Exploring Cosmology Worksheet Cameron Smith

2a)
The three truly maximally symmetric spacetimes are

1. Minkowski (flat) space (zero curvature)
2. de Sitter space (positive curvature)
3. anti-de Sitter space (negative curvature)

b)
The metric for a maximally symmetric spacetime can be written as:

$$ds^2 = -dt^2 + R^2(t)\,d\sigma^2,$$

$$d\sigma^2 = \gamma_{ij}(u)\,du^i du^j$$

maximally symmetric (3D) metric

u^i are the comoving coordinates. They are the coordinates that "move with the universe". An observer who is at constant u^i will see the universe as spherically symmetric, centred about them (isotropic).

c)
An observer on Earth is not well described by a fixed co-moving coordinate. We are in fact moving relative to the Universe, as can be seen from the dipole anisotropy in the CMB (one hemisphere is blue shifted, the other hemisphere is red shifted - we are moving in the direction of blue shift (and opposite red shift))

2d)

$$^{(3)}R_{ij} = {}^{(3)}R^{k}{}_{ikj}$$

$$= {}^{(3)}\gamma^{kl}\, R_{likj} \quad (\text{Since } \gamma^{kl} \text{ is the 3D metric})$$

but $^{(3)}R_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk})$

$$\Rightarrow {}^{(3)}R_{likj} = k(\gamma_{lk}\gamma_{ij} - \gamma_{lj}\gamma_{ik})$$

$$\Rightarrow {}^{(3)}R_{ij} = k\gamma^{kl}(\gamma_{lk}\gamma_{ij} - \gamma_{lj}\gamma_{ik})$$

$$= k(\gamma^{kl}\gamma_{lk}\gamma_{ij} - \gamma^{kl}\gamma_{lj}\gamma_{ik})$$

$$= k(3\gamma_{ij} - \delta^{k}{}_{j}\gamma_{ik})$$

$$= k(3\gamma_{ij} - \gamma_{ij})$$

$$= 2k\gamma_{ij}$$

2e)

We have:

$$d\sigma^2 = \gamma_{ij}\,du^i du^j = e^{2\beta(r)}d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

$$\Rightarrow \gamma_{ij} = \begin{pmatrix} e^{2\beta(r)} & 0 & 0 \\ 0 & \bar{r}^2 & 0 \\ 0 & 0 & \bar{r}^2\sin\theta \end{pmatrix}$$

2e) cont

Using Mathematica (Section 2e), we find Ricci tensor components:

$$^{(3)}R_{rr} = \frac{2\beta'(\bar{r})}{\bar{r}}, \quad ^{(3)}R_{\theta\theta} = e^{-2\beta(\bar{r})}\left(e^{2\beta(\bar{r})} + \bar{r}\beta'(\bar{r}) - 1\right),$$

$$^{(3)}R_{\varphi\varphi} = e^{-2\beta(\bar{r})}\sin^2\theta\left(e^{2\beta(\bar{r})} + r\beta'(\bar{r})\right)$$

But from 2d, we have:

$$R^{(3)}_{ij} = 2k\gamma_{ij}$$

$$\Rightarrow R_{rr} = \frac{2\beta'(\bar{r})}{\bar{r}} = 2k e^{2\beta(\bar{r})}$$

$$\Rightarrow \boxed{\beta'(\bar{r}) = k r e^{2\beta(\bar{r})}} \quad (1)$$

Similarly,

$$R_{\theta\theta} = e^{-2\beta(\bar{r})}\left(e^{2\beta(\bar{r})} + r\beta'(\bar{r}) - 1\right) = 2k\bar{r}^2$$

Sub in (1) and expand:

$$\Rightarrow 1 + k\bar{r}^2 - e^{-2\beta(\bar{r})} = 2k\bar{r}^2$$

$$\Rightarrow e^{-2\beta(\bar{r})} = 1 - k\bar{r}^2$$

$$\Rightarrow -2\beta(\bar{r}) = \ln(1 - k\bar{r}^2)$$

$$\Rightarrow \boxed{\beta(\bar{r}) = -\frac{1}{2}\ln(1 - k\bar{r}^2)}$$

2f)

Subbing this back in, we get:

$$ds^2 = -dt^2 + R^2(t)\, e^{2\beta(\bar{r})} d\bar{r}^2 + R^2(t)\bar{r}^2 d\Omega^2$$

$$\boxed{= -dt^2 + \frac{R^2(t)\, d\bar{r}^2}{1-k\bar{r}^2} + R^2(t)\bar{r}^2 d\Omega^2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$

2g)

make the substitution

$$k^{(new)} = \frac{k^{(old)}}{|k^{(old)}|}, \quad k^{(new)} = \text{sign}(k^{(old)}) = \begin{cases} 1 & k^{(old)} > 0 \\ 0 & k^{(old)} = 0 \\ -1 & k^{(old)} < 0 \end{cases}$$

$$\Rightarrow k^{(old)} = c\, k^{(new)}, \quad \text{where } c = |k^{(old)}|$$

$$\Rightarrow ds^2 = -dt^2 + \frac{R^2(t)\, d\bar{r}^2}{1-ck\bar{r}^2} + R^2(t)\bar{r}^2 d\Omega^2$$

make the change of coords:

$$\bar{r} \to \frac{\bar{r}}{\sqrt{c}} \Rightarrow d\bar{r} \to \frac{d\bar{r}}{\sqrt{c}}, \quad \text{and } R(t) \to \sqrt{c}\, R(t)$$

$$\Rightarrow ds^2 = -dt^2 + \frac{\cancel{c} R^2(t)\, d\bar{r}^2/\cancel{c}}{1-\cancel{c}k\bar{r}^2/\cancel{c}} + \cancel{c}R^2(t)\frac{\bar{r}^2}{\cancel{c}} d\Omega^2$$

$$\Rightarrow ds^2 = -dt^2 + R^2(t)\left[\frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2 d\Omega^2\right]$$

3a)

We define χ such that:

$$d\chi = \frac{d\bar{r}}{\sqrt{1-k\bar{r}^2}}$$

$$\Rightarrow \int d\chi = \int \frac{1}{\sqrt{1-k\bar{r}^2}} d\bar{r}$$

$$\Rightarrow \chi = \frac{\sin^{-1}(\sqrt{k}\,\bar{r})}{\sqrt{k}} + c$$ (Mathematica, 3a)

choose integration constant $c=0$ (this is fine as we define χ)

$$\Rightarrow \chi = \frac{\sin^{-1}(\sqrt{k}\,\bar{r})}{\sqrt{k}}$$

Inverting this in mathematica (3a), we find

$$\bar{r} = \begin{cases} \sin(\chi), & k=1 \\ \chi, & k=0 \\ \sinh(\chi), & k=-1 \end{cases}$$

3b)

for $k=1$,

$$ds^2 = -dt^2 + R^2(t)\left[d\chi^2 + \sin^2(\chi)\,d\Omega^2\right]$$

i.e. we get the metric for a sphere (see Assignment 1)

3b)

Therefore $K=1$ corresponds to spherical space, i.e. positive curvature.

for $k=0$, we get metric

$$ds^2 = -dt^2 + R^2(t)\left(d\chi^2 + \chi^2 d\Omega^2\right)$$

i.e. we get the minkowski metric (with scale factor R), i.e flat space

$\therefore$ $k=0$ corresponds to zero curvature

for $k=-1$, we get metric

$$ds^2 = -dt^2 + R^2(t)\left(d\chi^2 + \sinh^2(\chi)\,d\Omega^2\right)$$

which corresponds to hyperbolic space, i.e. negative curvature

Using the metric given in 2g, we can find the Christoffel symbols, and solve the geodesic equation:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0.$$

We do this in mathematica (3b), for initially parallel rays of light confined to the plane $\theta = \frac{\pi}{2}$

$\therefore$ we have initial conditions

$\theta(0) = \frac{\pi}{2}$, $\dot\theta(0)=0$

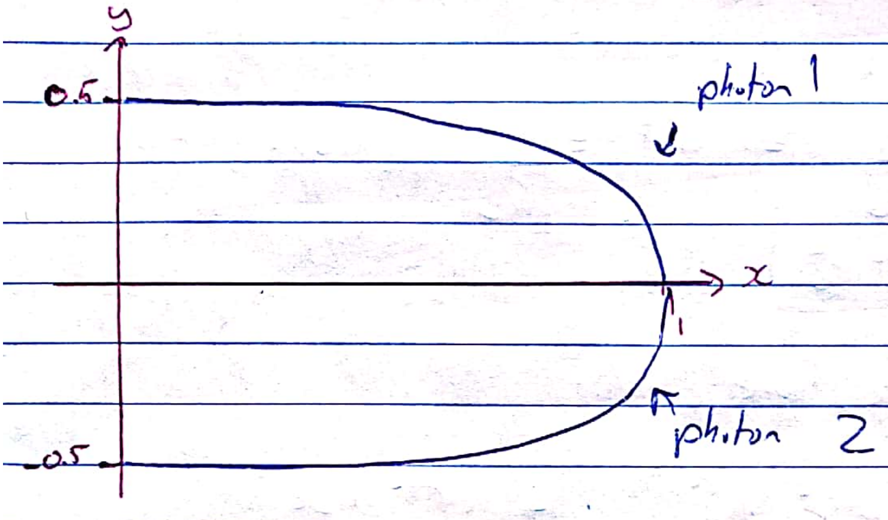
$r(0)=0.5$, $\dot r(0)=0$

$\varphi(0) = \pm\frac{\pi}{2}$, $\dot\varphi(0) = \mp C = \mp 1$

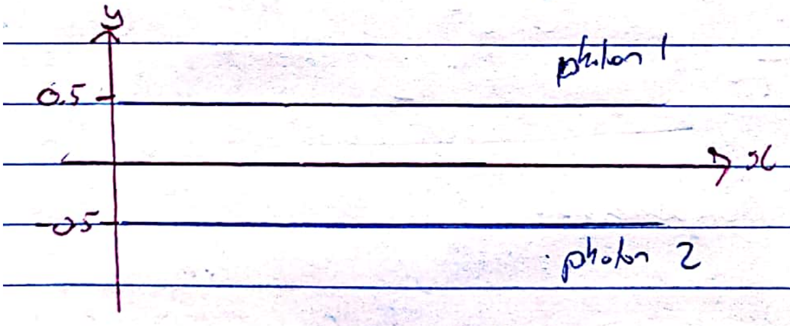
3b) Cont

We find: (I would print this, but don't have one at home)

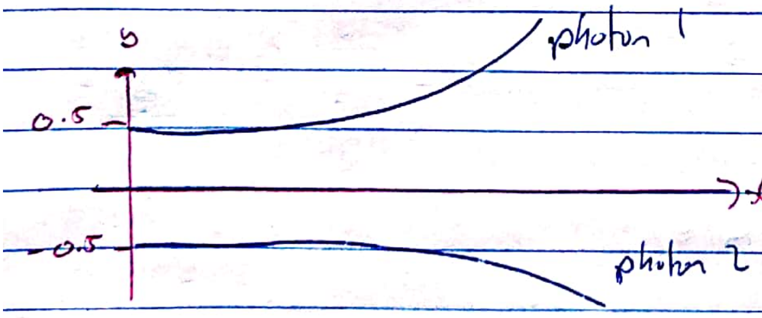
$K=1$:



$K=0$



$K=-1$



See Mathematica 3b for more details.

3c)

We have

$$a(t) = \frac{R(t)}{R_0}, \quad \Rightarrow R(t) = R_0 a(t)$$

$$r = R_0 \bar{r} \quad \Rightarrow \bar{r} = \frac{r}{R_0} \quad \Rightarrow d\bar{r} = \frac{dr}{R_0} \quad (R_0 \text{ is Const})$$

$$K \equiv \frac{k}{R_0^2} \quad \Rightarrow k = R_0^2 K$$

Subbing this into our metric, we find:

$$ds^2 = -dt^2 + \cancel{R_0^2} a^2(t) \left[\frac{dr^2/\cancel{R_0^2}}{1 - \cancel{R_0^2} K r^2/\cancel{R_0^2}} + \frac{r^2 d\Omega^2}{\cancel{R_0^2}} \right]$$

$$\Rightarrow ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - K r^2} + r^2 d\Omega^2 \right]$$

4a)

We find non-zero Ricci tensor components:

$$R_{tt} = -3 \frac{\ddot{a}(t)}{a(t)}$$

$$R_{rr} = \frac{2K + 2\dot{a}(t)^2 + a(t)\ddot{a}(t)}{1 - Kr^2}$$

$$R_{\theta\theta} = r^2 \left(2K + 2\dot{a}^2(t) + a(t)\ddot{a}(t) \right)$$

$$R_{\varphi\varphi} = r^2 \sin^2\theta \left(2K + 2\dot{a}(t)^2 + a(t)\ddot{a}(t) \right),$$

And Ricci Scalar

$$R = \frac{6\left(K + \dot{a}(t)^2 + a(t)\ddot{a}(t)\right)}{a(t)^2}$$

4b)

Consider moving with the fluid. From our point of view (reference frame), the fluid would not be moving relative to us. We would measure the velocity of the fluid through space to be:

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$$

(note for non-undegrads, we have $t = \tau$)

in spherical coordinates, this is equivalent to:

$$\frac{dr}{dt} = \frac{d\theta}{dt} = \frac{d\varphi}{dt} = 0$$

However every object (with mass) must move through time, and so we have

$$\frac{dt}{d\tau} = \frac{dt}{dt} = 1$$

(τ is defined as the time measured by the fluid - and therefore us)

the four - velocity is defined as

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right)$$

$$= \left(\frac{dt}{dt}, \frac{dr}{dt}, \frac{d\theta}{dt}, \frac{d\varphi}{dt}\right)$$

$$= (1, 0, 0, 0)$$

4c)

The stress energy tensor is given by

$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$

however, $U_\mu = (1, 0, 0, 0)$ for the comoving fluid,

$\therefore T^\mu{}_\nu = g^{\mu\kappa} T_{\kappa\nu}$

$= g^{\mu\sigma}\left((p+\rho) U_\sigma U_\nu + P g_{\sigma\nu} \right)$

$= (\rho + P) g^{\mu\sigma} U_\sigma U_\nu + P g^{\mu\sigma} g_{\sigma\nu}$

but $g^{\mu\sigma} U_\sigma U_\nu = U^\mu g_{\nu\gamma} U^\gamma$

$= g_{\nu\gamma} \delta^\mu_0 \delta^\gamma_0$

$= g_{\nu 0} \delta^\mu_0$

$= g_{00} \delta^\mu_0$ (since g is diagonal)

$= -\delta^\mu_0$

additionally,

$g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu{}_\nu$

$\Rightarrow T^\mu{}_\nu = -(\rho + P) \delta^\mu_0 + P \delta^\mu{}_\nu$

$\Rightarrow T^\mu{}_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} = \mathrm{diag}(-\rho, P, P, P).$

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4d)

Conservation of energy implies that

$$\nabla_\mu T^\mu{}_0 = 0$$

$$\Rightarrow \partial_\mu T^\mu{}_0 + \Gamma^\mu{}_{\mu\sigma} T^\sigma{}_0 - \Gamma^\sigma{}_{\mu 0} T^\mu{}_\sigma = 0$$

$$\Rightarrow \partial_0 T^0{}_0 + \Gamma^\mu{}_{\mu 0} T^0{}_0 - \Gamma^\mu{}_{\mu 0} T^\mu{}_\mu = 0 \quad (\text{Since } T^\mu{}_\nu \text{ is diagonal})$$

but $T^0{}_0 = -\rho$ and

$$\Gamma^\mu{}_{\mu 0} T^0{}_0 = -(\underbrace{\Gamma^0{}_{00}}_{=0} + \underbrace{\Gamma^1{}_{10}}_{=\frac{\dot{a}(t)}{a(t)}} + \underbrace{\Gamma^2{}_{20}}_{=\frac{\dot{a}(t)}{a(t)}} + \underbrace{\Gamma^3{}_{30}}_{=\frac{\dot{a}(t)}{a(t)}})\rho$$

(See mathematica Section 4d)

$$= -3\left(\frac{\dot{a}(t)}{a(t)}\right)\rho$$

$$-\Gamma^\mu{}_{\mu 0} T^\mu{}_\mu = -\left(0 + \frac{\dot{a}}{a}p + \frac{\dot{a}}{a}p + \frac{\dot{a}}{a}p\right) = -3\left(\frac{\dot{a}(t)}{a(t)}\right)p$$

$\therefore$ we find:

$$-\partial_0\rho - 3\left(\frac{\dot{a}(t)}{a(t)}\right)(\rho + p) = 0$$

4e)

If we have:

$p = w\rho$ we find:

$$-\partial_0\rho - 3\left(\frac{\dot{a}(t)}{a(t)}\right)(w+1)\rho = 0$$

4e) Cont

$$\Rightarrow \frac{1}{\rho}\frac{d\rho}{dt} = \frac{-3(1+w)}{a}\frac{da}{dt}$$

integrate both sides w.r.t. t:

$$\int \frac{1}{\rho} d\rho = \int \frac{-3(1+w)\, da}{a}$$

$$\ln(\rho) = -3(1+w)\ln(a) + c_1$$

$$\ln(\rho) = \ln\left(c_2 a^{-3(1+w)}\right), \quad \text{where } c_2 = \pm e^{c}$$

$$\Rightarrow \rho = c_2\, a^{-3(1+w)}$$

$$\Rightarrow \rho \propto a^{-3(1+w)}$$

4f)

i) Dust (matter) has no pressure (to a good approximation), however has a non-zero density (since it has mass). Therefore, since pressure relates to density by the factor w, we require $w=0$.

ii)

for radiation, we have $w = \frac{1}{3}$

$\Rightarrow \rho_R \propto a^{-4}$

Vacuum energy has a value of $w = -1$

$\Rightarrow \rho_V \propto a^0 = 1$

for matter we have $w=0$, giving:

$\rho_M \propto a^{-3}$

~~4(ii) Cont~~

4(iii)

We have

$\rho_V \propto a^0$, $\rho_m \propto a^{-3}$, $\rho_R \propto a^{-4}$

The universe has been expanding since the big bang, begining as a singularity (not 100% sure but very small), and expanding to it's current size today.

The size of the early universe (distant past) was very small, and so our scale factor $a(t)$ was also very small. for $a \ll 1$, we have

$$\rho_R \gg \rho_m \gg \rho_V$$

i.e. the early universe was dominated by radiation, with matter still dominating vacuum energy.

In the distant future, we will have a much larger universe (and therefore scale factor). for $a \gg 1$, we have

$$\rho_V \gg \rho_m \gg \rho_R$$

And so in the distant future vacuum energy will dominate, with the mass and radiation "spreading out" with the expansion of the universe.

5a)

The Einstein equations are given by:

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

the $\mu=0$, $\nu=0$ component is:

$$R_{00} = 8\pi G\left(T_{00} - \frac{1}{2} g_{00} T\right)$$

We have

$$T_{\mu\nu} = (\rho+p) U_\mu U_\nu + p g_{\mu\nu}, \quad \text{with } U^\mu = (1,0,0,0)$$

$$= (\rho+p) g_{\mu\sigma} g_{\nu\lambda} U^\sigma U^\lambda + p g_{\mu\nu} \quad (1)$$

$$\Rightarrow T_{00} = (\rho+p) g_{00} g_{00} U^0 U^0 + p g_{00} \quad (\text{since } g \text{ is diagonal})$$

$$= \rho + p - p = \rho$$

from 4c), we have

$$T^\mu{}_\nu = \mathrm{diag}(-\rho, p, p, p), \quad \text{therefore:}$$

$$T = T^\mu{}_\mu = 3p - \rho$$

Finally, from 4a), we have

$$R_{00} = -3\frac{\ddot{a}(t)}{a(t)}, \quad \text{giving:}$$

$$-3\frac{\ddot{a}(t)}{a(t)} = 8\pi G\left(\rho + \frac{1}{2}(3p - \rho)\right)$$

5 a) Cont

$$\Rightarrow \boxed{\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\left(\rho + 3p\right)}$$

For $\mu = i$, $\nu = j$ (i.e. $\mu \neq 0$, $\nu \neq 0$), we have:

from ①, we have:

$$T_{ij} = (\rho + p) g_{i\sigma} g_{j\nu} U^\sigma U^\nu + p g_{ij}$$

$$= p g_{ij} \quad (\text{Since } g_{ij} \text{ is diagonal, and } U^i = 0)$$

$$\therefore R_{ij} = 8\pi G \left(p g_{ij} - \tfrac{1}{2} g_{ij} (3p - \rho) \right)$$

$$R_{ij} = 4\pi G (\rho - p) g_{ij}$$

However, comparing R_{ij} with g_{ij}, we see

$R_{ij} = g_{ij} = 0$ $(i \neq j)$, (i.e. both diagonal), and:

$$\frac{R_{ii}}{g_{ii}} = \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2K}{a^2} \quad \text{for all } i$$

(See mathematica 5a)

$$\Rightarrow \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2K}{a^2} = 4\pi G (\rho - p)$$

but $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$, and so:

5a) Cont

$$\cancel{2}\left(\frac{\ddot{a}}{a}\right)^2 + \cancel{2}\frac{K}{a^2} = \cancel{4}^{2}\pi G\left(\rho - \cancel{p} + \frac{\rho}{3} + \cancel{p}\right)$$

$$\Rightarrow \boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}}$$

5b)

Consider a flat Universe with just one source of energy. we have:

$$a \propto t^{2/n}$$

for a matter dominated universe, we have

$$n = n_M = 3$$

$$\Rightarrow a \propto t^{2/3}$$

at $t=0$, we have $a=0$ i.e. the size of the universe is 0, and we therefore have a singularity at $t=0$

Similarly, for a radiation dominated universe, we have $n = n_R = 4$

$$\Rightarrow a \propto t^{2/4} = t^{1/2}$$

at $t=0$ we again have a singularity with $a=0$

5b) Cont

In either case, the universe begins as a singularity ($a=0$) and expands at an initially rapid, but continuously decreasing rate: