From Data to Insights - Exercise sheet 7

discussed June 06 and June 07

May 31, 2024

1 Calculating Evidence and Posterior with pen and paper (no computers)

Consider a measured data vector

$$\mathbf{d} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{1}$$

which we know to be drawn from a Gaussian likelihood with covariance matrix

$$\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \tag{2}$$

and with expectation value

$$\mu(\alpha) = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \ , \tag{3}$$

where α is a free parameter. To constrain that parameter we perform a Bayesian analysis, employing a flat prior over the interval $[\alpha_{\min}, \alpha_{\max}]$.

Assume that this prior is much wider than the final posterior. Under this assumption:

A) Calculate the Bayesian evidence $p(\mathbf{d})$. How does it depend on the difference $(\alpha_{\max} - \alpha_{\min})$?

B) Calculate the posterior $p(\alpha|\mathbf{d})$. What are the mean μ_α and the standard deviation σ_α of that posterior?

2 A metric for the tension between datasets

Assume that two data vectors $\mathbf{d}_A$ and $\mathbf{d}_B$ have been measured. Our likelihood models

$$p_A(\mathbf{d}_A|\boldsymbol{\pi}) \ , \ p_B(\mathbf{d}_B|\boldsymbol{\pi}) \tag{4}$$

for the data vectors share the same parameter space, and we assume the sampe prior distribution $p(\boldsymbol{\pi})$ for both measurements.

To determine, whether both measurements can indeed be described by the same values for the parameters $\boldsymbol{\pi}$, let us consider two different likelihood models for the combined data vector

$$\mathbf{d} \equiv \begin{pmatrix} \mathbf{d}_A \\ \mathbf{d}_B \end{pmatrix} \ . \tag{5}$$

The first model assumes that both measurements are indeed described by the same parameters, i.e.

$$p_1(\mathbf{d}|\boldsymbol{\pi}) = p_A(\mathbf{d}_A|\boldsymbol{\pi}) p_B(\mathbf{d}_B|\boldsymbol{\pi}) \ , \tag{6}$$

such that the posterior becomes

$$p_1(\boldsymbol{\pi}|\mathbf{d}) = \frac{p_A(\mathbf{d}_A|\boldsymbol{\pi}) p_B(\mathbf{d}_B|\boldsymbol{\pi}) p(\boldsymbol{\pi})}{p_1(\mathbf{d})} \ . \tag{7}$$

The 2nd model assumes that both measurements need completely independent values of their respective parameters. This leads to the likelihood

$$p_2(\mathbf{d}|\boldsymbol{\pi}_A, \boldsymbol{\pi}_B) = p_A(\mathbf{d}_A|\boldsymbol{\pi}_A) p_B(\mathbf{d}_B|\boldsymbol{\pi}_B) \tag{8}$$

and to the posterior

$$p_2(\boldsymbol{\pi}|\mathbf{d}) = \frac{p_A(\mathbf{d}_A|\boldsymbol{\pi}_A) p_B(\mathbf{d}_B|\boldsymbol{\pi}_B) p(\boldsymbol{\pi}_A) p(\boldsymbol{\pi}_B)}{p_2(\mathbf{d})} \ . \tag{9}$$

A) Write down an expression for the Bayes ratio between these two models.

B) Assume that $p(\boldsymbol{\pi})$ is a flat prior over some region of parameters space with volume V. Assume also, that this region is much larger than the bulk of the respective posteriors (e.g. the prior may be much wider than the 6-σ contour of the posteriors).

How does the Bayes ratio depend on the prior volume V?

C) In the limit of very large V, which model will be preferred according to the Bayes factor?

3 Distribution of best-fit χ^2

Consider a Gaussian likelihood model with covariance matrix

$$\mathbf{C} = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix} \tag{10}$$

and with expectation value

$$\boldsymbol{\mu}(\alpha) = \begin{pmatrix} \alpha \\ \alpha^2 \end{pmatrix} . \tag{11}$$

A) As in the last exercise sheet, put yourself in 'god mode': assume that the true parameter value is $\alpha_{\mathrm{true}} = 0$ and draw many (>10000) data vectors $\mathbf{d}_1$, $\mathbf{d}_2$, ... Back in human mode, calculate the best-fitting value of χ^2 for each of these measurements. Is the mean best-fit χ^2 equal to the number of degrees-of-freedom $N_{\mathrm{dof}} = N_{\mathrm{data}} - N_{\mathrm{param}} = 1$?

B) Plot a (normalised) histogram of the χ^2 values you obtained and compare it to the χ^2-distribution with $N_{\mathrm{dof}} = 1$.

C) Repeat A) and B) but changing the expectation value of the likelihood model to

$$\boldsymbol{\mu}(\alpha) = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} . \tag{12}$$

4 Discussion question

- Bring up whatever questions you may have from the lecture.