课堂练习

练习1：证明式$Tr(\mathbf{AB}) = Tr(\mathbf{BA})$

证明：设$\mathbf{A} = (a_{ij})$，$\mathbf{B} = (b_{ij})$,其中$\mathbf{A}$、$\mathbf{B}$为n级矩阵，则有：

$$Tr(\mathbf{AB}) = \sum_{i=1}^{n}(\mathbf{AB})_{ii} = \sum_{i=1}^{n}(\sum_{k=1}^{n} a_{ik}b_{ki}) \text{，} \quad Tr(\mathbf{BA}) = \sum_{k=1}^{n}(\mathbf{BA})_{kk} = \sum_{k=1}^{n}(\sum_{i=1}^{n} b_{ki}a_{ik}) = \sum_{i=1}^{n}(\sum_{k=1}^{n} a_{ik}b_{ki})$$

（交换求和顺序不影响最终结果）
因此$Tr(\mathbf{AB}) = Tr(\mathbf{BA})$

练习2：证明如果$\mathbf{T}^{-1}\mathbf{BT} = \mathbf{A}$，则有$Tr(\mathbf{B}) = Tr(\mathbf{A})$

证明：由练习1的结论得：

$$Tr(\mathbf{T}^{-1}\mathbf{BT}) = Tr(\mathbf{T}^{-1}(\mathbf{BT})) = Tr((\mathbf{BT})\mathbf{T}^{-1}) = Tr(\mathbf{B}(\mathbf{TT}^{-1})) = Tr(\mathbf{B})$$

又由题可知$\mathbf{T}^{-1}\mathbf{BT} = Tr(\mathbf{A})$，故$Tr(\mathbf{B}) = Tr(\mathbf{A})$

练习3：用3×3矩阵的行列式验证式(18)，其中式(18)的形式为

$$\begin{aligned} det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{vmatrix} &\equiv \sum_{I}^{n!}(-1)^{P_I}\hat{P}_I A_{11}A_{22}\dots A_{nn} \\ = \sum_{I}^{n!}(-1)^{P_I}A_{I_1 1}A_{I_2 2}\dots A_{I_n n} &= \sum_{I}^{n!}(-1)^{P_I}A_{1I_1}A_{2I_2}\dots A_{nI_n} \end{aligned}$$

证明：自然数1~3的排列为$(1,2,3)$，$(1,3,2)$，$(2,1,3)$，$(2,3,1)$，$(3,1,2)$，$(3,2,1)$；相应的，互换次数为$P_{(1,2,3)} = 0$，$P_{(1,3,2)} = 1$，$P_{(2,1,3)} = 1$，$P_{(2,3,1)} = 2$，$P_{(3,1,2)} = 2$，$P_{(3,2,1)} = 1$。因此对3×3矩阵的行列式，有：

$$\begin{aligned} det(\mathbf{A}) = |\mathbf{A}| &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \\ &= (-1)^{P(1,2,3)}A_{11}A_{22}A_{33} + (-1)^{P(1,3,2)}A_{11}A_{23}A_{32} + (-1)^{P(2,1,3)}A_{12}A_{21}A_{33} \\ &\quad + (-1)^{P(2,3,1)}A_{12}A_{23}A_{31} + (-1)^{P(3,1,2)}A_{13}A_{21}A_{32} + (-1)^{P(3,2,1)}A_{13}A_{22}A_{31} \\ &= A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31} \end{aligned}$$

这与3×3矩阵的行列式定义（即式(15)）一致

习题1.1

1.考虑由四个复数a, b, c, d构成的如下2×2矩阵$\mathbf{A} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

1)满足什么条件时A是个厄米矩阵？2)满足什么条件时A是个幺正矩阵？3)满足什么条件时A可逆（存在逆矩阵）？写出A的逆矩阵具体表达式。

解：1)若$\mathbf{A}$是个厄米矩阵，即$\mathbf{A} = \mathbf{A}^\dagger$，其中$\mathbf{A}^\dagger = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}$，则$\begin{cases} a = a^* \\ c = b^* \\ b = c^* \\ d = d^* \end{cases}$，故当$a$，$d$皆为实数，而$b$，$c$互为共轭复数时，$\mathbf{A}$是个厄米矩阵

2)若$\mathbf{A}$是个幺正矩阵，则$\mathbf{AA}^\dagger = \mathbf{A}^\dagger\mathbf{A} = I$，又$\mathbf{AA}^\dagger = \begin{bmatrix} aa^* + cc^* & ab^* + cd^* \\ ba^* + dc^* & bb^* + dd^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$，$\mathbf{A}^\dagger\mathbf{A} = \begin{bmatrix} a^*a + b^*b & a^*c + b^*d \\ c^*a + d^*b & c^*c + d^*d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
故由一一对应得，当$\begin{cases} aa^* = dd^* \Leftrightarrow |a| = |d| \\ bb^* = cc^* \Leftrightarrow |b| = |c| \\ aa^* + bb^* = 1, cc^* + dd^* = 1 \\ ba^* + dc^* = 0, ac^* + bd^* = 0 \end{cases}$时，$\mathbf{A}$是个幺正矩阵。

3)若$\mathbf{A}$可逆，则存在矩阵$\mathbf{B}$，使得$\mathbf{AB} = \mathbf{BA} = \mathbf{I}$，容易验证，当$\mathbf{B} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$，其中$|\mathbf{A}| = det(\mathbf{A}) = ad - bc$时，有

$\mathbf{AB}=\mathbf{BA}=\frac{1}{|\mathbf{A}|}\begin{bmatrix}ad-bc & 0\\ 0 & ad-bc\end{bmatrix}=\frac{1}{ad-bc}\begin{bmatrix}ad-bc & 0\\ 0 & ad-bc\end{bmatrix}$，若要进一步变为单位矩阵，需要$ad-bc\neq 0$，否则原式无意义

因此，当$ad-bc\neq 0$时，$\mathbf{A}$可逆，此时逆矩阵为$\mathbf{A}^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d & -c\\ -b & a\end{bmatrix}$

2.证明：如果两个厄米矩阵$\mathbf{A}$和$\mathbf{B}$的乘积$\mathbf{C}=\mathbf{AB}$也是厄米矩阵，那么$\mathbf{A}$和$\mathbf{B}$一定对易

证明：因为$\mathbf{C}=\mathbf{C}^\dagger$，其中$\mathbf{C}=\mathbf{AB}$，$\mathbf{C}^\dagger=(\mathbf{AB})^\dagger=\mathbf{B}^\dagger\mathbf{A}^\dagger$，故$\mathbf{AB}=\mathbf{B}^\dagger\mathbf{A}^\dagger$
又由题意知$\mathbf{A}=\mathbf{A}^\dagger$，$\mathbf{B}=\mathbf{B}^\dagger$，故结合得$\mathbf{AB}=\mathbf{BA}$，即$[\mathbf{A},\mathbf{B}]=\mathbf{AB}-\mathbf{BA}=0$，从而$\mathbf{A}$与$\mathbf{B}$一定对易，证毕

3.从行列式一般定义（或2×2矩阵）出发证明（验证）上面行列式的性质

证明：在证明前，我们定义矩阵$\mathbf{A}=(A_{ij})$，$\mathbf{B}=(B_{ij})$，其中$\mathbf{A}$、$\mathbf{B}$为n级矩阵，此外，我们还会利用行列式的定义，以及行列式的余子式展开
$|\mathbf{A}|=\sum\limits_{i=1}^{n}A_{ij}cof(A_{ij}),\forall j=1,2,\ldots,n$
定理1的证明：若矩阵的某一行矩阵元都为零，如矩阵$\mathbf{A}$的第i行为零，则有：

$$|\mathbf{A}|=\begin{vmatrix}A_{11} & A_{12} & \ldots & A_{1n}\\ A_{21} & A_{22} & \ldots & A_{2n}\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \ldots & 0\\ \vdots & \vdots & & \vdots\\ A_{n1} & A_{n2} & \ldots & A_{nn}\end{vmatrix}=0\times cof(A_{i1})+0\times cof(A_{i2})+\cdots+0\times cof(A_{in})=0$$

同理可得，矩阵的某一列矩阵元均为零
定理2的证明：以上三角矩阵为例，其对应的行列式为：

$$|\mathbf{A}|=\begin{vmatrix}A_{11} & A_{12} & A_{13} & \ldots & A_{1n}\\ 0 & A_{22} & A_{23} & \ldots & A_{2n}\\ 0 & 0 & A_{33} & \ldots & A_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & A_{nn}\end{vmatrix}=A_{11}\begin{vmatrix}A_{22} & A_{23} & \ldots & A_{2n}\\ 0 & A_{33} & \ldots & A_{3n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & A_{nn}\end{vmatrix}+\sum_{i=1}^{n}0\times cof(A_{i1})=A_{11}\begin{vmatrix}A_{22} & A_{23} & \ldots & A_{2n}\\ 0 & A_{33} & \ldots & A_{3n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & A_{nn}\end{vmatrix}$$
$$=A_{11}(A_{22}\begin{vmatrix}A_{33} & \ldots & A_{3n}\\ \vdots & \ddots & \vdots\\ 0 & \ldots & A_{nn}\end{vmatrix}+\sum_{i=2}^{n}0\times cof(A_{i2}))=A_{11}A_{22}\begin{vmatrix}A_{33} & \ldots & A_{3n}\\ \vdots & \ddots & \vdots\\ 0 & \ldots & A_{nn}\end{vmatrix}=\cdots=\prod_{i=1}^{n}A_{ii}$$

下三角矩阵对应的行列式的计算方法同理，特别的，对角矩阵是上（下）三角矩阵的特殊情形
定理3的证明：以下只讨论交换两行的情形，设有如下行列式：

$$|\mathbf{A}|=\begin{vmatrix}A_{11} & A_{12} & \ldots & A_{1n}\\ \vdots & \vdots & & \vdots\\ A_{l1} & A_{l2} & \ldots & A_{ln}\\ \vdots & \vdots & & \vdots\\ A_{m1} & A_{m2} & \ldots & A_{mn}\\ \vdots & \vdots & & \vdots\\ A_{n1} & A_{n2} & \ldots & A_{nn}\end{vmatrix},\quad |\mathbf{A}_{\text{swap}}|=\begin{vmatrix}A_{11} & A_{12} & \ldots & A_{1n}\\ \vdots & \vdots & & \vdots\\ A'_{l1} & A'_{l2} & \ldots & A'_{ln}\\ \vdots & \vdots & & \vdots\\ A'_{m1} & A'_{m2} & \ldots & A'_{mn}\\ \vdots & \vdots & & \vdots\\ A_{n1} & A_{n2} & \ldots & A_{nn}\end{vmatrix}$$

其中$A'_{li}=A_{mi},A'_{mi}=A_{li},\forall i=1,2,\ldots,n$
对$|\mathbf{A}|$和$|\mathbf{A}_{\text{swap}}|$分别按行展开，有$|\mathbf{A}|=\sum\limits_{I}^{n!}(-1)^{P_I}A_{1I_1}\ldots A_{lI_l}\ldots A_{mI_m}\ldots A_{nI_n}$，$|\mathbf{A}_{\text{swap}}|=\sum\limits_{I}^{n!}(-1)^{P_I}A_{1I_1}\ldots A'_{lI_l}\ldots A'_{mI_m}\ldots A_{nI_n}$，
结合前述性质知，$|\mathbf{A}|$中$A_{1I_1}\ldots A_{lI_l}\ldots A_{mI_m}\ldots A_{nI_n}$这一项在$|\mathbf{A}_{\text{swap}}|$中为$A_{1I_1}\ldots A'_{lI_m}\ldots A'_{mI_l}\ldots A_{nI_n}$，而对应的置换操作满足
$\hat{P}_{(1\ldots I_l\ldots I_m\ldots I_n)}=\hat{P}_{(1\ldots I_m\ldots I_l\ldots I_n)}\hat{P}_{I_mI_l}$，即操作数相差1，故$A_{1I_1}\ldots A_{lI_l}\ldots A_{mI_m}\ldots A_{nI_n}$这一项在$|\mathbf{A}|$和$|\mathbf{A}_{\text{swap}}|$中系数相反，从而
$|\mathbf{A}_{\text{swap}}|=-|\mathbf{A}|$，证毕。同理可得该定理对列交换也成立
定理4的证明：利用定理3可知，若$|\mathbf{A}|$存在相同的两行（或两列），则相互交换后，有$|\mathbf{A}_{\text{swap}}|=-|\mathbf{A}|$，但由于相互交换的两行（或两列）相同，因此交换后行列式不变，即$|\mathbf{A}_{\text{swap}}|=|\mathbf{A}|$，联立可得$|\mathbf{A}|=-|\mathbf{A}|$，即$|\mathbf{A}|=0$，证毕
定理5的证明：为讨论方便，我们令$\mathbf{B}=(B_{ij})=(A_{ji})$，此时$\mathbf{B}=\mathbf{A}^T$
由行列式定义，对$|\mathbf{A}|$按行展开，有$|\mathbf{A}|=\sum\limits_{I}^{n!}(-1)^{P_I}A_{1I_1}A_{2I_2}\ldots A_{nI_n}$；按列展开，有$|\mathbf{A}|=\sum\limits_{I}^{n!}(-1)^{P_I}A_{I_11}A_{I_22}\ldots A_{I_nn}$。这两种展开是相同的。
另一方面，对$|\mathbf{B}|$按行展开，有$|\mathbf{B}|=|\mathbf{A}^T|=\sum\limits_{I}^{n!}(-1)^{P_I}B_{1I_1}B_{2I_2}\ldots B_{nI_n}=\sum\limits_{I}^{n!}(-1)^{P_I}A_{I_11}A_{I_22}\ldots A_{I_nn}$，恰好为对$|\mathbf{A}|$按列展开，因此有

$|\mathbf{A}| = |\mathbf{B}| = |\mathbf{A}^T|$

定理6的证明：仿照定理5的证明，令$\mathbf{B} = (B_{ij}) = (A^*_{ji})$，此时$\mathbf{B} = \mathbf{A}^\dagger$。分别展开展开$|\mathbf{A}|$和$|\mathbf{B}|$得：

$$|\mathbf{A}| = \sum_{I}^{n!}(-1)^{P_I} A_{1I_1}A_{2I_2}\dots A_{nI_n} = \sum_{I}^{n!}(-1)^{P_I} A_{I_11}A_{I_22}\dots A_{I_nn}\text{，}\quad |\mathbf{B}| = |\mathbf{A}^\dagger| = \sum_{I}^{n!}(-1)^{P_I} B_{1I_1}B_{2I_2}\dots B_{nI_n} = \sum_{I}^{n!}(-1)^{P_I} A^*_{I_11}A^*_{I_22}\dots A^*_{I_nn}$$

让$|\mathbf{A}|$取复共轭，得$|\mathbf{A}|^* = [\sum_{I}^{n!}(-1)^{P_I} A_{I_11}A_{I_22}\dots A_{I_nn}]^* = \sum_{I}^{n!}(-1)^{P_I} A^*_{I_11}A^*_{I_22}\dots A^*_{I_nn}$，从而得$|\mathbf{A}|^* = |\mathbf{B}| = |\mathbf{A}^\dagger|$

定理7的证明：此处要用到Laplace定理，记$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}$，$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \dots & B_{nn} \end{bmatrix}$，再记对角矩阵为$\mathbf{I}$，零矩阵为$\mathbf{O}$，则我们可以将以上矩阵拼接为$\begin{bmatrix} \mathbf{A} & \mathbf{O} \\ -\mathbf{I} & \mathbf{B} \end{bmatrix}$，其对应的行列式为：

$$\begin{vmatrix} \mathbf{A} & \mathbf{O} \\ -\mathbf{I} & \mathbf{B} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} & 0 & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & A_{2n} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} & 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & B_{11} & B_{12} & \dots & B_{1n} \\ 0 & -1 & \dots & 0 & B_{21} & B_{22} & \dots & B_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & -1 & B_{n1} & B_{n2} & \dots & B_{nn} \end{vmatrix}$$

根据Laplace定理，对该行列式按前n行展开，则因该行列式中前n行除去最左上角的n级子式外，其余的n级子式均含有一个全零列（即其余n级子式均等于零），因此$\begin{vmatrix} \mathbf{A} & \mathbf{O} \\ -\mathbf{I} & \mathbf{B} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{vmatrix} \cdot \begin{vmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \dots & B_{nn} \end{vmatrix} = |\mathbf{A}||\mathbf{B}|$

接下来，根据定理4，我们可以将行列式的一行（或一列）乘上一个系数，加在行列式的另一行（或另一列）上，而行列式的值不变。从而，将第$(n+1)$行乘以A_{11}，再加在第一行上；将第$(n+2)$行乘以A_{12}，再加在第一行上；……；以此类推，直至将第$2n$行乘以A_{1n}，再加在第一行上。由此可得：

$$\begin{vmatrix} \mathbf{A} & \mathbf{O} \\ -\mathbf{I} & \mathbf{B} \end{vmatrix} = \begin{vmatrix} 0 & 0 & \dots & 0 & \sum_{i=1}^{n} A_{1i}B_{i1} & \sum_{i=1}^{n} A_{1i}B_{i2} & \dots & \sum_{i=1}^{n} A_{1i}B_{in} \\ A_{21} & A_{22} & \dots & A_{2n} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} & 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & B_{11} & B_{12} & \dots & B_{1n} \\ 0 & -1 & \dots & 0 & B_{21} & B_{22} & \dots & B_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & -1 & B_{n1} & B_{n2} & \dots & B_{nn} \end{vmatrix}$$

同理，将第$(n+1)$行乘以A_{21}，再加在第二行上；将第$(n+2)$行乘以A_{22}，再加在第二行上；……；以此类推，直至将第$2n$行乘以A_{2n}，再加在第二行上。由此可得：

$$\begin{vmatrix} \mathbf{A} & \mathbf{O} \\ -\mathbf{I} & \mathbf{B} \end{vmatrix} = \begin{vmatrix} 0 & 0 & \dots & 0 & \sum_{i=1}^{n} A_{1i}B_{i1} & \sum_{i=1}^{n} A_{1i}B_{i2} & \dots & \sum_{i=1}^{n} A_{1i}B_{in} \\ 0 & 0 & \dots & 0 & \sum_{i=1}^{n} A_{2i}B_{i1} & \sum_{i=1}^{n} A_{2i}B_{i2} & \dots & \sum_{i=1}^{n} A_{2i}B_{in} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} & 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & B_{11} & B_{12} & \dots & B_{1n} \\ 0 & -1 & \dots & 0 & B_{21} & B_{22} & \dots & B_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & -1 & B_{n1} & B_{n2} & \dots & B_{nn} \end{vmatrix}$$

继续如此迭代，最终结合Laplace定理，得到：

$$
\begin{aligned}
\begin{vmatrix} \mathbf{A} & \mathbf{O} \\ -\mathbf{I} & \mathbf{B} \end{vmatrix} &= \begin{vmatrix}
0 & 0 & \dots & 0 & \sum\limits_{i=1}^{n} A_{1i}B_{i1} & \sum\limits_{i=1}^{n} A_{1i}B_{i2} & \dots & \sum\limits_{i=1}^{n} A_{1i}B_{in} \\
0 & 0 & \dots & 0 & \sum\limits_{i=1}^{n} A_{2i}B_{i1} & \sum\limits_{i=1}^{n} A_{2i}B_{i2} & \dots & \sum\limits_{i=1}^{n} A_{2i}B_{in} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \dots & 0 & \sum\limits_{i=1}^{n} A_{ni}B_{i1} & \sum\limits_{i=1}^{n} A_{ni}B_{i2} & \dots & \sum\limits_{i=1}^{n} A_{ni}B_{in} \\
-1 & 0 & \dots & 0 & B_{11} & B_{12} & \dots & B_{1n} \\
0 & -1 & \dots & 0 & B_{21} & B_{22} & \dots & B_{2n} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \dots & -1 & B_{n1} & B_{n2} & \dots & B_{nn}
\end{vmatrix} \\
&= \begin{vmatrix}
\sum\limits_{i=1}^{n} A_{1i}B_{i1} & \sum\limits_{i=1}^{n} A_{1i}B_{i2} & \dots & \sum\limits_{i=1}^{n} A_{1i}B_{in} \\
\sum\limits_{i=1}^{n} A_{2i}B_{i1} & \sum\limits_{i=1}^{n} A_{2i}B_{i2} & \dots & \sum\limits_{i=1}^{n} A_{2i}B_{in} \\
\vdots & \vdots & & \vdots \\
\sum\limits_{i=1}^{n} A_{ni}B_{i1} & \sum\limits_{i=1}^{n} A_{ni}B_{i2} & \dots & \sum\limits_{i=1}^{n} A_{ni}B_{in}
\end{vmatrix} \cdot \begin{vmatrix}
-1 & 0 & \dots & 0 \\
0 & -1 & \dots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \dots & -1
\end{vmatrix} \cdot (-1)^{(\sum\limits_{i=1}^{n} i)+\sum\limits_{i=1}^{n}(n+i)} \\
&= \begin{vmatrix}
\sum\limits_{i=1}^{n} A_{1i}B_{i1} & \sum\limits_{i=1}^{n} A_{1i}B_{i2} & \dots & \sum\limits_{i=1}^{n} A_{1i}B_{in} \\
\sum\limits_{i=1}^{n} A_{2i}B_{i1} & \sum\limits_{i=1}^{n} A_{2i}B_{i2} & \dots & \sum\limits_{i=1}^{n} A_{2i}B_{in} \\
\vdots & \vdots & & \vdots \\
\sum\limits_{i=1}^{n} A_{ni}B_{i1} & \sum\limits_{i=1}^{n} A_{ni}B_{i2} & \dots & \sum\limits_{i=1}^{n} A_{ni}B_{in}
\end{vmatrix} \cdot (-1)^n \cdot (-1)^{(\sum\limits_{i=1}^{n} i)+\sum\limits_{i=1}^{n}(n+i)} \\
&= \begin{vmatrix}
\sum\limits_{i=1}^{n} A_{1i}B_{i1} & \sum\limits_{i=1}^{n} A_{1i}B_{i2} & \dots & \sum\limits_{i=1}^{n} A_{1i}B_{in} \\
\sum\limits_{i=1}^{n} A_{2i}B_{i1} & \sum\limits_{i=1}^{n} A_{2i}B_{i2} & \dots & \sum\limits_{i=1}^{n} A_{2i}B_{in} \\
\vdots & \vdots & & \vdots \\
\sum\limits_{i=1}^{n} A_{ni}B_{i1} & \sum\limits_{i=1}^{n} A_{ni}B_{i2} & \dots & \sum\limits_{i=1}^{n} A_{ni}B_{in}
\end{vmatrix} = |\mathbf{AB}|
\end{aligned}
$$

因此$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$
定理8的证明：将题中行列式按第i列（即和式出现的那一列）展开余子式得：

$$\begin{vmatrix} A_{11} & A_{12} & \dots & \sum_{k=1}^{m} c_k B_{1k} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & \sum_{k=1}^{m} c_k B_{2k} & \dots & A_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & \sum_{k=1}^{m} c_k B_{nk} & \dots & A_{nn} \end{vmatrix} = \sum_{k=1}^{m} c_k B_{1k}(-1)^{1+i} \begin{vmatrix} A_{21} & A_{22} & \dots & A_{2n} \\ A_{31} & A_{32} & \dots & A_{3n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{vmatrix} + \sum_{k=1}^{m} c_k B_{2k}(-1)^{2+i} \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{31} & A_{32} & \dots & A_{3n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{vmatrix}$$

$$+ \dots + \sum_{k=1}^{m} c_k B_{nk}(-1)^{n+i} \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n-1,1} & A_{n-1,2} & \dots & A_{n-1,n} \end{vmatrix}$$

$$= \sum_{k=1}^{m} c_k [B_{1k}(-1)^{1+i} \begin{vmatrix} A_{21} & A_{22} & \dots & A_{2n} \\ A_{31} & A_{32} & \dots & A_{3n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{vmatrix} + B_{2k}(-1)^{2+i} \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{31} & A_{32} & \dots & A_{3n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{vmatrix}$$

$$+ \dots + B_{nk}(-1)^{n+i} \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n-1,1} & A_{n-1,2} & \dots & A_{n-1,n} \end{vmatrix}]$$

$$= \sum_{k=1}^{m} c_k \begin{vmatrix} A_{11} & A_{12} & \dots & B_{1k} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & B_{2k} & \dots & A_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & B_{nk} & \dots & A_{nn} \end{vmatrix}$$

该命题也适用于行中出现相同形式和式的情形

4. 证明将矩阵的任一行（列）加上另外一行（列）乘以一个常数所得新的矩阵的行列式与原矩阵行列式相等，以3×3矩阵为例， $\begin{vmatrix} A_{11}+aA_{12} & A_{12} & A_{13} \\ A_{21}+aA_{22} & A_{22} & A_{23} \\ A_{31}+aA_{32} & A_{32} & A_{33} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$

证明：根据第3题第8点性质，可得：

$$\begin{aligned} \begin{vmatrix} A_{11}+aA_{12} & A_{12} & A_{13} \\ A_{21}+aA_{22} & A_{22} & A_{23} \\ A_{31}+aA_{32} & A_{32} & A_{33} \end{vmatrix} &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} aA_{12} & A_{12} & A_{13} \\ aA_{22} & A_{22} & A_{23} \\ aA_{32} & A_{32} & A_{33} \end{vmatrix} \\ &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} + a\begin{vmatrix} A_{12} & A_{12} & A_{13} \\ A_{22} & A_{22} & A_{23} \\ A_{32} & A_{32} & A_{33} \end{vmatrix} \end{aligned}$$

再由第3题第4点性质，得$a\begin{vmatrix} A_{12} & A_{12} & A_{13} \\ A_{22} & A_{22} & A_{23} \\ A_{32} & A_{32} & A_{33} \end{vmatrix} = 0$，故有$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} + a\begin{vmatrix} A_{12} & A_{12} & A_{13} \\ A_{22} & A_{22} & A_{23} \\ A_{32} & A_{32} & A_{33} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$，证毕