Stern-Gerlach Shutters

— The Problem —

1. The Stern-Gerlach Scientific Contraption Company makes a device called a Stern-Gerlach Shutter. One of these is depicted in Figure 1.

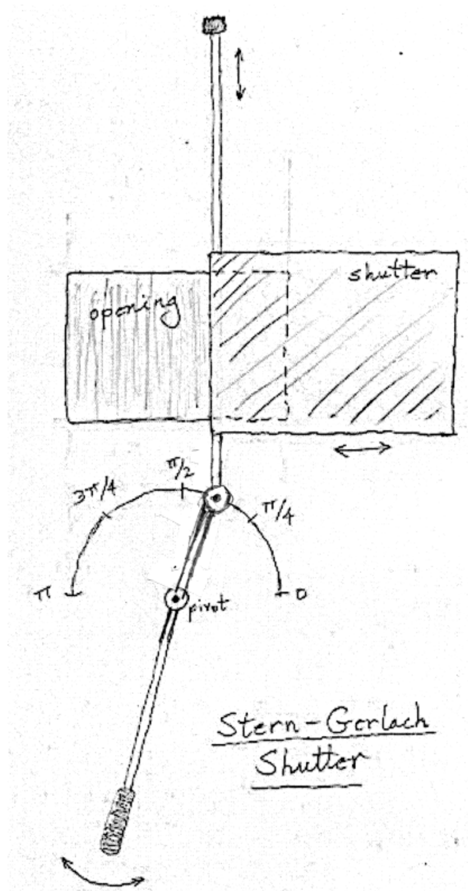

Figure 1 Stern-Gerlach Shutter

2. The settings gauge at the bottom of the device is set at a desired angle between zero and π. The shutter sheet slides horizontally, guided by a long bar that is connected to the rotating gauge, and which can slide vertically through a tube at the left edge of the shutter sheet.

3. The shutter sheet is used to cover a rectangular opening that is centered over the settings gauge's pivot. The width of the opening is equal to the diameter of the semicircle swept out by the settings gauge.

4. We have two of these Stern-Gerlach Shutters, and for each one a corresponding Stern-Gerlach Shutter Shooter. A Stern-Gerlach Shutter Shooter fires a pellet at a random point inside the boundaries of the opening of a Stern-Gerlach Shutter. The pellet may either pass through the opening or bounce off the shutter sheet. The Shutter Shooter may fire either a black pellet or a white pellet, at the discretion of the experimenter.

5. Shutter 1 is set to ϕ_1 and Shutter 2 is set to ϕ_2.

6. According to the flip of a fair coin, we shoot either a black pellet at Shutter 1 and a white pellet at Shutter 2, or the other way around.

7. Let $+1$ represent either a black pellet passing through an opening or a white pellet bouncing off a shutter sheet. Let -1 represent pellets doing it the other way around.

8. Compute, for the two Stern-Gerlach shutters, the correlation coefficient ρ as a function of $\phi_1 - \phi_2$.

* * *

— The Solution —

1. Figure 2 depicts important dimensions of a Stern-Gerlach Shutter.

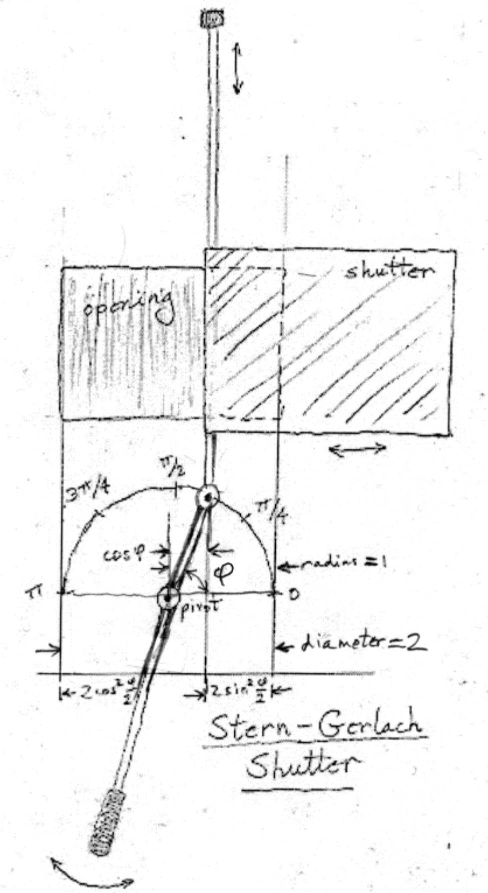

Figure 2 Dimensions of a Stern-Gerlach Shutter

2. For convenience, let the radius of the settings gauge equal one. Therefore the diameter equals two and the width of the opening is also two.

3. Refer to the settings gauge's setting as ϕ. The projection of the gauge onto horizontal is $\cos\phi$.

4. By double-angle identities in the *CRC Handbook of Mathematical Sciences*,

$$\cos\phi = \cos^2(\phi/2) - \sin^2(\phi/2) = 2\cos^2(\phi/2) - 1 = 1 - 2\sin^2(\phi/2)$$

5. As shown in Figure 2, these identities imply the shutter sheet divides the width of the opening into $2\cos^2(\phi/2)$ and $2\sin^2(\phi/2)$.

6. Thus the shutter sheet divides the area of the opening into proportions $\cos^2(\phi/2)$ and $\sin^2(\phi/2)$.

7. The probability of a pellet passing through the opening is therefore $\cos^2(\phi/2)$, and the probability of it bouncing off the shutter sheet is $\sin^2(\phi/2)$.

8. Let us solve for the case $\phi_1 = \phi_{01} \in [0, \pi]$, $\phi_2 = \phi_{02} = 0$. Shutter 2 is fully open, so one merely counts black pellets as $+1$ and white pellets as -1. It is as if there were no shutter sheet at all, and in fact one need not bother shooting at Shutter 2.

9. For this special case,

$$\rho = \rho^{++} + \rho^{+-} + \rho^{-+} + \rho^{--}$$

   where

$$\begin{aligned}
\rho^{++} &= (+1)\ (+1)\sin^2(\phi_{01}/2)/2 \\
\rho^{+-} &= (+1)\ (-1)\cos^2(\phi_{01}/2)/2 \\
\rho^{-+} &= (-1)\ (+1)\cos^2(\phi_{01}/2)/2 \\
\rho^{--} &= (-1)\ (-1)\sin^2(\phi_{01}/2)/2
\end{aligned}$$

   The calculation above is for the covariance. The correlation coefficient is covariance with its range normalized to $[-1, +1]$, but this covariance already has that.

10. Using the double-angle identities again, ρ simplifies to

$$\rho = -(\cos^2\frac{\phi_{01}}{2} - \sin^2\frac{\phi_{01}}{2}) = -\cos\phi_{01} = -\cos(\phi_{01} - \phi_{02})$$

11. Suppose one is given ϕ_1 and ϕ_2 as the respective settings. Assume $\phi_1 \geq \phi_2$. (Otherwise you can simply reverse their roles, because cosine is symmetric around the origin.) Then let $\phi_{01} = \phi_1 - \phi_2$ and $\phi_{02} = 0 = \phi_2 - \phi_2$, and add ϕ_2 to both sides of the minus sign in $\phi_{01} - \phi_{02}$.

$$\begin{aligned}
\rho &= -\cos(\phi_{01} - \phi_{02}) \\
&= -\cos((\phi_1 - \phi_2) - (\phi_2 - \phi_2)) \\
&= -\cos(\phi_1 - \phi_2)
\end{aligned}$$

12. And there we have it, the general solution: $\rho = -\cos(\phi_1 - \phi_2)$. It is invariant under in-unison rotations of the settings.

— Alternative Solution by Quantum Mechanics —

1. A solution by quantum mechanics is much more complicated, so no wonder quantum mechanics is considered arcane and difficult. Because all mathematical methods[1] must reach the same result, quantum mechanics is never necessary to solve a problem. Thus one might consider not using it.

2. Nevertheless, let us use it here. I have not had coursework in quantum mechanics and so will try to figure this out for myself. Otherwise, one may look up an analysis of the EPR-Bohm problem for Stern-Gerlach magnets. A similar analysis ought to apply to Stern-Gerlach Shutters.

3. Possible states of our calculation will form a linear space of ordered pairs of vectors. We attribute no physical meaning to this space. The ordered pairs in the space represent states in a symbolic calculation of the correlation coefficient.

4. Either of the two pellets has its own linear space that is more primitive than the 'full' linear space. Let e^{b} represent the basis vector for a black pellet and e^{w} the basis vector for a white pellet. These vectors induce a linear space. So-called 'superpositions' can be formed from linear combinations of the two basis vectors.

5. However, for a linear combination to be a valid 'superposition', there is a constraint on it. The scalar multipliers of the components may be real or complex numbers, but the sums of the squares of the absolute values of those multipliers must equal one.

6. Now, let us clarify that the state space represents *the state of a calculation*, and *not* the physical state of the experiment. There is no constraint on the shooting of Pellet 1 and Pellet 2 being done simultaneously. If a settings gauge be set at zero, no shooting is necessary at all.[2] Nevertheless, the result of the flip of the fair coin will be the superposition $e^{i\pi/4}(\mathrm{e}^{\mathrm{b}}, \mathrm{e}^{\mathrm{w}}) + e^{i\pi/4}(\mathrm{e}^{\mathrm{w}}, \mathrm{e}^{\mathrm{b}})$. We

7. NOPE. I DON'T KNOW HOW TO DO THIS, AND IT IS IMMENSELY COMPLICATED. I GIVE UP. □

[1] *When employed correctly*!

[2] Similar observations are true of experiments with photons, electrons, etc., although physicists have not realized this. Conservation laws, not 'superposition', account for particles occurring in pairs.