A study of heterogeneous materials and wave propagation in phononic media

Bryan Chem

May 14th, 2018

Contents

1 Introduction

Heterogeneous materials appear in all kinds of forms and can be found naturally and in industrial applications. Recently, interest has grown in artificially constructed phononic media which can alter the propagation of acoustic and elastic waves. A phononic medium is a heterogeneous material which possesses spatial periodicity in the arrangement of its constituents, its structure, or the boundary conditions applied to it. As a result of this periodic structure, it has been shown that there exists band gaps for which certain frequencies of waves cannot propagate. These frequency band gaps have been utilized in many engineering applications related to vibrations and the control of acoustic and elastic waves.

There were three objectives for this independent study:

1. Study heterogeneous materials and the associated mathematical methods used to model their behavior
2. Study wave propagation in phononic media and the associated computational methods (mainly finite elements)
3. Complete a computational project related to the above studies

In this report, the focus will be on the later two items as the study of heterogeneous materials was completed via a set of exercises from the lecture notes of Prof. Ponte-Castañeda [1]. To this end, phononic media and their applications are presented, and the interesting behavior that is the presence of frequency band gaps where waves cannot propagate is described. After introducing these concepts, the history and state of active research is outlined in the literature review section. Dispersion relations and their relation to studies in phononic media are introduced and will be central to further studies in the report. While it is favorable to have exact analytical expressions for these dispersion relations, it is not possible to calculate this by hand for more complicated materials. As such we must utilize computational methods which are presented in varying levels of detail. For future research, it is planned to use the finite element method, and this is the computational method that is used in the independent study. Finite elements is used in the 1D and 2D case in order to obtain the dispersion relation for a variety of materials. These studies will be integral for further research that will advance the field.

1.1 Phononic media

As mentioned before, the field of phononic media has grown tremendously in the past two decades and continues to grow today. Examples of phononic media include mass spring systems, laminates, and materials embedded with a periodic lattice of inclusions (or voids). There are several levels of classification within phononic media. In the literature a distinction, is made between phononic materials and phononic structures. Phononic materials are infinite in extent while phononic structures are finite in extent [2]. Phononic crystals refer to a heterogenous material or non-uniform material the material phases that may be fluid or solid which have a periodic arrangement in space [3]. The constituents may be arranged in such a way as to have a band gap in a certain frequency range or may even be tunable such as by modulating an outside field the material properties may be changed in order to move the band gap. A further class of phononic media that is of much interest is the metamaterial. Metamaterials possess "local resonance" which is a different way for producing band gaps that is not seen in phononic crystals.

The existence of frequency band gaps in these materials has resulted in many practical applications such as:

- *Vibration control*: The simplest application would be to incorporate these phononic media in the construction of buildings or vehicles with band gaps in the range of frequencies of acoustic or elastic waves found in the operating environment in order to provide insulation.

- *Acoustic/elastic waveguides*: By simply removing inclusions in a phononic crystal it has been found that wave propagation can be localized to certain areas within the material [4].

- *Acoustic diodes*: It has been found that sonic crystals, which are phononic crystals where one or more of the material phases is a liquid, can be used in 1D acoustic diodes which allow certain waves to propagate in one direction but not in the opposite direction [5].

- *Subwavelength imaging*:

- *Cloaking*:

This, of course, is a non-exhaustive list with new studies in materials with tunable properties, damping, non-linear affects, and disorder other novel and useful applications will arise.

1.2 Band gaps

All of the above applications rely on presence of band gaps in these phononic media. There are two mechanisms which cause band gaps to form:

1. *Bragg scattering*: At certain frequencies, waves can interact with the structure of a material in such a way that reflections off of inclusions or the structure of material itself destructively interfere with the incident wave. Here, the behavior is dependent on the periodicity in the arrangement of the material phases or structure of the material [6].

2. *Local resonance*: When resonators are placed inside of a materials, interactions occur at the resonant frequency of the resonator. At these frequencies, the resonators begin to absorb energy from incident waves and produce a band gap.

These two mechanisms have predictable effects on the frequency band structure of phononic media.

2 Literature review

Here the field of phononic media is described. Much of the work as been applied and experimental in nature. There is plenty of room for theoretical studies building off of first principles in wave propagation and mechanics. These kinds of studies are much needed in order to fully and accurately characterize phononic media. First, past studies related to phononic media are reviewed in order to examine already explored avenues of research. Afterwards, more open fields of study are presented including damping, nonlinear systems, disordered phononic media, and dynamic effective properties.

2.1 History of phononic media

The development of photonic crystals in 1987 by Yablonovitch and John can be viewed as a jumping off point for the development of phonic media, since five years later Sigalas and Economou

2.2 Active research areas and open fields

2.2.1 Damping

In applications, energy is often lost through damping. This will affect the propagation of waves, and thus models that consider this effect will better represent the actual behavior of a material. Within the body of research on damping, studies focus on either harmonic wave propagation or free wave propagation. A study of harmonic wave propagation entails one in which the wave numbers are confined to be real while free wave propagation allows for complex wave numbers (and as a result evanescent waves). Standard computational methods as discussed in Section 4 are used in both types of studies.

In [3], dispersion relations are first obtained for a 1D diatomic lattice with linear viscous damping. The method of analysis for this example is then extended to a 1D lattice with internal resonators and also a 2D phononic crystal with square unit cells embedded with a single, denser square inclusion. The general result of including damping is the collapase of the band gap between the optical and acoustic branches of the dispersion relation. This occurs such that the optical branch lowers proportional to the amount of damping.

2.2.2 Nonlinear systems

Relative to linear studies, there have been fewer studies of nonlinear phononic materials. Due to nonlinearities, Bloch's theorem cannot not be applied in the same way to obtain an eigenvalue problem for the dispersion relation -at least not immediately. Perturbation techniques have been developed as early as 1994 in order to study weakly non-linear systems. These techniques "expand" the governing equations and produce a linear equation while instead considering the nonlinearities to be a part of forcing term.

2.2.3 Disordered materials

Just as sparse as nonlinear systems, the study of phononic media with disorder has not been widely researched. Incorporating even small amounts of disorder can increase the predictablility of a model much like damping. In the manufacturing process perfect periodicity may not be able to be acheived. The difficulty in disordered phononic media is that the conventional method of Bloch wave analysis does not work. A requirement of Floquet-Bloch theory is that the material properties vary periodically with a known period. A single unit cell can no longer model the behavior of the entire system if there is even mild disorder.

Recently, there has been research done on so called hyperuniform phononic crystals. Hyperuniformity is a way of categorizing point distributions. It specifically refers to distributions with the fluctuations in density completly vanish as the window of observation increases. Despite not having any periodicity, and therefore no Bragg peaks, there still exists band gaps.

An earlier paper in 2010 by, examined a 2d phononic medium with circular inclusions arranged in a lattice. Disorder was added to the system by increasing or decreasing the size of the inclusions or displacing them by some small amount. Here they used the plane-wave expansion method. Of course, due to the added disorder they were no longer able to look at a single unit cell and instead define a 'supercell' for which the analysis was carried out.

2.2.4 Dynamic effective properties

With any heterogeneous material, it is of interest to be able to define effective properties and model the material as one homogenous material. This reduces the computational cost of modeling. Most analysis in homogenization has been done in the long wavelength limit. Trying to perform a similar type of analysis on a material with dispersive properties results in the loss of this behavior. The problem here lies in the volume averages used to calculate the effective properties. The richness in behavior of these phononic media is due to the relative motion of contituents as such the behavior is said to be nonlocal and this must be taken into affect when trying to formulate dynamic effective properties.

Michael Frazier has done some work calculating the dynamic effective viscosity in damped 1D layered structures.

3 Dispersion relations

Dispersion relations provide all of the information necessary to characterize the behavior of waves propagating in a phononic medium. Mathematically, they are a relationship between the frequency of a wave and its wavenumber. From this relationship, it is possible to determine the phase velocity and group velocity. Dispersion relationships immediately indicate where the frequency band gaps of a system are. For this reason, they are also called frequency band diagrams or frequency band structures.

As will be shown, there are a variety of waves to calculate dispersion relations, however, there is a commonality between each type of analysis. This common feature is the application of Floquet-Bloch theory.

3.1 Floquet-Bloch theorem

Given the equation

$$\frac{\partial^2 u}{\partial x^2} + c(x)y = 0 \tag{1}$$

where c(x) is periodic in x, Floquet's theorem says that (1) has two solutions

$$\begin{aligned} u(x) &= \tilde{u}_1(x)e^{ikx} \\ &= \tilde{u}_2(x)e^{-ikx} \end{aligned}$$

In the above, $\tilde{f}_1$ and $\tilde{f}_2$ are also periodic in x [7].

Bloch's theorem is similar to Floquet's theorem but applies to higher dimensions [6].

Given

$$-\nabla \cdot (c(\mathbf{r})\nabla u(\mathbf{r})) = \omega^2 u(\mathbf{r}) \tag{2}$$

where $c(\mathbf{r})$ is periodic in space, Bloch's theorem says equation (2) has eigensolutions

$$u(\mathbf{r}) = \tilde{u}(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{3}$$

These eigensolutions are called *Bloch waves* and equation (2) is called the periodic Helmholtz equation.

3.2 First principles and example problems

3.2.1 Scalar wave equation

The scalar wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{4}$$

where u is the displacement and c is the wave speed. Assume a solution of the form

$$u(x,t) = A\sin(kx - \omega t) \tag{5}$$

In fact, we can assume $u(x,t) = f(kx - \omega t)$ and the following results will still hold. k is the wave number which describes the "spatial" frequency of the wave for a fixed time and ω is the angular frequency that describes the temporal frequency of the wave. Substituting this solution into the wave equation gives

$$\omega^2 = c^2 k^2 \tag{6}$$

Taking the square root of the above

$$\omega = \pm ck \tag{7}$$

This equation gives a relationship between the frequency of the wave and its wave number. We call relations of this type dispersion relations. In general, dispersion relations can be written in the form

$$\omega = \omega(k) \tag{8}$$

To get a flavor for what dispersion entails, we need to look at the phase velocities waves. The phase velocity of a wave is

$$v_p = \frac{\omega}{k} \tag{9}$$

Thus, for the scalar wave equation discussed in this introductory paragraph

$$v_p = c \tag{10}$$

This relation holds for all waves admissible according to (4). Here, every wave propagates at the same velocity regardless of frequency. In this case, we say that there is no dispersion. In systems with dispersion we expect a relation in the form (8) and also

$$v_p = v_p(\omega) = v_p(\omega(k)) \tag{11}$$

3.2.2 Monatomic lattice

Take a system of n masses and springs like in Figure 1 Lets write the equations of motion for the nth mass

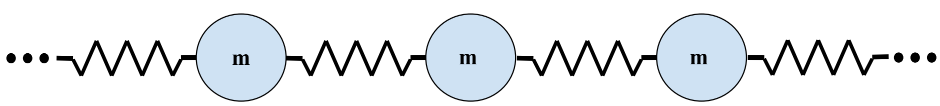

Figure 1: A system of masses connected by springs extending infinitely to the left and right.

system.

$$m\frac{\partial^2 u_n}{\partial t^2} = k(u_{n-1} - u_n) - k(u_n - u_{n+1}) = ku_{n-1} - 2ku_n + ku_{n+1} \tag{12}$$

where u_n gives the displacement of the nth node and k is the spring constant of each spring. To acquire, the dispersion relation for this system we assume a solution of the form

$$u_n = Ae^{i(kx_n - \omega t)} \tag{13}$$

A is the amplitude of the wave and $x_n = nl$ where l is the distance between the masses. Substituting the above into the equation of motion

$$\begin{aligned}
-m\omega^2 u_n &= ke^{i(k(n-1)l-\omega t)} - 2ku_n + ke^{i(k(n+1)l-\omega t)} \\
&= ke^{-ikl}u_n + ke^{ikl}u_n - 2ku_n \\
&= k\cos(kl)u_n - 2ku_n
\end{aligned}$$

Thus,

$$\left[2\omega_0^2(1-\cos(kl))-\omega^2\right]u_n = 0 \tag{14}$$

where $\omega_0 = \sqrt{k/m}$ The above has a trivial solution when $u_n = 0$. Not so trivial solutions exist when

$$2\omega_0^2(1-\cos(kl))-\omega^2 = 0 \tag{15}$$

We can write the dispersion relation for waves propagating through our mass spring system as

$$\omega = \pm\sqrt{2\omega_0^2(1-\cos(kl))} \tag{16}$$

It is now becoming apparent that in a system of n masses and springs that there is dispersion. To see this explicitly, write the phase velocity

$$v_p = \frac{w(k)}{k} = \frac{\pm\sqrt{2\omega_0^2(1-\cos(kl))}}{k} \tag{17}$$

Clearly, the phase velocity is a function of wave number/frequency which indicates dispersivity of the system.

Take our dispersion relation (16) and nondimensionalize by letting $\Omega = \frac{\omega}{\omega_0}$ and $\mu = kl$. Then,

$$\Omega = \pm\sqrt{2(1-cos(\mu))} \tag{18}$$

The positive part is plotted in Figure 2. The dispersion relation characterizes many features of the mass-

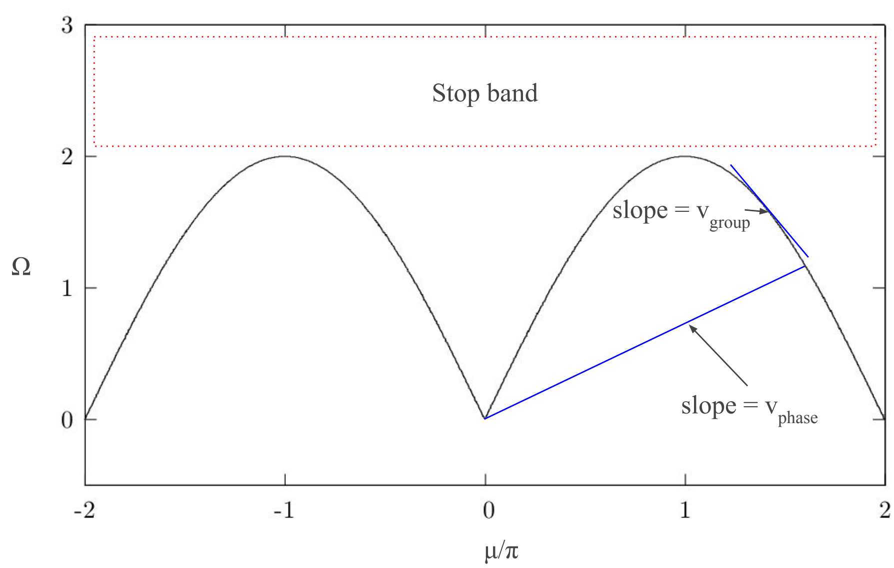

Figure 2: The non-dimensional dispersion relation for the mass spring system

spring system. In Figure 2, it is easy to identify the phase and group velocities along the curve for different waves propagating through the system.

At first glance, it appears that there are no frequencies above $\Omega = 2$. In this region (called the stop band), waves experience attenuation and exponentially die off. Wave numbers in the stop band are imaginary.

Looking at (13), it becomes obvious where the attenuation comes from. Waves whose wave numbers are imaginary are called *evanescent* waves.

Solving for wavenumber in terms of a given frequency

$$\mu = \arccos\left(1 - \frac{\Omega^2}{2}\right) \tag{19}$$

3.2.3 Diatomic lattice

Next, we consider a diatomic lattice as shown in Figure 3.

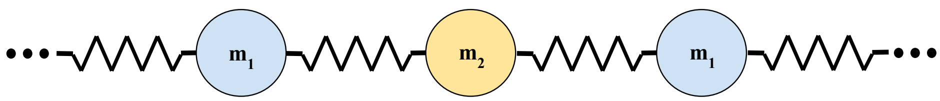

Figure 3: A system of alternating masses connected by springs extending infinitely to the left and right.

After assuming a harmonic motion

$$u_n = \tilde{u}(x_n)e^{i\omega t} \tag{20}$$

The equations of motion for a diatomic lattice can be written

$$(-\omega^2 m_2 + 2k)u_{2n} - k(u_{2n-1} + u_{2n+1}) = 0 \tag{21}$$

$$(-\omega^2 m_1 + 2k)u_{2n+1} - k(u_{2n} + u_{2n+2}) = 0 \tag{22}$$

We can write this in a matrix for

$$(\mathbf{K}_n - \omega^2\mathbf{M})\mathbf{u}_n + \mathbf{K}_{n-1}\mathbf{u}_{n-1} + \mathbf{K}_{n+1}\mathbf{u}_{n+1} = 0 \tag{23}$$

where

$$\mathbf{u}_n = \begin{bmatrix} u_{2n} \\ u_{2n+1} \end{bmatrix} \tag{24}$$

We will use the inverse method to solve for the dispersion relation. We use Bloch's theorem to write

$$\mathbf{u_n} = \hat{\mathbf{u}}(\mu)e^{in\mu} \tag{25}$$

Finally, we can write

$$\left[\mathbf{K}_n + \mathbf{K}_{n-1}e^{-i\mu} + \mathbf{K}_{n+1}e^{i\mu} - \omega^2\mathbf{M}\right]\hat{\mathbf{u}}(\mu)e^{in\mu} = 0 \tag{26}$$

$$\left[\bar{\mathbf{K}}(\mu) - \omega^2\mathbf{M}\right]\hat{\mathbf{u}}(\mu)e^{in\mu} = 0 \tag{27}$$

Calculating the determinant of the matrix on the left we get the relation

$$\omega = \pm\sqrt{k\frac{m_1+m_2}{m_1m_2} \pm k\sqrt{\left(\frac{m_1+m_2}{m_1m_2}\right) - \frac{4(\sin ka)^2}{m_1m_2}}} \tag{28}$$

3.2.4 Transfer matrix method

The transfer matrix method enforces continuity between layers of a system. At the interface between two unit cells the displacements must be equal and the forces or tractions must be as well. First, a relationship between these variables must be derived from one end of the unit cell to the other end. This results in the formulation of a so called transfer matrix. This type of analysis is completed by of course applying Bloch's theorem to relate the displacements and forces in one cell to another.

4 Computational methods for phononic media

4.1 Plane-wave expansion method

According to [8], the plane-wave method is widely used because of its convenience. Apparently, the drawback of the method is slow convergence.

1. To start expand the displacements and material properties in the wave equation in terms of a Fourier series (assert solutions in the form of Bloch waves)
2. Substitute these expansions back into the wave equation
3. After forming some kind of inner products, an eigenvalue problem can be found
4. In order to actually find the dispersion relations, one must solve the eigenvalue problem for a range of wave numbers

4.2 Finite difference method

Finite difference methods are similar to the plane wave expansion method with the advantage that the solution convergence is faster for materials with multiple phases

1. Instead of expanding the solution on material properties, the differential operators of the wave equation are expanded using standard finite differences
2. The equations of motion can be written as a matrix equation
3. An eigenvalue problem appears after asserting a Bloch wave type solution
4. Once again the eigenvalue problem must be solved for a range of wave numbers

4.3 Finite element method

The finite element method is useful for complex material geometries. When applied to periodic systems, the domain of computation is greatly reduced via application of Bloch's theorem. The standard outline for the finite element method is as follows

1. Formulate the weak form of the wave function by multiplying by a trial function and integrating
2. Write the equation in matrix form by allowing the solution to be expressed as a combination of weight functions
3. Apply the Bloch's theorem
4. Solve the resulting eigenvalue problem for a range of wave numbers

4.4 Multiple scattering method

Compared to the above methods, the multiple scattering method has an advantage when the material under consideration contains scattering elements (resonators?). The method relies on separating the solution into fields associated with the incident wave and then waves scattered from each scattering element in the system. Once the solution, at a single scattering element a system of equations can be solved for the rest of the scattered fields. The dispersion relations are then found by application of Bloch's theorem.

5 Analysis of lattice structures via real time evolution

5.1 Monatomic lattice

Here we verify the results of Section 3.2.2. The system is modeled using the equations of motion (12) and the Verlet integration method. Figure 4 displays the dispersion relation for the mass spring system which has been calculated from the real time evolution of the system. The system is excited in a range of frequencies between $\Omega = 0$ and $\Omega = 2$. For each of the frequencies, the system is allowed to evolve and the Fourier transform of the resulting wave is computed. In the reciprocal space, there is an obvious peak at one wave number. This wave number is then plotted with its corresponding frequency.

Applying a sinusoidal, forcing term to the first mass it is easy to explore the attenuation effect at the stop band as seen in Figure 2. Figures 5 and 6 display snapshots of the system at excitation frequencies above and below $\Omega = 2$. There is an obvious drop in amplitude inside and outside of the stop band.

5.2 Diatomic lattice

Here the results for a diatomic lattice are presented from a similar simulation to the above.

6 Analysis of 1D continuum materials via finite element method

6.1 Uniform bar

As a first exercise in using finite elements to find the dispersion properties of materials, we will solve for the real time evolution of a waves in a bar and find the dispersion relation in a similar manner as in the

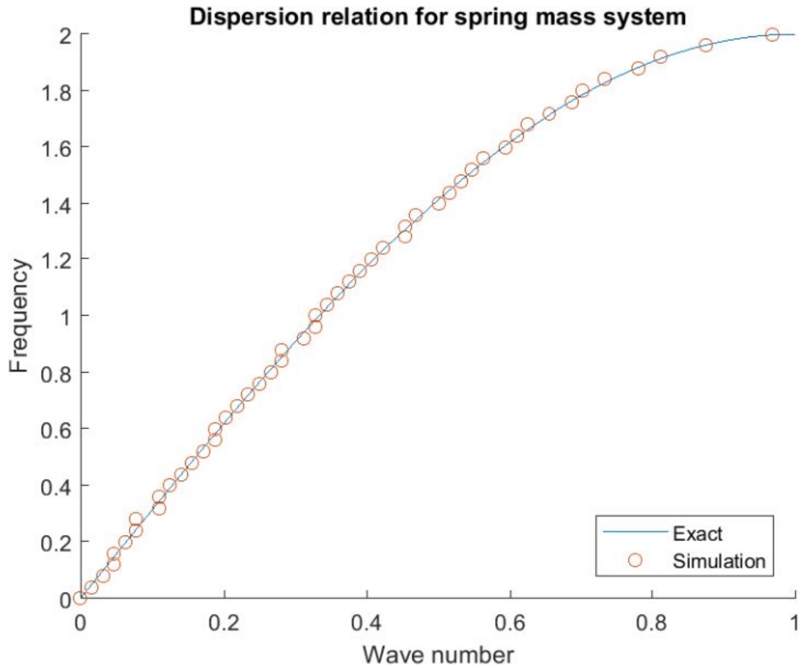

Figure 4: Using a Fourier transform in space, wave lengths for different excitation frequencies are identified and plotted

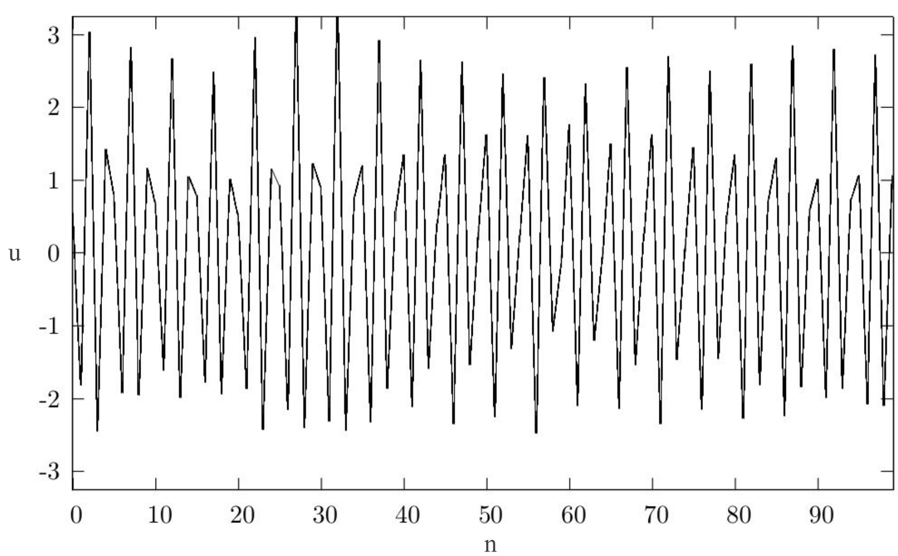

Figure 5: Snapshot of mass spring system with $\Omega = 1.9$

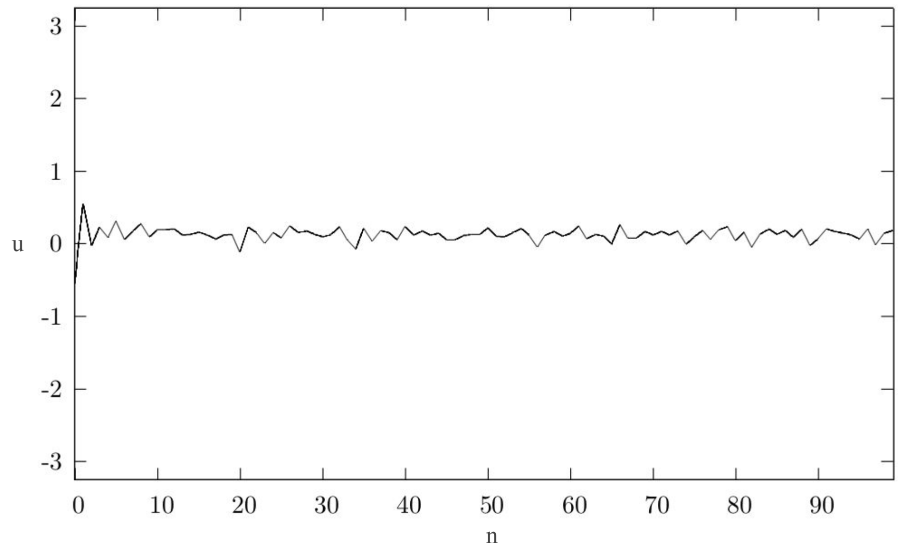

Figure 6: Snapshot of mass spring system with $\Omega = 2.1$

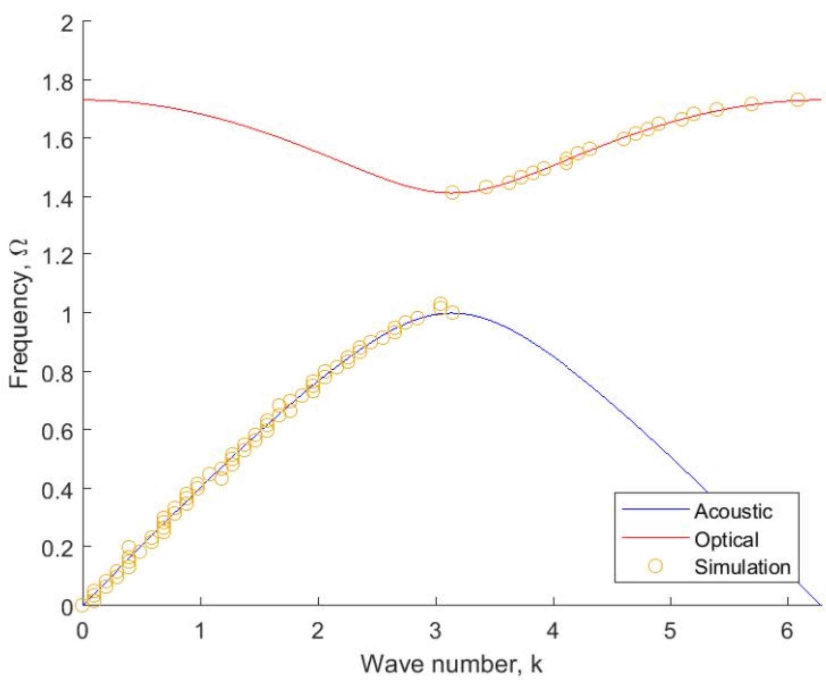

Figure 7: Dispersion relation for the diatomic lattice

mass-spring system (using the Fourier transform in space). For the one dimensional problem, the elasticity equations we want to solve are.

$$\frac{\partial \sigma_x}{\partial x} + b_i = \rho \frac{\partial^2 u_x}{\partial t^2}, \quad x \in [0, L] \quad \text{(Equilibrium)} \tag{29}$$

$$\sigma_x = E\varepsilon_x \quad \text{(Constitutive relation)} \tag{30}$$

$$\varepsilon_x = \frac{\partial u_x}{\partial x} \quad \text{(Strain-displacement)} \tag{31}$$

$$u_x(x = 0, t) = A_0 sin(\omega t) \tag{32}$$

$$\sigma_x(x = L, t) = 0 \tag{33}$$

Note that the finite element solution with n nodes is analogous to the mass spring system with n masses. Here, we can relate the material properties via $\rho = \frac{M}{AL}$ and $E = \frac{kL}{A}$. Thus, for a system of n masses, the analogous system in finite elements has material properties

$$\rho = \frac{n}{d(n-1)} \tag{34}$$

and

$$E = \frac{k_{eff} \times d(n-1)}{1} = (1/k + \cdots)^{-1} \times d(n-1) = \frac{d(n-1)}{n} \tag{35}$$

where $L = d(n-1)$ and we require that the nodes have the spacing with respect to each other as the springs to with each other.

In the absence of body forces, we can find the weak form of the equilibrium equation by multiplying by a test function φ and integrating of the domain.

$$\int_0^L \varphi \frac{\partial \sigma_x}{\partial x} dx = \int_0^L \varphi\rho \frac{\partial^2 u_x}{\partial t^2} dx \tag{36}$$

Using integration by parts on the LHS of the above we find that

$$\varphi(L)\sigma_x(L) - \varphi(0)\sigma_x(0) - \int_0^1 \frac{\partial \varphi}{\partial x} \sigma_x dx = \int_0^L \varphi\rho \frac{\partial^2 u_x}{\partial t^2} dx \tag{37}$$

Applying boundary conditions, constitutive relation, and the strain-displacement equation we arrive at the weak form of the equation

$$\int_0^1 \frac{\partial \varphi}{\partial x} E \frac{\partial u_x}{\partial x} dx + \int_0^L \varphi\rho \frac{\partial^2 u_x}{\partial t^2} dx = 0 \tag{38}$$

Now, we approximate our test function and solution using linear shape functions

$$\varphi = N_i(x) \tag{39}$$

$$u_x = \sum_j a_j(t) N_j(x) \tag{40}$$

Then,

$$\int_0^1 \frac{\partial N_i(x)}{\partial x} E \sum_j \frac{\partial N_j(x)}{\partial x} a_j dx + \int_0^L N_i \rho \sum_j N_j \frac{\partial^2 a_j(t)}{\partial t^2} dx = 0 \tag{41}$$

Rearranging,

$$\int_0^1 E \sum_j \frac{\partial N_i(x)}{\partial x} \frac{\partial N_j(x)}{\partial x} a_j dx + \int_0^L \rho \sum_j N_i N_j \frac{\partial^2 a_j(t)}{\partial t^2} dx = 0 \tag{42}$$

Finally, we write the above equation in matrix form

$$Ka + M\ddot{a} = 0 \tag{43}$$

where K is called the stiffness matrix, and M is called the mass matrix.

We now assume a solution of the form

$$a = ae^{i(kx - \omega t)} \tag{44}$$

such that the second order differential equation (43) becomes a complex eigenvalue problem:

$$Ka - \omega^2 Ma = (K - \omega^2 M)a = 0 \tag{45}$$

Note that as of now K and M are not functions of the wave number and solving the eigenvalue problem as is does not yield any information about the dispersion properties of the system. To obtain this dependence the Bloch condition must be enforced.

The Bloch condition is similar to the periodic boundary condition

$$a(x + h) = a(x) \quad \text{(Periodic boundary condition)} \tag{46}$$

where h is the size of the unit cell. The difference between the two conditions is a single multiplicative factor:

$$a(x + h) = e^{ik_x h} a(x) \quad \text{(Bloch condition)} \tag{47}$$

The multiplicative factor in the boundary condition shifts is such that when the wave reaches a boundary it is phase shifted to match the wave at the opposite boundary such that there is no interference.

Enforcing the condition is done via a linear transformation which in one dimension is

$$T = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ e^{ik_x h} & \mathbf{0} \end{bmatrix} \tag{48}$$

where **I** is an identity matrix with dimensions equal to the number of interior nodes (in the one dimensional case this is the total number of nodes minus two). Using this transformation matrix we reduce the number of degrees of freedoms in our system and write

$$a = \begin{bmatrix} a_{left} \\ a_{int} \\ a_{right} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ e^{ik_x h} & \mathbf{0} \end{bmatrix} \begin{bmatrix} a_{left} \\ a_{int} \end{bmatrix} = T\hat{a} \tag{49}$$

Our eigenvalue problem is now

$$(K - \omega^2 M)T\hat{a} = 0 \tag{50}$$

To make it so our matrix is square, and therefore invertible, we premultiply the equation by T^T

$$(\hat{K} - \omega^2 \hat{M})\hat{a} = 0 \tag{51}$$

where

$$\hat{K}(k_x) = T^T K T \tag{52}$$

and

$$\hat{M}(k_x) = T^T M T \tag{53}$$

We are now in position to extract the dispersion properties of the uniform bar. The eigenvalue problem is solved several times for values of k_x, and we obtain several corresponding values for ω.

6.2 Layered composite

7 Analysis of 2D continuum materials via finite element method

7.1 Uniform material

7.2 Laminate

7.3 Material with inclusions

8 Future work

9 Conclusion

10 Additional studies of heterogeneous materials

In addition to this study of phononic media, much time and effort was put into the notes of Prof. Ponte-Castañeda. Attached to this document, are solutions to various problems in the lecture notes.

11 Acknowledgements

I would like to thank my advisors Prof. Ponte-Castañeda and Prof. Reina for guidance this semester. Additionally, I would like to thank Chenchen Liu of Prof. Reina's group for helping me get a grip of using finite elements and Bloch analysis for this phononic media.

A Appendix

A.1 Verlet integration

For a single particle, the Verlet integration method is used to calculate its trajectory given that the forces (and therefore the acceleration) are known. The algorithm is

$$x_n^{t+1} = 2x_n^t - x_n^{t-1} + a(x_n^t)\Delta t^2 \tag{54}$$

where t denotes the current time step, x is the particle position, $a(x_n)$ is the acceleration, and Δt is suitably small time step. b

A.2 Finite difference in time for real time evolution FEM

For equation (43), it is possible to apply a finite difference method in time in order to get the real time evolution of the uniform bar. First, in order to apply the displacement boundary conditions at the left boundary, we need to modify (43). We partition the matrices

$$\begin{bmatrix} K_{11} & K_{1a} \\ K_{a1} & K_{aa} \end{bmatrix} \begin{bmatrix} a_1 \\ a \end{bmatrix} + \begin{bmatrix} M_{11} & M_{1a} \\ M_{a1} & M_{aa} \end{bmatrix} \begin{bmatrix} \ddot{a_1} \\ \ddot{a} \end{bmatrix} = 0 \tag{55}$$

Here, K_{11} and M_{11} denote the entry in the first row and first column. K_{1a} and M_{1a} are row vectors of the rest of the entries in the first row and K_{a1} and M_{a1} are column vectors of the remaining elements of the first column. K_{aa} and M_{aa} are then matrices of the remaining elements of the matrices K and M. Since we know a_1 and $\ddot{a_1}$ (these are the displacement and acceleration at the first node), we only wish to solve the bottom row of (55).

$$K_{aa}a + M_{aa}\ddot{a} = -a_1 K_{a1} - \ddot{a_1} M_{a1} \tag{56}$$

To solve the above we discretize $\ddot{a}$ in time using an explicit finite difference scheme.

$$K_{aa}a^t + M_{aa}\frac{a^{t+1} - 2a^t + a^{t-1}}{\Delta t^2} = -a_1^{t+1} K_{a1} - \ddot{a_1}^{t+1} M_{a1} \tag{57}$$

Solving for a^{t+1}

$$a^{t+1} = M_{aa}^{-1}(-a_1^{t+1} K_{a1} - \ddot{a_1}^{t+1} M_{a1} - K_{aa}a^t)\Delta t^2 + 2a^t - a^{t-1} \tag{58}$$

References

[1] Pedro Ponte Castañeda. Heterogeneous materials. Lecture Notes, Ecole Polytechnique, Department of Mechanics, ISBN 2 - 7302 - 1267 - 1.

[2] Mahmoud Hussein and Michael Frazier. Band structure of phononic crystals with general damping. *Journal of Applied Physics*, 108, 2010.

[3] Mahmoud I. Hussein, Michael J. Leamy, and Massimo Ruzzene. Dynamics of phononic materials and structures: Historical origins, recent progress, and future outlook. *Applied Mechanics Reviews*, 66, 2014.

[4] M. Badreddine Assouar, Matteo Senesi, Mourad Oudich, Massimo Ruzzene, and Zhilin Hou. Broadband plate-type acoustic metamaterial for low-frequency sound attenuation. *Appl. Phys. Lett.*, 101, 2012.

[5] B. Liang, X. S. Guo, J. Tu, D. Zhang, and J. C. Cheng.

[6] Vincent Laude. *Phononic Crystals : Artificial Crystals for Sonic, Acoustic, and Elastic Waves*. 2015.

[7] Wilhelm Magnus and Stanley Winkler. *Hill's Equation*. 1979.

[8] Yongjun Cao, Zhilin Hou, and Youyan Liu. Convergence problem of plane-wave expansion method for phononic crystals. *Physics Letters A*, 327(2):247 – 253, 2004.