Comp P-set 1

1a) we have:

$$\frac{\partial \rho}{\partial t} + (\vec{v} \cdot \nabla)\rho = -\rho(\nabla \cdot \vec{v}) \quad (1)$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla P}{\rho} - \nabla \phi \quad (2)$$

Assume small perturbations around some reference state:

$$\rho = \rho_0 + \delta\rho,$$
$$\vec{v} = \cancel{\vec{v}_0}^{\,=0} + \delta\vec{v} \qquad \text{(assume medium at rest)}$$
$$P = P_0 + \delta P$$

So (1) becomes:

assume const density

$$\frac{\partial(\overset{\text{const in time}}{\cancel{\rho_0}} + \delta\rho)}{\partial t} + (\delta\vec{v} \cdot \nabla)(\overset{\text{assume const density}}{\cancel{\rho_0}} + \overset{\text{second order term}}{\cancel{\delta\rho}}) = -(\rho_0 + \cancel{\delta\rho})(\nabla \cdot \delta\vec{v})$$

$$\Rightarrow \frac{\partial \delta\rho}{\partial t} = -\rho_0 \nabla \cdot \delta\vec{v} \quad (3)$$

(2) becomes:

$$\frac{\partial \delta\vec{v}}{\partial t} + \overset{\text{Second order}}{\cancel{(\delta\vec{v} \cdot \nabla)\delta\vec{v}}} = -\frac{\nabla(\overset{\text{Const in space}}{\cancel{P_0}} + \delta P)}{\rho_0 + \delta\rho} - \nabla\phi$$

$$(\rho_0 + \overset{\text{Second order}}{\cancel{\delta\rho}})\frac{\partial \delta\vec{v}}{\partial t} = -\nabla \delta P - (\rho_0 + \delta\rho)\nabla\phi$$

1a) cont We have:

$$c_s^2 = \frac{\delta P}{\delta \rho} \Rightarrow \delta P = c_s^2 \, \delta \rho$$

furthermore, assume small perturbation in c_s:

$$c_s = c_{s_0} + \delta c_s ,$$

$$\Rightarrow \delta P = c_{s_0}^2 \delta \rho \quad (\text{ignoring} \geq 2\text{nd order terms})$$

$$\therefore \rho_0 \frac{\partial(\delta \vec{v})}{\partial t} = -\overset{\nearrow \text{const in space}}{c_{s_0}^2} \nabla \delta \rho - (\rho_0 + \delta \rho) \nabla \phi$$

Assume
$$\phi = \phi_0 + \delta \phi , \quad \text{with } \phi_0 \text{ const in space}$$

$$\Rightarrow \nabla \phi = \nabla \delta \phi$$

$$\Rightarrow \rho_0 \frac{\partial(\delta \vec{v})}{\partial t} = -c_{s_0}^2 \nabla \delta \rho - \rho_0 \nabla \delta \phi \quad (\text{ignoring 2nd order terms})$$

to match (3), take div:

$$\nabla \cdot \left(\overset{\nearrow \text{const in space}}{\rho_0} \frac{\partial(\delta \vec{v})}{\partial t}\right) = \nabla \cdot \left(-\overset{\nearrow \text{const in space}}{c_{s_0}^2} \nabla \delta \rho\right) - \nabla \cdot \left(\rho_0 \nabla \delta \phi\right)$$

$$\Rightarrow \rho_0 \frac{\partial(\nabla \cdot \delta \vec{v})}{\partial t} = -c_{s_0}^2 \nabla^2 \delta \rho - \rho_0 \nabla^2 \delta \phi$$

and take time derivative of ③:

$$\frac{\partial^2 \delta\rho}{\partial t^2} = -\overset{\nearrow \text{const in time}}{\rho_o} \, \partial \left(\frac{\nabla \cdot \delta\vec{v}}{\partial t} \right)$$

And so we get:

$$-\frac{\partial^2 \delta\rho}{\partial t^2} = -c_{s_o}^2 \nabla^2 \delta\rho - \rho_o \nabla^2 \delta\phi$$

Assuming background state obeys Poissons eqn; i.e:

$\nabla^2 \phi_o = 4\pi G \rho_o$,

we have

$\nabla^2 \phi = 4\pi G \rho$

$\Rightarrow \cancel{\nabla^2 \phi_o} + \nabla^2 \delta\phi = \cancel{4\pi G \rho_o} + 4\pi G \delta\rho$

$\Rightarrow \nabla^2 \delta\phi = 4\pi G \, \delta\rho$

$$\Rightarrow \frac{\partial^2 \delta\rho}{\partial t^2} = c_{s_o}^2 \nabla^2 \delta\rho + \rho_o (4\pi G \, \delta\rho)$$

$$\Rightarrow \frac{1}{c_{s_o}^2} \frac{\partial^2 \delta\rho}{\partial t^2} - \nabla^2 \delta\rho = \overbrace{\frac{4\pi G \rho_o}{c_{s_o}^2}}^{k_J^2} \delta\rho \qquad ④$$

Assume solution of the form

$$\delta\rho = D e^{i(\kappa \cdot \vec{x} - \omega t)}$$

$$\frac{\partial \delta\rho}{\partial t} = -i\omega \, \delta\rho$$

$$\frac{\partial^2 \delta\rho}{\partial t^2} = -\omega^2 \delta\rho$$

working:

$$\omega^2 = c_s^2(k^2 - k_J^2)$$

$$-\omega^2 + c_s^2 k^2 = c_s^2 k_J^2$$

$$-\omega^2 \delta\rho + c_s^2 k^2 \delta\rho = c_s^2 k_J^2 \delta\rho$$

$$\vec{k} \cdot \vec{x} = k_1 x + k_2 y + k_3 z$$

$$\nabla(\vec{k} \cdot \vec{x}) = \vec{k}$$

$$\nabla \delta\rho = -i\vec{k}\,\delta\rho$$

$$\nabla^2 \delta\rho = -k^2 \delta\rho$$

(4) becomes:

$$\frac{1}{c_{s_0}^2}\left(-\omega^2 \cancel{\delta\rho}\right) + k^2 \cancel{\delta\rho} = k_J^2 \cancel{\delta\rho}$$

$$\omega^2 = c_{s_0}^2 (k^2 - k_J^2)$$

where $k_J^2 = \dfrac{4\pi G \rho_0}{c_{s_0}^2}$

P. Sheet 1 cont

1b)

If we have:

$k_J > k$,

then

$\omega^2 = c_s^2 (k^2 - k_J^2) < 0$

$\Rightarrow$ ω is imaginary

$\Rightarrow$ $\omega = |\omega| i$

and so our perturbation in density is given by:

$$\delta\rho = D e^{ikx} e^{-i\omega t}$$

$$= D e^{ikx} e^{|\omega| t},$$

and so our perturbation grows exponentially with time.

$\therefore$ unstable perturbations for

$$k < k_J = \sqrt{\frac{4\pi G \rho_0}{c_s^2}}$$

and so the critical wavenumber is

$$k_c = k_J = \sqrt{\frac{4\pi G \rho_0}{c_s^2}}$$

1c) we have

$$\lambda = \frac{2\pi}{K}$$

$$\Rightarrow \lambda_c = \frac{2\pi}{K_c} = \frac{2\pi}{K_J}$$

$$= \sqrt{\frac{\pi c_s^2}{G \rho_0}}$$

1d)

Maximum growth occurs at max $|\omega|$ (or equivalently max ω^2)
(Since our exponential factor is $e^{|\omega| t}$

this occurs at

$$\frac{\partial \omega^2}{\partial k} = 0$$

$\Rightarrow 2 k c_s^2 = 0$ (Since $\omega^2 = c_s^2 (k^2 - k_J^2)$)
$\Rightarrow k = 0$

and so max growth at $\lambda \to \infty$ (Since $\lambda = \frac{2\pi}{k}$)

Problem Sheet 1

2)

a) $$\frac{d\rho}{dt} = -\rho(\nabla\cdot\vec{v})$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + (\vec{v}\cdot\nabla)\rho = -\rho(\nabla\cdot\vec{v})$$

$$\frac{\partial\rho}{\partial t} + (\vec{v}\cdot\nabla)\rho + \rho(\nabla\cdot\vec{v}) = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \quad \text{(product rule)}$$

b)

$$\frac{d\vec{v}}{dt} = -\frac{\nabla P}{\rho}$$

$$\Rightarrow \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = \frac{-\nabla P}{\rho}$$

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla P$$

from a), we have:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

$$\Rightarrow \vec{v}\frac{\partial\rho}{\partial t} + \vec{v}(\nabla\cdot(\rho\vec{v})) = 0.$$

adding this to our equation, we get:

$$\underbrace{\rho\frac{\partial \vec{v}}{\partial t} + \vec{v}\frac{\partial \rho}{\partial t}}_{=\frac{\partial(\rho\vec{v})}{\partial t}} + \underbrace{\rho(\vec{v}\cdot\nabla)\vec{v} + \vec{v}(\nabla\cdot(\rho\vec{v}))}_{=\nabla\cdot(\rho\vec{v}\vec{v})} = \underbrace{-\nabla P}_{=-\nabla\cdot(PI)}$$

$$\Rightarrow \frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot\left(PI + \rho\vec{v}\vec{v}\right) = 0$$

c)

$$\frac{du}{dt} = -\frac{P}{\rho}(\nabla\cdot\vec{v})$$

$$\Rightarrow \frac{\partial u}{\partial t} + (\vec{v}\cdot\nabla)u = -\frac{P}{\rho}(\nabla\cdot\vec{v})$$

we have:

$$e = \frac{1}{2}v^2 + u$$

$$\rho e = \frac{1}{2}\rho v^2 + \rho u$$

$$\frac{\partial(\rho e)}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial t}(\rho u)$$

$$= \frac{1}{2}\vec{v}\cdot\frac{\partial\rho\vec{v}}{\partial t} + \frac{1}{2}\rho\vec{v}\cdot\frac{\partial\vec{v}}{\partial t} + u\frac{\partial\rho}{\partial t} + \rho\frac{\partial u}{\partial t}$$

c) We have

$$e = \frac{1}{2} v^2 + u,$$

$$\Rightarrow \rho e = \frac{1}{2} \rho v^2 + \rho u$$

$$\frac{d(\rho e)}{dt} = \frac{1}{2} v^2 \frac{d\rho}{dt} + \rho \frac{d(\frac{1}{2}v^2)}{dt} + \rho \frac{du}{dt} + u \frac{d\rho}{dt}$$

but we have:

$$\frac{d(\rho e)}{dt} = \frac{\partial(\rho e)}{\partial t} + (\vec{v} \cdot \nabla)(\rho e),$$

$$\frac{d\rho}{dt} = -\rho(\nabla \cdot \vec{v}) \Rightarrow \frac{1}{2} \vec{v}^2 \frac{d\rho}{dt} = -\frac{1}{2} \vec{v}^2 \rho (\nabla \cdot \vec{v})$$

$$\rho \frac{d(\frac{1}{2}v^2)}{dt} = \rho \frac{d(\frac{1}{2}\vec{v}^2)}{d\vec{v}} \cdot \frac{d\vec{v}}{dt}$$

$$= \rho \vec{v} \left(-\frac{\nabla P}{\rho} \right) = -\vec{v} \cdot \nabla P,$$

$$\rho \frac{du}{dt} = -P(\nabla \cdot \vec{v}),$$

$$u \frac{d\rho}{dt} = -u\rho(\nabla \cdot \vec{v})$$

$$= \frac{1}{2} \vec{v}^2 \rho (\nabla \cdot \vec{v}) - e\rho(\nabla \cdot \vec{v})$$

putting this all together, we get:

$$\frac{\partial(\rho e)}{\partial t} + (\vec{v}\cdot\nabla)(\rho e) + (\rho e)(\nabla\cdot\vec{v})$$

$$= \cancel{-\frac{1}{2}\vec{v}^2\rho(\nabla\cdot\vec{v})} - \vec{v}\cdot\nabla P - P(\nabla\cdot\vec{v}) + \cancel{\frac{1}{2}\vec{v}^2\rho(\nabla\cdot\vec{v})}$$

$$\Rightarrow \frac{\partial(\rho e)}{\partial t} + \nabla\cdot(\rho e\vec{v}) = -\nabla\cdot(P\vec{v})$$

$$\Rightarrow \frac{\partial(\rho e)}{\partial t} + \nabla\cdot[(\rho e + P)\vec{v}] = 0$$

as required.

3a) We have the following jump conditions:

$$\rho_1 v_1 = \rho_2 v_2 \quad (1)$$

$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2 \quad (2)$$

$$\frac{1}{2} v_1^2 + \frac{\gamma P_1}{(\gamma-1)\rho_1} = \frac{1}{2} v_2^2 + \frac{\gamma P_2}{(\gamma-1)\rho_2} \quad (3)$$

from (1):

$$v_2^2 = \frac{\rho_1^2}{\rho_2^2} v_1^2 \quad (4)$$

Sub into (2):

$$\rho_1 v_1^2 + P_1 = \cancel{\rho_2}\left(\frac{\rho_1^2}{\rho_2^{\cancel{2}}} v_1^2\right) + P_2$$

$$\Rightarrow P_2 = P_1 + \rho_1 v_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right) \quad (5)$$

from (3):

$$\frac{1}{2} v_1^2 - \frac{1}{2} v_2^2 + \frac{c_{s_1}^2}{\gamma - 1} = \gamma \frac{P_2}{(\gamma-1)\rho_2} \quad \Big(\text{where } ($$

Sub in (4) and (5):

$$\frac{1}{2} v_1^2 \left(1 - \frac{\rho_1^2}{\rho_2^2}\right) + \frac{c_{s_1}^2}{\gamma-1} = \frac{\rho_1}{\rho_2(\gamma-1)} c_s^2 + \frac{\gamma}{(\gamma-1)} \frac{\rho_1 v}{\rho_2}$$

multiply by $\frac{2(\gamma-1)}{c_{s_1}^2}$; and define $M_1^2 \equiv \frac{v_1^2}{c_{s_1}^2}$

$$M_1^2(\gamma-1)\left(1-\frac{\rho_1^2}{\rho_2^2}\right)+2 = \frac{2\rho_1}{\rho_2} + 2\gamma\frac{\rho_1}{\rho_2}M_1^2\left(1-\frac{\rho_1}{\rho_2}\right)$$

$$\Rightarrow M_1^2(\gamma-1)\cancel{\left(1-\frac{\rho_1}{\rho_2}\right)}\left(1+\frac{\rho_1}{\rho_2}\right)+2\cancel{\left(1-\frac{\rho_1}{\rho_2}\right)} + 2\gamma\frac{\rho_1}{\rho_2}M_1^2\cancel{\left(1-\frac{\rho_1}{\rho_2}\right)}$$

$$\therefore (\gamma-1)M_1^2\left(1+\frac{\rho_1}{\rho_2}\right)+2 = 2\gamma M_1^2\frac{\rho_1}{\rho_2}$$

$$\Rightarrow M_1^2(\gamma-1)+2 = \frac{\rho_1}{\rho_2}M_1^2\left[2\gamma-(\gamma-1)\right]$$

$$\Rightarrow \frac{\rho_2}{\rho_1} = \frac{M_1^2(\gamma+1)}{M_1^2(\gamma-1)+2}$$

3b) from (5) in a), we have

$$\frac{P_2}{P_1} = 1 + \frac{\rho_1 v_1^2}{P_1}\left(1 - \frac{\rho_1}{\rho_2}\right)$$

but $C_{s_1}^2 = \frac{\gamma_1 P_1}{\rho_1}$

$$\Rightarrow \frac{v_1^2}{C_{s_1}^2} = M_1^2 = \frac{1}{\gamma} \frac{\rho_1 v_1^2}{P_1}$$

$$\therefore \frac{P_2}{P_1} = 1 + \gamma M_1^2\left(1 - \frac{\rho_1}{\rho_2}\right)$$

but

$$\frac{\rho_1}{\rho_2} = \frac{2 + (\gamma - 1) M_1^2}{M_1^2(\gamma+1)}$$

$$\Rightarrow \frac{P_2}{P_1} = 1 + \gamma \cancel{M_1^2}\left(\frac{M_1^2(\gamma+1) - 2 - (\gamma-1)M_1^2}{\cancel{M_1^2}(\gamma+1)}\right)$$

$$= 1 + \gamma\left(\frac{M_1^2(\cancel{\gamma}+1-\cancel{\gamma}+1) - 2}{\gamma+1}\right)$$

$$= \frac{\gamma + 1 + 2\gamma M_1^2 - 2}{\gamma+1}$$

$$= \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}$$

3c)

We have:

$$c_s^2 = \frac{k_B T}{\mu M_H} = \frac{\gamma P}{\rho}$$

$$\Rightarrow \frac{T_2}{T_1} = \frac{P_2 \rho_1}{P_1 \rho_2}$$

$$= \frac{[2\gamma M_1^2 - (\gamma-1)][2 + (\gamma-1)M_1^2]}{(\gamma+1)^2 M_1^2}$$

4a)

if $\sigma_{ij} = \sigma_{ji}$,

$$\sum_i^n \sum_j^n \sigma_{ij}(x_i - x_j) =$$

$$= \frac{1}{2}\sum_i \sum_j \sigma_{ij}(x_i - x_j) + \frac{1}{2}\sum_i \sum_j \sigma_{ij}(x_i - x_j)$$

$$= \frac{1}{2}\sum_i \sum_j \sigma_{ij}(x_i - x_j) + \underbrace{\frac{1}{2}\sum_i \sum_j \sigma_{ji}(x_j - x_i)}_{\text{switch } i \leftrightarrow j}$$

but $\sigma_{ji} = \sigma_{ij}$

$$\Rightarrow \sum_i \sum_j \sigma_{ij}(x_i - x_j) = \frac{1}{2}\sum_i \sum_j \sigma_{ij}(x_i - x_j) - \frac{1}{2}\sum_i \sum_j \sigma_{ij}(x_i - x_j)$$

$$= 0$$

And so in the case of $\vec{r}$, we have

$$\sum_i \sum_j \sigma_{ij}(\vec{r}_i - \vec{r}_j) = 0$$

4b) If linear momentum is conserved:

$$\frac{d}{dt}\left(\sum_i m_i \vec{v}_i \right) = 0$$

we have

$$\frac{d}{dt}\left(\sum_i m_i \vec{v}_i \right) = \sum_i m_i \frac{d\vec{v}_i}{dt}$$

$w_{ij} = (r_i - r_j)$ f

$$= -\sum_i \sum_j m_i m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla_i W_{ij}$$

We have

$$\nabla_i W_{ij} = \underbrace{(\vec{r}_i - \vec{r}_j)}_{\text{antisymetric}} \underbrace{f_{ij}}_{\text{Symetric}}$$

$$\Rightarrow \nabla_i W_{ij} = -\nabla_j W_{ji}$$

Since $m_i m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right)$ is symmetric,

symetric,

$$\underbrace{m_i m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j} \right)}_{\text{symmetric}} \underbrace{\nabla_i W_{ij}}_{\text{antisymetric}}$$ is anti symetric

is anti symetric

$\therefore$ we have $\frac{d}{dt}\left(\sum_i m_i \vec{v}_i \right) = \vec{0}$, ii)

$$\frac{d}{dt}\left(\sum_i m_i \vec{r}_i \times \vec{v}_i\right)$$

$$= \sum_i m_i \frac{d}{dt}\left(\vec{r}_i \times \vec{v}_i\right)$$

$$= \sum_i m_i \left(\vec{r}_i \times \frac{d\vec{v}_i}{dt} + \frac{d\vec{r}_i}{dt} \times \vec{v}_i\right)$$

but $\frac{d\vec{r}_i}{dt} \times \vec{v}_i = \vec{v}_i \times \vec{v}_i$

$$= \vec{0}$$

$$\Rightarrow \frac{d}{dt}\left(\sum_i m_i \vec{r}_i \times \vec{v}_i\right) = \sum_i m_i \vec{r}_i \times \frac{dv_i}{dt}$$

$$= -\sum_i \sum_j m_i m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right) \vec{r}_i \times \nabla_i W_{ij}$$

but,

$$\vec{r}_i \times \nabla_i W_{ij} = \vec{r}_i \times (\vec{r}_i - \vec{r}_j) f_{ij}$$

$$= \cancel{\vec{r}_i \times \vec{r}_i}^{\,0} f_{ij} - \underbrace{\vec{r}_i \times \vec{r}_j}_{\text{anti symetric}} \underbrace{f_{ij}}_{\text{Symmetric}}$$

Since every other factor in the term is symmetric, the whole term is anti symmetric, and so the sum is $\vec{0}$.

$\therefore$ Angular Momentum is conserved.

5) $$\frac{de_i}{dt} = \frac{d\left(\frac{1}{2}v_i^2 + u_i\right)}{dt}$$

$$= \vec{v}_i \cdot \frac{d\vec{v}_i}{dt} + \frac{du_i}{dt}$$

$$\vec{v}_i \cdot \frac{d\vec{v}_i}{dt} = -\sum_j m_j \left(\frac{P_i \vec{v}_i}{\rho_i^2} + \frac{P_j \vec{v}_i}{\rho_j^2} \right) \cdot \nabla_i W_{ij}$$

$$\frac{du_i}{dt} = \sum_j m_j \left(\frac{P_i (\vec{v}_i - \vec{v}_j)}{\rho_i^2} \right) \cdot \nabla_i W_{ij}$$

$$\therefore \frac{de_i}{dt} = -\sum_j m_j \left(\frac{P_i \cancel{\vec{v}_i}}{\rho_i^2} + \frac{P_j \vec{v}_i}{\rho_j^2} - \frac{P_i (\cancel{\vec{v}_i} - \vec{v}_j)}{\rho_i^2} \right) \cdot$$

$$= -\sum_j m_j \left(\frac{P_i \vec{v}_j}{\rho_i^2} + \frac{P_j \vec{v}_i}{\rho_j^2} \right) \cdot \nabla_i W_{ij}$$

as required.

b)

$$\frac{dE}{dt} = \sum_i m_i \frac{de_i}{dt}$$

$$= -\sum_i \sum_j \underbrace{m_i m_j \left(\frac{P_i \vec{v}_j}{\rho_i^2} + \frac{P_j \vec{v}_i}{\rho_j^2} \right)}_{\text{symmetric}} \underbrace{\cdot \nabla_i W_{ij}}_{\text{antisymmetric}}$$

$$\therefore \frac{dE}{dt} = 0$$

And so total energy is conserved.