ASP4200/ASP4020 Computational Astrophysics

Assignment 2

1. Consider a test particle in orbit around a central point mass, with equations of motion in the form

$$\ddot{x} = -\frac{GMx}{r^3}; \qquad \ddot{y} = -\frac{GMy}{r^3};$$

where $r = \sqrt{x^2 + y^2}$ and using code units such that $G = M = 1$. Starting with initial conditions corresponding to the test particle at pericentre, i.e.

$$x = 1 - e; \;\; y = 0; \qquad \dot{x} = 0; \;\; \dot{y} = \sqrt{\frac{1+e}{1-e}};$$

write a program to solve for the resulting orbital motion using

   (a) The Leapfrog/Velocity Verlet scheme (2nd order)

   (b) The 4th order Runge-Kutta method

Compare the solutions for an eccentric orbit ($e = 0.7$) for each integrator up after 5000 timesteps using $\Delta t = 0.01, 0.05$ and 0.1. In each case plot the trajectory of the orbit in the x-y plane. Finally, plot a graph showing angular momentum $L = x\dot{y} - y\dot{x}$ and total energy $e = 1/2(\dot{x}^2 + \dot{y}^2) - 1/r$ as a function of time for the two integrators in the case $\Delta t = 0.05$. Explain your results.

[8 marks]

2. Work your way through the MHD example problems supplied with the NDSPMHD code (download both the code and the examples from my website).

   (a) Run the shock tube example corresponding to a "1.75D" magnetised shock with initial conditions $[\rho, P, v_x, v_y, v_z, B_x, B_y, B_z] = [1.08, 0.95, 1.2, 0.01, 0.5, 2/\sqrt{4\pi}, 3.6/\sqrt{4\pi}, 2/\sqrt{4\pi}]$ for $x < 0$ and $[\rho, P, v_x, v_y, v_z, B_x, B_y, B_z] = [1, 1, 0, 0, 0, 2/\sqrt{4\pi}, 4/\sqrt{4\pi}, 2/\sqrt{4\pi}]$ for $x > 0$. Plot a graph of your numerical solution at t=0.2 and on this graph correctly label which MHD wave type each of the discontinuities corresponds to. Justify your choices in terms of the characteristics of each type of wave — specifically whether or not the wave is compressive, and the relative ordering of the wave speeds in the initial conditions.

(b) Perform the "2D MHD rotor test" involving a rotating 'disc' of high density gas ($\rho = 10$ for $R < 0.1$) embedded in a low density $\rho = 1$ surrounding medium with $[B_x, B_y, B_z] = [5/\sqrt{4\pi}, 0, 0]$ with $\gamma = 1.4$. Use SPLASH to plot a rendering of the density, pressure and magnetic energy and Mach number in the domain at $t = 0.15$. Calculate the initial Alfvén speed in the low density medium and use this to estimate the distance that a disturbance travelling at the Alfvén speed would travel. Using this estimate along with a similar estimate for the fast and slow MHD wave speeds in directions perpendicular and parallel to the field, correctly identify the 'wavefront' from all three MHD wave types in the density field. *Hint: when computing distances, recall that the initial discontinuity is at $R = 0.1$. Also, you can turn the axes labels back on in the p)age menu.*

[4 marks]

3. Use the example DUSTYWAVE problem supplied with NDSPMHD to simulate linear waves in a gas-dust mixture, by solving either the 'two fluid' equations

$$\frac{\partial \rho_{\rm g}}{\partial t} + \nabla \cdot (\rho_{\rm g} \boldsymbol{v}_{\rm g}) = 0, \tag{1}$$

$$\frac{\partial \rho_{\rm d}}{\partial t} + \nabla \cdot (\rho_{\rm d} \boldsymbol{v}_{\rm d}) = 0, \tag{2}$$

$$\rho_{\rm g} \left[\frac{\partial \boldsymbol{v}_{\rm g}}{\partial t} + (\boldsymbol{v}_{\rm g} \cdot \nabla) \boldsymbol{v}_{\rm g} \right] = -\nabla P_{\rm g} + K (\boldsymbol{v}_{\rm d} - \boldsymbol{v}_{\rm g}), \tag{3}$$

$$\rho_{\rm d} \left[\frac{\partial \boldsymbol{v}_{\rm d}}{\partial t} + (\boldsymbol{v}_{\rm d} \cdot \nabla) \boldsymbol{v}_{\rm d} \right] = -K (\boldsymbol{v}_{\rm d} - \boldsymbol{v}_{\rm g}) \tag{4}$$

or by solving the 'one fluid' equations describing the mixture:

$$\frac{{\rm d}\rho}{{\rm d}t} = -\rho (\nabla \cdot \boldsymbol{v}), \tag{5}$$

$$\frac{{\rm d}\epsilon}{{\rm d}t} = -\frac{1}{\rho} \nabla \cdot \left(\frac{\rho_{\rm g}\rho_{\rm d}}{\rho} \Delta \boldsymbol{v} \right), \tag{6}$$

$$\frac{{\rm d}\boldsymbol{v}}{{\rm d}t} = -\frac{\nabla P_{\rm g}}{\rho} - \frac{1}{\rho} \nabla \cdot \left(\frac{\rho_{\rm g}\rho_{\rm d}}{\rho} \Delta \boldsymbol{v} \Delta \boldsymbol{v} \right), \tag{7}$$

$$\frac{{\rm d}\Delta \boldsymbol{v}}{{\rm d}t} = -\frac{\Delta \boldsymbol{v}}{t_{\rm s}} + \frac{\nabla P_{\rm g}}{\rho_{\rm g}} - (\Delta \boldsymbol{v} \cdot \nabla) \boldsymbol{v} + \frac{1}{2} \nabla \left(\frac{\rho_{\rm d} - \rho_{\rm g}}{\rho} \Delta \boldsymbol{v}^2 \right). \tag{8}$$

(a) Run the DUSTYWAVE examples as described above, and use SPLASH to plot the numerical solution for the gas and dust velocities after 5 wave periods for the simulations using the 5 different values of K given in the example input files, i.e. $K = 0.001, 0.01, 1.0, 100$ and 1000, alongside the exact solution. *Submit two plots, one for the 'two fluid method' and one for the 'one fluid method'* *showing $v_x(x)$ at $t = 5$ with different rows corresponding to different values of the drag coefficient (this is the default plot supplied with the examples).*

[1 + 1 = 2 marks]

(b) Derive the dispersion relation for dust-gas mixtures by completing Q8 of Problem Sheet 2

[2 marks]

(c) Compute the value of the stopping time for each value of K used in part a). Use the dispersion relation for dust-gas mixtures (e.g. as derived above) to explain the behaviour of the linear wave solutions.

[1 + 1 = 2 marks]

(d) In which regime does the two fluid method give accurate results compared to the analytic solution? Assuming an Epstein drag prescription, what kind of grains would this correspond to? Similarly, in which regime does the one fluid method give accurate results and what kind of grains would this correspond to?

[1 + 1 = 2 marks]

Please hand in your answers by scanning and uploading to moodle.