GR Assignment 1 Cameron Smith

1) a

Consider a sphere ~~embed~~ inside Euclidean three space in spherical coordinates, the euclidean metric is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

On the surface of the sphere, we have constant radius $r = R$, and so we have:

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2$$

In matrix form, we have:

$$\cancel{g_{ij}}\; g_{\mu\nu} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix},$$

with coordinates (θ, ϕ)

For a unit sphere $(R=1)$, this becomes:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

Since the metric is diagonal, the inverse metric is:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$$

The connection coefficients are given by:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma} \right)$$

1a) Con't

however, since our metric is diagonal, we have

$g^{\lambda\sigma} = 0$ for $\lambda \neq \sigma$

Futhermore, the only non-vanishing derivative is:

$$g_{\varphi\varphi,\theta} = \frac{\partial}{\partial\theta}\left(\sin^2\theta\right)$$

$$= 2\sin\theta\cos\theta$$

Therefore, the only non-zero connection coefficients are:

$$\Gamma^{\theta}_{\varphi\varphi} = \frac{1}{2} g^{\theta\theta}\left(\cancel{g_{\theta\varphi,\varphi}}^{0} + \cancel{g_{\theta\varphi,\varphi}}^{0} - g_{\varphi\varphi,\theta}\right)$$

$$= -\frac{1}{2}(1)(2\sin\theta\cos\theta)$$

$$= -\sin\theta\cos\theta$$

and,

$$\Gamma^{\varphi}_{\varphi,\theta} = \Gamma^{\varphi}_{\theta,\varphi} = \frac{1}{2} g^{\varphi\varphi}\left(\cancel{g_{\varphi\theta,\varphi}}^{0} + g_{\varphi\varphi,\theta} - \cancel{g_{\theta\varphi,\varphi}}^{0}\right)$$

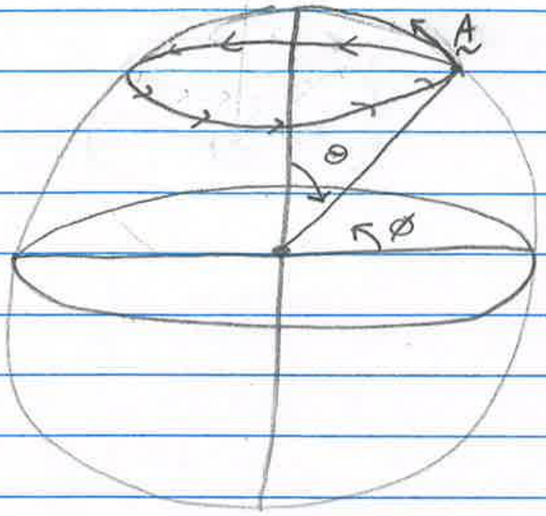
$$= \frac{1}{2}\left(\frac{1}{\sin^2\theta}\right)(2\sin\theta\cos\theta)$$

$$= \frac{\cos\theta}{\sin\theta}$$

With all other connection coefficients going to zero.

1b)

Consider transporting the tangent vector A around a parallel with fixed latitude θ:



On the manifold, $\underset{\sim}{A}$ can be expressed as:

$$\underset{\sim}{A} = A^{\varphi}\partial_{\varphi} + A^{\theta}\partial_{\theta}$$

As we <u>parallel transport</u> $\underset{\sim}{A}$ around the path, we have:

$$\nabla_{\varphi}\underset{\sim}{A} = 0$$

and so,

$$\partial_{\varphi} A^{\mu} + \Gamma^{\mu}_{\varphi\nu} A^{\nu} = 0$$

$$\Rightarrow \partial_{\varphi} A^{\theta} + \cancelto{0}{\Gamma^{\theta}_{\varphi\theta}} A^{\theta} + \Gamma^{\theta}_{\varphi\varphi} A^{\varphi} = 0$$

$$\Rightarrow \partial_{\varphi} A^{\theta} - \sin\theta\cos\theta\, A^{\varphi} = 0 \qquad (1)$$

and:

$$\partial_{\varphi} A^{\varphi} + \Gamma^{\varphi}_{\varphi\theta} A^{\theta} + \cancelto{0}{\Gamma^{\varphi}_{\varphi\varphi}} A^{\varphi} = 0$$

$$\Rightarrow \partial_{\varphi} A^{\varphi} + \frac{\cos\theta}{\sin\theta} A^{\theta} = 0 \qquad (2)$$

1b) Cont

(1) gives:

$$A^\varphi = \frac{1}{\sin\theta\cos\theta}\partial_\varphi A^\theta$$

Sub into (2):

$$\partial_\varphi\left(\frac{1}{\sin\theta\cos\theta}\partial_\varphi A^\theta\right) + \frac{\cos\theta}{\sin\theta}A^\theta = 0$$

$$\Rightarrow \frac{\partial^2}{\partial\varphi^2}A^\theta + \cos^2\theta A^\theta = 0$$

$$\Rightarrow A^\theta = a\cos(\cos\theta\varphi) + b\sin(\cos\theta\varphi) \quad (3)$$

for some constants a, b.

Similarly (2) gives:

$$A^\theta = -\frac{\sin\theta\partial_\varphi A^\varphi}{\cos\theta}$$

Sub into (1):

$$\partial_\varphi\left(-\frac{\sin\theta}{\cos\theta}A^\varphi\right) - \sin\theta\cos\theta A^\varphi = 0$$

$$\Rightarrow \frac{\partial^2 A^\varphi}{\partial\varphi^2} + \cos^2\theta A^\varphi = 0$$

1b) Cont

$$\Rightarrow A^{\varphi} = c\cos(\cos\theta\,\varphi) + d\sin(\cos\theta\,\varphi) \quad (4)$$

Arbitrarily ~~setting our s~~ choosing $\varphi = 0$ to be our starting point, we can find our constants a, b, c and d with respect to the starting point:

$$\underset{\sim}{A}(\varphi=0) = \begin{pmatrix} A^{\varphi}(0) \\ A^{\theta}(0) \end{pmatrix}$$

From (4), we get:

$$c = A^{\varphi}(0)$$

And from (3), we get:

$$a = A^{\theta}(0)$$

Subbing (3), (4) into equations (1) and (2), we get:

$$-A^{\theta}(0)\cancel{\cos}\theta\sin(\cos\theta\,\varphi) + b\cancel{\cos}\theta\cos(\cos\theta\varphi)$$
$$-\sin\theta\,\cancel{\cos}\theta\left(A^{\varphi}(0)\cos(\cos\theta\varphi) + d\sin(\cos\theta\,\varphi)\right) = 0$$

at $\varphi = 0$,

$$b - A^{\varphi}(0)\sin\theta = 0$$

$$\Rightarrow b = A^{\varphi}(0)\sin\theta$$

1b) Cont

And,

$$-A^{\varphi}(0)\cancel{\cos}\Theta \sin(\cos\Theta\,\varphi) + d\,\cancel{\cos}\Theta \cos(\cos\Theta\,\varphi)$$

$$+\frac{\cancel{\cos}\Theta}{\sin\Theta}\left(A^{\Theta}(0)\cos(\cos\Theta\,\varphi) + b\sin(\cos\Theta\,\varphi)\right) = 0$$

at $\varphi = 0$, this becomes:

$$d + \frac{A^{\Theta}(0)}{\sin\Theta} = 0$$

$$\Rightarrow d = -\frac{A^{\Theta}(0)}{\sin\Theta}$$

and so (3) and (4) become:

$$A^{\varphi}(\varphi) = A^{\varphi}(0)\cos(\cos\Theta\,\varphi) - \frac{A^{\Theta}(0)}{\sin\Theta}\sin(\cos\Theta\,\varphi),$$

$$\Rightarrow \sin\Theta\, A^{\varphi}(\varphi) = \sin\Theta\, A^{\varphi}(0)\cos(\cos\Theta\,\varphi) - A^{\Theta}(0)\sin(\cos\Theta\,\varphi)$$

and,

$$A^{\Theta}(\varphi) = A^{\varphi}(0)\sin\Theta\,\sin(\cos\Theta\,\varphi) + A^{\Theta}(0)\cos(\cos\Theta\,\varphi)$$

Making a change of variables: $\hat{A}^{\Theta} = A^{\Theta}$ and $\hat{A}^{\varphi} = A^{\varphi}\sin\Theta$, we get:

$$\hat{A}^{\varphi} = \hat{A}^{\varphi}(0)\cos(\cos\Theta\,\varphi) - \hat{A}^{\Theta}(0)\sin(\cos\Theta\,\varphi)$$

$$\hat{A}^{\Theta}(\varphi) = \hat{A}^{\varphi}(0)\sin(\cos\Theta\,\varphi) + \hat{A}^{\Theta}(0)\cos(\cos\Theta\,\varphi)$$

1b) Cont

This can be represented by the following matrix ~~rotat~~ equation:

$$\begin{pmatrix} \hat{A}^{\varphi} \\ \hat{A}^{\theta} \end{pmatrix} = \begin{pmatrix} \cos(\cos\theta\varphi) & -\sin(\cos\theta\varphi) \\ \sin(\cos\theta\varphi) & \cos(\cos\theta\varphi) \end{pmatrix} \begin{pmatrix} \hat{A}^{\varphi}(0) \\ \hat{A}^{\theta}(0) \end{pmatrix}$$

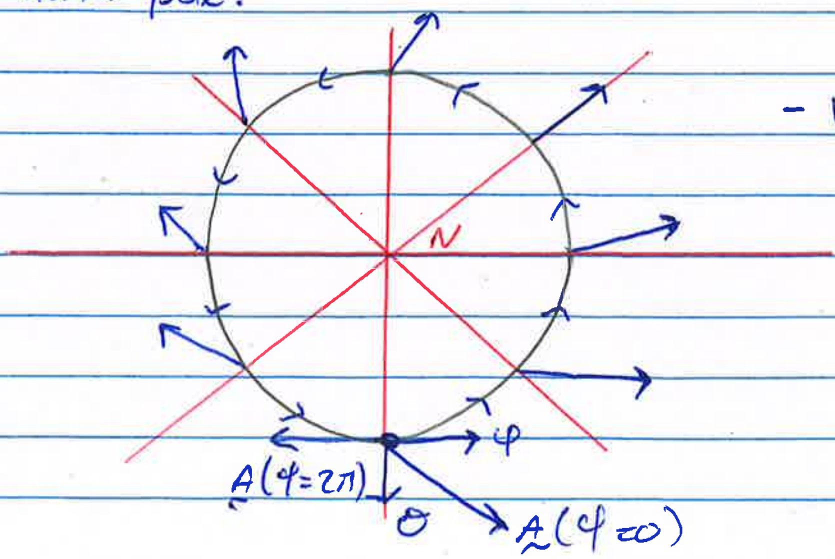
which represents a rotation ~~of~~ by angle $\varphi\cos\theta$ from our starting point.

$\therefore$ the vector rotates on the manifold by an angle of $2\pi\cos\theta$ while being transported around the sphere. However, since the manifold itself rotates along the circle anti clockwise, the vector will rotate by a total angle of

$$2\pi - 2\pi\cos\theta = 2\pi(1-\cos\theta)$$

In other words, the vector needs to "catch up" to the rotation of the sphere and will fall behind ~~hind~~ by the angle : $2\pi(1-\cos\theta)$

near the north pole:



- Vector is rotating but lags behind rotation of sphere

2a)

We have the coordinate transformation:

$$x = uv\cos\phi$$
$$y = uv\sin\phi$$
$$z = \tfrac{1}{2}(u^2 - v^2)$$

where $(x^1, x^2, x^3) = (x, y, z)$ and

$$(x'^1, x'^2, x'^3) = (u, v, \phi)$$

The Jacobian is given by:

$$\frac{\partial x^\alpha}{\partial x'^\beta} = \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^1}{\partial x'^3} \\ \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^2}{\partial x'^3} \\ \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^2} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix}$$

$$= \begin{pmatrix} v\cos\phi & u\cos\phi & -uv\sin\phi \\ v\sin\phi & u\sin\phi & uv\cos\phi \\ u & -v & 0 \end{pmatrix}$$

Using mathematica, we can find the inverse Jacobian:

$$\frac{\partial x'^\alpha}{\partial x^\beta} = \begin{pmatrix} \frac{v\cos\phi}{u^2+v^2} & \frac{v\sin\phi}{u^2+v^2} & \frac{u}{u^2+v^2} \\ \frac{u\cos\phi}{u^2+v^2} & \frac{u\sin\phi}{u^2+v^2} & -\frac{v}{u^2+v^2} \\ -\frac{\sin\phi}{uv} & \frac{\cos\phi}{uv} & 0 \end{pmatrix}$$

2b)

We have:

$$\frac{\partial}{\partial x'^{\beta}} = \frac{\partial x^{\alpha}}{\partial x'^{\beta}} \frac{\partial}{\partial x^{\alpha}}$$

we have

$$\frac{\partial}{\partial x^{\alpha}} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),$$ and

$$\frac{\partial}{\partial x'^{\beta}} = \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial \phi} \right),$$ and so we get:

$$\frac{\partial}{\partial u} = v\cos\phi \frac{\partial}{\partial x} + v\sin\phi \frac{\partial}{\partial y} + u \frac{\partial}{\partial z},$$

$$\frac{\partial}{\partial v} = u\cos\phi \frac{\partial}{\partial x} + u\sin\phi \frac{\partial}{\partial y} - v \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial \phi} = -uv\sin\phi \frac{\partial}{\partial x} + uv\cos\phi \frac{\partial}{\partial y}$$

Where we have used the Jacobian as found in 2a).

c)

The metric in Euclidean three-space is given by:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To get the metric in the primed coordinates, we use:

$$g_{\alpha'\beta'} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} g_{\mu\nu}$$

2c) Cont

$\therefore$ we find:

$$g_{1'1'} = \frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{1'}} g_{11} + \frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{1'}} g_{22} + \frac{\partial x^3}{\partial x^{1'}}\frac{\partial x^3}{\partial x^{1'}} g_{33}$$

$$= v^2\cos^2\phi + v^2\sin^2\phi + u^2$$

$$= u^2 + v^2$$

similarly, the other components of $g_{\mu'\nu'}$ are:

$$g_{1'2'} = vu\cos^2\phi + vu\sin^2\phi - vu$$

$$= 0$$

$$g_{1'3'} = -uv^2\cos\phi\sin\phi + uv^2\cos\phi\sin\phi + 0$$

$$= 0$$

$$g_{2'1'} = g_{1'2'} = 0$$

$$g_{2'2'} = u^2\cos^2\phi + u^2\sin^2\phi + v^2$$

$$= u^2 + v^2$$

$$g_{2'3'} = -u^2v\cos\phi\sin\phi + u^2v\cos\phi\sin\phi + 0$$

$$= 0$$

2c) Cont

$$g_{3'1'} = g_{1'3'} = 0$$

$$g_{3'2'} = g_{2'3'} = 0$$

$$g_{3'3'} = u^2v^2 \sin^2\phi + u^2v^2\cos^2\phi + 0$$

$$= u^2 + v^2$$

So we find:

$$g_{\mu'\nu'} = \begin{pmatrix} u^2+v^2 & 0 & 0 \\ 0 & u^2+v^2 & 0 \\ 0 & 0 & u^2v^2 \end{pmatrix}$$

Since it's diagonal, we also have inverse metric:

$$g^{\mu'\nu'} = \begin{pmatrix} \frac{1}{u^2+v^2} & 0 & 0 \\ 0 & \frac{1}{u^2+v^2} & 0 \\ 0 & 0 & \frac{1}{u^2v^2} \end{pmatrix}$$

2d)

The Divergence is given by:

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda,$$

but we have

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{|g|}} \partial_\lambda \sqrt{|g|}$$

2d) Cont

Using the metric for the new Coordinates we calculated in 2c), we get

$$\sqrt{|g|} = \sqrt{|\det(g_{\mu\nu})|}$$

$$= \sqrt{(u^2+v^2)^2 u^2 v^2}$$

$$= (u^2+v^2)uv$$

$$\therefore \nabla_\mu V^\mu = \partial_\mu V^\mu + \frac{1}{(u^2+v^2)uv} \partial_\lambda [(u^2+v^2)uv] V^\lambda$$

$$= \partial_u V^u + \partial_v V^v + \partial_\phi V^\phi + \frac{3u^2v+v^3}{(u^2+v^2)uv} V^u$$

$$+ \frac{u^3+3uv^2}{(u^2+v^2)uv} V^v + 0$$

$$\Rightarrow \nabla_\mu V^\mu = \partial_\mu V^\mu + \frac{3u^2+v^2}{(u^2+v^2)u} V^u + \frac{u^2+3v^2}{(u^2+v^2)v} V^v$$

ii)

We now want to find the laplacian:

$$\nabla_\mu \nabla^\mu J$$

We have:

$$\nabla^\mu J = g^{\mu\nu} \nabla_\nu J$$

2d) Cont

But J is a Scalar,

$\Rightarrow \nabla^\mu J = g^{\mu\nu}\partial_\nu J$

$\therefore$ from part i), we have:

$$\nabla_\mu \nabla^\mu J = \partial_\mu\left(g^{\mu\nu}\partial_\nu J\right) + \frac{3u^2+v^2}{(u^2+v^2)u} g^{u\mu}\partial_\mu J + \frac{u^2+3v^2}{(u^2+v^2)v} g^{v\mu}\partial_\mu J$$

but off diagonal components of $g^{\mu\nu}$ are 0, so this reduces to:

$$\nabla_\mu \nabla^\mu J = \partial_u\left(g^{uu}\partial_u J\right) + \partial_v\left(g^{vv}\partial_v J\right) + \partial_\phi\left(g^{\phi\phi}\partial_\phi J\right)$$

$$+ \frac{3u^2+v^2}{(u^2+v^2)u} g^{uu}\partial_u J + \frac{u^2+3v^2}{(u^2+v^2)v} g^{vv}\partial_v J$$

Using the product rule, we find:

$$\nabla_\mu \nabla^\mu J = \frac{-2u}{(u^2+v^2)^2}\partial_u J + \frac{1}{u^2+v^2}\frac{\partial^2 J}{\partial u^2} - \frac{2v}{(u^2+v^2)^2}\partial_v J + \frac{1}{u^2+v^2}\frac{\partial^2 J}{\partial v^2}$$

$$+ 0 + \frac{1}{u^2v^2}\frac{\partial^2 J}{\partial \phi^2} + \frac{3u^2+v^2}{(u^2+v^2)^2 u}\partial_u J + \frac{u^2+3v^2}{(u^2+v^2)^2 v}\partial_v J$$

$$= \frac{u^2+v^2}{(u^2+v^2)^2 u}\partial_u J + \frac{u^2+v^2}{(u^2+v^2)^2 v}\partial_v J + \frac{1}{u^2+v^2}\left(\frac{\partial^2 J}{\partial u^2} + \frac{\partial^2 J}{\partial v^2}\right) + \frac{1}{u^2v^2}\frac{\partial^2 J}{\partial \phi^2}$$

$$= \frac{\partial_u J}{(u^2+v^2)u} + \frac{\partial_v J}{(u^2+v^2)v} + \frac{1}{u^2+v^2}\left(\frac{\partial^2 J}{\partial u^2} + \frac{\partial^2 J}{\partial v^2}\right) + \frac{1}{u^2v^2}\frac{\partial^2 J}{\partial \phi^2}$$