



Figure 1

For each small rectangle, the Width is:

$$W_i = \frac{b - a}{n} \quad (\text{Divided evenly into } n \text{ parts})$$

And the Height is:

1. Take the right endpoint x_i ($i = 1, 2, 3, \dots, n$)

$$H_i = f(x_i) = f(a + i \cdot W) = f(a + i \cdot \frac{b-a}{n})$$

2. Take the left endpoint x_{i-1} ($i = 1, 2, 3, \dots, n$)

$$H_i = f(x_{i-1}) = f(a + (i-1) \cdot W) = f(a + (i-1) \cdot \frac{b-a}{n})$$

3. Take the middle point c_i ($i = 1, 2, 3, \dots, n$)

$$c_i = \frac{x_{i-1} + x_i}{2} = \frac{a + (i-1) \cdot \frac{b-a}{n} + a + i \cdot \frac{b-a}{n}}{2} = a + \frac{2i-1}{2} \cdot \frac{b-a}{n}$$

$$H_i = f(c_i) = f(a + \frac{2i-1}{2} \cdot \frac{b-a}{n})$$

So, the Area of the small rectangle is:

$$A_i = W_i \cdot H_i \quad (i = 1, 2, 3, \dots, n)$$

$$= \begin{cases} \frac{b-a}{n} \cdot f(a + i \cdot \frac{b-a}{n}) & \text{Take the right endpoint } x_i \\ \frac{b-a}{n} \cdot f(a + (i-1) \cdot \frac{b-a}{n}) & \text{Take the left endpoint } x_{i-1} \\ \frac{b-a}{n} \cdot f(a + \frac{2i-1}{2} \cdot \frac{b-a}{n}) & \text{Take the middle point } c_i \end{cases}$$

So, the Definite Integral is:

$$\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n A_i$$

$$= \begin{cases} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b-a}{n} \cdot f\left(a + \frac{i(b-a)}{n}\right) & \text{Take the right endpoint } x_i \\ \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b-a}{n} \cdot f\left(a + \frac{(i-1)(b-a)}{n}\right) & \text{Take the left endpoint } x_{i-1} \\ \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{b-a}{n} \cdot f\left(a + \frac{(2i-1)(b-a)}{2n}\right) & \text{Take the middle point } c_i \end{cases}$$

Specifically, when $a = 0$ and $b = 1$:

$$\int_0^1 f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n A_i$$

$$= \begin{cases} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \cdot f\left(\frac{i}{n}\right) & \text{Take the right endpoint } x_i \\ \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \cdot f\left(\frac{i-1}{n}\right) & \text{Take the left endpoint } x_{i-1} \\ \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \cdot f\left(\frac{2i-1}{2n}\right) & \text{Take the middle point } c_i \end{cases}$$