3b)

We have:

$$\rho_g = \rho_{g0} + \rho_{g1}, \quad \rho_d = \rho_{d0} + \rho_{d1}, \quad P_g = P_{g0} + P_{g1}, \quad P_d = 0,$$

$$\vec{v}_g = \vec{v}_{g1}, \quad \vec{v}_d = \vec{v}_{d1}$$

Where subscript "0" denotes background value, and "1" denotes <u>small</u> perturbation.

Our fluid equations are:

$$\frac{\partial \rho_g}{\partial t} + \nabla \cdot (\rho_g \vec{v}_g) = 0 \qquad (1)$$

$$\frac{\partial \rho_d}{\partial t} + \nabla \cdot (\rho_d \vec{v}_d) = 0 \qquad (2)$$

$$\rho_g \left[\frac{\partial \vec{v}_g}{\partial t} + (\vec{v}_g \cdot \nabla) \vec{v}_g \right] = -\nabla P_g + K(\vec{v}_d - \vec{v}_g) \qquad (3)$$

$$\rho_d \left[\frac{\partial \vec{v}_d}{\partial t} + (\vec{v}_d \cdot \nabla) \vec{v}_d \right] = -K(\vec{v}_d - \vec{v}_g) \qquad (4)$$

After the perturbation, we have:

(1):

$$\frac{\partial \rho_{g1}}{\partial t} + \nabla \cdot (\rho_{g0} \vec{v}_{g1}) = 0 \Rightarrow \frac{\partial \rho_{g1}}{\partial t} + \rho_{g0} \nabla \cdot (\vec{v}_{g1}) = 0 \qquad (5)$$

(2):

$$\frac{\partial \rho_{d1}}{\partial t} + \rho_{d0} \nabla \cdot (\vec{v}_{d1}) = 0 \qquad (6)$$

$$\frac{\partial \vec{v}_{g_1}}{\partial t} = -\frac{\nabla P_{g_1}}{\rho_{g_0}} + \frac{K}{\rho_{g_0}}(\vec{v}_{d_1} - \vec{v}_{g_1}) \quad (7)$$

$$\frac{\partial \vec{v}_{d_1}}{\partial t} = \frac{-K}{\rho_{d_0}}(\vec{v}_{d_1} - \vec{v}_{g_1}) \quad (8)$$

(ignoring second order terms)

taking the div of (7) and (8), we get

$$\rho_{g_0} \frac{\partial}{\partial t}(\nabla \cdot \vec{v}_{g_1}) = -c_s^2 \nabla^2 \rho_{g_1} + K \nabla \cdot (\Delta \vec{v}) \quad (9)$$

and

$$\rho_{d_0} \frac{\partial}{\partial t}(\nabla \cdot \vec{v}_{d_1}) = -K \nabla \cdot (\Delta \vec{v}) \quad (10)$$

(Since $P_{g_1} = c_s^2 \rho_{g_1}$)

(9) + (10):

$$\rho_{g_0} \frac{\partial}{\partial t}(\nabla \cdot \vec{v}_{g_1}) + \rho_{d_0} \frac{\partial}{\partial t}(\nabla \cdot \vec{v}_{d_1}) = -c_{s_0}^2 \nabla^2 \rho_{g_1}$$

Sub in $\frac{\partial}{\partial t}$ of (5) and (6):

$$\frac{\partial^2 \rho_{g_1}}{\partial t^2} + \frac{\partial^2 \rho_{d_1}}{\partial t^2} = c_{s_0}^2 \nabla^2 \rho_{g_1} \quad (11)$$

On the other hand:

(9) − (10)

$$\rho_{g_0} \frac{\partial}{\partial t}(\nabla \cdot \vec{v}_{g_1}) - \rho_{d_0} \frac{\partial}{\partial t}(\nabla \cdot \vec{v}_{d_1}) = -c_{s_0}^2 \nabla^2 \rho_{g_1} + 2K \nabla \cdot (\Delta \vec{v})$$

$$= -c_{s_0}^2 \nabla^2 \rho_{g_1} + 2K \nabla \cdot (\vec{v}_{d_1}) - 2K \nabla \cdot (\vec{v}_{g_1})$$

Substitute $\frac{\partial}{\partial t}$(5) and $\frac{\partial}{\partial t}$(6) :

$$\frac{\partial^2 \rho_{d_1}}{\partial t^2} - \frac{\partial^2 \rho_{g_1}}{\partial t^2} = -C_{s_0}^2 \nabla^2_{\rho_{g_1}} - \frac{2k}{\rho_{d_0}} \frac{\partial \rho_{d_1}}{\partial t} + \frac{2k}{\rho_{g_0}} \frac{\partial \rho_{g_1}}{\partial t}$$

$$\frac{\partial^2 \rho_{d_1}}{\partial t^2} = \frac{\partial^2 \rho_{g_1}}{\partial t^2} \cancel{-} C_{s_0}^2 - \frac{2k}{\rho_{d_0}} \frac{\partial \rho_{d_1}}{\partial t} + \frac{2k}{\rho_{g_0}} \frac{\partial \rho_{g_1}}{\partial t}$$

Sub in (11) :

$$2 \frac{\partial^2 \rho_{g_1}}{\partial t} - C_s^2 \nabla^2 \rho_{g_1} - \frac{2k}{\rho_{d_0}} \frac{\partial \rho_{d_1}}{\partial t} + \frac{2k}{\rho_{g_0}} \frac{\partial \rho_{g_1}}{\partial t} = C_{s_0}^2 \nabla^2 \rho_{g_1}$$

$$\frac{\partial^2 \rho_{g_1}}{\partial t} = C_s^2 \nabla^2 \rho_{g_1} + \frac{k}{\rho_{d_0}} \frac{\partial \rho_{d_1}}{\partial t} - \frac{k}{\rho_{g_0}} \frac{\partial \rho_{g_1}}{\partial t}$$

$$\Rightarrow \frac{\partial^3 \rho_{g_1}}{\partial t^3} = C_s^2 \frac{\partial}{\partial t}(\nabla \rho_{g_1}) + \frac{k}{\rho_{d_0}}\left[C_s^2 \nabla^2_{\rho_{g_1}} - \frac{\partial^2 \rho_{g_1}}{\partial t} \right] - \frac{k}{\rho_{g_0}} \frac{\partial^2 \rho_{g_1}}{\partial t^2}$$

$$\Rightarrow \frac{\partial^3 \rho_{g_1}}{\partial t^3} + k\left(\frac{1}{\rho_{d_0}} + \frac{1}{\rho_{g_0}}\right) \frac{\partial^2 \rho_{g_1}}{\partial t^2} - \frac{k}{\rho_{d_0}}\left[C_{s_0}^2 \nabla^2 \rho_{g_1} \right] - C_s^2 \frac{\partial}{\partial t}(\nabla^2 \rho_{g_1}) = 0$$

Assume perturbations of the form:

$$\rho_{g_1} = D e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

we have:

$$\nabla^2 \rho_{g_1} = -k^2 \rho_{g_1}, \quad \frac{\partial^2}{\partial t^2}(\rho_{g_1}) = -\omega^2 \rho_{g_1}, \quad \frac{\partial^3}{\partial t^3}(\rho_{g_1}) = i\omega^3 \rho_{g_1}$$

$$\Rightarrow i\omega^3 \cancel{\rho_{g_1}} - \omega^2 \cancel{\rho_{g_1}}\left(k\left[\frac{1}{\rho_{d_0}} + \frac{1}{\rho_{g_0}}\right]\right) + \frac{k}{\rho_{d_0}} C_s^2 k^2 \cancel{\rho_{g_1}} - C_s^2 i\omega k^2 \cancel{\rho_{g_1}} = 0$$

But $k\left[\frac{1}{\rho_{d_0}} + \frac{1}{\rho_{g_0}}\right] = k\left[\frac{\rho_{d_0}+\rho_{g_0}}{\rho_{d_0}\rho_{g_0}}\right] = \frac{1}{t_s}$

And

$$\frac{k c_s^2}{\rho_{d_0}} = \frac{\rho_g c_s^2}{t_s(\rho_g+\rho_d)} = \frac{c_s^2}{t_s(1+\rho_d/\rho_g)} = \frac{\tilde{c}_s^2}{t_s}$$

$$\Rightarrow \quad \omega i(\omega^2 - c_{s_0}^2 k^2) - \frac{1}{t_s}(\omega^2 - \tilde{c}_{s_0}^2 k^2) = 0$$

$$\Rightarrow (\omega^2 - k^2 c_s^2) + \frac{i}{\omega t_s}(\omega^2 - \tilde{c}_s^2 k^2) = 0$$

As required