Cookies for Alice and Bob

— The Problem —

1. You have one box of Chocolate Cookies and one box of Vanilla Cookies.

2. You will deliver one of the boxes to Alice and one of the boxes to Bob. Neither has a preference of flavor.

3. You flip a coin to decide who gets which box.[1]

4. Alice has a favorite angle, α. Bob also has a favorite angle, β. You write down the numbers $\cos^2\alpha$, $\sin^2\alpha$, $\cos^2\beta$, and $\sin^2\beta$. Also, you have a pair of ten-sided dice. You can roll these dice to get a number between 0.00 and 0.99, inclusive.

5. Suppose Alice is getting the box of Chocolate Cookies. You roll the dice. If the number you roll is less than $\cos^2\alpha$, then you write $+1$ on the box. Otherwise you write -1.[2]

6. Suppose, on the other hand, that Alice is getting the box of Vanilla Cookies. You roll the dice. If the number you roll is less than $\sin^2\alpha$, then you write $+1$ on the box. Otherwise you write -1.

7. Suppose Bob is getting the box of Chocolate Cookies. You roll the dice. If the number you roll is less than $\cos^2\beta$, then you write $+1$ on the box. Otherwise you write -1.

8. Suppose, on the other hand, that Bob is getting the box of Vanilla Cookies. You roll the dice. If the number you roll is less than $\sin^2\beta$, then you write $+1$ on the box. Otherwise you write -1.

9. Calculate the correlation coefficient of the numbers written on the cookie boxes, as a function of $\alpha - \beta$.[3]

— The Solution —

[1] This step is equivalent to a light source generating a pair of perpendicularly polarized photons.
[2] This is equivalent to a photon being detected in one of the channels of a polarizing beam splitter.
[3] The answer from quantum mechanics is $-\cos 2(\alpha - \beta)$, and physicists have assured us that only by traversal of states in a Hilbert space can one derive this answer. Of course, this claim defies the principles of mathematics and is preposterous.

1. First let us solve the problem for $\alpha_0 \in (-\infty, +\infty)$ and $\beta_0 = 0$. Then

$$\cos^2 \beta_0 = 1$$
$$\sin^2 \beta_0 = 0$$

and so the $\beta_0 = 0$ setting does nothing except tell us to mark Chocolate Cookies with a $+1$ and Vanilla Cookies with a -1.

2. Thanks to our choice of markings, the correlation coefficient equals the covariance (that is, there is no need to normalize), and is

$$\rho = \rho^{++} + \rho^{+-} + \rho^{-+} + \rho^{--}$$

where

$$\rho^{++} = \frac{1}{2}\,(+1)\,(+1)\sin^2 \alpha_0$$
$$\rho^{+-} = \frac{1}{2}\,(+1)\,(-1)\cos^2 \alpha_0$$
$$\rho^{-+} = \frac{1}{2}\,(-1)\,(+1)\cos^2 \alpha_0$$
$$\rho^{--} = \frac{1}{2}\,(-1)\,(-1)\sin^2 \alpha_0$$

3. Thus (by a double-angle identity found in reference books)

$$\rho = -(\cos^2 \alpha_0 - \sin^2 \alpha_0) = -\cos 2\alpha_0 = -\cos 2(\alpha_0 - \beta_0)$$

and ρ is the solution for the special case α_0 and β_0.

4. Now set $\alpha_0 = \alpha - \beta$. That makes $\alpha = \alpha_0 + \beta$ and $\beta = \beta_0 + \beta$. Add $\beta - \beta = 0$ to $\alpha_0 - \beta_0$ in the last expression for ρ above. This gives

$$\rho = -\cos 2(\alpha - \beta)$$

as the general solution.

5. Interestingly, although the particular 'scattering' of $+1$ and -1 marks (were a person to do the operation many times) depends on the specific values of α and β, the correlation coefficient depends only on the difference $\alpha - \beta$. That is, there exists a relationship between $+1$ and -1 marks that is invariant under in-unison rotation of the angles.[4] □

[4] Of course, the same rotational invariance was always true of Bell test experiments, but must not have been widely noticed, for otherwise Clauser would not have received a Nobel Prize.