Correlation Coefficient of a Two-Channel Bell Test by Probability Theory

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***Abstract*—It is widely believed that only quantum mechanics can characterize the correlations of a Bell test. In fact, quantum mechanics is superfluous to the problem, and a correlation coefficient may be derived by probability theory.**

I. Introduction

John S. Bell famously proposed a test supposed to prove that quantum mechanics was, in essence, magical [1]. I put the matter this way because the outcome of such an experiment is soluble by probability theory, and yet Bell seems to have claimed quantum mechanics alone could provide the One True Answer that surpassed all other solutions. At the time of this writing, Bell's test is considered passed and the matter resolved in favor of the One True Answer.

Of course, what actually happened is that Bell *did not know* the test could be analyzed by means other than quantum mechanics, and also surely *already entertained the notion* that quantum mechanics might indeed be magical. Quantum mechanics has widely been regarded as some kind of "magic" for a very long time. However, some of us do not—at least when supposedly engaged in the scientific endeavor—entertain the notion that magic—or, indeed, anything remotely resembling it—exists. Furthermore, the Bell test problem is not, in fact, one of quantum mechanics at all. It is more properly a problem in random process analysis. It merely happens that quantum mechanics "knows" how to solve such a problem, without the "user" of quantum mechanics having to understand one iota of the subject of random process analysis.

You and I, however, wish to understand—and so shall we recast the problem as one of random signal analysis. We shall state a problem in random signal analysis, and the reader can confirm for themself that it is equivalent to a two-channel Bell test as described in [1]. Then we shall derive a correlation coefficient, using probability theory rather than quantum mechanics. Because the result can be reached at all, of course it will be the same as that of quantum mechanics. (Otherwise mathematics would be inconsistent.)

II. The Problem

There is a transmitter that sends a *signal* randomly from the set

$$S = \{\curvearrowleft,\ \curvearrowright\} \tag{1}$$

The transmission goes into both of two channels. Each channel attaches a *tag* to the signal, according to an algorithm to be specified below, and re-transmits the tagged signal. The tag comes from the set

$$T = \{\oplus,\ \ominus\} \tag{2}$$

The tagging algorithm works as follows. The channel is "tuned" by an angular setting $\zeta \in [0, 2\pi]$. Let r represent a number chosen uniformly from $[0, 1]$. Now suppose the signal σ is $\curvearrowleft$. In that case, if $r < \cos^2 \zeta$ then re-transmit $(\oplus,\ \curvearrowleft)$. Otherwise re-transmit $(\ominus,\ \curvearrowleft)$. On the other hand, suppose the signal is σ is $\curvearrowright$. Then, if $r < \sin^2 \zeta$ re-transmit $(\oplus,\ \curvearrowright)$, else re-transmit $(\ominus,\ \curvearrowright)$.

At the end of both channels is a receiver-recorder, which makes a record of received pairs of tagged signals, for some pair of "tunings" (ζ_1, ζ_2) for the two channels. For example, one hundred thousand or one million pairs of tagged signals might be recorded.

Now suppose we map tags to numbers, $T \to T' = \{-1, +1\}$, thus:

$$\oplus \mapsto +1 \tag{3}$$
$$\ominus \mapsto -1 \tag{4}$$

The problem is to use these numbers to calculate a correlation coefficient ρ, characterizing the correlation between tags in the received signal pairs.

III. Solution

We will use subscripts to refer to channel numbers. Thus, for example, ζ_2 may refer to a ζ parameter for channel 2, τ_1 to a tag value for channel 1, and so on. An unsubscripted letter may refer to either channel. Thus, for instance, τ may stand in for either τ_1 or τ_2. And so on like that.

We will use more or less conventional probability notation, though also always adding the letter "λ" as a condition, meaning something such as "any relevant information we may so far have neglected". It pays to be cautious.

Let $\phi_{01}, \phi_{02} \in [0, 2\pi]$ be the *landmarks* (or *origins*) for settings of the respective ζ parameters, and let ϕ_0 represent either or both of them. Let ϕ_1 and ϕ_2 be respective settings, and ϕ represent either or both of them. Also introduce the notation

$$\Delta\phi_1 = \phi_1 - \phi_{01} \tag{5}$$
$$\Delta\phi_2 = \phi_2 - \phi_{02} \tag{6}$$
$$\Delta\phi = \phi - \phi_0 \tag{7}$$

By the problem definition, one immediately gets

$$P(\sigma = \curvearrowleft \mid \lambda) = 1/2 \tag{8}$$
$$P(\sigma = \curvearrowright \mid \lambda) = 1/2 \tag{9}$$

and

$$P(\tau = \oplus \,|\, \sigma = \curvearrowleft,\ \zeta = \Delta\phi,\ \phi_0 \in [0, 2\pi],\ \lambda) = \cos^2 \Delta\phi \tag{10}$$
$$P(\tau = \ominus \,|\, \sigma = \curvearrowleft,\ \zeta = \Delta\phi,\ \phi_0 \in [0, 2\pi],\ \lambda) = \sin^2 \Delta\phi \tag{11}$$
$$P(\tau = \oplus \,|\, \sigma = \curvearrowright,\ \zeta = \Delta\phi,\ \phi_0 \in [0, 2\pi],\ \lambda) = \sin^2 \Delta\phi \tag{12}$$
$$P(\tau = \ominus \,|\, \sigma = \curvearrowright,\ \zeta = \Delta\phi,\ \phi_0 \in [0, 2\pi],\ \lambda) = \cos^2 \Delta\phi \tag{13}$$

Here we are taking a liberty: actually, for both ϕ_{01} and ϕ_{02} there should be a probability density function (pdf) specified in the conditions. However, we shall introduce each pdf only when it is about to be used. For now, consider them implicitly specified.

We want to construct a table of probabilities of tagged signal pairs received by the receiver-recorder, so let us start by finding an expression for the following.

$$P_1 = P(\sigma = \curvearrowleft,\ \tau_1 = \oplus,\ \tau_2 = \oplus \,| \\ \zeta_1 = \Delta\phi_1,\ \zeta_2 = \Delta\phi_2,\ \phi_0 \in [0, 2\pi],\ \lambda) \tag{14}$$

However, because that does not fit well into a line of text, let us first introduce a shorthand, by writing something like

$$P_1 = P(\curvearrowleft\ \oplus_1\ \oplus_2 \,|\, \phi_1,\ \phi_2,\ [0, 2\pi],\ \lambda) \tag{15}$$

to mean the same thing. Then, by probability theory, and taking into account that $\Delta\phi_1$ and $\Delta\phi_2$ respectively pertain exclusively to channel 1 or channel 2 (so conditionality on the opposite channel's $\Delta\phi$ may be dropped),

$$\begin{aligned} P_1 &= P(\curvearrowleft\ \oplus_1\ \oplus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) && (16)\\ &= P(\curvearrowleft\ \mid \lambda) P(\oplus_1\ \oplus_2\ \mid\ \curvearrowleft,\ \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) && (17)\\ &= P(\curvearrowleft\ \mid \lambda) P_{11} P_{12} = \frac{1}{2} P_{11} P_{12} && (18) \end{aligned}$$

where

$$\begin{aligned} P_{11} &= P(\oplus_1 \mid\ \curvearrowleft,\ \phi_1\,,\ [0, 2\pi]\,,\ \lambda) = \cos^2 \Delta\phi_1 && (19)\\ P_{12} &= P(\oplus_2 \mid\ \curvearrowleft,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) = \cos^2 \Delta\phi_2 && (20) \end{aligned}$$

By that and similar calculations the following table may be constructed.

$$P(\curvearrowleft\ \oplus_1\ \oplus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) = \frac{1}{2} \cos^2 \Delta\phi_1\ \cos^2 \Delta\phi_2 \tag{21}$$

$$P(\curvearrowleft\ \oplus_1\ \ominus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) = \frac{1}{2} \cos^2 \Delta\phi_1\ \sin^2 \Delta\phi_2 \tag{22}$$

$$P(\curvearrowleft\ \ominus_1\ \oplus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) = \frac{1}{2} \sin^2 \Delta\phi_1\ \cos^2 \Delta\phi_2 \tag{23}$$

$$P(\curvearrowleft\ \ominus_1\ \ominus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) = \frac{1}{2} \sin^2 \Delta\phi_1\ \sin^2 \Delta\phi_2 \tag{24}$$

$$P(\curvearrowright\ \oplus_1\ \oplus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) = \frac{1}{2} \sin^2 \Delta\phi_1\ \sin^2 \Delta\phi_2 \tag{25}$$

$$P(\curvearrowright\ \oplus_1\ \ominus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) = \frac{1}{2} \sin^2 \Delta\phi_1\ \cos^2 \Delta\phi_2 \tag{26}$$

$$P(\curvearrowright\ \ominus_1\ \oplus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) = \frac{1}{2} \cos^2 \Delta\phi_1\ \sin^2 \Delta\phi_2 \tag{27}$$

$$P(\curvearrowright\ \ominus_1\ \ominus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) = \frac{1}{2} \cos^2 \Delta\phi_1\ \cos^2 \Delta\phi_2 \tag{28}$$

Disregarding the signal values gives (by addition of probabilities)

$$\begin{aligned} &P(\oplus_1\ \oplus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda)\\ &\qquad = P(\ominus_1\ \ominus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda)\\ &\qquad = \frac{1}{2} \cos^2 \Delta\phi_1\ \cos^2 \Delta\phi_2\\ &\qquad\qquad + \frac{1}{2} \sin^2 \Delta\phi_1\ \sin^2 \Delta\phi_2 \end{aligned} \tag{29}$$

and

$$\begin{aligned} &P(\oplus_1\ \ominus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda)\\ &\qquad = P(\ominus_1\ \oplus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda)\\ &\qquad = \frac{1}{2} \cos^2 \Delta\phi_1\ \sin^2 \Delta\phi_2\\ &\qquad\qquad + \frac{1}{2} \sin^2 \Delta\phi_1\ \cos^2 \Delta\phi_2 \end{aligned} \tag{30}$$

We want to calculate the correlation coefficient

$$\rho = \frac{E(\tau'_1 \tau'_2)}{\sqrt{E({\tau'_1}^2)}\sqrt{E({\tau'_2}^2)}} \tag{31}$$

where $\tau'_1, \tau'_2 \in T'$ and the expectations E are calculated with respect to the conditional probabilities derived above. The numerator is the covariance and the denominator is the product of the standard deviations.

The choice of values for the elements of T' makes it so the standard deviations in (31) equal one, and thus the correlation coefficient simplifies to the covariance

$$\rho = E(\tau'_1 \tau'_2) \tag{32}$$

which we now must calculate. To do so, not only must we compute a sum weighted by the probabilities in (29) and (30), but we must also eliminate the angular landmarks

First let us calculate an average of $\tau'_1 \tau'_2$ weighted by the probabilities in (29) and (30), and call that sum ρ'.

$$\begin{aligned} \rho' = &(+1)(+1) P(\oplus_1\ \oplus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda)\\ &+ (+1)(-1) P(\oplus_1\ \ominus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda)\\ &+ (-1)(+1) P(\ominus_1\ \oplus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda)\\ &+ (-1)(-1) P(\ominus_1\ \ominus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) \end{aligned} \tag{33}$$

Substituting for the probabilities gives

$$\begin{aligned} \rho' = &\cos^2 \Delta\phi_1\ \cos^2 \Delta\phi_2 - \cos^2 \Delta\phi_1\ \sin^2 \Delta\phi_2\\ &- \sin^2 \Delta\phi_1\ \cos^2 \Delta\phi_2 + \sin^2 \Delta\phi_1\ \sin^2 \Delta\phi_2 \end{aligned} \tag{34}$$

and so

$$\begin{aligned} \rho' &= (\cos^2 \Delta\phi_2 - \sin^2 \Delta\phi_2)(\cos^2 \Delta\phi_1 - \sin^2 \Delta\phi_1) && (35)\\ &= \cos(2\Delta\phi_2)\ \cos(2\Delta\phi_1) && (36) \end{aligned}$$

with the last step by one of the double-angle identities.

To eliminate the landmark ϕ_{01}, we will compute the definite integral of ρ' with respect to $\Delta\phi_1$, weighted by a uniform pdf—which means just a simple integral, with no special weighting factor. Using a uniform pdf is the same as saying we do not not care where ϕ_1 and ϕ_{01} lie on the protractor. The result of this integration we will call ρ''. We run into this complication, however: we want to treat all landmarks (and thus angular coordinate systems) alike, yet the cosine sometimes is reversed in sign, leading to ρ'' being reversed in sign. Luckily, the sign of a correlation coefficient is arbitrary, so we can simply, as a convention, reverse the sign in some quadrants.

$$\rho'' = \int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \cos(2\Delta\phi_2)\cos(2\Delta\phi_1)\, d\Delta\phi_1 \qquad \Delta\phi_1 \in [-\frac{\pi}{4}, +\frac{\pi}{4}] \tag{37}$$

$$-\rho'' = \int_{+\frac{\pi}{4}}^{+\frac{3\pi}{4}} \cos(2\Delta\phi_2)\cos(2\Delta\phi_1)\, d\Delta\phi_1 \qquad \Delta\phi_1 \in [+\frac{\pi}{4}, +\frac{3\pi}{4}] \tag{38}$$

$$\rho'' = \int_{+\frac{3\pi}{4}}^{+\frac{5\pi}{4}} \cos(2\Delta\phi_2)\cos(2\Delta\phi_1)\, d\Delta\phi_1 \qquad \Delta\phi_1 \in [+\frac{3\pi}{4}, +\frac{5\pi}{4}] \tag{39}$$

$$-\rho'' = \int_{+\frac{5\pi}{4}}^{+\frac{7\pi}{4}} \cos(2\Delta\phi_2)\cos(2\Delta\phi_1)\, d\Delta\phi_1 \qquad \Delta\phi_1 \in [+\frac{5\pi}{4}, +\frac{7\pi}{4}] \tag{40}$$

In each case the result is the same, so

$$\rho'' = \cos(2\Delta\phi_2) = \cos(2(\phi_2 - \phi_{02})) \tag{41}$$

regardless of quadrant.

Now we must eliminate ϕ_{02}, again by (generalized) integration. This time, however, we *do* care where it should fall on the protractor. We are trying to calculate an in-unison rotation-invariant correlation coefficient, that depends only on the difference in settings of ζ_1 and ζ_2, and thus the landmark for ϕ_2 is ϕ_1 with probability one.

We will use a Dirac delta function as the pdf, and this time the result will be ρ itself.

$$\rho = \int_0^{2\pi} \cos(2(\phi_2 - \phi_{02}))\delta(\phi_{02} - \phi_1)d\phi_{02} \tag{42}$$

$$= \cos(2(\phi_2 - \phi_1)) \tag{43}$$

$$= \cos^2(\phi_2 - \phi_1) - \sin^2(\phi_2 - \phi_1) \tag{44}$$

This result accords with quantum mechanics.

IV. Discussion

I should not belabor this note with too many details of how Bell went wrong in his arguments. His writings simply are devoid of *correct* methods of logical inference. Instead we are subjected to *imitations* of logical inference.

For example, Bell famously introduces a mathematical contradiction with the goal of producing a physical absurdity [2]. In fact, this is the crux of his argument in its best known form. But this is not a sound method a logical inference. One cannot deduce *anything* from a mathematical contradiction, except that the mathematical assumptions behind it are not sound. In other words, all Bell proved is that he made a mathematical error! Indeed, he made what is probably the most common error made in probability theory, which is to factor a joint probability incorrectly. However, Bell was *imitating* methods of logical inference, not actually employing them. For his naïve audience, this was good enough.[1]

Perhaps, finally, we have reached the limit of what we should tolerate. Perhaps, now that millions and millions of dollars have been wasted, and graduate students' lives increasingly are being wasted, it is time to put our foot down and say it is enough. When it came to physics, Bell was a dunderhead, unable to reason like a scientist. And now an entire batch of fields related to his work is full of dunderheads and incompetents who are supposed to be our society's greatest "geniuses," yet who fail to visualize that a two-channel Bell test is a totally ordinary, causal, contact-action random process that is, mathematically, shaped like a pair of wheels, which may have imposed upon them any angular coordinate system one wishes.

Among the corollaries of that "shape" is that, if you rotate θ_1 and θ_2 in unison, the frequencies of tag values change (also in unison)[2] *but the correlation coefficient remains invariant*. Furthermore, it should be experimentally *impossible* to violate the so-called "CHSH inequality" [3], if the experiment be properly designed. This inequality applies a test that itself is not invariant under rotations of the apparatus. For Bell-test angles, in the case of photon experiments, a rotation by $\pi/8$ should see a drop in magnitude of the CHSH "quantum correlation" E from $\sqrt{1/2}$ to zero, so $|S|$ (in the ideal) adds up to $\sqrt{2}$ instead of $2\sqrt{2}$. The reason is that E, in the CHSH formulation, is *not* actually the expectation of experimental outcomes, as claimed. Or, to put it another way, it is an expectation conditional on a particular coordinate system—but this is not what quantum mechanics gives, and it is not what one wants. Physicists have left out the steps that took *our* calculation from ρ' to ρ proper, and which gave us the *correct*, *rotation-invariant* correlation coefficient. Yet experimenters report positive results. Therefore not only is the inequality itself a failure of scientific methods, so must be the experiments.

[1] Perhaps Bell confused mathematical "*reductio ad absurdum*"—more properly called proof by contradiction—with the more general method of argumentation called *reductio ad absurdum*. But one cannot use a mathematical contradiction to prove an absurdity, because a mathematical contradiction is vacuous.

[2] Which could be computer-animated in numerous interesting ways.

Furthermore, derivation of the CHSH inequality merely mimics logical reasoning, much as Bell does in deriving his inequality. For mathematics to be consistent, all methods must reach the same result. Thus, when the CHSH "quantum correlation" does not reach the same result as quantum mechanics, the conclusion should *not* be that quantum mechanics is "different" from classical physics. The *correct* conclusion is that there is an error in the CHSH calculations, and that the researcher must keep working at it. Quantum physicists, instead, publish a paper "confirming quantum mechanics," and then await the accolades. It is a cushy job, insofar as intellect is concerned.

At this point I shall state unequivocally that there is no such thing as "entanglement." I write not for the journals but directly to the reader and so may speak freely. It is obvious that a "superposition state" or any such stuff must always be simply different notation for some probability expression, wave coherence expression, or the like. The real reason anyone believes there is magic involved seems not to be that the magic exists, but that the Fathers of the early Quantum Church (although perhaps not the great Emissary himself [4]) forbade treatment of quantum mechanics in the classical fashion. Physicists have been true to that stricture since, and simply do not *permit* the heresy of a solution by classical methods.[3] One is permitted *only* to reach some solution different from that of quantum mechanics, which then is used to "prove" that quantum mechanics is "different." Otherwise physicists would have discovered long ago what we discovered above: that there actually *is no* "modern physics"—just "classical physics" that no one has been allowed to investigate.

Quantum physicists obviously do not even teach themselves and their students *how* to reach solutions of random process problems by classical methods. *I* knew how because *I* majored not in physics at all, but in the very closely related subject of *electrical engineering* (where mistakes are punishable by corporate bankruptcy, and even prison). Peer review thus serves only to reinforce the orthodoxy. Orthodox papers in quantum physics are reviewed by orthodox quantum physicists for publication in orthodox journals. Peer review is useless, in this instance, to serve the scientific mission. The papers never are subjected to analysis by experts in random process analysis, who would tear the arguments to shreds, but only by ignorant persons who delight in the conclusions. Shining careers are constructed as the peers see each others' papers published in the most elite of journals. Eventually, the Nobel Prize in Physics for 2022 is awarded to some of these quantum physicists, for their superlative achievements in the pursuit of this balderdash [6].

The author of *this* paper would tell a Nobel Committee to take their prize and dump it in *Östersjön*, and recommends the same for all Nobels in science and mathematics. Such prizes are unethical and destructive at best [7], [8], but now also shown to be awarded without distinction to both actual science and pseudoscientific claptrap. Is it worth giving up on the scientific enterprise, just to be famous and also insulted with a cash reward?

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[3] Indeed, some Bell tests are soluble *by electromagnetic wave theory* [5], which is as classical a physics as one can imagine, but this fact is dismissed one way or another by the orthodoxy.

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