Two Oscillators

— The Problem —

1. You have two oscillators, $a(t) = \sin(\omega t + \phi_a)$ and $b(t) = \sin(\omega t + \phi_b)$, where

$$(\phi_a, \phi_b) = \begin{cases} (0, \frac{\pi}{2}) & \text{with probability } \frac{1}{2} \\ (\frac{\pi}{2}, 0) & \text{with probability } \frac{1}{2} \end{cases}$$

2. The oscillator outputs are subjected to phase shifts, giving $a'(t) = \sin(\omega t + \phi_a + \theta_a)$ and $b'(t) = \sin(\omega t + \phi_b + \theta_b)$.

3. Define the function

$$g(x) = \begin{cases} +1 & \text{with probability } x^2 \\ -1 & \text{with probability } 1 - x^2 \end{cases}$$

4. Calculate the correlation coefficient of $g(a'(t))$ and $g(b'(t))$ as a function of $\theta_a - \theta_b$.

— The Solution —

1. By change of the time variable and letting $\theta_{ab} = \theta_a - \theta_b$, one can rewrite $a'(t)$ and $b'(t)$ as

$$(\alpha(\tau), \beta(\tau)) = \begin{cases} (\sin(\omega\tau + \theta_{ab}), \cos\,\omega\tau) & \text{with probability } \frac{1}{2} \\ (\cos(\omega\tau + \theta_{ab}), \sin\,\omega\tau) & \text{with probability } \frac{1}{2} \end{cases}$$

2. At $\tau = 0$ this gives

$$(\alpha(0), \beta(0)) = \begin{cases} (\sin\theta_{ab}, 1) & \text{with probability } \frac{1}{2} \\ (\cos\theta_{ab}, 0) & \text{with probability } \frac{1}{2} \end{cases}$$

3. The correlation coefficient for $\tau = 0$, in terms of the difference of angles θ_{ab} and zero, $\theta_{ab} - 0 = \theta_{ab}$, is

$$\rho = \rho^{++} + \rho^{+-} + \rho^{-+} + \rho^{--}$$

where

$$\rho^{++} = \frac{1}{2}(+1)\,(+1)\sin^2\theta_{ab}$$
$$\rho^{+-} = \frac{1}{2}(+1)\,(-1)\cos^2\theta_{ab}$$
$$\rho^{-+} = \frac{1}{2}(-1)\,(+1)\cos^2\theta_{ab}$$
$$\rho^{--} = \frac{1}{2}(-1)\,(-1)\sin^2\theta_{ab}$$

4. Using a double-angle identity, this simplifies to

$$\rho = -(\cos^2\theta_{ab} - \sin^2\theta_{ab}) = -\cos 2\theta_{ab}$$

or, in other words,

$$\rho = -\cos 2(\theta_a - \theta_b)$$

which would seem to be the answer to the problem. However, we derived it under the assumption that $\tau = 0$.

5. Note, though, that the solution is invariant under in-unison rotations of θ_a and θ_b. Any change of the time variable is absorbed by such a rotation. Therefore ρ is indeed the general solution to the problem.[1] □

[1] It is also, as it must be, the correlation coefficient calculated by quantum mechanics for a two-channel optical Bell test experiment. The two problems are equivalent, and all valid methods of solution must come to the same result. Be forewarned, though: 'doctors of philosophy' might suggest the problems *are not* equivalent, for reasons beyond the realm of mathematics. For instance, the oscillators were not created by a calcium radiative cascade and therefore were not 'quantum'. If they *had been* created by cascade, *then* only quantum mechanics (they might suggest) could have solved the problem. This would be circular argument, however, because the Bell test is what supposedly 'proves' that the cascade is 'quantum'! It is an argument enforceable (*and in the actual world enforced*) by authority, not logic. There is actually no special quality associated with 'quantum' physics. 'Quantum mechanics' refers to a patchwork of obfuscated methods for solving physics problems. Here we solved the same problem by other, in this case better means.