Stern-Gerlach Shutters

— The Problem —

1. The Stern-Gerlach Scientific Contraption Company makes a device called a Stern-Gerlach Shutter. One of these is depicted in Figure 1.

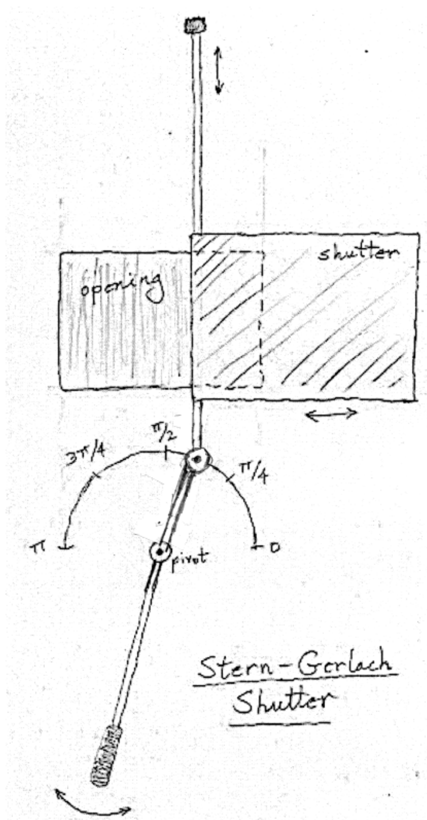

Figure 1
Stern-Gerlach Shutter

2. The settings gauge at the bottom of the device is set at a desired angle between zero and π. The shutter sheet slides horizontally, guided by a long bar that is connected to the rotating handle, and which can slide vertically through a tube at the left edge of the shutter sheet.

3. The shutter sheet is used to cover a rectangular opening that is centered over the settings gauge's pivot. The width of the opening is equal to the diameter of the semicircle swept out by the settings gauge.

4. We have two of these Stern-Gerlach Shutters, and for each one a corresponding Stern-Gerlach Shutter Shooter. A Stern-Gerlach Shutter Shooter fires a pellet at a random point inside the boundaries of the opening of a Stern-Gerlach Shutter. The pellet may either pass through the opening or bounce off the shutter sheet. The Shutter Shooter may fire either a black pellet or a white pellet, at the discretion of the experimenter.

5. Shutter 1 is set to ϕ_1 and Shutter 2 is set to ϕ_2.

6. According to the flip of a fair coin, we shoot either a black pellet at Shutter 1 and a white pellet at Shutter 2, or the other way around.

7. Let $+1$ represent either a black pellet passing through an opening or a white pellet bouncing off a shutter sheet. Let -1 represent pellets doing it the other way around.

8. Compute, for the two Stern-Gerlach shutters, the correlation coefficient ρ as a function of $\phi_1 - \phi_2$.

— The Solution —

1. Figure 2 depicts important dimensions of a Stern-Gerlach Shutter.

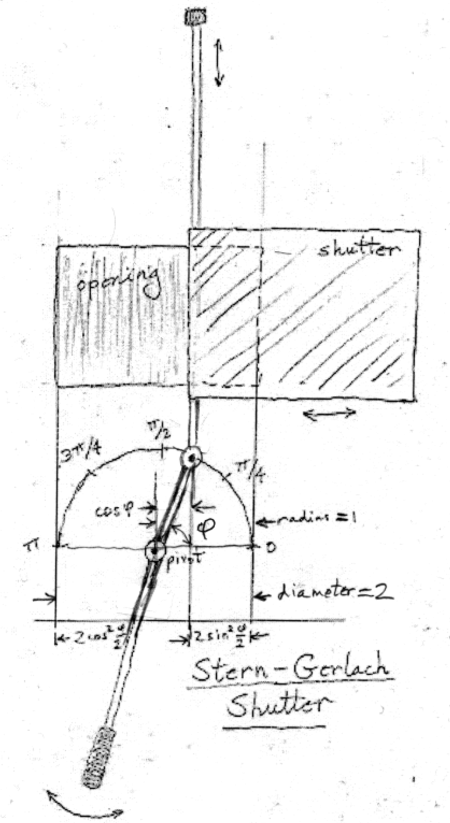

Figure 2 Dimensions of a Stern-Gerlach Shutter

2. For convenience, let the radius of the settings gauge equal one. Therefore the diameter equals two and the width of the opening is also two.

3. Refer to the setting gauge's setting as ϕ. The projection of the gauge onto horizontal is $\cos\phi$.

4. By double-angle identities in the *CRC Handbook of Mathematical Sciences*,

$$\cos\phi = \cos^2(\phi/2) - \sin^2(\phi/2) = 2\cos^2(\phi/2) - 1 = 1 - 2\sin^2(\phi/2)$$

5. As shown in Figure 2, these identities imply the shutter sheet divides the width of the opening into $2\cos^2(\phi/2)$ and $2\sin^2(\phi/2)$.

6. Thus the shutter sheet divides the area of the opening into proportions $\cos^2(\phi/2)$ and $\sin^2(\phi/2)$.

7. The probability of a pellet passing through the opening is therefore $\cos^2(\phi/2)$, and the probability of it bouncing off the shutter sheet is $\sin^2(\phi/2)$.

8. Let us solve for the case $\phi_1 = \phi_{01} \in [0, \pi]$, $\phi_2 = \phi_{02} = 0$. Shutter 2 is fully open, so one merely counts black pellets as $+1$ and white pellets as -1. It is as if there were no shutter at all, and in fact one need not bother shooting at Shutter 2.

9. For this special case,

$$\rho = \rho^{++} + \rho^{+-} + \rho^{-+} + \rho^{--}$$

where

$$\begin{aligned}
\rho^{++} &= \frac{1}{2}\,(+1)\,(+1)\sin^2(\phi_{01}/2)\\
\rho^{+-} &= \frac{1}{2}\,(+1)\,(-1)\cos^2(\phi_{01}/2)\\
\rho^{-+} &= \frac{1}{2}\,(-1)\,(+1)\cos^2(\phi_{01}/2)\\
\rho^{--} &= \frac{1}{2}\,(-1)\,(-1)\sin^2(\phi_{01}/2)
\end{aligned}$$

10. Using the double-angle identities again, this simplifies to

$$\rho = -(\cos^2\frac{\phi_{01}}{2} - \sin^2\frac{\phi_{01}}{2}) = -\cos\phi_{01} = -\cos(\phi_{01} - \phi_{02})$$

11. Suppose one is given ϕ_1 and ϕ_2 as the respective settings. Assume $\phi_1 \geq \phi_2$. (Otherwise you can simply reverse their roles, because the cosine is symmetric around the origin.) Then let $\phi_{01} = \phi_1 - \phi_2$ and $\phi_{02} = \phi_2 - \phi_2 = 0$, and add ϕ_2 to both sides of the minus sign in $\phi_{01} - \phi_{02}$.

$$\begin{aligned}
\rho &= -\cos(\phi_{01} - \phi_{02})\\
&= -\cos\{(\phi_1 - \phi_2) - (\phi_2 - \phi_2)\}\\
&= -\cos(\phi_1 - \phi_2)
\end{aligned}$$

12. And there we have it, the general solution: $\rho = -\cos(\phi_1 - \phi_2)$. It is invariant under in-unison rotations of the settings. □