GR Worksheet 1 Cameron Smith

2.2)

The metric for the surface of a 2-sphere of radius r is:

$$ds^2 = r^2 d\theta + r^2 \sin^2\theta \, d\varphi$$

$$\Rightarrow \quad g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{pmatrix}$$

The Ricci scalar of this space is given by:

$$R = \frac{2}{r^2} = \text{Const}$$ (See App 2.2)

$\Rightarrow$ Constant Curvature

The Einstein tensor is given by:

$$G^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ (See App 2.2)

It is not possible to get a non-zero einstein tensor with a 2 dimensional metric, since even a completely general metric:

$$g_{ii} = \begin{pmatrix} f_1(x,y), & f_2(x,y) \\ f_2(x,y), & f_3(x,y) \end{pmatrix}$$

has a vanishing Einstein tensor (See App 2.2)

While a 2 dimensional manifold is not required to be flat (i.e. sphere, cylinder), it must be conformaly flat. I.e. it can be projected onto flat space. Equivalently, for an observer on the 2D manifold, space would appear flat - there is no prefered direction.

2.1) Cont

R is constant on a sphere, but this does not hold true in general for 2D manifolds.

For example:

$$g_{ij} = \begin{pmatrix} r^2\cos^2\phi & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix}$$

has Ricci scalar:

$$R = -\frac{2}{R^2}\left(\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\phi}\right) \qquad \text{(See App 2.2)}$$

which is not constant curvature.

The metric

$$g_{ij} = \begin{pmatrix} -r^2 & 0 \\ 0 & -r^2\sin^2\theta \end{pmatrix}$$

has Ricci scalar:

$$R = -\frac{2}{R^2}$$

i.e. Constant negative curvature.

3.1)

We have metric:

$$ds^2 = -a(t)^2 dt^2 + dx^2 + dy^2 + dz^2$$

$$\Rightarrow g_{\mu\nu} = \begin{pmatrix} -a(t)^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

However, this is just a change of coordinates from the flat space given the minkowski metric,

i.e. $t \to A(t) \Rightarrow dt \to a(t)$; where $a(t) = A'(t)$

This is further backed up by mathematica, which tells us we are in flat space (App 3.1)

Consider instead:

$$ds^2 = -dt^2 + a(t)^2 dx^2 + dy^2 + dz^2$$

$$\Rightarrow g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Mathematica, we find einstein tensor:

$$G_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{a''(t)}{a(t)} & 0 \\ 0 & 0 & 0 & -\frac{a''(t)}{a(t)} \end{pmatrix} \quad \text{(App 3.1)}$$

3.1) Cont

$\therefore$ Since we have a non-vanishing Einstein tensor, this metric is <u>not</u> a vacuum solution.

The metric is:

- Symmetric under time reversal $t \to -t$
- Spatially symmetric in y and z

For observers in this "universe" space would be flat in all directions except the x direction. Distances between two objects in the x-direction would depend on $a(t)$ (i.e. if at some time t, $a(t)$ is small, the universe would be "squished" in the x direction, while if $a(t)$ was large, things would be "spread out" in x).

Using Mathematica, we find the non-vanishing christoffel symbols:

$$\Gamma^0_{11} = a(t)\, a'(t),$$

$$\Gamma^1_{01} = \Gamma^1_{10} = a'(t)/a(t) \qquad \text{(App 3.1)}$$

The geodesic equation is:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

For a particle initially at rest, we have

$$\frac{dx}{d\lambda} = \frac{dy}{d\lambda} = \frac{dz}{d\lambda} = 0$$

31) Cont

And so our acceleration is:

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{00}\frac{dt}{d\lambda}\frac{dt}{d\lambda}$$

$\therefore$ Using our Christoffel symbols, we find:

$$\frac{d^2x^\mu}{d\lambda^2} = 0 \qquad (\text{for all } \mu).$$

i.e. a test particle initially at rest stays at rest.

For photon's moving perpendicular to the x direction, we have:

$$\frac{dx^\mu}{d\lambda} = \left(0, 0, \frac{dy}{d\lambda}, \frac{dz}{d\lambda}\right)^T$$

$\therefore$ Using our christoffel symbols, we find:

$$\frac{d^2t}{d\lambda^2} = -a(t)a'(t)\frac{dx}{d\lambda}\frac{dx}{d\lambda}$$

$$= 0,$$

$$\frac{d^2x}{d\lambda^2} = \frac{a'(t)}{a(t)}\frac{dt}{d\lambda}\cancelto{0}{\frac{dx}{d\lambda}} + \frac{a'(t)}{a(t)}\frac{dx}{d\lambda}\cancelto{0}{\frac{dt}{d\lambda}}$$

$$= 0$$

$$\frac{d^2y}{d\lambda^2} = \frac{d^2z}{d\lambda^2} = 0$$

3.1) Cont

i.e. our photon's aren't accelerating.

However if we consider the distance between two photons, say at $x=0$ and $x=1$, we find that the distance between them is:

$$\Delta x = \int_{x=0}^{x=1} \sqrt{ds^2}$$

Since we are integrating over x, we have $dt = dy = dz = 0$

$$\Rightarrow \Delta x = \int_{x=0}^{x=1} \sqrt{a(t)^2 dx^2}$$

$$= a(t) \int_{x=0}^{x=1} dx$$

$$= a(t)$$

i.e. the photon's aren't accelerating but the distance between them changes as $a(t)$.

For photon's moving in arbitrary directions, we have

$$\frac{dt^2}{d\lambda^2} = -a(t)a'(t)\left(\frac{dx}{d\lambda}\right)^2, \quad \frac{d^2x}{d\lambda^2} = \frac{d^2y}{d\lambda^2} = \frac{d^2z}{d\lambda^2} = 0$$

$\Rightarrow$ photon is not accelerating in space, despite the expansion/contraction in the x direction.

If we have:

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2),$$

3.1) Cont

Then space is expanding / contracting in all directions isotropically with $a(t)$.

3.2)

We have metric:

$$ds^2 = -a(x)^2 dt^2 + dx^2 + dy^2 + dz^2$$

$$\Rightarrow g_{\mu\nu} = \begin{pmatrix} -a(x)^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Mathematica, we find the following Christoffel symbols:

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{a'(x)}{a(x)},$$ (App 3.2)

$$\Gamma^1_{00} = a(x)\, a'(x)$$

With all other christoffel symbols going to 0.

$\therefore$ We get the following geodesic equation:

$$\frac{d^2 x^\mu}{d\lambda^2} = -\Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda}$$

$$\Rightarrow \frac{d^2 t}{d\lambda^2} = -2\frac{a'(x)}{a(x)} \frac{dx}{d\lambda} \frac{dt}{d\lambda}$$

3.2) Cont

$$\frac{d^2x}{d\lambda^2} = -a(x)\,a'(x)\left(\frac{dt}{d\lambda}\right)^2,$$

$$\frac{d^2y}{d\lambda^2} = \frac{d^2z}{d\lambda^2} = 0 \quad (1)$$

for a particle initially at rest, we find:

$$\frac{d^2t}{d\lambda^2} = \frac{d^2y}{d\lambda^2} = \frac{d^2z}{d\lambda^2} = 0,$$

$$\frac{d^2x}{d\lambda^2} = -a(x)a'(x)\left(\frac{dt}{d\lambda}\right)^2$$

$\therefore$ particles initially at rest will accelerate in the x direction.

integrating equations (1), we find:

$$y(\lambda) = y'(0)\lambda + y(0), \quad z(\lambda) = z'(0)\lambda + z(0)$$

i.e. the particles move in the y or z direction at a constant velocity (their initial velocity).

This is a result of the symmetry between y and z, i.e. the evolution of a particle should not depend on it's y or z coordinate, since we have spatial symmetry in y and z.

3.3) Cont

The metric:

$$ds^2 = -x^2 dt^2 + dx^2 + dy^2 + dz^2$$

describes a flat space, with no curvature (the Rieman tensor is 0 everywhere), while the metric:

$$ds^2 = -x^4 dt^2 + dx^2 + dy^2 + dz^2$$

has

$$R^0{}_{110} = -R^0{}_{101} = \frac{2}{x^2} \qquad \text{(App 3.3)}$$

i.e. Infinite curvature (Singularity) at $x=0$

So by going from $-x^2dt^2 \to -x^4dt^2$, we go from no curvature to a Singularity at $x=0$.
Futhermore our Ricci scaler is $R = -4/x^2$ which also goes to infinity at $x=0$, and so we have a physical singularity.
Consider the metric:

$$ds^2 = -dt^2 + e^{-(x^2+y^2+z^2)}(dx^2+dy^2+dz^2)$$

$$\Rightarrow g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & e^{-R^2} & 0 & 0 \\ 0 & 0 & e^{-R^2} & 0 \\ 0 & 0 & 0 & e^{-R^2} \end{pmatrix}$$

where $R = \sqrt{x^2+y^2+z^2}$

3.3) cont

From Mathematica we see this does not contain any singularities. (App 3.2)

We can try and find a "radius" for this universe by integrating between $R=0$ and $R=\infty$. We find:

$$\Delta R = \int_{R=0}^{R=\infty} \sqrt{ds^2}$$

Considering only the variation in space (not time), we have $dt=0$

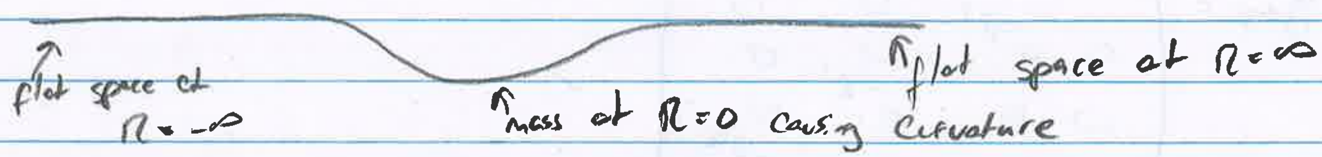
$$\Rightarrow \Delta R = \int_{R=0}^{R=\infty} \sqrt{e^{-R^2}\, dR^2}$$

$$= \int_{R=0}^{R=\infty} e^{-\frac{1}{2}R^2}\, dR$$

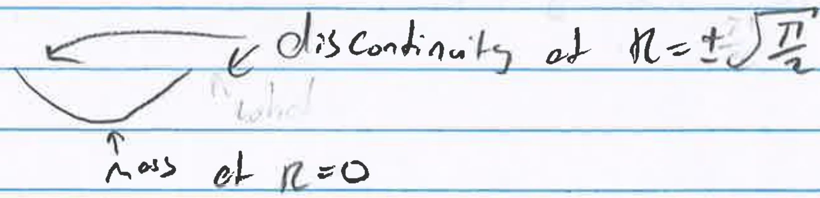
$$= \sqrt{\frac{\pi}{2}}$$

And so the universe is finite.

This would therefore not be suitable for a mass configuration in flat space. To see this consider how we would like our universe to behave with mass:



However with a finite universe we get the following



3.2)

This is very clearly <u>not</u> well behaved.

3.3)

We want to find solutions for waves in a vacuum. Consider the following metric:

$$g_{\mu\nu} = \begin{pmatrix} -1+f(x-t) & 0 & 0 & 0 \\ 0 & 1+f(x-t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

While this appears to be a reasonable solution for a wave travelling in the x-direction, we find

$R_{00} \neq 0$, and $R_{11} \neq 0$ (App 3.3)

And so this does <u>not</u> satisfy the vacuum field equations ($R_{\mu\nu}=0$) (we also have $G^{\mu}{}_{\nu} \neq 0$)

Instead consider:

$$g_{\mu\nu} = \begin{pmatrix} -1+f(x-t) & -f(x-t) & 0 & 0 \\ -f(x-t) & 1+f(x-t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is flat space ($R_{\mu\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} = 0$), And so clearly satisfies the vacuum solution, (Since any mass would curve space.

3.3) Cont

We now generalise to the metric:

$$g_{\mu\nu} = \begin{pmatrix} -1+f(x-t,y,z) & -f(x-t) & 0 & 0 \\ -f(x-t,y,z) & 1+f(x-t,y,z) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This gives Ricci Tensor:

$$R_{\mu\nu} = \begin{pmatrix} -\frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right) & \frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right) & 0 & 0 \\ \frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right) & -\frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This clearly goes to zero for any harmonic functions of y and z:

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Indicating that harmonic functions f result in a flat space vacuum solution.

Consider the following two harmonic solutions:

$$f_1(x-t,y,z) = e^{-y}\sin z \sin(x-t)$$

$$f_2(x-t,y,z) = \ln(y^2+z^2)\sin(x-t)$$

3.3) Cont

Both these functions satisfy :

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0, \qquad \text{(See App 3.3)}$$

And so represent Vacuum solutions with

$R_{\mu\nu} = 0$ (See App 3.3)

The factor of $\sin(x-t)$ ensures that both f_1 and f_2 are wavelike in the x direction.

f_1 is additionally wavelike in the z direction, however the e^{-y} factor means that $f_1 \to \infty$ as $y \to -\infty$ which is not what we would expect for a wave like solution. (Ideally, we would have $f_1 \to 0$ as $y \to \pm\infty$ indicating that the wave is "spreading out" and disapating).

It is reasonably well behaved for $y \geq 0$.

f_2 is also not well behaved, as it has a singularity at $y=0$, $z=0$ and also goes to ∞ as $y \to \pm\infty$ or $z \to \pm\infty$. Once again this is not a reasonable solution, as we would expect a wave to spread out and get smaller as it gets further from the origin.

To summarise, we would like a wave solution to approach flat space in the limit as $y \to \pm\infty$, $z \to \pm\infty$,

3.3)

i.e. we would like

$$\lim_{\substack{z \to \infty \\ y \to \pm\infty}} g_{\mu\nu} = \eta_{\mu\nu}$$

This is not the case for either of our two examples.

2 RG&TC-Code

Appendix

0.1

In[54]:=
```
xCoord = {t, χ, θ, φ};
g = {
    {-x y, 0, 0, 0},
    {0, x y t, 0, 0},
    {0, 0, z, 0},
    {0, 0, 0, x t}
  };
RGtensors[g, xCoord]
```

$$\text{gdd} = \begin{pmatrix} -x\,y & 0 & 0 & 0 \\ 0 & t\,x\,y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & t\,x \end{pmatrix}$$

$$\text{LineElement} = -x\,y\,d[t]^2 + z\,d[\theta]^2 + t\,x\,d[\varphi]^2 + t\,x\,y\,d[\chi]^2$$

$$\text{gUU} = \begin{pmatrix} -\frac{1}{x\,y} & 0 & 0 & 0 \\ 0 & \frac{1}{t\,x\,y} & 0 & 0 \\ 0 & 0 & \frac{1}{z} & 0 \\ 0 & 0 & 0 & \frac{1}{t\,x} \end{pmatrix}$$

```
gUU computed in 0.00122 sec

Gamma computed in 0.002656 sec

Riemann(dddd) computed in 0.00472 sec

Riemann(Uddd) computed in 0.002312 sec

Ricci computed in 0.000301 sec

Weyl computed in 0.003843 sec

Einstein computed in 0.00016 sec
```

Out[56]=
```
All tasks completed in 0.020573
```

In[57]:=
```
(* Ricci Scalar *)
```

In[58]:=
```
R
```

Out[58]=

$$-\frac{1}{2\,t^2\,x\,y}$$

In[59]:=
```
(* Einstein Tensor *)
```

In[60]:=
```
EUd
```

Out[60]=
$$\left\{\left\{-\frac{1}{4\,t^2\,x\,y},0,0,0\right\},\left\{0,\frac{1}{4\,t^2\,x\,y},0,0\right\},\left\{0,0,\frac{1}{4\,t^2\,x\,y},0\right\},\left\{0,0,0,\frac{1}{4\,t^2\,x\,y}\right\}\right\}$$

In[61]:=
```
(* Christoffel Symbol *)
```

In[62]:=
```
GUdd // MatrixForm
```

Out[62]//MatrixForm=
$$\begin{pmatrix} \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\\frac{1}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\\frac{1}{2y}\end{pmatrix} \\ \begin{pmatrix}0\\\frac{1}{2t}\\0\\0\end{pmatrix} & \begin{pmatrix}\frac{1}{2t}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \\ \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \\ \begin{pmatrix}0\\0\\0\\\frac{1}{2t}\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}\frac{1}{2t}\\0\\0\\0\end{pmatrix} \end{pmatrix}$$

In[63]:=
```
Part[GUdd, 1, 2, 2]
Part[GUdd, 2, 2, 1]
```

Out[63]=
$$\frac{1}{2}$$

Out[64]=
$$\frac{1}{2\,t}$$

In[65]:=
```
(* Riemann tensor *)
```

In[66]:= RUddd

Out[66]=
$$\left\{\left\{\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\left\{\left\{0,-\frac{1}{4\,t},0,0\right\},\left\{\frac{1}{4\,t},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\left\{\left\{0,0,0,-\frac{1}{4\,t\,y}\right\},\{0,0,0,0\},\{0,0,0,0\},\left\{\frac{1}{4\,t\,y},0,0,0\right\}\right\}\right\},$$
$$\left\{\left\{\left\{0,-\frac{1}{4\,t^2},0,0\right\},\left\{\frac{1}{4\,t^2},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\left\{\{0,0,0,0\},\left\{0,0,0,\frac{1}{4\,t\,y}\right\},\{0,0,0,0\},\left\{0,-\frac{1}{4\,t\,y},0,0\right\}\right\}\right\},$$
$$\{\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\},$$
$$\left\{\left\{\left\{0,0,0,-\frac{1}{4\,t^2}\right\},\{0,0,0,0\},\{0,0,0,0\},\left\{\frac{1}{4\,t^2},0,0,0\right\}\right\},\left\{\{0,0,0,0\},\left\{0,0,0,-\frac{1}{4\,t}\right\},\{0,0,0,0\},\left\{0,\frac{1}{4\,t},0,0\right\}\right\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\right\}\right\}$$

In[67]:= (* Ricci Tensor *)

In[68]:= Rdd

Out[68]=
$$\left\{\left\{\frac{1}{2\,t^2},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\right\}$$

In[69]:= Part[Rdd, 1, 1]

Out[69]=
$$\frac{1}{2\,t^2}$$

2.1

In[70]:= xCoord = {ρ, ϕ};
g = {{1, 0}, {0, ρ^2}}

Out[71]=
$$\{\{1,0\},\{0,\rho^2\}\}$$

In[72]:= RGtensors[g, xCoord]

gdd = $\begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}$

LineElement = $d[\rho]^2 + \rho^2\, d[\phi]^2$

gUU = $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\rho^2} \end{pmatrix}$

gUU computed in 0.000357 sec

Gamma computed in 0.000264 sec

Riemann (dddd) computed in 0.000135 sec

Flat Space!

Out[72]= Aborted after 0.006891

In[73]:= **GUdd // MatrixForm**

Out[73]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\rho \end{pmatrix} \\ \begin{pmatrix} 0 \\ \frac{1}{\rho} \end{pmatrix} & \begin{pmatrix} \frac{1}{\rho} \\ 0 \end{pmatrix} \end{pmatrix}$$

In[74]:= **Part[gUU, 2, 2]**

Out[74]= $\frac{1}{\rho^2}$

Since we’re in flat space, Mathematica doesn’t calculate the other tensors

2.2

In[75]:= **xCoord = {θ, ϕ};**
g = {{r^2, 0}, {0, r^2 Sin[θ]^2}}

Out[76]= $\{\{r^2, 0\}, \{0, r^2 \operatorname{Sin}[\theta]^2\}\}$

In[77]:= **RGtensors[g, xCoord]**

gdd = $\begin{pmatrix} r^2 & 0 \\ 0 & r^2 \operatorname{Sin}[\theta]^2 \end{pmatrix}$

LineElement = $r^2 \, d[\theta]^2 + r^2 \, d[\phi]^2 \operatorname{Sin}[\theta]^2$

gUU = $\begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{\operatorname{Csc}[\theta]^2}{r^2} \end{pmatrix}$

gUU computed in 0.002156 sec

Gamma computed in 0.000793 sec

Riemann (dddd) computed in 0.000404 sec

Riemann (Uddd) computed in 0.000229 sec

Ricci computed in 0.000155 sec

Weyl computed in 0.000019 sec

Conformally Flat

Einstein computed in 0.000057 sec

Einstein Space

Out[77]= All tasks completed in 0.006174

We have einstein tensor:

In[78]:= **EUd**

Out[78]= {{0, 0}, {0, 0}}

And Ricci Scalar

In[79]:= **R**

Out[79]= $\frac{2}{r^2}$

Can we find a non-vanishing Einstein tensor?

In[80]:= **RGtensors[{{f1[x, y], f2[x, y]}, {f2[x, y], f3[x, y]}}, {x, y}];**

$$gdd = \begin{pmatrix} f1[x, y] & f2[x, y] \\ f2[x, y] & f3[x, y] \end{pmatrix}$$

$$LineElement = d[x]^2\, f1[x, y] + 2\, d[x] \times d[y] \times f2[x, y] + d[y]^2\, f3[x, y]$$

$$gUU = \begin{pmatrix} \frac{f3[x,y]}{-f2[x,y]^2+f1[x,y]\times f3[x,y]} & \frac{f2[x,y]}{f2[x,y]^2-f1[x,y]\times f3[x,y]} \\ \frac{f2[x,y]}{f2[x,y]^2-f1[x,y]\times f3[x,y]} & \frac{f1[x,y]}{-f2[x,y]^2+f1[x,y]\times f3[x,y]} \end{pmatrix}$$

gUU computed in 0.002126 sec

Gamma computed in 0.004489 sec

Riemann (dddd) computed in 0.004371 sec

Riemann (Uddd) computed in 0.008025 sec

Ricci computed in 0.010105 sec

Weyl computed in 0.000024 sec

Conformally Flat

Einstein computed in 0.004559 sec

Einstein Space

In[81]:=
```
EUd
```

Out[81]= {{0, 0}, {0, 0}}

So no, any 2d metric has a vanishing Einstein tensor

non constant Ricci Scalar:

In[82]:=
```
xCoord = {θ, ϕ};
g = {{-Cos[ϕ]^2 r^2, 0}, {0, -r^2 Sin[θ]^2}};
RGtensors[g, xCoord]
```

gdd = $\begin{pmatrix} -r^2\,\text{Cos}[\phi]^2 & 0 \\ 0 & -r^2\,\text{Sin}[\theta]^2 \end{pmatrix}$

LineElement = $-r^2\,\text{Cos}[\phi]^2\,\text{d}[\theta]^2 - r^2\,\text{d}[\phi]^2\,\text{Sin}[\theta]^2$

gUU = $\begin{pmatrix} -\frac{\text{Sec}[\phi]^2}{r^2} & 0 \\ 0 & -\frac{\text{Csc}[\theta]^2}{r^2} \end{pmatrix}$

gUU computed in 0.00301 sec

Gamma computed in 0.000838 sec

Riemann (dddd) computed in 0.000354 sec

Riemann (Uddd) computed in 0.000628 sec

Ricci computed in 0.000705 sec

Weyl computed in 0.000014 sec

Conformally Flat

Einstein computed in 0.000333 sec

Einstein Space

Out[84]= All tasks completed in 0.00851

In[85]:= **R // Simplify**

Out[85]= $-\frac{2\,(\text{Csc}[\theta]^2 + \text{Sec}[\phi]^2)}{r^2}$

Can we make a 2d manifold with constant negative curvature?

In[86]:=
```
xCoord = {θ, ϕ};
g = {{-r^2, 0}, {0, -r^2 Sin[θ]^2}};
RGtensors[g, xCoord]
```

gdd = $\begin{pmatrix} -r^2 & 0 \\ 0 & -r^2 \text{Sin}[\theta]^2 \end{pmatrix}$

LineElement = $-r^2\, d[\theta]^2 - r^2\, d[\phi]^2\, \text{Sin}[\theta]^2$

gUU = $\begin{pmatrix} -\frac{1}{r^2} & 0 \\ 0 & -\frac{\text{Csc}[\theta]^2}{r^2} \end{pmatrix}$

gUU computed in 0.000492 sec

Gamma computed in 0.000451 sec

Riemann (dddd) computed in 0.000194 sec

Riemann (Uddd) computed in 0.000157 sec

Ricci computed in 0.000116 sec

Weyl computed in 0.000013 sec

Conformally Flat

Einstein computed in 0.000051 sec

Einstein Space

Out[88]= All tasks completed in 0.003855

In[89]:= **R // Simplify**

Out[89]= $-\frac{2}{r^2}$

so yes, we can.

In[90]:=

3.1

In[91]:=
```
g = {{-a[t]^2 , 0 , 0, 0},
  {0, 1, 0, 0},
  {0, 0, 1, 0},
  {0, 0, 0, 1}}
```

Out[91]= $\{\{-a[t]^2, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}\}$

In[92]:= **coord = {t, x, y, z}**

Out[92]= {t, x, y, z}

In[93]:= **RGtensors[g, coord]**

$$gdd = \begin{pmatrix} -a[t]^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

LineElement $= -a[t]^2\, d[t]^2 + d[x]^2 + d[y]^2 + d[z]^2$

$$gUU = \begin{pmatrix} -\frac{1}{a[t]^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gUU computed in 0.000483 sec

Gamma computed in 0.006737 sec

Riemann (dddd) computed in 0.00187 sec

Flat Space!

Out[93]= Aborted after 0.011206

let's instead try:

```
In[94]:= g = {{-1 , 0 , 0, 0},
   {0, a[t]^2, 0, 0},
   {0, 0, 1, 0},
   {0, 0, 0, 1}}
```

Out[94]= $\{\{-1, 0, 0, 0\}, \{0, a[t]^2, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}\}$

```
In[95]:= coord = {t, x, y, z}
```

Out[95]= {t, x, y, z}

```
In[96]:= RGtensors[g, coord]
```

$$\text{gdd} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a[t]^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{LineElement} = -d[t]^2 + a[t]^2\, d[x]^2 + d[y]^2 + d[z]^2$$

$$\text{gUU} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{a[t]^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gUU computed in 0.000444 sec

Gamma computed in 0.001094 sec

Riemann (dddd) computed in 0.001814 sec

Riemann (Uddd) computed in 0.001374 sec

Ricci computed in 0.000218 sec

Weyl computed in 0.002341 sec

Einstein computed in 0.00009 sec

Out[96]= All tasks completed in 0.009561

In[97]:= **EUd // MatrixForm**

Out[97]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{a''[t]}{a[t]} & 0 \\ 0 & 0 & 0 & -\frac{a''[t]}{a[t]} \end{pmatrix}$$

Not a vacuum solution, einstein tensor is not 0.

In[98]:= **GUdd // MatrixForm**

Out[98]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ a[t]\, a'[t] \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ \frac{a'[t]}{a[t]} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{a'[t]}{a[t]} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

In[99]:=
```
Part[GUdd, 1, 2, 2]
```
Out[99]= $a[t]\, a'[t]$

In[100]:=
```
Part[GUdd, 2, 1, 2]
```
Out[100]= $\frac{a'[t]}{a[t]}$

In[101]:=
```
Part[GUdd, 2, 2, 1]
```
Out[101]= $\frac{a'[t]}{a[t]}$

3.2

In[102]:=
```
g = {{-a[x]^2 , 0 , 0, 0},
   {0, 1, 0, 0},
   {0, 0, 1, 0},
   {0, 0, 0, 1}}
```
Out[102]= $\{\{-a[x]^2, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}\}$

In[103]:=
```
coord = {t, x, y, z}
```
Out[103]= {t, x, y, z}

In[104]:=
```
RGtensors[g, coord]
```
$$gdd = \begin{pmatrix} -a[x]^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$LineElement = -a[x]^2\, d[t]^2 + d[x]^2 + d[y]^2 + d[z]^2$$

$$gUU = \begin{pmatrix} -\frac{1}{a[x]^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gUU computed in 0.00047 sec

Gamma computed in 0.000952 sec

Riemann(dddd) computed in 0.001817 sec

Riemann(Uddd) computed in 0.001241 sec

Ricci computed in 0.000185 sec

Weyl computed in 0.002388 sec

Einstein computed in 0.000078 sec

Out[104]= All tasks completed in 0.009187

In[105]:= `GUdd // MatrixForm`

Out[105]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 \\ \frac{a'[x]}{a[x]} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{a'[x]}{a[x]} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} a[x]\,a'[x] \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

In[106]:=
```
g = {{-X^2 , 0 , 0, 0},
  {0, 1, 0, 0},
  {0, 0, 1, 0},
  {0, 0, 0, 1}}
```

Out[106]= $\{\{-X^2, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}\}$

In[107]:= `coord = {t, x, y, z}`

Out[107]= {t, x, y, z}

In[108]:= `RGtensors[g, coord]`

$$\text{gdd} = \begin{pmatrix} -X^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

LineElement $= -X^2\, d[t]^2 + d[x]^2 + d[y]^2 + d[z]^2$

$$\text{gUU} = \begin{pmatrix} -\frac{1}{X^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gUU computed in 0.000668 sec

Gamma computed in 0.001673 sec

Riemann (dddd) computed in 0.002794 sec

Flat Space!

Out[108]= Aborted after 0.007828

In[109]:=
```
Rdddd
```

Out[109]=
```
{{{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}},
 {{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}},
 {{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}},
 {{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
   {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}}
```

In[110]:=

In[111]:=
```
g = {{-x^4 , 0 , 0, 0},
   {0, 1, 0, 0},
   {0, 0, 1, 0},
   {0, 0, 0, 1}}
```

Out[111]= $\{\{-x^4, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}\}$

In[112]:=
```
coord = {t, x, y, z}
```

Out[112]=
```
{t, x, y, z}
```

In[113]:=
```
RGtensors[g, coord]
```

$$gdd = \begin{pmatrix} -x^4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

LineElement = $-x^4\, d[t]^2 + d[x]^2 + d[y]^2 + d[z]^2$

$$gUU = \begin{pmatrix} -\frac{1}{x^4} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
gUU computed in 0.000577 sec

Gamma computed in 0.00096 sec

Riemann (dddd) computed in 0.001984 sec

Riemann (Uddd) computed in 0.001498 sec

Ricci computed in 0.000183 sec

Weyl computed in 0.002395 sec

Einstein computed in 0.000082 sec
```

Out[113]= All tasks completed in 0.00965

In[114]:=

In[115]:= **RUddd // MatrixForm**

Out[115]//MatrixForm=

$$\begin{pmatrix}
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&-\frac{2}{x^2}&0&0\\\frac{2}{x^2}&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\
\begin{pmatrix} 0&-2x^2&0&0\\2x^2&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}
\end{pmatrix}$$

In[116]:= **Part[RUddd, 1, 2, 1, 2]**

Out[116]= $-\frac{2}{x^2}$

In[117]:= `Part[RUddd, 1, 2, 2, 1]`

Out[117]= $\frac{2}{x^2}$

In[118]:= `R`

Out[118]= $-\frac{4}{x^2}$

In[119]:=
```
g = {{-1 , 0 , 0, 0},
  {0, Exp[-(x^2 + y^2 + z^2)], 0, 0},
  {0, 0, Exp[-(x^2 + y^2 + z^2)], 0},
  {0, 0, 0, Exp[-(x^2 + y^2 + z^2)]}}
```

Out[119]= $\{\{-1, 0, 0, 0\}, \{0, e^{-x^2-y^2-z^2}, 0, 0\}, \{0, 0, e^{-x^2-y^2-z^2}, 0\}, \{0, 0, 0, e^{-x^2-y^2-z^2}\}\}$

In[120]:= `coord = {t, x, y, z}`

Out[120]= {t, x, y, z}

In[121]:= `RGtensors[g, coord]`

$$\text{gdd} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & e^{-x^2-y^2-z^2} & 0 & 0 \\ 0 & 0 & e^{-x^2-y^2-z^2} & 0 \\ 0 & 0 & 0 & e^{-x^2-y^2-z^2} \end{pmatrix}$$

$$\text{LineElement} = -d[t]^2 + e^{-x^2-y^2-z^2}\, d[x]^2 + e^{-x^2-y^2-z^2}\, d[y]^2 + e^{-x^2-y^2-z^2}\, d[z]^2$$

$$\text{gUU} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & e^{x^2+y^2+z^2} & 0 & 0 \\ 0 & 0 & e^{x^2+y^2+z^2} & 0 \\ 0 & 0 & 0 & e^{x^2+y^2+z^2} \end{pmatrix}$$

gUU computed in 0.002237 sec

Gamma computed in 0.003972 sec

Riemann (dddd) computed in 0.004701 sec

Riemann (Uddd) computed in 0.00167 sec

Ricci computed in 0.001064 sec

Weyl computed in 0.006724 sec

Einstein computed in 0.002555 sec

Out[121]= All tasks completed in 0.026086

In[122]:= `RUddd // MatrixForm`

Out[122]//MatrixForm=

$$\begin{pmatrix}
\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\\
\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&2-z^2&y\,z\\0&-2+z^2&0&-x\,z\\0&-y\,z&x\,z&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&y\,z&2-y^2\\0&-y\,z&0&x\,y\\0&-2+y^2&-x\,y&0\end{pmatrix}\\
\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&-2+z^2&-y\,z\\0&2-z^2&0&x\,z\\0&y\,z&-x\,z&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&-x\,z&x\,y\\0&x\,z&0&2-x^2\\0&-x\,y&-2+x^2&0\end{pmatrix}\\
\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&-y\,z&-2+y^2\\0&y\,z&0&-x\,y\\0&2-y^2&x\,y&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&x\,z&-x\,y\\0&-x\,z&0&-2+x^2\\0&x\,y&2-x^2&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}
\end{pmatrix}$$

^^ And so no singularities

In[123]:= `R`

Out[123]= $-2\,e^{x^2+y^2+z^2}\left(-6+x^2+y^2+z^2\right)$

In[124]:= `EUd // MatrixForm`

Out[124]//MatrixForm=

$$\begin{pmatrix}
e^{x^2+y^2+z^2}\left(-6+x^2+y^2+z^2\right) & 0 & 0 & 0\\
0 & e^{x^2+y^2+z^2}\left(-2+x^2\right) & e^{x^2+y^2+z^2}\,x\,y & e^{x^2+y^2+z^2}\,x\,z\\
0 & e^{x^2+y^2+z^2}\,x\,y & e^{x^2+y^2+z^2}\left(-2+y^2\right) & e^{x^2+y^2+z^2}\,y\,z\\
0 & e^{x^2+y^2+z^2}\,x\,z & e^{x^2+y^2+z^2}\,y\,z & e^{x^2+y^2+z^2}\left(-2+z^2\right)
\end{pmatrix}$$

Integrating from R=0 to R=∞, we get:

In[125]:= $\texttt{Integrate}\left[\texttt{Exp}\left[\frac{-1}{2}\,r^2\right],\ \{r,\ 0,\ \infty\}\right]$

Out[125]= $\sqrt{\frac{\pi}{2}}$

3.3

In[126]:=
```
g = {{-1 + f[x - t] , 0 , 0, 0},
    {0, 1 + f[x - t], 0, 0},
    {0, 0, 1, 0},
    {0, 0, 0, 1}};
g // MatrixForm
```

Out[127]//MatrixForm=

$$\begin{pmatrix} -1+f[-t+x] & 0 & 0 & 0 \\ 0 & 1+f[-t+x] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In[128]:=
```
coord = {t, x, y, z}
```

Out[128]= {t, x, y, z}

In[129]:=
```
RGtensors[g, coord]
```

$$\text{gdd} = \begin{pmatrix} -1+f[-t+x] & 0 & 0 & 0 \\ 0 & 1+f[-t+x] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{LineElement} = d[y]^2 + d[z]^2 + d[t]^2 (-1 + f[-t+x]) + d[x]^2 (1 + f[-t+x])$$

$$\text{gUU} = \begin{pmatrix} \frac{1}{-1+f[-t+x]} & 0 & 0 & 0 \\ 0 & \frac{1}{1+f[-t+x]} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gUU computed in 0.000903 sec

Gamma computed in 0.001293 sec

Riemann (dddd) computed in 0.001973 sec

Riemann (Uddd) computed in 0.001473 sec

Ricci computed in 0.000773 sec

Weyl computed in 0.0032 sec

Einstein computed in 0.000361 sec

Out[129]= All tasks completed in 0.012435

In[130]:=
```
Rdd // MatrixForm
```

Out[130]//MatrixForm=

$$\begin{pmatrix} -\frac{-f[-t+x]\, f'[-t+x]^2 - f''[-t+x] + f[-t+x]^2\, f''[-t+x]}{(-1+f[-t+x])\,(1+f[-t+x])^2} & 0 & 0 & 0 \\ 0 & -\frac{-f[-t+x]\, f'[-t+x]^2 - f''[-t+x] + f[-t+x]^2\, f''[-t+x]}{(-1+f[-t+x])^2\,(1+f[-t+x])} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[131]:=
```
EUd // MatrixForm
```

Out[131]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-f[-t+x]\, f'[-t+x]^2 - f''[-t+x] + f[-t+x]^2\, f''[-t+x]}{(-1+f[-t+x])^2\,(1+f[-t+x])^2} & 0 \\ 0 & 0 & 0 & \frac{-f[-t+x]\, f'[-t+x]^2 - f''[-t+x] + f[-t+x]^2\, f''[-t+x]}{(-1+f[-t+x])^2\,(1+f[-t+x])^2} \end{pmatrix}$$

We now consider the following metric:

In[132]:=
```
g = {{-1 + f[x - t] , - f[x - t], 0, 0},
    {- f[x - t], 1 + f[x - t], 0, 0},
    {0, 0, 1, 0},
    {0, 0, 0, 1}};
g // MatrixForm
```

Out[133]//MatrixForm=

$$\begin{pmatrix} -1+f[-t+x] & -f[-t+x] & 0 & 0 \\ -f[-t+x] & 1+f[-t+x] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In[134]:=
```
coord = {t, x, y, z}
```

Out[134]= {t, x, y, z}

In[135]:=
```
RGtensors[g, coord]
```

$$\text{gdd} = \begin{pmatrix} -1+f[-t+x] & -f[-t+x] & 0 & 0 \\ -f[-t+x] & 1+f[-t+x] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{LineElement} = d[y]^2 + d[z]^2 + d[t]^2\,(-1+f[-t+x]) - 2\,d[t]\times d[x]\times f[-t+x] + d[x]^2\,(1+f[-t+x])$$

$$\text{gUU} = \begin{pmatrix} -1-f[-t+x] & -f[-t+x] & 0 & 0 \\ -f[-t+x] & 1-f[-t+x] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gUU computed in 0.000596 sec

Gamma computed in 0.001291 sec

Riemann (dddd) computed in 0.001803 sec

Flat Space!

Out[135]= Aborted after 0.006029

In[136]:=
```
Rdddd // MatrixForm
```

Out[136]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \\ \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \end{pmatrix}$$

Generalising, we find:

In[137]:=
```
g = {{-1 + f[x - t, y, z], -f[x - t, y, z], 0, 0},
   {-f[x - t, y, z], 1 + f[x - t, y, z], 0, 0},
   {0, 0, 1, 0},
   {0, 0, 0, 1}};
g // MatrixForm
```

Out[138]//MatrixForm=

$$\begin{pmatrix} -1 + f[-t + x, y, z] & -f[-t + x, y, z] & 0 & 0 \\ -f[-t + x, y, z] & 1 + f[-t + x, y, z] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In[139]:=
```
coord = {t, x, y, z}
```

Out[139]= {t, x, y, z}

In[140]:=
```
RGtensors[g, coord]
```

$$gdd = \begin{pmatrix} -1+f[-t+x,y,z] & -f[-t+x,y,z] & 0 & 0 \\ -f[-t+x,y,z] & 1+f[-t+x,y,z] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

LineElement =

$d[y]^2 + d[z]^2 + d[t]^2 (-1 + f[-t+x, y, z]) - 2\, d[t] \times d[x] \times f[-t+x, y, z] + d[x]^2 (1 + f[-t+x, y, z])$

$$gUU = \begin{pmatrix} -1-f[-t+x,y,z] & -f[-t+x,y,z] & 0 & 0 \\ -f[-t+x,y,z] & 1-f[-t+x,y,z] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gUU computed in 0.000624 sec

Gamma computed in 0.001971 sec

Riemann (dddd) computed in 0.00323 sec

Riemann (Uddd) computed in 0.002458 sec

Ricci computed in 0.000348 sec

Weyl computed in 0.003267 sec

Einstein computed in 0.000335 sec

Out[140]= All tasks completed in 0.014753

In[141]:= **Rdd // MatrixForm**

Out[141]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\left(-f^{(0,0,2)}[-t+x,y,z] - f^{(0,2,0)}[-t+x,y,z]\right) & \frac{1}{2}\left(f^{(0,0,2)}[-t+x,y,z] + f^{(0,2,0)}[-t+x,y,z]\right) & 0 & 0 \\ \frac{1}{2}\left(f^{(0,0,2)}[-t+x,y,z] + f^{(0,2,0)}[-t+x,y,z]\right) & \frac{1}{2}\left(-f^{(0,0,2)}[-t+x,y,z] - f^{(0,2,0)}[-t+x,y,z]\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[142]:= **f1[xminust_ , y_, z_] = Exp[-y] Sin[z] Sin[xminust]**

f2[xminust_ , y_, z_] = Log[y^2 + z^2] Sin[xminust]

Out[142]= e^{-y} Sin[xminust] Sin[z]

Out[143]= Log[$y^2 + z^2$] Sin[xminust]

In[144]:= **D[D[f1[x - t, y, z], y], y] + D[D[f1[x - t, y, z], z], z] // Simplify**

Out[144]= 0

In[145]:= **D[D[f2[x - t, y, z], y], y] + D[D[f2[x - t, y, z], z], z] // Simplify**

Out[145]= 0

So both f1 and f2 are harmonic functions. Let’s double check they satisfy the vacuum field equaitons:

In[146]:=
```
g = {{-1 + f1[x - t, y, z], -f1[x - t, y, z], 0, 0},
   {-f1[x - t, y, z], 1 + f1[x - t, y, z], 0, 0},
   {0, 0, 1, 0},
   {0, 0, 0, 1}};
g // MatrixForm
```

Out[147]//MatrixForm=

$$\begin{pmatrix} -1 - e^{-y}\,\mathrm{Sin}[t - x]\,\mathrm{Sin}[z] & e^{-y}\,\mathrm{Sin}[t - x]\,\mathrm{Sin}[z] & 0 & 0 \\ e^{-y}\,\mathrm{Sin}[t - x]\,\mathrm{Sin}[z] & 1 - e^{-y}\,\mathrm{Sin}[t - x]\,\mathrm{Sin}[z] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In[148]:=
```
coord = {t, x, y, z}
```

Out[148]= {t, x, y, z}

In[149]:=
```
RGtensors[g, coord]
```

$$\mathrm{gdd} = \begin{pmatrix} -1 - e^{-y}\,\mathrm{Sin}[t - x]\,\mathrm{Sin}[z] & e^{-y}\,\mathrm{Sin}[t - x]\,\mathrm{Sin}[z] & 0 & 0 \\ e^{-y}\,\mathrm{Sin}[t - x]\,\mathrm{Sin}[z] & 1 - e^{-y}\,\mathrm{Sin}[t - x]\,\mathrm{Sin}[z] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

LineElement =

$$d[y]^2 + d[z]^2 + 2\,e^{-y}\,d[t] \times d[x]\,\mathrm{Sin}[t - x]\,\mathrm{Sin}[z] + e^{-y}\,d[x]^2\,(e^{y} - \mathrm{Sin}[t - x]\,\mathrm{Sin}[z]) - e^{-y}\,d[t]^2\,(e^{y} + \mathrm{Sin}[t - x]\,\mathrm{Sin}[z])$$

$$\mathrm{gUU} = \begin{pmatrix} -e^{-y}\,(e^{y} - \mathrm{Sin}[t - x]\,\mathrm{Sin}[z]) & e^{-y}\,\mathrm{Sin}[t - x]\,\mathrm{Sin}[z] & 0 & 0 \\ e^{-y}\,\mathrm{Sin}[t - x]\,\mathrm{Sin}[z] & e^{-y}\,(e^{y} + \mathrm{Sin}[t - x]\,\mathrm{Sin}[z]) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gUU computed in 0.002438 sec

Gamma computed in 0.013297 sec

Riemann (dddd) computed in 0.008017 sec

Riemann (Uddd) computed in 0.014566 sec

Ricci computed in 0.00035 sec

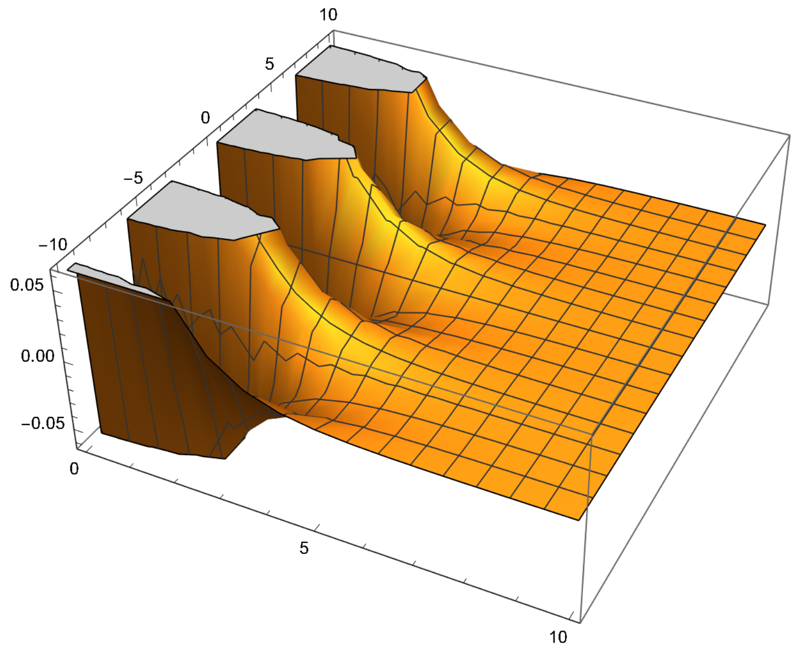
Weyl computed in 0.000031 sec

Ricci Flat

Out[149]= All tasks completed in 0.044088

In[150]:=
```
Plot3D[Evaluate[f1[xt, y, z] /. {xt → π / 2}], {y, 0, 10}, {z, -10, 10}]
```

Out[150]=



In[151]:=
```
Rdd // MatrixForm
```

Out[151]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[152]:=
```
g = {{-1 + f2[x - t, y, z] , - f2[x - t, y, z], 0, 0},
    {- f2[x - t, y, z], 1 + f2[x - t, y, z], 0, 0},
    {0, 0, 1, 0},
    {0, 0, 0, 1}};
g // MatrixForm
```

Out[153]//MatrixForm=

$$\begin{pmatrix} -1 - \mathrm{Log}[y^2 + z^2]\, \mathrm{Sin}[t - x] & \mathrm{Log}[y^2 + z^2]\, \mathrm{Sin}[t - x] & 0 & 0 \\ \mathrm{Log}[y^2 + z^2]\, \mathrm{Sin}[t - x] & 1 - \mathrm{Log}[y^2 + z^2]\, \mathrm{Sin}[t - x] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In[154]:=
```
coord = {t, x, y, z}
```

Out[154]= {t, x, y, z}

In[155]:=
```
RGtensors[g, coord]
```

$$\text{gdd} = \begin{pmatrix} -1-\text{Log}[y^2+z^2]\,\text{Sin}[t-x] & \text{Log}[y^2+z^2]\,\text{Sin}[t-x] & 0 & 0 \\ \text{Log}[y^2+z^2]\,\text{Sin}[t-x] & 1-\text{Log}[y^2+z^2]\,\text{Sin}[t-x] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{LineElement} = d[y]^2 + d[z]^2 + 2\,d[t] \times d[x]\,\text{Log}[y^2+z^2]\,\text{Sin}[t-x] + d[t]^2\,(-1-\text{Log}[y^2+z^2]\,\text{Sin}[t-x]) + d[x]^2\,(1-\text{Log}[y^2+z^2]\,\text{Sin}[t-x])$$

$$\text{gUU} = \begin{pmatrix} -1+\text{Log}[y^2+z^2]\,\text{Sin}[t-x] & \text{Log}[y^2+z^2]\,\text{Sin}[t-x] & 0 & 0 \\ \text{Log}[y^2+z^2]\,\text{Sin}[t-x] & 1+\text{Log}[y^2+z^2]\,\text{Sin}[t-x] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
gUU computed in 0.001261 sec

Gamma computed in 0.012206 sec

Riemann (dddd) computed in 0.009286 sec

Riemann (Uddd) computed in 0.01675 sec

Ricci computed in 0.000369 sec

Weyl computed in 0.000063 sec
```

Ricci Flat

Out[155]= All tasks completed in 0.044019

In[156]:=
```
Rdd // MatrixForm
```

Out[156]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[157]:=
```
Plot3D[Evaluate[f2[xt, y, z] /. {xt → π / 2}], {y, -10, 10}, {z, -10, 10}]
```

Out[157]=