课堂练习

练习1：证明$\hat{D}^{-1}(\boldsymbol{ds})\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{ds})=\hat{\boldsymbol{x}}+\boldsymbol{ds}$

证明：（方法1）对任意位置表象的态矢$|x\rangle$，有：

$$\begin{aligned}\hat{D}^{-1}(\boldsymbol{ds})\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{ds})|\boldsymbol{x}\rangle&=\hat{D}^{-1}(\boldsymbol{ds})\hat{\boldsymbol{x}}|\boldsymbol{x}+\boldsymbol{ds}\rangle=\hat{D}^{-1}(\boldsymbol{ds})[(\boldsymbol{x}+\boldsymbol{ds})|\boldsymbol{x}+\boldsymbol{ds}\rangle]=\hat{D}^{-1}(\boldsymbol{ds})[(\boldsymbol{x}+\boldsymbol{ds})|\boldsymbol{x}+\boldsymbol{ds}\rangle]\\&=(\boldsymbol{x}+\boldsymbol{ds})\hat{D}(-\boldsymbol{ds})|\boldsymbol{x}+\boldsymbol{ds}\rangle=(\boldsymbol{x}+\boldsymbol{ds})|\boldsymbol{x}\rangle\end{aligned}$$

而$(\hat{\boldsymbol{x}}+\boldsymbol{ds})|\boldsymbol{x}\rangle=\hat{\boldsymbol{x}}|\boldsymbol{x}\rangle+\boldsymbol{ds}|\boldsymbol{x}\rangle=\boldsymbol{x}|\boldsymbol{x}\rangle+\boldsymbol{ds}|\boldsymbol{x}\rangle=(\boldsymbol{x}+\boldsymbol{ds})|\boldsymbol{x}\rangle$，因此
$\hat{D}^{-1}(\boldsymbol{ds})\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{ds})|\boldsymbol{x}\rangle=(\hat{\boldsymbol{x}}+\boldsymbol{ds})|\boldsymbol{x}\rangle$，即$\hat{D}^{-1}(\boldsymbol{ds})\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{ds})=\hat{\boldsymbol{x}}+\boldsymbol{ds}$
（方法2）根据位置算符$\hat{\boldsymbol{x}}$与坐标平移算符$\hat{D}(\boldsymbol{ds})$的对易关系$[\hat{\boldsymbol{x}},\hat{D}(\boldsymbol{ds})]=\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{ds})-\hat{D}(\boldsymbol{ds})\hat{\boldsymbol{x}}=\boldsymbol{ds}$，有：

$$\begin{aligned}\hat{D}^{-1}(\boldsymbol{ds})\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{ds})&=\hat{D}^{-1}(\boldsymbol{ds})\{[\hat{\boldsymbol{x}},\hat{D}(\boldsymbol{ds})]+\hat{D}(\boldsymbol{ds})\hat{\boldsymbol{x}}\}=\hat{D}^{-1}(\boldsymbol{ds})\{\boldsymbol{ds}+\hat{D}(\boldsymbol{ds})\hat{\boldsymbol{x}}\}=\hat{D}^{-1}(\boldsymbol{ds})\boldsymbol{ds}+\hat{D}^{-1}(\boldsymbol{ds})\hat{D}(\boldsymbol{ds})\hat{\boldsymbol{x}}\\&=\hat{D}(-\boldsymbol{ds})\boldsymbol{ds}+[\hat{D}^{-1}(\boldsymbol{ds})\hat{D}(\boldsymbol{ds})]\hat{\boldsymbol{x}}=(1-\mathrm{i}\hat{\boldsymbol{K}}\cdot\boldsymbol{ds})\boldsymbol{ds}+\hat{I}\,\hat{\boldsymbol{x}}=\boldsymbol{ds}-\mathrm{i}\hat{\boldsymbol{K}}\cdot\boldsymbol{ds}^2+\hat{\boldsymbol{x}}\approx\hat{\boldsymbol{x}}+\boldsymbol{ds}\end{aligned}$$

练习2：证明$\langle\boldsymbol{x}|\hat{D}(\boldsymbol{ds})|u\rangle=\langle x-\boldsymbol{ds}|u\rangle$

证明：因为
$\langle\boldsymbol{x}|\hat{D}(\boldsymbol{ds})|u\rangle=\langle u|\hat{D}^{\dagger}(\boldsymbol{ds})|\boldsymbol{x}\rangle^*=\langle u|\hat{D}^{-1}(\boldsymbol{ds})|\boldsymbol{x}\rangle^*=\langle u|\hat{D}(-\boldsymbol{ds})|\boldsymbol{x}\rangle^*=\langle u|\boldsymbol{x}-\boldsymbol{ds}\rangle^*=\langle\boldsymbol{x}-\boldsymbol{ds}|u\rangle$
，故原式得证

练习3：在坐标表象中证明，坐标算符和动量算符满足基本对易关系$[\hat{x},\hat{p}]=\mathrm{i}\hbar$

证明：对任意的态矢$|\psi\rangle,|\phi\rangle$，有：

$$\begin{aligned}\langle\psi|[\hat{x},\hat{p}]|\phi\rangle&=\langle\psi|(\hat{x}\hat{p}-\hat{p}\hat{x})|\phi\rangle=\langle\psi|\hat{x}\hat{p}|\phi\rangle-\langle\psi|\hat{p}\hat{x}|\phi\rangle=\int\langle\psi|x\rangle\langle x|\hat{x}\hat{p}|\phi\rangle dx-\int\langle\psi|x\rangle\langle x|\hat{p}\hat{x}|\phi\rangle dx\\&=\int\langle\psi|x\rangle(\langle x|\hat{x})\hat{p}|\phi\rangle dx-\int\langle\psi|x\rangle\langle x|\hat{p}(\hat{x}|\phi\rangle)dx=\int\langle\psi|x\rangle(\langle x|x)\hat{p}|\phi\rangle dx-\int\langle\psi|x\rangle\{-\mathrm{i}\hbar\nabla[\langle x|(\hat{x}|\phi\rangle)]\}dx\\&=\int x\langle\psi|x\rangle\langle x|\hat{p}|\phi\rangle dx-\int\langle\psi|x\rangle\{-\mathrm{i}\hbar\nabla\langle x|\hat{x}|\phi\rangle\}dx=\int x\langle\psi|x\rangle[-\mathrm{i}\hbar\nabla\langle x|\phi\rangle]dx-\int\langle\psi|x\rangle\{-\mathrm{i}\hbar\nabla(x\langle x|\phi\rangle)\}dx\\&=-\mathrm{i}\hbar\int x\langle\psi|x\rangle\nabla\langle x|\phi\rangle dx+\mathrm{i}\hbar\int\langle\psi|x\rangle\{\langle x|\phi\rangle+x\nabla\langle x|\phi\rangle\}dx=\mathrm{i}\hbar\int\langle\psi|x\rangle\langle x|\phi\rangle dx=\mathrm{i}\hbar\langle\psi|\phi\rangle=\langle\psi|(\mathrm{i}\hbar\hat{I})|\phi\rangle\end{aligned}$$

因此$[\hat{x},\hat{p}]=\mathrm{i}\hbar\hat{I}=\mathrm{i}\hbar$，证毕

练习4：证明角动量的对易关系$[\hat{L}_i,\hat{L}_j]=\mathrm{i}\hbar\sum_k\varepsilon_{ijk}\hat{L}_k$，其中$\varepsilon_{ijk}$为Levi-Civita符号，若$ijk$由1,2,3的偶置换变成则$\varepsilon_{ijk}=1$，若$ijk$由1,2,3的奇置换变成则$\varepsilon_{ijk}=-1$，若$ijk$中任意一对相等则$\varepsilon_{ijk}=0$

证明：

第二章习题

1.证明δ函数的下列性质：1）$\delta(ax)=\frac{\delta(x)}{|a|}\ (a\neq 0)$；2）$\delta(x)=\delta(-x)$，即$\delta(x)$为偶函数；3）定义$\delta$函数的导数为$\delta'(x-x')\equiv\frac{d}{dx}\delta(x-x')$，则有$\delta'(x-x')=\delta(x-x')\frac{d}{dx}$；4）$\delta(f(x))=\sum_i\frac{\delta(x_i)}{|f'(x_i)|}$，其中$x_i$是方程的第$i$个根，$f'(x)$表示对$f(x)$的一阶导数，这里要求$f(x)$是个光滑函数，并且$f'(x_i)\neq 0$

2.求出波函数$\psi(x)=A\mathrm{e}^{-\frac{x^2}{2\sigma^2}}$的归一化因子，然后求出动量空间的波函数形式。你发现什么特征?

3.令$|a\rangle=|s_z+\rangle$，$|b\rangle=|s_x+\rangle$，1）写出算符$|a\rangle\langle b|$以$\hat{S}_z$的本征态为基矢的矩阵表示；2）计算态矢$|u\rangle=\alpha(|a\rangle+|b\rangle)$中的归一化因子$\alpha$；3）对状态$|u\rangle$测量得到$s_z=\frac{1}{2}\hbar$和$s_z=-\frac{1}{2}\hbar$的概率分别是多少?

4.证明对应于有限平移$\boldsymbol{s}$，存在如下恒等式$\hat{D}^{-1}(\boldsymbol{s})\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{s})=\hat{\boldsymbol{x}}+\boldsymbol{s}$（看看你能用几种方法证明？）

5.假定函数$F(x)$和$G(x)$关于$x=0$泰勒展开收敛，证明如下对易关系恒等式：$[\hat{x},F(\hat{p})]=\mathrm{i}\hbar\frac{\partial F}{\partial\hat{p}}$，$[\hat{p},G(\hat{x})]=-\mathrm{i}\hbar\frac{\partial G}{\partial\hat{x}}$

证明：我们首先证明$[\hat{x},\hat{p}^n]=\mathrm{i}\hbar n\hat{p}^{n-1},[\hat{p},\hat{x}^n]=-\mathrm{i}\hbar n\hat{x}^{n-1}$，根据对易关系$[\hat{x},\hat{p}]=\hat{x}\hat{p}-\hat{p}\hat{x}=\mathrm{i}\hbar$，得：

$$\begin{aligned}[\hat{x},\hat{p}^n]&=\hat{x}\hat{p}^n-\hat{p}^n\hat{x}=(\hat{x}\hat{p})\hat{p}^{n-1}-\hat{p}^n\hat{x}=([\hat{x},\hat{p}]+\hat{p}\hat{x})\hat{p}^{n-1}-\hat{p}^n\hat{x}=(\mathrm{i}\hbar+\hat{p}\hat{x})\hat{p}^{n-1}-\hat{p}^n\hat{x}=\mathrm{i}\hbar\hat{p}^{n-1}+\hat{p}\hat{x}\hat{p}^{n-1}-\hat{p}^n\hat{x}\\&=\mathrm{i}\hbar\hat{p}^{n-1}+\hat{p}(\hat{x}\hat{p})\hat{p}^{n-2}-\hat{p}^n\hat{x}=\mathrm{i}\hbar\hat{p}^{n-1}+\hat{p}([\hat{x},\hat{p}]+\hat{p}\hat{x})\hat{p}^{n-2}-\hat{p}^n\hat{x}=\mathrm{i}\hbar\hat{p}^{n-1}+\hat{p}(\mathrm{i}\hbar+\hat{p}\hat{x})\hat{p}^{n-2}-\hat{p}^n\hat{x}\\&=\mathrm{i}\hbar\hat{p}^{n-1}+(\mathrm{i}\hbar\hat{p}+\hat{p}^2\hat{x})\hat{p}^{n-2}-\hat{p}^n\hat{x}=\mathrm{i}\hbar\hat{p}^{n-1}+\mathrm{i}\hbar\hat{p}^{n-1}+\hat{p}^2\hat{x}\hat{p}^{n-2}-\hat{p}^n\hat{x}=2\mathrm{i}\hbar\hat{p}^{n-1}+\hat{p}^2\hat{x}\hat{p}^{n-2}-\hat{p}^n\hat{x}\\&=\cdots=\mathrm{i}\hbar n\hat{p}^{n-1}+\hat{p}^n\hat{x}-\hat{p}^n\hat{x}=\mathrm{i}\hbar n\hat{p}^{n-1}\end{aligned}$$

$$\begin{aligned}[\hat{p},\hat{x}^n]&=\hat{p}\hat{x}^n-\hat{x}^n\hat{p}=(\hat{p}\hat{x})\hat{x}^{n-1}-\hat{x}^n\hat{p}=(-[\hat{x},\hat{p}]+\hat{x}\hat{p})\hat{x}^{n-1}-\hat{x}^n\hat{p}=(-\mathrm{i}\hbar+\hat{x}\hat{p})\hat{x}^{n-1}-\hat{x}^n\hat{p}=-\mathrm{i}\hbar\hat{x}^{n-1}+\hat{x}\hat{p}\hat{x}^{n-1}-\hat{x}^n\hat{p}\\&=-\mathrm{i}\hbar\hat{x}^{n-1}+\hat{x}(\hat{p}\hat{x})\hat{x}^{n-2}-\hat{x}^n\hat{p}=-\mathrm{i}\hbar\hat{x}^{n-1}+\hat{x}(-[\hat{x},\hat{p}]+\hat{x}\hat{p})\hat{x}^{n-2}-\hat{x}^n\hat{p}=-\mathrm{i}\hbar\hat{x}^{n-1}+\hat{x}(-\mathrm{i}\hbar+\hat{x}\hat{p})\hat{x}^{n-2}-\hat{x}^n\hat{p}\\&=-\mathrm{i}\hbar\hat{x}^{n-1}+(-\mathrm{i}\hbar\hat{x}+\hat{x}^2\hat{p})\hat{x}^{n-2}-\hat{x}^n\hat{p}=-\mathrm{i}\hbar\hat{x}^{n-1}-\mathrm{i}\hbar\hat{x}^{n-1}+\hat{x}^2\hat{p}\hat{x}^{n-2}-\hat{x}^n\hat{p}=-2\mathrm{i}\hbar\hat{x}^{n-1}+\hat{x}^2\hat{p}\hat{x}^{n-2}-\hat{x}^n\hat{p}\\&=\cdots=-\mathrm{i}\hbar n\hat{x}^{n-1}+\hat{x}^n\hat{p}-\hat{x}^n\hat{p}=-\mathrm{i}\hbar n\hat{x}^{n-1}\end{aligned}$$

接下来，我们让$F(\hat{p})$和$G(\hat{x})$在原点处进行泰勒展开，得$F(\hat{p})=\sum\limits_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}\hat{p}^k,G(\hat{x})=\sum\limits_{k=0}^{\infty}\frac{G^{(k)}(0)}{k!}\hat{x}^k$，因此：

$$[\hat{x},F(\hat{p})]=[\hat{x},\sum_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}\hat{p}^k]=\sum_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}[\hat{x},\hat{p}^k]=\sum_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}(\mathrm{i}\hbar k\hat{p}^{k-1})=\mathrm{i}\hbar\sum_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}\frac{\partial\hat{p}^k}{\partial\hat{p}}=\mathrm{i}\hbar\frac{\partial\sum\limits_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}\hat{p}^k}{\partial\hat{p}}=\mathrm{i}\hbar\frac{\partial F}{\partial\hat{p}}$$

$$[\hat{p},G(\hat{x})]=[\hat{p},\sum_{k=0}^{\infty}\frac{G^{(k)}(0)}{k!}\hat{x}^k]=\sum_{k=0}^{\infty}\frac{G^{(k)}(0)}{k!}[\hat{p},\hat{x}^k]=\sum_{k=0}^{\infty}\frac{G^{(k)}(0)}{k!}(-\mathrm{i}\hbar k\hat{x}^{k-1})=-\mathrm{i}\hbar\sum_{k=0}^{\infty}\frac{G^{(k)}(0)}{k!}\frac{\partial\hat{x}^k}{\partial\hat{x}}=-\mathrm{i}\hbar\frac{\partial\sum\limits_{k=0}^{\infty}\frac{G^{(k)}(0)}{k!}\hat{x}^k}{\partial\hat{x}}=-\mathrm{i}\hbar\frac{\partial G}{\partial\hat{x}}$$