}}$$
$$\Rightarrow\frac{-\mathrm{i}k_1+k_2}{k_2}B\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}}=\frac{\mathrm{i}k_1+k_2}{k_2}S\mathrm{e}^{\frac{(\mathrm{i}k_1-2k_2)a}{2}}-\frac{\mathrm{i}k_1+k_2}{k_2}\mathrm{e}^{-\frac{\mathrm{i}k_1a}{2}}\quad ⑤$$

$\frac{①-②}{2}\cdot\mathrm{e}^{-\frac{k_2a}{2}}$，得$D=\frac{1}{2}\mathrm{e}^{\frac{-k_2a}{2}}(\frac{-\mathrm{i}k_1+k_2}{k_2}\mathrm{e}^{-\frac{\mathrm{i}k_1a}{2}}+\frac{\mathrm{i}k_1+k_2}{k_2}B\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}})$，$\frac{③-④}{2}\cdot\mathrm{e}^{\frac{k_2a}{2}}$，得$D=\frac{1}{2}\mathrm{e}^{\frac{k_2a}{2}}\cdot\frac{-\mathrm{i}k_1+k_2}{k_2}S\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}}=\frac{-\mathrm{i}k_1+k_2}{2k_2}S\mathrm{e}^{\frac{(\mathrm{i}k_1+k_2)a}{2}}$，代入可得

$$\frac{1}{2}\mathrm{e}^{\frac{-k_2a}{2}}(\frac{-\mathrm{i}k_1+k_2}{k_2}\mathrm{e}^{-\frac{\mathrm{i}k_1a}{2}}+\frac{\mathrm{i}k_1+k_2}{k_2}B\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}})=\frac{-\mathrm{i}k_1+k_2}{2k_2}S\mathrm{e}^{\frac{(\mathrm{i}k_1+k_2)a}{2}}$$
$$\Rightarrow\frac{\mathrm{i}k_1+k_2}{k_2}B\mathrm{e}^{\frac{\mathrm{i}k_1a}{2}}=\frac{-\mathrm{i}k_1+k_2}{k_2}S\mathrm{e}^{\frac{(\mathrm{i}k_1+2k_2)a}{2}}-\frac{-\mathrm{i}k_1+k_2}{k_2}\mathrm{e}^{-\frac{\mathrm{i}k_1a}{2}}\quad ⑥$$

故联立⑤与⑥得

$(-\mathrm{i}k_1+k_2)[(-\mathrm{i}k_1+k_2)S\mathrm{e}^{\frac{(\mathrm{i}k_1+2k_2)a}{2}}-(-\mathrm{i}k_1+k_2)\mathrm{e}^{-\frac{\mathrm{i}k_1a}{2}}]=(\mathrm{i}k_1+k_2)[(\mathrm{i}k_1+k_2)S\mathrm{e}^{\frac{(\mathrm{i}k_1-2k_2)a}{2}}-(\mathrm{i}k_1+k_2)\mathrm{e}^{-\frac{\mathrm{i}k_1a}{2}}]$

经化简得$S=\frac{4\mathrm{i}k_1k_2\mathrm{e}^{-\mathrm{i}k_1a}}{(\mathrm{i}k_1+k_2)^2\mathrm{e}^{-k_2a}-(-\mathrm{i}k_1+k_2)^2\mathrm{e}^{k_2a}}$，与题中原式一致。

（2）由（1）知$k_2=\pm\sqrt{\frac{2m(V_0-E)}{\hbar^2}}=\pm\sqrt{\frac{2m(\frac{\gamma}{a}-E)}{\hbar^2}}$，且$k_2^2-k_1^2=\frac{2m(\frac{\gamma}{a}-2E)}{\hbar^2}$，因此$\frac{dk_2}{da}=\pm\frac{-\frac{2m\gamma}{a^2\hbar^2}}{2\sqrt{\frac{2m(\frac{\gamma}{a}-E)}{\hbar^2}}}=\mp\frac{m\gamma}{a^2\hbar^2k_2}$，$\frac{d(k_2^2-k_1^2)}{da}=\frac{d(k_2^2)}{da}=-\frac{2m\gamma}{a^2\hbar^2}$。

取k_2为正值，即$k_2=\sqrt{\frac{2m(\frac{\gamma}{a}-E)}{\hbar^2}}\to+\infty$，$\frac{dk_2}{da}=-\frac{m\gamma}{a^2\hbar^2k_2}$，此时有$k_2a=\sqrt{\frac{2m(\gamma a-Ea^2)}{\hbar^2}}\to0$，

$\frac{d(k_2 a)}{da} = -\frac{m\gamma}{a\hbar^2 k_2} + k_2$。

接下来，我们稍作变形，得到

$$\lim_{a\to 0} S = \lim_{a\to 0} \frac{4\mathrm{i}k_1 \mathrm{e}^{-\mathrm{i}k_1 a}}{(k_2 - \frac{k_1^2}{k_2})(\mathrm{e}^{-k_2 a} - \mathrm{e}^{k_2 a}) + 2\mathrm{i}k_1(\mathrm{e}^{-k_2 a} + \mathrm{e}^{k_2 a})}$$

而$\lim\limits_{a\to 0} 4\mathrm{i}k_1 \mathrm{e}^{-\mathrm{i}k_1 a} = 4\mathrm{i}k_1$，$\lim\limits_{a\to 0} 2\mathrm{i}k_1(\mathrm{e}^{-k_2 a} + \mathrm{e}^{k_2 a}) = 2\mathrm{i}k_1 \cdot 2 = 4\mathrm{i}k_1$，因此只需要讨论分母另一项是否收敛即可，而根据洛必达法则，有:

$$\begin{aligned}
\lim_{a\to 0}(k_2 - \frac{k_1^2}{k_2})(\mathrm{e}^{-k_2 a} - \mathrm{e}^{k_2 a}) &= \lim_{a\to 0} \frac{\mathrm{e}^{-k_2 a} - \mathrm{e}^{k_2 a}}{\frac{k_2}{k_2^2 - k_1^2}} = \lim_{a\to 0} \frac{(\mathrm{e}^{-k_2 a} - \mathrm{e}^{k_2 a})'}{(\frac{k_2}{k_2^2 - k_1^2})'} = \lim_{a\to 0} \frac{(\mathrm{e}^{-k_2 a} + \mathrm{e}^{k_2 a})(\frac{m\gamma}{a\hbar^2 k_2} - k_2)}{\frac{-\frac{m\gamma}{a^2\hbar^2 k_2}(k_2^2 - k_1^2) - k_2(-\frac{2m\gamma}{a^2\hbar^2})}{(k_2^2 - k_1^2)^2}} \\
&= \lim_{a\to 0} \frac{(\mathrm{e}^{-k_2 a} + \mathrm{e}^{k_2 a})(\frac{m\gamma}{a\hbar^2 k_2} - k_2)(k_2^2 - k_1^2)^2}{\frac{m\gamma}{a^2\hbar^2 k_2}\frac{2m\gamma}{a\hbar^2}} = \lim_{a\to 0} \frac{(\mathrm{e}^{-k_2 a} + \mathrm{e}^{k_2 a})(1 - \frac{a\hbar^2 k_2^2}{m\gamma})[\frac{2m(\frac{\gamma}{a} - 2E)}{\hbar^2}]^2}{\frac{2m\gamma}{a^2\hbar^2}} \\
&= \lim_{a\to 0} \frac{(\mathrm{e}^{-k_2 a} + \mathrm{e}^{k_2 a})[1 - \frac{\hbar^2}{m\gamma}\frac{2m(\gamma - Ea)}{\hbar^2}][\frac{2m(\gamma - 2Ea)}{\hbar}]^2}{2m\gamma} = \frac{2\cdot(1 - \frac{\hbar^2}{m\gamma}\frac{2m\gamma}{\hbar^2})\cdot(\frac{2m\gamma}{\hbar})^2}{2m\gamma} = -\frac{4m\gamma}{\hbar^2}
\end{aligned}$$

因此$a \to 0$时，有$\lim\limits_{a\to 0} S = \frac{4\mathrm{i}k_1}{-\frac{4m\gamma}{\hbar^2} + 4\mathrm{i}k_1} = \frac{\mathrm{i}k_1\hbar^2}{\mathrm{i}k_1\hbar^2 - m\gamma}$，从而相应的透射系数为:

$$|S|^2 = \frac{\mathrm{i}k_1\hbar^2}{\mathrm{i}k_1\hbar^2 - m\gamma} \cdot \frac{-\mathrm{i}k_1\hbar^2}{-\mathrm{i}k_1\hbar^2 - m\gamma} = \frac{k_1^2\hbar^4}{m^2\gamma^2 + k_1^2\hbar^4} = \frac{1}{1 + \frac{m^2\gamma^2}{k_1^2\hbar^4}} = \frac{1}{1 + \frac{m^2\gamma^2}{\frac{2mE}{\hbar^2}\cdot\hbar^4}} = \frac{1}{1 + \frac{m\gamma^2}{2E\hbar^2}}$$