From Data to Insights - Exercise sheet 1

discussed in week after 1st lecture

April 22, 2024

Supplement on random variables / estimators:

In the 1st lecture I have used the terms "random variable" and "estimator" somewhat interchangeably. And indeed: every estimator can also be seen as a random variable.

Furthermore, at the end of the lecture some people were confused about the difference between PDFs and random variables. Here is an attempt at clarifying this:

- Your can think of a random variable X (or sometimes denoted $\hat{X}$) as "a way of measuring something" / "a measurement procedure".
- If that measurement yields an outcome x, then we can write this as $X = x$.
- If many copies of yourself repeat this measurement in different parallel universes, then each copy might obtain a different measurement outcome x, even though they all use the same measurement procedure X. The PDF $p(x)$ then describes the probability with which different measurement outcomes x should appear among the set of parallel universes.

1 The uniform distribution

The PDF of a uniform random variable X in the interval $[a, b]$ is given by

$$p_{\text{uni}}(x|a,b) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a,b] \\ \\ 0 & \text{else} \end{cases} . \tag{1}$$

Calculate the values of a and b for which the expectation value and variance of this random variable take the values $\langle X \rangle = 1$ and $\text{Var}(X) \equiv \langle (X - \langle X \rangle)^2 \rangle = 0.16$.

2 The moment generating function of a Gaussian random variable

In the lecture, we defined the moment generating function (MGF) of a random variable X as

$$\phi(\lambda) = \sum_{n=0}^{\infty} \frac{\langle X^n \rangle}{n!} \lambda^n . \tag{2}$$

Show that the MGF can also be written as the following expectation value:

$$\phi(\lambda) = \langle e^{\lambda X} \rangle . \tag{3}$$

The PDF of a Gaussian random variable X with mean μ and standard deviation σ is given by

$$p_G(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right) . \tag{4}$$

Use this PDF to evaluate the above expression for $\phi(\lambda)$ and thus show that the MGF of a Gaussian random variable is given by

$$\phi_G(\lambda|\mu,\sigma) = \exp\left(\mu\lambda + \frac{1}{2}\sigma^2\lambda^2\right) . \tag{5}$$

3 Use a computer to convince yourself, that equation (5) is correct

On your computer (or using `jupyter.physik.uni-muenchen.de`), setup a numerical experiment to confirm the result of Exercise 2. Hint: You can e.g. use the `python` package `numpy` to write a `jupyter` notebook. The function `numpy.random.normal` allows you to draw sets of Gaussian random variables with identical μ and σ. Given such a set you can directly estimate the expectation value on the right hand side of Equation 3.

4 How non-linear functions turn unbiased estimators into biased ones

Let $\hat{X}$ be an unbiased estimator of some quantity μ, i.e. $\langle \hat{X} \rangle = \mu$. Also, let $f(x)$ be a non-linear function. Show that in general

$$\langle f(\hat{X}) \rangle \neq f(\mu) \ . \tag{6}$$

Assume you know the variance $\sigma^2 \equiv \langle (\hat{X} - \mu)^2 \rangle$ of $\hat{X}$. How can you use that to estimate the bias $\langle f(\hat{X}) \rangle - f(\mu)$?

5 Discussion

Think about what you would like to learn in this course, and discuss this with the others in your tutorial group.