ASP4200/ASP4020 Computational Astrophysics

Problem Sheet 1

1. (a) Starting with the equations of fluid dynamics expressing mass and momentum conservation for a compressible, self-gravitating gas,

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho = -\rho(\nabla \cdot \boldsymbol{v}), \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla P}{\rho} - \nabla \Phi, \tag{2}$$

together with the Poisson equation for the gravitational field

$$\nabla^2 \Phi = 4\pi G \rho, \tag{3}$$

show that the dispersion relation for linear perturbations is given by

$$\omega^2 = c_{\rm s}^2 (k^2 - k_J^2), \tag{4}$$

where the Jeans wavenumber is given by

$$k_J^2 = \frac{4\pi G \rho_0}{c_{\rm s}^2}. \tag{5}$$

(b) What is the critical wavenumber at which perturbations will become unstable (grow exponentially)?

(c) Express this critical wavenumber as a wavelength.

(d) Compute the wavelength for maximum growth of the Jeans instability.

2. Starting with the fluid equations in Lagrangian form:

$$\frac{{\rm d}\rho}{{\rm d}t} = -\rho(\nabla \cdot \boldsymbol{v}), \tag{6}$$

$$\frac{{\rm d}\boldsymbol{v}}{{\rm d}t} = -\frac{\nabla P}{\rho}, \tag{7}$$

$$\frac{{\rm d}u}{{\rm d}t} = -\frac{P}{\rho}(\nabla \cdot \boldsymbol{v}), \tag{8}$$

show that these are equivalent to the 'conservative form' expresssions

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{9}$$

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (P\mathbf{I} + \rho \boldsymbol{v}\boldsymbol{v}) = 0, \tag{10}$$

$$\frac{\partial (\rho e)}{\partial t} + \nabla \cdot [(\rho e + P)\boldsymbol{v}] = 0, \tag{11}$$

where $e \equiv \frac{1}{2}v^2 + u$ is the total specific energy.

3. Starting with the Rankine-Hugoniot jump conditions for a plane-parallel shock, derive the following expressions:

   (a)
   $$\frac{\rho_2}{\rho_1} = \frac{\mathcal{M}_1^2(\gamma+1)}{2+(\gamma-1)\mathcal{M}_1^2} \tag{12}$$

   (b)
   $$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{\gamma+1} \tag{13}$$

   (c)
   $$\frac{T_2}{T_1} = \frac{[2\gamma\mathcal{M}_1^2 - (\gamma-1)]\,[2+(\gamma-1)\mathcal{M}_1^2]}{(\gamma+1)^2\mathcal{M}_1^2} \tag{14}$$

4. (a) Show that if $\sigma_{ij} = \sigma_{ji}$, then

   $$\sum_i \sum_j \sigma_{ij}(x_i - x_j) = 0, \tag{15}$$

   and therefore that

   $$\sum_i \sum_j \sigma_{ij}(\boldsymbol{r}_i - \boldsymbol{r}_j) = 0. \tag{16}$$

   (b) Using the result (16), show that the SPH acceleration equation (with $W_{ij} \equiv W(\boldsymbol{r}_i - \boldsymbol{r}_j, h)$);

   $$\frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right) \nabla_i W_{ij}, \tag{17}$$

   explicitly conserves

   i. Linear momentum $\sum_i m_i \boldsymbol{v}_i$, and
   ii. Angular momentum $\sum_i m_i \boldsymbol{r}_i \times \boldsymbol{v}_i$.

5. Starting with the SPH thermal energy equation given by

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{P_i}{\rho_i^2}\frac{\mathrm{d}\rho_i}{\mathrm{d}t} = \frac{P_i}{\rho_i^2}\sum_j m_j\,(\boldsymbol{v}_i - \boldsymbol{v}_j)\cdot\nabla_i W_{ij}, \tag{18}$$

together with the SPH equations of motion (Eq. 17), show that the total energy per unit mass $e = \frac{1}{2}v^2 + u$ evolves according to

$$\frac{\mathrm{d}e_i}{\mathrm{d}t} = -\sum_j m_j \left(\frac{P_i \boldsymbol{v}_j}{\rho_i^2} + \frac{P_j \boldsymbol{v}_i}{\rho_j^2} \right) \cdot \nabla_i W_{ij}. \tag{19}$$

and, using (16), show therefore that the total energy $E = \sum_i m_i e_i$ is conserved.

6. (a) Show that, if one calculates the density via a summation depending on the particle's own smoothing length:

$$\rho_i = \sum_j m_j W_{ij}(h_i), \tag{20}$$

and that if one assumes that the smoothing length is in turn a function of density $h = h(\rho)$, that the co-moving time derivative of density ($\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla$) is given by

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} = \frac{1}{\Omega_i} \sum_j m_j \left(\boldsymbol{v}_i - \boldsymbol{v}_j \right) \cdot \nabla W_{ij}(h_i), \tag{21}$$

where

$$\Omega_i = 1 - \frac{\partial h_i}{\partial \rho_i} \sum_j m_j \frac{\partial W_{ij}(h_i)}{\partial h_i}. \tag{22}$$

(b) Similar to the above, starting with the density summation for a particle b,

$$\rho_j = \sum_k m_k W_{jk}(h_j), \tag{23}$$

show that the gradient taken with respect to the position of particle a is given by

$$\frac{\partial \rho_j}{\partial \boldsymbol{r}_i} = \frac{1}{\Omega_j} \sum_k m_k \frac{\partial W_{jk}(h_j)}{\partial \boldsymbol{r}_i} (\delta_{ji} - \delta_{ki}), \tag{24}$$

(c) Finally, starting with the Lagrangian;

$$L = \sum_j m_j \left(\frac{1}{2} v_j^2 - u_j \right), \tag{25}$$

the Euler-Lagrange equations for particle a,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \boldsymbol{v}_i} \right) - \frac{\partial L}{\partial \boldsymbol{r}_i} = 0, \tag{26}$$

and the first law of thermodynamics written in the form

$$\mathrm{d}u = \frac{P}{\rho^2} \mathrm{d}\rho, \tag{27}$$

show that the SPH equations of motion are given by

$$\frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\sum_j m_j \left[\frac{P_i}{\Omega_i \rho_i^2} \nabla_i W_{ij}(h_i) + \frac{P_j}{\Omega_j \rho_j^2} \nabla_i W_{ij}(h_j) \right]. \tag{28}$$

7. Based on the standard kernel interpolation formula for a vector quantity $\boldsymbol{A}$:

$$\boldsymbol{A}(\boldsymbol{r}_i) = \sum_j \frac{m_j}{\rho_j} \boldsymbol{A}_j W(|\boldsymbol{r}_i - \boldsymbol{r}_j|, h), \tag{29}$$

and its derivatives, e.g.

$$\nabla \cdot \boldsymbol{A}(\boldsymbol{r}_i) = \sum_j \frac{m_j}{\rho_j} \boldsymbol{A}_j \cdot \nabla W(|\boldsymbol{r}_i - \boldsymbol{r}_j|, h), \tag{30}$$

$$\nabla \times \boldsymbol{A}(\boldsymbol{r}_i) = -\sum_j \frac{m_j}{\rho_j} \boldsymbol{A}_j \times \nabla W(|\boldsymbol{r}_i - \boldsymbol{r}_j|, h), \tag{31}$$

(a) Write down SPH interpolants for the following quantities which vanish when $\boldsymbol{v}$ is constant. In each case, prove that your discrete formula is valid by translating each term using (30), (31) or similar formulae derived from (29).

i. $\nabla \cdot \boldsymbol{v}$

ii. $\nabla \times \boldsymbol{v}$

iii. $\frac{\partial v^\alpha}{\partial x^\beta}$

iv. $(\boldsymbol{B} \cdot \nabla)\boldsymbol{v}$

(b) Show that the following are all discretisations of the continuity equation

i.

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} = \sum_j m_j (\boldsymbol{v}_i - \boldsymbol{v}_j) \cdot \nabla_i W_{ij},$$

ii.

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} = \rho_i \sum_j \frac{m_j}{\rho_j} (\boldsymbol{v}_i - \boldsymbol{v}_j) \cdot \nabla_i W_{ij},$$

iii.

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t} = \phi_i \sum_j \frac{m_j}{\phi_j} (\boldsymbol{v}_i - \boldsymbol{v}_j) \cdot \nabla W_{ij} \quad \text{(where } \phi \text{ is arbitrary scalar)}$$

(c) Show that the equation

$$\frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} \frac{\phi_i}{\phi_j} + \frac{P_j}{\rho_j^2} \frac{\phi_j}{\phi_i} \right) \nabla_i W_{ij}, \tag{32}$$

is a discretisation of

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho}, \tag{33}$$

for any differentiable function ϕ.

8. Starting with the second derivative of the standard kernel interpolation formula, i.e.

$$\nabla^2 A_i = \sum_j \frac{m_j}{\rho_j} A_j \nabla^2 Y_{ij}, \tag{34}$$

for an arbitrary kernel function Y, show that the SPH expression

$$-\sum_j \frac{m_j}{\rho_j}\frac{1}{2}(\kappa_i + \kappa_j)(A_i - A_j)\nabla^2 Y_{ij}, \tag{35}$$

is a representation of a second derivative term of the form

$$\nabla \cdot (\kappa \nabla A), \tag{36}$$

where κ is an arbitrary, differentiable scalar function.

9. Using the general expressions for vector second derivative terms in SPH, given by

$$\nabla^2 \boldsymbol{A} = 2\sum_j \frac{m_j}{\rho_j}(\boldsymbol{A}_i - \boldsymbol{A}_j)\frac{F_{ij}}{|r_{ij}|}, \tag{37}$$

$$\nabla(\nabla \cdot \boldsymbol{A}) = \sum_j \frac{m_j}{\rho_j}\left[(\delta^k_k + 2)(\boldsymbol{A}_{ij} \cdot \hat{\boldsymbol{r}}_{ij})\hat{\mathbf{r}}_{ij} - \boldsymbol{A}_{ij}\right]\frac{F_{ij}}{|r_{ij}|}, \tag{38}$$

where $\delta^k_k \equiv n$ i.e., the number of spatial dimensions and here $\nabla W_{ij} \equiv \hat{\boldsymbol{r}}_{ij} F_{ij}$;

(a) Write down an SPH expression for a physical shear viscosity term of the form

$$\nu \nabla^2 \boldsymbol{v}. \tag{39}$$

(b) Show that in 3D the SPH artificial viscosity term

$$\sum_j m_j \frac{\alpha \bar{c}_s \bar{h}_{ij}}{\bar{\rho}_{ij}} \frac{(\boldsymbol{v}_{ij} \cdot \boldsymbol{r}_{ij})}{r^2_{ij} + \epsilon \bar{h}^2_{ij}} \nabla_i W_{ij}, \tag{40}$$

can be translated into the continuum form,

$$\frac{f}{5}\nabla(\nabla \cdot \boldsymbol{v}) + \frac{f}{10}\nabla^2 \boldsymbol{v} \tag{41}$$

where $f = \alpha c_s h$ (Assume for simplicity that $\epsilon = 0$ and that α, h and c_s are constant over the kernel radius; if you wish you may also assume that $\bar{\rho}_{ij} \approx \rho_j$, though this is not strictly necessary).

(c) Comparing the above result to the compressible Navier Stokes equations (assuming constant coefficients)

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho} + \nu\nabla^2\boldsymbol{v} + \left(\zeta + \frac{\nu}{3}\right)\nabla(\nabla\cdot\boldsymbol{v}), \tag{42}$$

show that the shear and bulk viscosity coefficients are given by

$$\nu \approx \frac{1}{10}\alpha c_{\rm s} h, \tag{43}$$

$$\zeta \approx \frac{1}{6}\alpha c_{\rm s} h. \tag{44}$$

10. Consider an SPH representation of a shear viscosity term in the form

$$\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_{visc} = \nu\nabla^2\boldsymbol{v}, \tag{45}$$

given by

$$\left(\frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t}\right)_{visc} \approx 2\nu\sum_j \frac{m_j}{\bar{\rho}_{ij}}(\boldsymbol{v}_i - \boldsymbol{v}_j)\frac{F_{ij}}{|r_{ij}|}. \tag{46}$$

Show that the required contribution to the thermal energy equation $(du/dt)_{visc}$ such that the total energy $\sum_i m_i e_i$ is conserved, is given by

$$\left(\frac{\mathrm{d}u_i}{\mathrm{d}t}\right)_{visc} = -\nu\sum_j \frac{m_j}{\bar{\rho}_{ij}}(\boldsymbol{v}_i - \boldsymbol{v}_j)^2\frac{F_{ij}}{|r_{ij}|}, \tag{47}$$

such that, for positive, monotonic kernels (F_{ij} negative) the contribution to the thermal energy is always positive. *Hint: Start with the following:*

$$\sum_i m_i \frac{\mathrm{d}e_i}{\mathrm{d}t} = \sum_i m_i\left[\boldsymbol{v}_i\cdot\left(\frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t}\right)_{visc} + \left(\frac{\mathrm{d}u_i}{\mathrm{d}t}\right)_{visc}\right] = 0, \tag{48}$$

and use a rearrangement of the double summations via the identity

$$\sum_i\sum_j \sigma_i(\sigma_i - \sigma_j) = \frac{1}{2}\sum_i\sum_j(\sigma_i - \sigma_j)^2. \tag{49}$$