From Data to Insights - Exercise sheet 3

discussed in week after 3rd lecture

May 3, 2024

1 Bayesian inference in 1D and 2D parameter spaces

At the following url you can find a data file containing the measurement of an output signal $\xi(\theta)$ as a function of different inputs θ (cf. the last slides of lecture 3): `https://cloud.physik.lmu.de/index.php/s/eKp2eeqAmtFtz83` . The first column of that file contains the values of θ at which ξ was measured, the second column contains the corresponding values of ξ (this is the data vector $\mathbf{d}$), and the third column contains error bars for these measurements. In particular, we assume that our measurement uncertainties are described by Gaussian noise with a diagonal covariance matrix $\mathbf{C} = \sigma^2 \cdot \mathbf{1}$, $\sigma = 1.5$ (as in the lecture).

A) For the model $\xi(\theta) = A \cdot \theta^{0.5}$, reproduce the posterior density $p(A|\mathbf{d})$ on the parameter A that was obtained in lecture 3 when assuming a flat prior

$$p(A) = \frac{1}{4} \cdot \begin{cases} 1 & \text{for } A \in [0, 4] \\ \\ 0 & \text{else} \end{cases} . \tag{1}$$

B) Find the 68.3% confidence interval of this posterior, i.e. the smallest interval containing 68.3% of the probability of $p(A|\mathbf{d})$. Don't give up, if this sounds hard to you! Be creative, and don't shy away from messi solutions (though there is one quick & clean solution).

C) On the planet Vulcan, scientists found out that it is much more logical to parametrize the above model for ξ as

$$\xi(\theta) = B^2 \cdot \theta^{0.5} , \tag{2}$$

where $B = \sqrt{A}$ replaces A as a free parameter of the model. Derive a new posterior $p_{\text{Vulcan}}(B|\mathbf{d})$, assuming the flat prior

$$p(B) = \frac{1}{2} \cdot \begin{cases} 1 & \text{for } B \in [0, 2] \\ \\ 0 & \text{else} \end{cases} . \tag{3}$$

Plot this posterior!

D) We can turn the Vulcan density for B into a density for A via the usual rules of changing variables. This gives:

$$\tilde{p}_{\text{Vulcan}}(A|\mathbf{d}) \;=\; p_{\text{Vulcan}}(B(A)|\mathbf{d}) \, \frac{\mathrm{d}B}{\mathrm{d}A} \;=\; p_{\text{Vulcan}}(\sqrt{A}|\mathbf{d}) \, \frac{1}{2\sqrt{A}} . \tag{4}$$

Compare this to the original posterior $p(A)$ we Humans derived in part A).

- If the two densities are the same: explain why!
- If the two densities are the NOT the same: explain why!

Is the Vulcan analysis better than the Human one?

E) Not let us consider the more complicated model

$$\xi(\theta) = A \cdot \theta^{n} , \tag{5}$$

which has a second free parameter n . Derive a 2-dimensional posterior density $p(A, n|\mathbf{d})$, assuming the flat prior

$$p(A, n) = \frac{1}{4} \cdot \begin{cases} 1 & \text{for } A \in [0, 4] \text{ and } n \in [0, 1] \\ \\ 0 & \text{else} \end{cases} . \tag{6}$$

Visualise your posterior with `python` . You are free to choose any package you would like - but make sure that you indicate the $1 - \sigma$ contour of the posterior.

F) Is the point $(A, n) = (2.3, 0.2)$ inside the 1-σ contour? And how about the point $(A, n) = (2.3, 0.5)$?

2 Calculating a Fisher matrix

In exercise 1 you assumed that the measurement uncertainties were described by Gaussian noise with a diagonal covariance matrix $\mathbf{C} = \sigma^2 \cdot \mathbf{1}$, $\sigma = 1.5$ and you had considered two models for that measurement:

A) $\xi(\theta) = A \cdot \theta^{0.5}$

B) $\xi(\theta) = A \cdot \theta^{n}$.

Compute the Fisher matrix and the parameter covariance matrix in both of these situations (for model A those will just be 1-dimensional numbers). Since these models are non-linear in their parameters, the Fisher matrix will depend on the parameters. You can assume that $A_{\text{true}} = 2.3$ and $n_{\text{true}} = 0.5$.

How does the parameter covariance compare to the width of the posterior distributions you had calculated?

3 Discussion question

- If you only know the posterior $p(\boldsymbol{\pi}|\mathbf{d})$, can you tell whether the best-fitting parameters $\boldsymbol{\pi}_{\text{best}}$ actually provide a good fit to the measured data $\mathbf{d}$?