Schwarzschild Metric Work Sheet Cameron Smith

1a)

The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

Where $d\Omega^2$ describes the change in solid angle Ω and is given by:

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

3a)

We require a metric that satisfies the following:

1. The metric is spherically symmetric
2. The metric is static (no time dependence)
3. The metric is invariant under time reversal

1. and 2. are satisfied if the metric depends only on the radius, r, (not t, θ or φ).

A metric that satisfies these assumptions is:

$$ds^2 = -f_t(r)dt^2 + f_r(r)dr^2 + f_\Omega(r)d\Omega^2,$$

$$\Rightarrow g_{\mu\nu} = \begin{pmatrix} -f_t(r) & 0 & 0 & 0 \\ 0 & f_r(r) & 0 & 0 \\ 0 & 0 & f_\Omega(r)r^2 & 0 \\ 0 & 0 & 0 & f_\Omega(r)\sin^2\theta \end{pmatrix}$$

3b)

We have set any off-diagonal elements to zero. to see why this is the case, first consider reversing time:

$t \to -t$

The off diagonal components would then transform like:

$dt\,dr \to -dt\,dr,$
$dt\,d\theta \to -dt\,d\theta$
$dt\,d\phi \to -dt\,d\phi$

However we require the metric to be invariant under time reversal, and so we require the coefficients:

$$g_{tr} = g_{rt} = g_{t\theta} = g_{\theta t} = g_{t\phi} = g_{\phi t} = 0$$

futhermore, spherical symmetry demands that we maintain the form of $d\Omega^2$, i.e. we don't have any cross terms:

$$g_{r\theta} = g_{\theta r} = g_{r\phi} = g_{\phi r} = g_{\theta\phi} = g_{\phi\theta} = 0$$

If any of these were non-zero, the metric would change under rotation, i.e. we would lose our spherical symmetry.

$\therefore$ Any cross terms must be zero.

3c)

the metric:

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + e^{2\gamma(r)} r^2 d\Omega^2$$

satisfies the following symmetries:

1. spherical symmetry. We have no cross terms involving θ or φ, and we have $g_{\varphi\varphi} = \sin^2\theta \, g_{\theta\theta}$. Additionally, there is no θ or φ dependence outside $d\Omega^2$.

2. static. The metric has no time dependence.

3. Invariant under time reversal. time reversal $(t \to -t)$ does not change the metric (no t dependence, no $dt\,dx^i$ terms).

Futhermore, this is completely general as $\alpha(r)$, $\beta(r)$ and $\gamma(r)$ can completely determine the metric.

3d)

Define the new coordinate:

$$\bar{r} = e^{\gamma(r)} r$$

This substitution allows us to reduce the number of free parameters (by removing $\gamma(r)$), and simplify the metric.

3e)

We have associated basis one-form:

$$d\bar{r} = \frac{\partial \bar{r}}{\partial r} dr + \frac{\partial \bar{r}}{\partial \gamma} d\gamma$$

3e) Cont

$\Rightarrow d\bar{r} = e^{\gamma} dr + r e^{\gamma} d\gamma$

but $d\gamma = \frac{d\gamma}{dr} dr$

$\Rightarrow d\bar{r} = \left(1 + r\frac{d\gamma}{dr}\right) e^{\gamma} dr$

3f)

We therefore have:

$dr = \left(1 + r\frac{d\gamma}{dr}\right)^{-1} e^{-\gamma} d\bar{r}$,

$r = e^{-\gamma} \bar{r}$

We therefore get the metric:

$$ds^2 = -e^{2\alpha(r)} dt^2 + \left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta - 2\gamma(r)} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

where $r = r(\bar{r})$

3g)

make the relabelling:

$\bar{r} \to r$,

and define

$$e^{2\beta'(r)} = \left(1 + r\frac{d\gamma(r)}{dr}\right)^{-2} e^{2\beta(r) - 2\gamma(r)}$$

we therefore get metric:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta'(r)}dr^2 + r^2 d\Omega^2$$

Therefore by making a few changes of variables, we can absorb γ without any loss of generality. We can therefore chose $\gamma = 0$ initially, giving:

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2 d\Omega^2$$

4a)

We want to find a vacuum solution, so we want to solve the vacuum field equations. To do this, we first need to write down the metric and calculate our christofel symbols.

b)

The first step is to write down our metric:

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

4c)

From mathematica (Part 4), we have non-zero Christoffel symbols:

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \partial_r \alpha(r)$$

$$\Gamma^r_{tt} = e^{2\alpha(r) - 2\beta(r)} \partial_r \alpha(r)$$

$$\Gamma^r_{rr} = \partial_r \beta(r)$$

$$\Gamma^r_{\theta\theta} = -r e^{-2\beta(r)}$$

$$\Gamma^r_{\phi\phi} = -r e^{-2\beta(r)} \sin^2\theta$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$

$$\Gamma^\theta_{\phi\phi} = -\cos\theta \sin\theta$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$$

4d)

We have

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right)$$

4d Cont

$$\therefore \Gamma^r_{rr} = \frac{1}{2} g^{rr} \left(\partial_r g_{rr} + \cancel{\partial_r g_{rr}} - \cancel{\partial_r g_{rr}} \right)$$

(Since $g^{rt} = g^{r\theta} = g^{r\varphi} = 0$)

but $g^{rr} = \frac{1}{g_{rr}}$ (Since $g_{\mu\nu}$ is symmetric)

$$= e^{-2\beta(r)}$$

$$\Rightarrow \Gamma^r_{rr} = \frac{1}{2} e^{-2\beta(r)} \left(\partial_r e^{2\beta(r)} \right)$$

$$= \frac{1}{2} \cancel{e^{-2\beta(r)}} \cdot \left(2 \cdot \frac{\partial \beta(r)}{\partial r} \cancel{e^{2\beta(r)}} \right)$$

$$= \partial_r \beta(r) \quad \text{as required.}$$

e)

We can now calculate the Riemann and Ricci tensors, and use them to solve the vacuum field equations.

f)

See Mathematica (Part 4) for Riemann tensor $R^\rho{}_{\sigma\mu\nu}$ and Ricci tensor $R_{\mu\nu}$ calculation. We get non-zero riemann tensor components:

$$R^t{}_{rtr} = \partial_r \alpha \cdot \partial_r \beta - \partial_r^2 \alpha - (\partial_r \alpha)^2$$
$$R^t{}_{\theta t\theta} = -r e^{-2\beta} \partial_r \alpha$$
$$R^t{}_{\varphi t\varphi} = -r e^{-2\beta} \sin^2\theta \, \partial_r \alpha$$
$$R^r{}_{\theta r\theta} = r e^{-2\beta} \partial_r \beta$$
$$R^r{}_{\varphi r\varphi} = r e^{-2\beta} \sin^2\theta \, \partial_r \beta$$
$$R^\theta{}_{\varphi\theta\varphi} = (1 - e^{-2\beta}) \sin^2\theta$$

4f) Cont

Where the remaining components can be generated by considering the anti symmetry:

$$R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$$

and

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$$

$$\Rightarrow g_{\rho\lambda} R^\lambda{}_{\sigma\mu\nu} = -g_{\sigma\lambda} R^\lambda{}_{\rho\mu\nu}$$

$$\Rightarrow g_{\rho\rho} R^\rho{}_{\sigma\mu\nu} = -g_{\sigma\sigma} R^\sigma{}_{\rho\mu\nu} \quad \text{(Since } g \text{ is diagonal)}$$

Similarly, we find Ricci tensor components:

$$\boxed{R_{tt} = e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta + \tfrac{2}{r}\partial_r\alpha\right]}$$

$$\boxed{R_{rr} = -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\,\partial_r\beta + \tfrac{2}{r}\partial_r\beta}$$

$$\boxed{R_{\theta\theta} = e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1}$$

$$R_{\phi\phi} = \sin^2\theta\left(e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1\right)$$

4g)

The Riemann tensor can be calculated from the christoffel symbols as follows:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

While the Ricci tensor is calculated by contracting the Riemann tensor:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$$

4h)

Consider spacetime outside earth. In a spherically symmetric approximation, the spacetime will be equivalent to a vacuum spacetime with an earth-mass singularity at $r=0$. Therefore, outside earth we can use a vacuum solution with $T_{\mu\nu}=0$.

i)

We have:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

but $T_{\mu\nu}=0$ (vacuum)

$$\Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \qquad (1)$$

multiply by $g^{\mu\nu}$

$$\Rightarrow \underbrace{g^{\mu\nu} R_{\mu\nu}}_{=R} - \frac{1}{2} R \underbrace{g^{\mu\nu} g_{\mu\nu}}_{\delta^{\mu}{}_{\mu}} = 0$$

$$\Rightarrow R - \frac{1}{2} R \underbrace{\delta^{\mu}{}_{\mu}}_{4} = 0$$

$$\Rightarrow R - 2R = 0 \quad \Rightarrow \quad \boxed{R = 0}$$

j)

Sub $R=0$ into (1). We find

$$R_{\mu\nu} - 0 = 0 \quad \Rightarrow \quad \boxed{R_{\mu\nu} = 0}$$

4k)

We have $R_{\mu\nu}=0 \Rightarrow R_{tt}=R_{rr}=0$

From our expressions for R_{tt} and R_{rr}, we have

$$R_{tt}e^{-2(\alpha-\beta)} = \partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta + \frac{2}{r}\partial_r\alpha = 0 \quad (2)$$

$$-R_{rr} = \partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta - \frac{2}{r}\partial_r\beta = 0 \quad (3)$$

(2)-(3):

$$\frac{2}{r}\partial_r\alpha + \frac{2}{r}\partial_r\beta = 0$$

$$\Rightarrow \partial_r\alpha = -\partial_r\beta$$

integrate w.r.t. r:

$$\int \partial_r\alpha\,dr = -\int\partial_r\beta\,dr + \underline{\text{Const}}$$

$$\Rightarrow \alpha = -\beta + \underline{\text{Const}}$$

l)

let $c = \text{Const}$, $(\alpha = -\beta + c)$

we have metric:

$$ds^2 = -e^{2\alpha}dt^2 + \ldots$$

$$= -e^{-2\beta+2c}dt^2 + \ldots$$

$$= -e^{-2\beta}e^{2c}dt^2 + \ldots$$

4c) Cont

making the change of coordinates:

$$t \to e^{-c} t$$

$$dt \to e^{-c} dt$$

$$dt^2 \to e^{-2c} dt^2$$

we have metric

$$ds^2 = -e^{-2\beta} e^{2c} e^{-2c} dt^2 + \dots$$

$$= -e^{-2\beta} dt^2 + \dots$$

Since there is no other α or t dependence in the metric, we can choose our constant c to be 0 without loss of generality. Therefore, we choose:

$$\alpha = -\beta$$

4n)

We have

$$R_{\theta\theta} = e^{-2\beta}\left(r(\partial_r \beta - \partial_r \alpha) - 1\right) + 1 = 0$$

sub in $\beta = -\alpha$, $\partial_r \beta = -\partial_r \alpha$

$$e^{2\alpha}\left(-2r\,\partial_r \alpha - 1\right) = -1$$

$$\Rightarrow \quad 2\,\partial_r \alpha\, e^{2\alpha} r + e^{2\alpha} = 1$$

4a) Cont

$$\Rightarrow \partial_r (r e^{2\alpha}) = 1$$

Integrating w.r.t r, we find:

$r e^{2\alpha} = r - R_s$, where $-R_s$ is the constant of integration.

$$\Rightarrow e^{2\alpha} = 1 - \frac{R_s}{r}$$

5a)

With $e^{2\alpha} = e^{-2\beta} = 1 - \frac{R_s}{r}$, we find:

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$\Rightarrow g_{\mu\nu} = \begin{pmatrix} -\left(1 - R_s/r\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{R_s}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

5b)

The Geodesic Equation is given by:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$$

5b)

The non-zero Christoffel symbols are: (See Mathematica Part 5)

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{R_s}{2r(r-R_s)},$$

$$\Gamma^r_{tt} = \frac{(r-R_s)R_s}{2r^3}$$

$$\Gamma^r_{rr} = \frac{-R_s}{2r(r-R_s)}$$

$$\Gamma^r_{\theta\theta} = -r+R_s$$

$$\Gamma^r_{\varphi\varphi} = -\left((r-R_s)\sin^2\theta\right)$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$

$$\Gamma^\theta_{\varphi\varphi} = -\cos\theta\sin\theta$$

$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}$$

$$\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \frac{\cos\theta}{\sin\theta}$$

in the limit as $r \gg R_s$ $(r\to\infty)$, we have

$$\Gamma^t_{tr} = \Gamma^t_{rt} \to \frac{R_s}{2r^2},$$

$$\Gamma^r_{tt} \to \frac{R_s}{2r^2}, \quad \Gamma^r_{rr} \to -\frac{R_s}{2r^2}$$

$$\Gamma^r_{\theta\theta} \to -r$$

$$\Gamma^r_{\varphi\varphi} \to -r\sin^2\theta$$

5b) cont

From the geodesic equation, we have:

$$\frac{d^2r}{d\tau^2} = -\Gamma^r_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

plugging in our non-zero christoffell symbols, we find:

$$\frac{d^2r}{d\tau^2} = -\frac{R_s}{2r^2}\left(\frac{dt}{d\tau}\right)^2 + \frac{R_s}{2r^2}\left(\frac{dr}{d\tau}\right)^2 + r\left(\frac{d\theta}{d\tau}\right)^2 + r\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2$$

However, we have a slow moving particle,

$$\Rightarrow \frac{dt}{d\tau} << \frac{dx^i}{d\tau}$$

and so this reduces to:

$$\frac{d^2r}{d\tau^2} = \frac{-R_s}{2r^2}\left(\frac{dt}{d\tau}\right)^2,$$

however in the limit of $\frac{dt}{d\tau} << \frac{dx^i}{d\tau}$, we have

$$\frac{dt}{d\tau} \to c = 1$$

$$\Rightarrow \frac{d^2r}{d\tau^2} = \frac{-R_s}{2r^2}$$

but in the newtonian limit, we have:

$$\frac{d^2r}{d\tau^2} = -\frac{GM}{r^2},$$ and so we require $R_s = 2GM$

5b)

We therefore have metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

as $M \to 0$, we find:

$$ds^2 \to -dt^2 + dr^2 + r^2 d\Omega^2$$

i.e. flat (minkowski) space. This is expected, as a spacetime with no mass to curve it should be flat.

5d)

In the limit as $r \to \infty$, we have:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

Again, flat (Minkowski) space. This is again expected, since we expect that space is flat infinitly far from the mass.