

Question:

$$\int_0^{+\infty} e^{-x^2} \, dx$$

Answer:

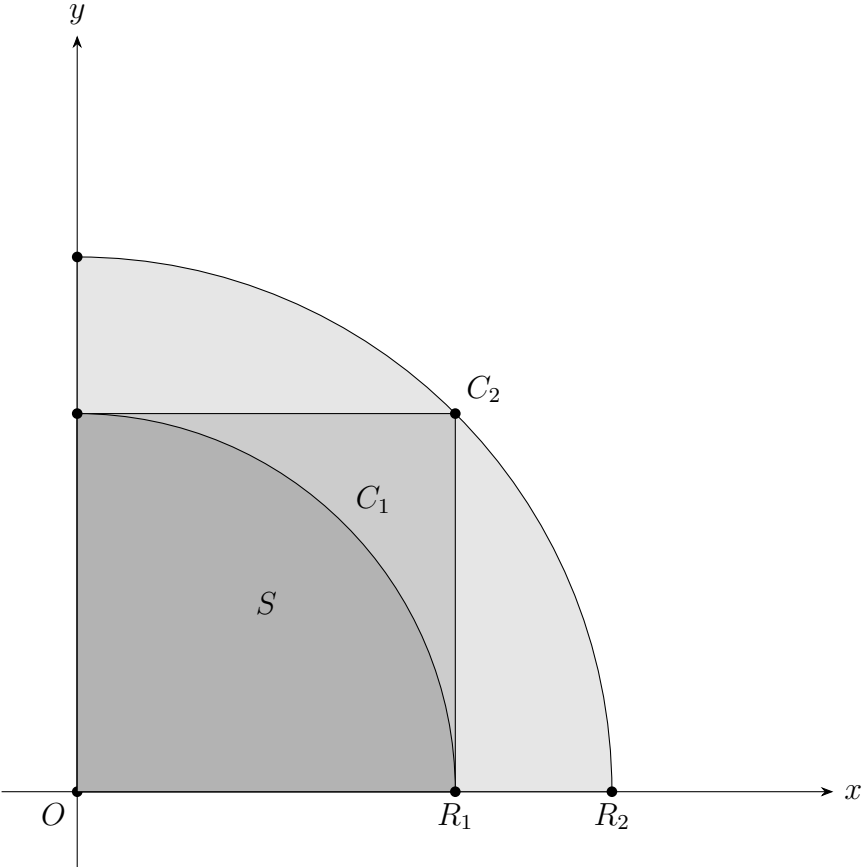


Figure 1: C_1 : small circle. C_2 : big circle. S : Square.

let

$$f(x, y) = e^{-x^2 - y^2}$$

because

$$R_2 = \sqrt{2}R_1$$

therefore

$$\begin{aligned} \iint_{C_1} f(x, y) \, dx \, dy &\leq \iint_S f(x, y) \, dx \, dy && \leq \iint_{C_2} f(x, y) \, dx \, dy \\ \iint_{C_1} e^{-x^2 - y^2} \, dx \, dy &\leq \int_0^{R_1} e^{-x^2} \, dx \cdot \int_0^{R_1} e^{-y^2} \, dy && \leq \iint_{C_2} e^{-x^2 - y^2} \, dx \, dy \\ \frac{\pi}{4}(1 - e^{-R_1^2}) &\leq \left(\int_0^{R_1} e^{-x^2} \, dx \right)^2 && \leq \frac{\pi}{4}(1 - e^{-2R_1^2}) \\ \lim_{R_1 \rightarrow +\infty} \frac{\pi}{4}(1 - e^{-R_1^2}) &\leq \lim_{R_1 \rightarrow +\infty} \left(\int_0^{R_1} e^{-x^2} \, dx \right)^2 && \leq \lim_{R_1 \rightarrow +\infty} \frac{\pi}{4}(1 - e^{-2R_1^2}) \\ \frac{\pi}{4} &\leq \left(\int_0^{+\infty} e^{-x^2} \, dx \right)^2 && \leq \frac{\pi}{4} \end{aligned}$$

therefore

$$\int_0^{+\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$$