On the probability of generating a primitive matrix

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Joint work with Yong Feng, Yang Liu and Wenyuan Wu

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What is a primitive matrix?

Primitive vector $\boldsymbol{x} \in \mathbb{Z}^n$:

- Definition: $\boldsymbol{x} = d\boldsymbol{y}$ for $\boldsymbol{y} \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ implies $d = \pm 1$.
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Primitive matrix $\boldsymbol{A} \in \mathbb{Z}^{k \times n}$ with $k \leq n$:

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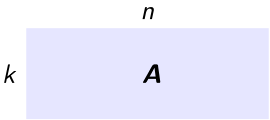
What is our problem?

- For a given primitive matrix $\boldsymbol{A} \in \mathbb{Z}^{k \times n}$ with $\|\boldsymbol{A}\| = \max_{i,j} |a_{i,j}| \le \lambda$



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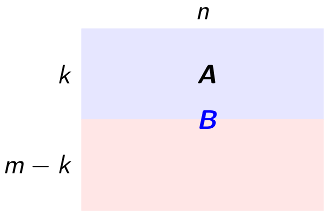

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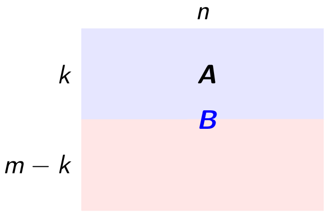

- **What is the probability** of that $\boldsymbol{B}$ is still primitive ?

Motivation

- Unimodular matrices has many applications.
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  - **Problem 1**: How many rows can we randomly choose ?
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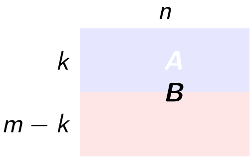
  - **Problem 3**: How fast is the algorithm ?

Our result on the probability

- A primitive matrix $\boldsymbol{A} \in \mathbb{Z}^{k \times n}$ with $\|\boldsymbol{A}\| \leq \lambda$
- An integer s with $0 \leq s \leq n - k - 2$
- $\boldsymbol{B} \in \mathbb{Z}^{(n-s-1) \times n}$: a completion of $\boldsymbol{A}$ with unif. rand. entries from Λ

Then the probability of that $\boldsymbol{B}$ is primitive is at least

$$1 - 4\left(\frac{2}{3}\right)^{s+1}\left(1 - \left(\frac{2}{3}\right)^{n-k-s-1}\right) - \frac{2(n-s-1)^2}{\lambda^{s+2}}\left(1 - \frac{1}{\lambda^{n-k-s-1}}\right).$$



$m = (n-1) - s$

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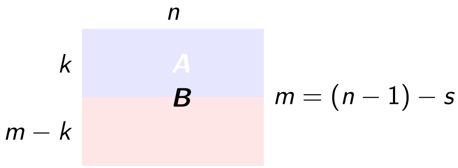
- When λ is large, the bound could be even simpler.
- E.g., if $s = 3$, then the probability is ≥ 0.2.
- The bound is **effective** only if $s \ge 3$!

Related work



- Maze, Rosenthal & Wagner '11: For $k = 0$, the natural density is

$$\prod_{j=s+2}^{n} \frac{1}{\zeta(j)} \quad (\lambda \to \infty),$$

  where $\zeta(\cdot)$ is the Riemann's zeta function.
- Fontein & Wocjan '14:
  - For $k \geq 2n+1$, a probability is rigorously proven.
  - For $n+1 \leq k < 2n+1$, a probability is conjectured.

Roadmap

1. **Proof of the result**

2. **Application to unimodular matrix completion**

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The idea of the proof

For $i = k, \ \ldots, \ n-s-1$, define

$$\boldsymbol{A}_i = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_i \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,n} \end{pmatrix}.$$

Idea: Give an upper bound on the probability of the event that $\boldsymbol{A}_{n-s-1}$ is not primitive under the assumption that $\boldsymbol{A}_k$ is primitive.

Tool: If $\boldsymbol{A}_i$ is not primitive, then there must be at least one prime p such that $\mathrm{rank}(\boldsymbol{A}_i) \le i-1$ over $\mathbb{Z}_p$.

Some events and their probabilities

MDep_i: There exists at least one prime p s.t. $\mathrm{rank}(\boldsymbol{A}_i) \leq i - 1$ over $\mathbb{Z}_p$.

$\neg\mathrm{MDep}_i$: $\boldsymbol{A}_i$ is a primitive matrix.

Goal: Give an upper bound on $\Pr[\mathrm{MDep}_{n-s-1} | \neg\mathrm{MDep}_k]$.

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$$\begin{aligned} & \Pr[\mathrm{MDep}_{n-s-1} | \neg\mathrm{MDep}_k] \\ \leq & \quad \cdots\cdots \\ \leq & \sum_{i=k+1}^{n-s-1} \Pr[\mathrm{MDep}_i \wedge \neg\mathrm{MDep}_{i-1}] \\ \leq & \sum_{i=k+1}^{n-s-1} \Pr[\mathrm{MDep}_i | \neg\mathrm{MDep}_{i-1}]. \end{aligned}$$

Bound $\Pr[\mathbf{MDep}_i | \neg \mathbf{MDep}_{i-1}]$

Let $\lambda \geq 2$ be an integer and $k+1 \leq i \leq n-3$.

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The case of $p < \lambda$

$$\Pr[\mathrm{MDep}_i | \neg \mathrm{MDep}_{i-1}] \leq \left(\frac{2}{3}\right)^{n-i+1} + \frac{3}{4}\left(\frac{1}{3}\right)^{n-i+1}.$$

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Remark. The analysis is adapted from Eberly, Giesbrecht & Villard (2000).

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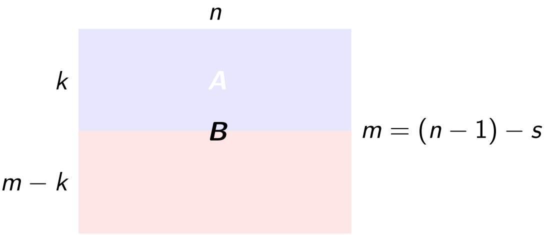

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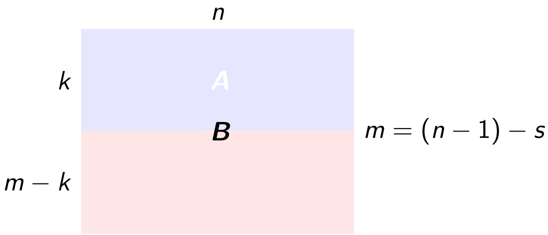

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A heuristic based on an extensively experimental study:

A constant lower bound on the probability exists for $s = 0, 1, 2$ as well.

Roadmap

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Hermite normal form

Non-singular matrix $\boldsymbol{H} \in \mathbb{Z}^{n\times n}$ is in Hermite normal form if

- $\boldsymbol{H}$ is upper triangular with non-negative entries,
- $h_{i,j} < h_{j,j}$.

$$\mathrm{HNF}(\boldsymbol{A}) = \begin{pmatrix} 1 & 0 & 0 & 10 & 260\,246\,748 \\ 0 & 1 & 0 & 2 & 292\,062\,707 \\ 0 & 0 & 1 & 7 & 244\,095\,302 \\ 0 & 0 & 0 & 14 & 342\,954\,195 \\ 0 & 0 & 0 & 0 & 344\,319\,363 \end{pmatrix}$$

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$$\boldsymbol{A} = \begin{pmatrix} -66 & -65 & 20 & -90 & 30 \\ 55 & 5 & -7 & -21 & 62 \\ 68 & 66 & 16 & -56 & -79 \\ 13 & -41 & -62 & -50 & 28 \\ 26 & -36 & -34 & -8 & -71 \end{pmatrix} \quad \mathrm{HNF}(\boldsymbol{A}) = \begin{pmatrix} 1 & 0 & 0 & 10 & 260\,246\,748 \\ 0 & 1 & 0 & 2 & 292\,062\,707 \\ 0 & 0 & 1 & 7 & 244\,095\,302 \\ 0 & 0 & 0 & 14 & 342\,954\,195 \\ 0 & 0 & 0 & 0 & 344\,319\,363 \end{pmatrix}$$

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Algorithm 1

Input: An integer matrix $\boldsymbol{A} \in \mathbb{Z}^{n\times n}$.

Output: A matrix $\boldsymbol{B} \in \mathbb{Z}^{n\times n}$, with $\boldsymbol{B}$ equal to $\boldsymbol{A}$ except for the last column, $\|\boldsymbol{B}\| \leq n^2\|\boldsymbol{A}\|$, and the last diagonal of HNF($\boldsymbol{B}$) equal to 1.

Proposition

Given an $n \times n$ integer matrix $\boldsymbol{A}$, Algorithm 1 is a correct Las Vegas algorithm and requires at most $O(n^{\omega+\varepsilon}\log^{1+\varepsilon}\|\boldsymbol{A}\|)$ bit operations.

Iterated determinant reduction (Storjohann '05)

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$$

$$
\mathrm{HNF}(BP) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 14 \end{pmatrix} \qquad \mathrm{HNF}(C) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

Unimodular matrix completion

Theorem

Given a primitive matrix $\boldsymbol{A} \in \mathbb{Z}^{k \times n}$, there exists a Las Vegas algorithm that completes $\boldsymbol{A}$ to an $n \times n$ unimodular matrix $\boldsymbol{U}$ such that

$$\|\boldsymbol{U}\| \leq n^{O(1)} \|\boldsymbol{A}\|$$

in an expected number of

$$O(n^{\omega+\varepsilon} \log^{1+\varepsilon} \|\boldsymbol{A}\|)$$

bit operations.

- The standard method: $O((n-k) n^{\omega+\varepsilon} \log^{1+\varepsilon} \|\boldsymbol{A}\|)$.

Conclusion

Given a primitive $\boldsymbol{A} \in \mathbb{Z}^{k\times n}$, consider to complete $\boldsymbol{A}$ to an $(n-s-1)\times n$ matrix with uniformly random integers in $[0, \|\boldsymbol{A}\|)$.

- We present a rigorous proof of the probability for $3 \le s \le n-k-2$.
  - Previously, only the limit probability when $\lambda \to \infty$ is known for $k = 0$.

Conclusion

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Conclusion

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Open problems

- A rigorous proof for $0 \leq s \leq 2$?
- And for $-n-2 < s < -1$?
- Other distributions?

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THANKS