Correlation Coefficient of a Two-Channel Bell Test by Probability Theory

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***Abstract*—It is widely believed that only quantum mechanics can characterize the correlations of a Bell test. In fact, quantum mechanics is superfluous to the problem, and a correlation coefficient may be derived by probability theory.**

I. Introduction

John S. Bell famously proposed a test **FILL IN REFERENCE HERE** supposed to prove that quantum mechanics was, in essence, magical. I put the matter this way because the proposed test is soluble by probability theory, and yet Bell seems to have claimed quantum mechanics alone could provide the One True Answer that surpassed all other solutions. At the time of this writing, Bell's test is considered passed and the matter resolved in favor of the One True Answer.

Of course, what actually happened is that Bell *did not know* the test could be analyzed by means other than quantum mechanics, and also surely *already entertained the notion* that quantum mechanics might indeed be magical. Quantum mechanics has widely been regarded as some kind of "magic" for a very long time. However, some of us do not—at least when supposedly engaged in the scientific endeavor—entertain the notion that magic—or, indeed, anything remotely resembling it—exists. Furthermore, the Bell test problem is not, in fact, one of quantum mechanics at all. It is more properly a problem in random process analysis. It merely happens that quantum mechanics "knows" how to solve such a problem, without the user having to understand the subject of random process analysis.

We, however, wish to understand—and so shall recast the problem as one of random signal analysis. We will state a problem in random signal analysis, and the reader can confirm for themself that it is equivalent to a Bell test. Then we shall derive a correlation coefficient.

II. The Problem

There is a transmitter that sends a *signal* randomly from the set

$$S = \{\curvearrowleft\,,\ \curvearrowright\} \tag{1}$$

The transmission goes into both of two channels. Each channel attaches a *tag* to the signal, according to an algorithm to be specified below, and re-transmits the tagged signal. The tag comes from the set

$$T = \{\oplus\,,\ \ominus\} \tag{2}$$

The tagging algorithm works as follows. The channel is "tuned" by an angular setting $\zeta \in [0, 2\pi]$. Let r represent a number chosen uniformly from $[0, 1]$. Now suppose the signal σ is $\curvearrowleft$. In that case, if $r < \cos^2\zeta$ then re-transmit $(\oplus\,,\ \curvearrowleft)$. Otherwise re-transmit $(\ominus\,,\ \curvearrowleft)$. On the other hand, suppose the signal is σ is $\curvearrowright$. Then, if $r < \sin^2\zeta$ re-transmit $(\oplus\,,\ \curvearrowright)$, else re-transmit $(\ominus\,,\ \curvearrowright)$.

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A PDF copy of this paper may be found at https://www.crudfactory.com/eprb_signal_correlations.pdf
The LaTeX source for this paper, and simulations based on its contents, reside in a Git repository at https://github.com/chemoelectric/eprb_signal_correlations

At the end of both channels is a receiver-recorder, which makes a record of received pairs of tagged signals, for some pair of "tunings" (ζ_1, ζ_2) for the two channels. For example, one hundred thousand or one million pairs of tagged signals might be recorded.

Now suppose we map tags to numbers, $T \to T' = \{-1, +1\}$, thus:

$$\oplus \mapsto +1 \tag{3}$$

$$\ominus \mapsto -1 \tag{4}$$

The problem is to use these numbers to calculate a correlation coefficient ρ, characterizing the correlation between tags in the received signal pairs.

III. Solution

We will use subscripts to refer to channel numbers. Thus, for example, ζ_2 may refer to a ζ parameter for channel 2, τ_1 to a tag value for channel 1, and so on. An unsubscripted letter may refer to either channel. Thus, for instance, τ may stand in for either τ_1 or τ_2. And so on like that.

We will use more or less conventional probability notation, though also always adding the letter "λ" as a condition, meaning something such as "any relevant information we may so far have neglected". It pays to be cautious.

Note also that all angles will be in some relation relative to each other, and positioned as a whole with respect to some "landmark." We will call the landmark ϕ_1 and associate it, purely for convenience, with the "calibration" of channel 1. The landmark may be any angle, and thus "$\phi_1 \in [0, 2\pi]$" will often appear as a condition.

By the problem definition, one immediately gets

$$P(\sigma = \curvearrowleft\ \mid \lambda) = \frac{1}{2} \tag{5}$$

$$P(\sigma = \curvearrowright\ \mid \lambda) = \frac{1}{2} \tag{6}$$

and

$$P(\tau =\ \oplus \mid \sigma = \curvearrowleft,\ \zeta = \phi,\ \phi_1 \in [0, 2\pi],\ \lambda) = \cos^2\phi \tag{7}$$

$$P(\tau =\ \ominus \mid \sigma = \curvearrowleft,\ \zeta = \phi,\ \phi_1 \in [0, 2\pi],\ \lambda) = \sin^2\phi \tag{8}$$

$$P(\tau =\ \oplus \mid \sigma = \curvearrowright,\ \zeta = \phi,\ \phi_1 \in [0, 2\pi],\ \lambda) = \sin^2\phi \tag{9}$$

$$P(\tau =\ \ominus \mid \sigma = \curvearrowright,\ \zeta = \phi,\ \phi_1 \in [0, 2\pi],\ \lambda) = \cos^2\phi \tag{10}$$

We want to construct a table of probabilities of tagged signal pairs received by the receiver-recorder, so let us start by finding an expression for the following.

$$\begin{aligned} P_1 = P(\sigma = \curvearrowleft,\ \tau_1 =\ \oplus,\ \tau_2 =\ \oplus \mid \zeta_1 = \phi_1\,,\ \zeta_2 = \phi_2 \\ \phi_1 \in [0, 2\pi]\,,\ \lambda) \end{aligned} \tag{11}$$

However, because that does not fit well into a line of text, let us first introduce a shorthand, by writing something like

$$P_1 = P(\curvearrowleft\ \oplus_1\ \oplus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) \tag{12}$$

to mean the same thing. Then, by probability theory, and taking into account that ϕ_1 and ϕ_2 respectively pertain exclusively to channel 1 or channel 2 (so conditionality on the opposite channel's ϕ may be dropped),

$$\begin{aligned} P_1 &= P(\curvearrowleft\ \oplus_1\ \oplus_2 \mid \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) && (13) \\ &= P(\curvearrowleft\ \mid \lambda) P(\oplus_1\ \oplus_2 \mid\ \curvearrowleft,\ \phi_1\,,\ \phi_2\,,\ [0, 2\pi]\,,\ \lambda) && (14) \\ &= P(\curvearrowleft\ \mid \lambda) P_{11} P_{12} = \frac{1}{2} P_{11} P_{12} && (15) \end{aligned}$$

where

$$P_{11} = P(\oplus_1 \mid \curvearrowleft, \ \phi_1 \ , \ [0, 2\pi] \ , \ \lambda) = \cos^2 \phi_1 \qquad (16)$$

$$P_{12} = P(\oplus_2 \mid \curvearrowleft, \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) = \cos^2 \phi_2 \qquad (17)$$

By that and similar calculations the following table may be constructed.

$$P(\curvearrowleft \ \oplus_1 \ \oplus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) = \frac{1}{2} \cos^2 \phi_1 \ \cos^2 \phi_2 \qquad (18)$$

$$P(\curvearrowleft \ \oplus_1 \ \ominus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) = \frac{1}{2} \cos^2 \phi_1 \ \sin^2 \phi_2 \qquad (19)$$

$$P(\curvearrowleft \ \ominus_1 \ \oplus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) = \frac{1}{2} \sin^2 \phi_1 \ \cos^2 \phi_2 \qquad (20)$$

$$P(\curvearrowleft \ \ominus_1 \ \ominus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) = \frac{1}{2} \sin^2 \phi_1 \ \sin^2 \phi_2 \qquad (21)$$

$$P(\curvearrowright \ \oplus_1 \ \oplus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) = \frac{1}{2} \sin^2 \phi_1 \ \sin^2 \phi_2 \qquad (22)$$

$$P(\curvearrowright \ \oplus_1 \ \ominus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) = \frac{1}{2} \sin^2 \phi_1 \ \cos^2 \phi_2 \qquad (23)$$

$$P(\curvearrowright \ \ominus_1 \ \oplus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) = \frac{1}{2} \cos^2 \phi_1 \ \sin^2 \phi_2 \qquad (24)$$

$$P(\curvearrowright \ \ominus_1 \ \ominus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) = \frac{1}{2} \cos^2 \phi_1 \ \cos^2 \phi_2 \qquad (25)$$

Disregarding the signal values gives (by addition of probabilities)

$$\begin{aligned} &P(\oplus_1 \ \oplus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) \\ &\qquad = P(\ominus_1 \ \ominus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) \\ &\qquad = \frac{1}{2} \cos^2 \phi_1 \ \cos^2 \phi_2 + \frac{1}{2} \sin^2 \phi_1 \ \sin^2 \phi_2 \qquad (26) \end{aligned}$$

and

$$\begin{aligned} &P(\oplus_1 \ \ominus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) \\ &\qquad = P(\ominus_1 \ \oplus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) \\ &\qquad = \frac{1}{2} \cos^2 \phi_1 \ \sin^2 \phi_2 + \frac{1}{2} \sin^2 \phi_1 \ \cos^2 \phi_2 \qquad (27) \end{aligned}$$

We want to calculate the correlation coefficient

$$\rho = \frac{E(\tau_1' \tau_2')}{\sqrt{E({\tau_1'}^2)}\sqrt{E({\tau_2'}^2)}} \qquad (28)$$

where $\tau_1', \tau_2' \in T'$ and the expectations E are calculated with respect to the conditional probabilities derived above. The numerator is the covariance and the denominator is the product of the standard deviations.

The choice of values for the elements of T' makes it so the standard deviations in (28) equal one, and thus the correlation coefficient simplifies to the covariance

$$\rho = E(\tau_1' \tau_2') \qquad (29)$$

which we now must calculate. To do so, not only must we compute a sum weighted by the probabilities in (26) and (27), but we must also eliminate the $\phi_1 \in [0, 2\pi]$ condition by integrating with respect to ϕ_1. We have no information ahead of time on what this angular landmark ϕ_1 will be, and so the probability distribution is uniform, which simplifies our integration. We will encounter a complication, but it will be minor: we will have to break the integration into pieces and solve the problem separately for each.

First let us calculate the sum for (26) and (27) (call the sum ρ'), and then we will integrate the result of that sum.

$$\begin{aligned} \rho' = &(+1)(+1)P(\oplus_1 \ \oplus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) \\ &+ (+1)(-1)P(\oplus_1 \ \ominus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) \\ &+ (-1)(+1)P(\ominus_1 \ \oplus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) \\ &+ (-1)(-1)P(\ominus_1 \ \ominus_2 \mid \phi_1 \ , \ \phi_2 \ , \ [0, 2\pi] \ , \ \lambda) \qquad (30) \end{aligned}$$

Substituting for the probabilities gives

$$\begin{aligned} \rho' = &\cos^2 \phi_1 \ \cos^2 \phi_2 - \cos^2 \phi_1 \ \sin^2 \phi_2 \\ &- \sin^2 \phi_1 \ \cos^2 \phi_2 + \sin^2 \phi_1 \ \sin^2 \phi_2 \qquad (31) \end{aligned}$$

and so

$$\rho' = (\cos^2 \phi_2 - \sin^2 \phi_2)(\cos^2 \phi_1 - \sin^2 \phi_1) \qquad (32)$$

$$= \cos(2\phi_2) \ \cos(2\phi_1) \qquad (33)$$

with the last step by one of the double-angle identities.

This expression must now be integrated with respect to ϕ_1 over all of $[0, 2\pi]$, using a uniform probability distribution function (pdf) as the weight. We run into this complication: we have to treat all landmark angles alike (a coordinate system does not care what quadrant its landmark is in), yet the cosine sometimes is reversed in sign, leading to ρ being reversed in sign. However, the sign of a correlation coefficient is arbitrary. We can deal with the problem by breaking the circle into pieces reversing the sign of the integral in those places, either by negating the integral explicitly or by reversing the direction of integration. Here we will integrate in the reverse direction.

$$\rho = \cos(2\phi_2) \int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \cos(2\phi_1) \, d\phi_1 \qquad \phi_1 \in [-\frac{\pi}{4}, +\frac{\pi}{4}] \qquad (34)$$

$$\rho = \cos(2\phi_2) \int_{+\frac{3\pi}{4}}^{+\frac{\pi}{4}} \cos(2\phi_1) \, d\phi_1 \qquad \phi_1 \in [+\frac{\pi}{4}, +\frac{3\pi}{4}] \qquad (35)$$

$$\rho = \cos(2\phi_2) \int_{+\frac{3\pi}{4}}^{+\frac{5\pi}{4}} \cos(2\phi_1) \, d\phi_1 \qquad \phi_1 \in [+\frac{3\pi}{4}, +\frac{5\pi}{4}] \qquad (36)$$

$$\rho = \cos(2\phi_2) \int_{+\frac{7\pi}{4}}^{+\frac{5\pi}{4}} \cos(2\phi_1) \, d\phi_1 \qquad \phi_1 \in [+\frac{5\pi}{4}, +\frac{7\pi}{4}] \qquad (37)$$

In each case the integral equals one, and so the expression is the same, regardless of quadrant. The correlation coefficient is

$$\rho = \cos(2\phi_2) = \cos^2 \phi_2 - \sin^2 \phi_2 \qquad (38)$$

Now suppose we have finished calibrating our transmission channels and wish to mark the dials. Where the landmark is on channel 1 we mark a zero. Where ϕ_2 is on channel 2 we mark the value of $\phi_2 - \phi_1$. Then we complete the two circles using a protractor or similar device. We label the dials "θ".

The general formula for the correlation coefficient of the calibrated channels is

$$\rho = \cos(2(\theta_2 - \theta_1)) = \cos^2(\theta_2 - \theta_1) - \sin^2(\theta_2 - \theta_1) \qquad (39)$$

because one could always redo the calculations from scratch, this time using $\phi_1 = \theta_1$ and $\phi_2 = \theta_2 - \theta_1$.

This result accords with quantum mechanics **CITATION**.

IV. Discussion

I should not belabor this note with too many details of how Bell went wrong in his arguments. His writings simply are devoid of *correct* methods of logical inference. Instead we are subjected to *imitations* of logical inference.

For example, Bell famously introduces a mathematical contradiction with the goal of producing a physical absurdity **CITATION**. In fact, this is the crux of his argument in its best known form. But this is not a sound method a logical inference. One cannot deduce *anything* from a mathematical contradiction, except that the mathematical assumptions behind it are not sound. In other words, all Bell proved is that he made a mathematical error! Indeed, he made what is probably the most common error made in probability theory, which is to factor a joint probability incorrectly. However, Bell was *imitating* methods of logical inference, not actually employing them. For his naïve audience, this was good enough.

Perhaps, finally, we have reached the limit of what we should tolerate. Perhaps, now that millions and millions of dollars have been wasted, and graduate students' lives increasingly are being wasted, it is time to put our foot down and say it is enough. When it came to physics, Bell was a dunderhead, unable to reason like a scientist. And now an entire batch of fields related to his work is full of dunderheads and incompetents who are supposed to be our society's greatest "geniuses," yet who fail to visualize that a two-channel Bell test is a totally ordinary, causal, contact-action random process that is, mathematically, shaped like a pair of wheels, which may have imposed upon them any angular coordinate system one wishes.

Among the corollaries of that "shape" is that, if you rotate θ_1 and θ_2 in unison, the frequencies of tag values change *but the correlations remains invariant*. A consequence of this is that it should actually be experimentally *impossible* to violate the so-called "CHSH inequality" **CITATION**, if the experiment be properly designed. This inequality specifically tests for non-invariance of correlations under in-unison rotation. Because there is no such non-invariance, there should be no positive results. Yet experimenters report positive results. Therefore not only is the inequality itself a failure of scientific methods, so must be the experiments.