Two Oscillators

— The Problem —

1. You have two oscillators, $a(t) = \sin(\omega t + \phi_a)$ and $b(t) = \sin(\omega t + \phi_b)$, where

$$(\phi_a, \phi_b) = \begin{cases} (0, \frac{\pi}{2}) & \text{with probability } \frac{1}{2} \\ (\frac{\pi}{2}, 0) & \text{with probability } \frac{1}{2} \end{cases}$$

2. The oscillator outputs are subjected to phase shifts, giving $a'(t) = \sin(\omega t + \phi_a + \theta_a)$ and $b'(t) = \sin(\omega t + \phi_b + \theta_b)$.

3. Define the function

$$g(x) = \begin{cases} +1 & \text{with probability } x^2 \\ -1 & \text{with probability } 1 - x^2 \end{cases}$$

4. Calculate the correlation coefficient of $g(a'(t))$ and $g(b'(t))$ as a function of $\theta_a - \theta_b$.

— The Solution —

1. By change of the time variable and letting $\theta_{ab} = \theta_a - \theta_b$, one can rewrite $a'(t)$ and $b'(t)$ as

$$(\alpha(\tau), \beta(\tau)) = \begin{cases} (\sin(\omega\tau + \theta_{ab}), \cos\,\omega\tau) & \text{with probability } \frac{1}{2} \\ (\cos(\omega\tau + \theta_{ab}), \sin\,\omega\tau) & \text{with probability } \frac{1}{2} \end{cases}$$

2. At $\tau = 0$ this gives

$$(\alpha(0), \beta(0)) = \begin{cases} (\sin\theta_{ab}, 1) & \text{with probability } \frac{1}{2} \\ (\cos\theta_{ab}, 0) & \text{with probability } \frac{1}{2} \end{cases}$$

3. The correlation coefficient for $\tau = 0$, in terms of the difference of angles θ_{ab} and zero, $\theta_{ab} - 0 = \theta_{ab}$, is

$$\rho = \rho^{++} + \rho^{+-} + \rho^{-+} + \rho^{--}$$

   where

$$\rho^{++} = \frac{1}{2}\,(+1)\,(+1)\,\sin^2\theta_{ab}$$
$$\rho^{+-} = \frac{1}{2}\,(+1)\,(-1)\,\cos^2\theta_{ab}$$
$$\rho^{-+} = \frac{1}{2}\,(-1)\,(+1)\,\cos^2\theta_{ab}$$
$$\rho^{--} = \frac{1}{2}\,(-1)\,(-1)\,\sin^2\theta_{ab}$$

4. Using a double-angle identity, this simplifies to

$$\rho = -(\cos^2\theta_{ab} - \sin^2\theta_{ab}) = -\cos\,2\theta_{ab}$$

or, in other words,

$$\rho = -\cos\,2(\theta_a - \theta_b)$$

which would seem to be the answer to the problem. However, we derived it under the assumption that $\tau = 0$.

5. Note, though, that the solution is invariant under in-unison rotations of θ_a and θ_b. Any change of the time variable is absorbed by such a rotation. Therefore ρ is indeed the general solution to the problem.[1] □

[1] It is also, as it must be, the correlation coefficient calculated by quantum mechanics for a two-channel optical Bell test experiment. The two problems are equivalent, and all valid methods of solution must come to the same result.