General Relativity – Formula Sheet

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Note that this formula sheet is not guaranteed to contain precisely the formulae and equations needed for the General Relativity exam. Beware of typos and čeck carefully before use!

I. BASIC TENSOR CALCULUS

Transformation of vectors:

$$V^{\mu'} = \left(\frac{\partial x'^{\mu'}}{\partial x^\alpha}\right) V^\alpha. \tag{1}$$

Transformation of one-forms (covectors, covariant vectors):

$$A_{\mu'} = \left(\frac{\partial x^\alpha}{\partial x'^{\mu'}}\right) A_\alpha. \tag{2}$$

Vectors and one-forms expressed in terms of coordinate basis vectors/one-forms:

$$\mathbf{X} = X^\mu \partial_\mu, \tag{3}$$
$$\mathbf{A} = A_\mu \,\mathrm{d}x^\mu. \tag{4}$$

Concepts to remember:

- Addition, tensor products, contraction, trace,
- Rank & type of tensors,
- Symmetry & antisymmetry of tensors,
- Einstein summation convention,
- Free indices on both sides of tensor equation must be the same,
- Can relabel summed indices to any other unused character ($A^\mu + B^{\mu\alpha} D_\alpha = C^\mu \to A^\mu + B^{\mu\lambda} D_\lambda = C^\mu$). ,
- To relabel free indices, must relabel *all* occurrences ($A^\mu + B^\mu = C^\mu \to A^\nu + B^\nu = C^\nu$).

II. FOUR-VELOCITY

The four-velocity u^μ of a particle is given by the derivative of its coordinates with respect to proper time/affine parameter:

$$u^\mu = \mathrm{d}x^\mu / \mathrm{d}\tau. \tag{5}$$

For massive point masses:

$$u^\mu u_\mu = -1. \tag{6}$$

For massless particles:

$$u^\mu u_\mu = 0. \tag{7}$$

III. THE METRIC

The metric is a rank $(0,2)$-tensor:

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \,\mathrm{d}x^\nu. \tag{8}$$

Length of curve

$$s = \int \sqrt{g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}. \tag{9}$$

Inverse $g^{\mu\nu}$ of the metric obeys

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu. \tag{10}$$

Raising and lowering of indices:

$$T^{\alpha\beta\ldots\mu}{}_{\nu\ldots} = g^{\mu\sigma} T^{\alpha\beta\ldots}{}_{\sigma\nu\ldots}. \tag{11}$$
$$T^{\alpha}{}_{\beta}{}^{\ldots}{}_{\mu\nu\ldots} = g_{\beta\sigma} T^{\alpha\sigma\ldots}{}_{\mu\nu\ldots}. \tag{12}$$

Norm of vector:

$$|\mathbf{u}| == \sqrt{g_{\mu\nu} u^\mu u^\nu}. \tag{13}$$

IV. PARALLEL TRANSPORT AND COVARIANT DERIVATIVES

Covariant derivative of scalar field:

$$\nabla_\nu f = f_{;\nu} = \partial_\nu f. \tag{14}$$

Covariant derivative of vector field:

$$\nabla_\nu X^\mu = X^\mu_{;\nu} = \partial_\nu X^\mu + \Gamma^\mu{}_{\nu\lambda} X^\lambda. \tag{15}$$

Covariant derivative of one-form:

$$\nabla_\nu A_\mu = A_{\mu;\nu} = \partial_\nu A_\mu - \Gamma^\lambda{}_{\nu\mu} A_\lambda. \tag{16}$$

Covariant derivative of higher-rank tensor:

$$\begin{aligned}\nabla_\lambda T^{\alpha\beta\ldots}{}_{\mu\nu\ldots} &= \partial_\lambda T^{\alpha\beta\ldots}{}_{\mu\nu\ldots} \\ &+ \Gamma^\alpha_{\lambda\sigma} T^{\sigma\beta\ldots}{}_{\mu\nu\ldots} + \Gamma^\beta_{\lambda\sigma} T^{\alpha\sigma\ldots}{}_{\mu\nu\ldots} + \ldots \\ &- \Gamma^\sigma_{\lambda\mu} T^{\alpha\beta\ldots}{}_{\sigma\nu\ldots} - \Gamma^\sigma_{\lambda\nu} T^{\alpha\beta\ldots}{}_{\mu\sigma\ldots} - \ldots.\end{aligned} \tag{17}$$

Directional derivative:

$$\nabla_{\mathbf{u}} T^{\alpha\beta\ldots}{}_{\mu\nu\ldots} = u^\lambda \nabla_\lambda T^{\alpha\beta\ldots}{}_{\mu\nu\ldots}. \tag{18}$$

Special case – directional derivative of vector field along tangent vector $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ of curve $x^\mu(\lambda)$:

$$\nabla_{\mathbf{u}}\mathbf{Y} = u^\lambda\left(\nabla_\lambda Y^\mu + \Gamma^\mu_{\lambda\nu}Y^\nu\right) = \frac{\mathrm{d}Y^\mu}{\mathrm{d}\lambda} + \Gamma^\mu_{\lambda\nu}Y^\nu u^\lambda. \quad (19)$$

Parallel transport of vector field $\mathbf{Y}$ along vector $\mathbf{u}$:

$$\nabla_{\mathbf{u}}\mathbf{Y} = u^\lambda \nabla_\lambda Y^\mu = 0. \quad (20)$$

Covariant derivative of the metric:

$$\nabla_\lambda g_{\mu\nu} = 0, \quad (21)$$
$$\nabla_\lambda g^{\mu\nu} = 0, \quad (22)$$
$$\nabla_\lambda \delta^\mu_\nu = 0. \quad (23)$$

Christoffel symbols in terms of metric:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right), \quad (24)$$

where commas denote partial derivatives.

Special case of a diagonal metric:

$$\Gamma^\lambda_{\mu\nu} = 0, \quad (25)$$
$$\Gamma^\lambda_{\mu\mu} = -\frac{1}{2g_{\lambda\lambda}}\partial_\lambda g_{\mu\mu}, \quad (26)$$
$$\Gamma^\lambda_{\lambda\mu} = \partial_\mu \ln\sqrt{|g_{\lambda\lambda}|} \quad (27)$$
$$\Gamma^\lambda_{\lambda\lambda} = \partial_\lambda \ln\sqrt{|g_{\lambda\lambda}|} \quad (28)$$

V. GEODESIC MOTION

Abstract form of geodesic equation:

$$\nabla_{\mathbf{u}}\mathbf{u} = 0. \quad (29)$$

Component form of geodesic equation:

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = 0, \quad (30)$$

where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, and τ is proper time (for time-like trajectories) or the affine parameter (for null trajectories of massless particles).

VI. CURVATURE

Riemann curvature tensor:

$$R^\mu{}_{\nu\alpha\beta} = \nabla_\alpha\Gamma^\mu_{\nu\beta} - \nabla_\beta\Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\nu\alpha}. \quad (31)$$

Symmetry properties of the Riemann tensor:

$$R^\mu{}_{\nu\alpha\beta} = -R^\mu{}_{\nu\beta\alpha} \quad (32)$$
$$R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha} \quad (33)$$
$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} \quad (34)$$
$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}, \quad (35)$$
$$R_{\mu\nu\alpha\beta} + R_{\mu\alpha\beta\nu} + R_{\mu\beta\nu\alpha} = 0. \quad (36)$$

Second Bianchi identity:

$$R_{\mu\nu\alpha\beta;\gamma} + R_{\mu\nu\beta\gamma;\alpha} + R_{\mu\nu\gamma\alpha;\beta} = 0. \quad (37)$$

Ricci tensor:

$$R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}. \quad (38)$$

Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = R^\mu{}_\mu. \quad (39)$$

Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \quad (40)$$

VII. FIELD EQUATIONS

Standard form:

$$G_{\mu\nu} = \kappa T_{\mu\nu}. \quad (41)$$

Alternative form:

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right). \quad (42)$$

Form with cosmological constant:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \quad (43)$$