GR Worksheet 1 Cameron Smith

2.2)

The metric for the surface of a 2-sphere of radius r is:

$$ds^2 = r^2 d\theta + r^2 \sin^2\theta \, d\varphi$$

$$\Rightarrow \quad g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{pmatrix}$$

The Ricci scalar of this space is given by:

$$R = \frac{2}{r^2} = \text{Const} \qquad \text{(See App 2.2)}$$

$\Rightarrow$ Constant Curvature

The Einstein tensor is given by:

$$G^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{(See App 2.2)}$$

It is <u>not</u> possible to get a non-zero einstein tensor with a 2 dimensional metric, since even a completely general metric:

$$g_{ij} = \begin{pmatrix} f_1(x,y), & f_2(x,y) \\ f_2(x,y), & f_3(x,y) \end{pmatrix}$$

has a vanishing Einstein tensor (See App 2.2)

While a 2 dimensional manifold is not required to be flat (i.e. sphere, cylinder), it must be <u>conformally</u> flat. i.e. it can be projected onto flat space. Equivalently, for an observer on the 2D manifold, space would appear flat - there is no prefered direction.

2.1) Cont

R is constant on a sphere, but this does not hold true in general for 2D manifolds.

For example:

$$g_{ij} = \begin{pmatrix} r^2\cos^2\phi & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix}$$

has Ricci scalar:

$$R = -\frac{2}{R^2}\left(\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\phi}\right)$$ (See App 2.2)

which is not constant curvature.

The metric

$$g_{ij} = \begin{pmatrix} -r^2 & 0 \\ 0 & -r^2\sin^2\theta \end{pmatrix}$$

has Ricci scalar:

$$R = -\frac{2}{R^2}$$

i.e. constant negative curvature.

3.1)

We have metric:

$$ds^2 = -a(t)^2 dt^2 + dx^2 + dy^2 + dz^2$$

$$\Rightarrow g_{\mu\nu} = \begin{pmatrix} -a(t)^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

However, this is just a change of coordinates from the flat space given the minkowski metric,

i.e. $t \to A(t) \Rightarrow dt \to a(t)$; where $a(t) = A'(t)$

This is further backed up by mathematica, which tells us we are in flat space (App 3.1)

Consider instead:

$$ds^2 = -dt^2 + a(t)^2 dx^2 + dy^2 + dz^2$$

$$\Rightarrow g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Mathematica, we find einstein tensor

$$G_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{a''(t)}{a(t)} & 0 \\ 0 & 0 & 0 & -\frac{a''(t)}{a(t)} \end{pmatrix} \quad \text{(App 3.1)}$$

3.1) Cont

$\therefore$ Since we have a non-vanishing Einstein tensor, this metric is not a vacuum solution.

The metric is:

- Symmetric under time reversal $t \to -t$
- Spatially symmetric in y and z

For observers in this "universe" space would be flat in all directions except the x direction. Distances between two objects in the x-direction would depend on $a(t)$ (i.e if at some time t, $a(t)$ is small, the universe would be "squished" in the x direction, while if $a(t)$ was large, things would be "spread out" in x).

Using Mathematica, we find the non-vanishing christoffel symbols:

$$\Gamma^{0}_{11} = a(t)\, a'(t),$$

$$\Gamma^{1}_{01} = \Gamma^{1}_{10} = a'(t)/a(t) \qquad \text{(App 3.1)}$$

The geodesic equation is:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$$

For a particle initially at rest, we have

$$\frac{dx}{d\lambda} = \frac{dy}{d\lambda} = \frac{dz}{d\lambda} = 0$$

31) Cont

And so our acceleration is:

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{00}\frac{dt}{d\lambda}\frac{dt}{d\lambda}$$

∴ Using our Christoffel symbols, we find:

$$\frac{d^2x^\mu}{d\lambda^2} = 0 \qquad (\text{for all } \mu).$$

i.e. a test particle initially at rest stays at rest.

For photon's moving perpendicular to the x direction, we have:

$$\frac{dx^\mu}{d\lambda} = \left(0, 0, \frac{dy}{d\lambda}, \frac{dz}{d\lambda}\right)^T$$

∴ Using our christoffel symbols, we find:

$$\frac{d^2t}{d\lambda^2} = -a(t)a'(t)\frac{dx}{d\lambda}\frac{dx}{d\lambda}$$

$$= 0,$$

$$\frac{d^2x}{d\lambda^2} = \frac{a'(t)}{a(t)}\cancel{\frac{dt}{d\lambda}}\overbrace{\frac{dx}{d\lambda}}^{0} + \frac{a'(t)}{a(t)}\frac{dx}{d\lambda}\overbrace{\cancel{\frac{dt}{d\lambda}}}^{0}$$

$$= 0$$

$$\frac{d^2y}{d\lambda^2} = \frac{d^2z}{d\lambda^2} = 0$$

3.1) Cont

i.e. our photon's aren't accelerating.

However if we consider the distance between two photons, say at $x=0$ and $x=1$, we find that the distance between them is:

$$\Delta x = \int_{x=0}^{x=1} \sqrt{ds^2}$$

Since we are integrating over x, we have $dt = dy = dz = 0$

$$\Rightarrow \Delta x = \int_{x=0}^{x=1} \sqrt{a(t)^2 dx^2}$$

$$= a(t) \int_{x=0}^{x=1} dx$$

$$= a(t)$$

i.e. the photon's aren't accelerating but the distance between them changes as $a(t)$.

For photon's moving in arbitrary directions, we have

$$\frac{d^2t}{d\lambda^2} = -a(t)a'(t)\left(\frac{dx}{d\lambda}\right)^2, \quad \frac{d^2x}{d\lambda^2} = \frac{d^2y}{d\lambda^2} = \frac{d^2z}{d\lambda^2} = 0$$

$\Rightarrow$ photon is not accelerating in space, despite the expansion/contraction in the x direction.

If we have:

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2),$$

3.1) Cont

Then space is expanding / Contracting in all directions isotropically with $a(t)$.

3.2)

We have metric:

$$ds^2 = -a(x)^2 dt^2 + dx^2 + dy^2 + dz^2$$

$$\Rightarrow g_{\mu\nu} = \begin{pmatrix} -a(x)^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Mathematica, we find the following Christoffel symbols:

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{a'(x)}{a(x)}, \qquad \text{(App 3.2)}$$

$$\Gamma^1_{00} = a(x)\,a'(x)$$

With all other christoffel symbols going to 0.

$\therefore$ We get the following geodesic equation:

$$\frac{d^2 x^\mu}{d\lambda^2} = -\Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda}$$

$$\Rightarrow \frac{d^2 t}{d\lambda^2} = -2\frac{a'(x)}{a(x)} \frac{dx}{d\lambda} \frac{dt}{d\lambda}$$

3.2) Cont

$$\frac{d^2x}{d\lambda^2} = -a(x)\,a'(x)\left(\frac{dt}{d\lambda}\right)^2,$$

$$\frac{d^2y}{d\lambda^2} = \frac{d^2z}{d\lambda^2} = 0 \qquad (1)$$

for a particle initially at rest, we find:

$$\frac{d^2t}{d\lambda^2} = \frac{d^2y}{d\lambda^2} = \frac{d^2z}{d\lambda^2} = 0,$$

$$\frac{d^2x}{d\lambda^2} = -a(x)a'(x)\left(\frac{dt}{d\lambda}\right)^2$$

$\therefore$ particles initially at rest will accelerate in the x direction.

integrating equations (1), we find:

$$y(\lambda) = y'(0)\lambda + y(0), \quad z(\lambda) = z'(0)\lambda + z(0)$$

i.e. the particles move in the y or z direction at a constant velocity (their initial velocity).

This is a result of the symmetry between y and z, i.e. the evolution of a particle should not depend on it's y or z coordinate, since we have spatial symmetry in y and z.

3.3) Cont

The Metric:

$$ds^2 = -x^2 dt^2 + dx^2 + dy^2 + dz^2$$

describes a flat space, with no curvature (the Rieman tensor is 0 everywhere), while the metric:

$$ds^2 = -x^4 dt^2 + dx^2 + dy^2 + dz^2$$

has

$$R^0{}_{110} = -R^0{}_{101} = \frac{2}{x^2} \qquad \text{(App 3.3)}$$

i.e. Infinite curvature (Singularity) at $x=0$

So by going from $-x^2dt^2 \to -x^4dt^2$, we go from no curvature to a Singularity at $x=0$.
Futhermore our Ricci scalar is $R = -4/x^2$ which also goes to infinity at $x=0$, and so we have a physical singularity.
Consider the metric:

$$ds^2 = -dt^2 + e^{-(x^2+y^2+z^2)}(dx^2 + dy^2 + dz^2)$$

$$\Rightarrow g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & e^{-R^2} & 0 & 0 \\ 0 & 0 & e^{-R^2} & 0 \\ 0 & 0 & 0 & e^{-R^2} \end{pmatrix}$$

where $R = \sqrt{x^2+y^2+z^2}$

3.2) cont

From Mathematica we see this does not contain any singularities. (App 3.2)

We can try and find a "radius" for this universe by integrating between $R=0$ and $R=\infty$. We find:

$$\Delta R = \int_{R=0}^{R=\infty} \sqrt{ds^2}$$

Considering only the variation in space (not time), we have $dt=0$

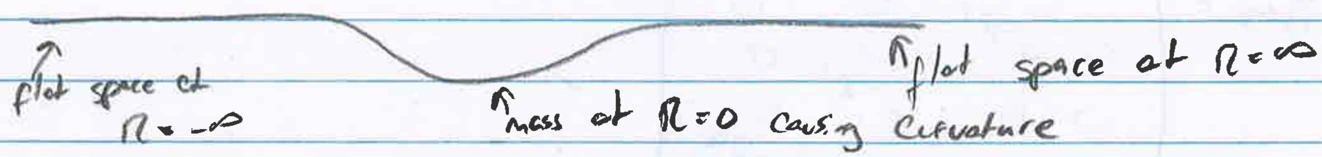
$$\Rightarrow \Delta R = \int_{R=0}^{R=\infty} \sqrt{e^{-R^2}\, dR^2}$$

$$= \int_{R=0}^{R=\infty} e^{-\frac{1}{2}R^2}\, dR$$

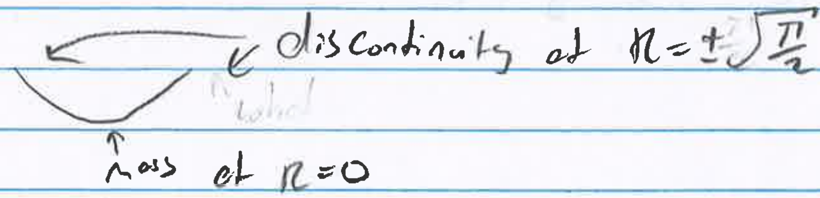
$$= \sqrt{\frac{\pi}{2}}$$

And so the universe is finite.

This would therefore not be suitable for a mass configuration in flat space. To see this consider how we would like our universe to behave with mass:



However with a finite universe we get the following



3.2)

This is very clearly <u>not</u> well behaved.

3.3)

We want to find solutions for waves in a vacuum

Consider the following metric:

$$g_{\mu\nu} = \begin{pmatrix} -1+f(x-t) & 0 & 0 & 0 \\ 0 & 1+f(x-t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

While this appears to be a reasonable solution for a wave travelling in the x-direction, we find

$R_{00} \neq 0$, and $R_{11} \neq 0$ (App 3.3)

And so this does <u>not</u> satisfy the vacuum field equations ($R_{\mu\nu}=0$) (we also have $G^{\mu}{}_{\nu} \neq 0$)

Instead consider:

$$g_{\mu\nu} = \begin{pmatrix} -1+f(x-t) & -f(x-t) & 0 & 0 \\ -f(x-t) & 1+f(x-t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is flat space ($R_{\mu\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} = 0$), And so clearly satisfies the vacuum solution, (Since any mass would curve space.

3.3) Cont

We now generalise to the metric:

$$g_{\mu\nu} = \begin{pmatrix} -1 + f(x-t,y,z) & -f(x-t) & 0 & 0 \\ -f(x-t,y,z) & 1 + f(x-t,y,z) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This gives Ricci Tensor:

$$R_{\mu\nu} = \begin{pmatrix} -\frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) & \frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) & 0 & 0 \\ \frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) & -\frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This clearly goes to zero for any harmonic functions of y and z:

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Indicating that harmonic functions f result in a flat space vacuum solution.

Consider the following two harmonic solutions:

$$f_1(x-t,y,z) = e^{-y} \sin z \sin(x-t)$$

$$f_2(x-t,y,z) = \ln(y^2+z^2) \sin(x-t)$$

3.3) Cont

Both these functions satisfy :

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0, \qquad \text{(See App 3.3)}$$

And so represent Vacuum solutions with

$$R_{\mu\nu} = 0 \qquad \text{(See App 3.3)}$$

The factor of $\sin(x-t)$ ensures that both f_1 and f_2 are wavelike in the x direction.

f_1 is additionally wavelike in the z direction, however the e^{-y} factor means that $f_1 \to \infty$ as $y \to -\infty$ which is not what we would expect for a wave like solution. (Ideally, we would have $f_1 \to 0$ as $y \to \pm\infty$ indicating that the wave is "spreading out" and disapating).

It is reasonably well behaved for $y \geq 0$.

f_2 is also not well behaved, as it has a singularity at $y=0$, $z=0$ and also goes to ∞ as $y \to \pm\infty$ or $z \to \pm\infty$. Once again this is not a reasonable solution, as we would expect a wave to spread out and get smaller as it gets further from the origin.

To summarise, we would like a wave solution to approach flat space in the limit as $y \to \pm\infty$, $z \to \pm\infty$,

3.3)

i.e. we would like

$$\lim_{\substack{z \to \pm\infty \\ y \to \pm\infty}} g_{\mu\nu} = \eta_{\mu\nu}$$

This is <u>not</u> the case for either of our two examples.