Homework1

Problem 1

For a uniform distribution, the expectation value is:

$$\langle X\rangle = \int_a^b X\frac{1}{b-a}dX = \frac{a+b}{2} = 1$$

The variance in this case is:

$$\begin{aligned} V(X) &= \left\langle X^2\right\rangle - \langle X\rangle^2 \\ &= \int_a^b X^2\frac{1}{b-a}dX - \langle X\rangle^2 \\ &= \frac{1}{3}\left(a^2+ab+b^2\right) - \langle X\rangle^2 = 0.16 \end{aligned}$$

So, we can calculate:

$$\begin{cases} a+b=2 \\ b>a \\ a^2+ab+b^2=3.48 \end{cases} \rightarrow \begin{cases} a=0.30718 \\ b=1.69282 \end{cases}$$

Problem 2

$$\begin{aligned} \phi(\lambda) = \left\langle e^{\lambda x}\right\rangle &= \left\langle \sum_{n=0}^{\infty}\frac{(\lambda X)^n}{n!}\right\rangle \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\langle X^n\rangle \end{aligned}$$

For a variable with Gaussian distribution, the MGF could be calculated by:

$$\phi(\lambda|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{\lambda x}e^{-\frac{(x-\mu)}{2\sigma^2}}dx$$

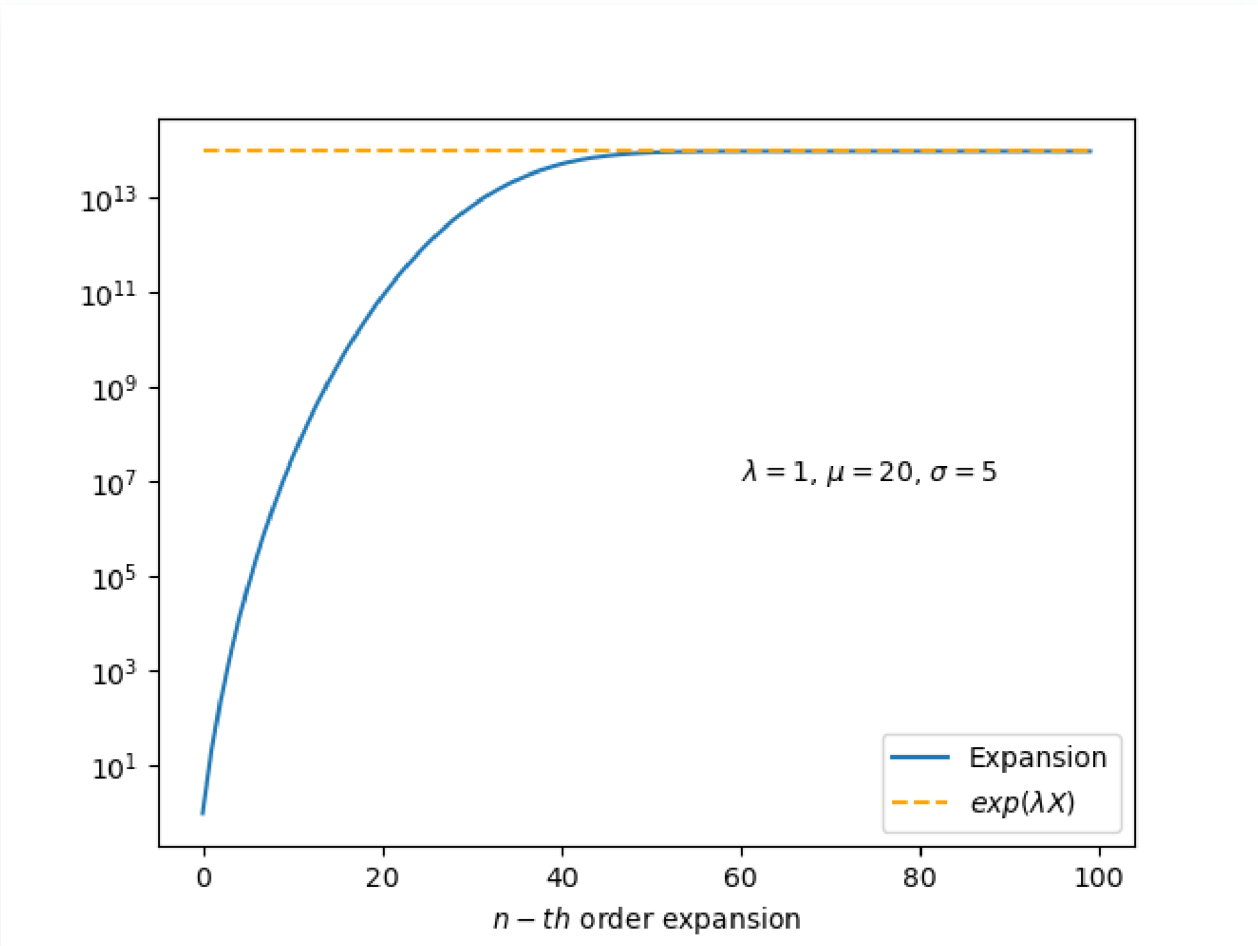
For the index term:

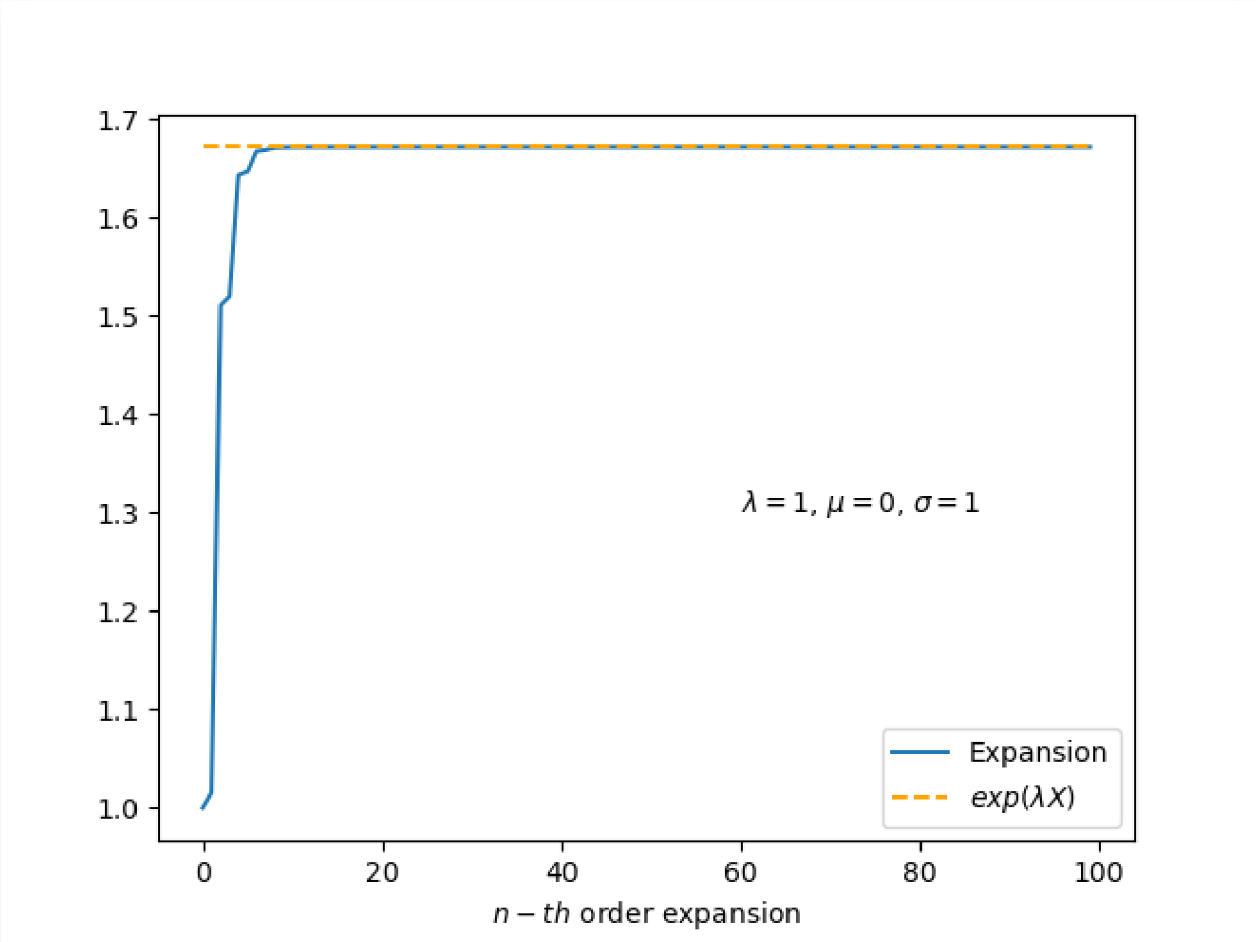
$$\begin{aligned}
-\frac{(x-\mu)^2}{2\sigma^2}+\lambda x &= -\frac{1}{2\sigma^2}\left(x^2-2\mu x+\mu^2\right)+\lambda x \\
&= -\frac{1}{2\sigma^2}x^2+\frac{\mu}{\sigma^2}x-\frac{\mu^2}{2\sigma^2}+\lambda x \\
&= -\frac{1}{2\sigma^2}x^2+\left(\frac{\mu}{\sigma^2}+\lambda\right)x-\frac{\mu^2}{2\sigma^2} \\
&= -\frac{1}{2\sigma^2}\left[x^2-\left(2\mu+2\lambda\sigma^2\right)x+\left(\mu+\lambda\sigma^2\right)^2\right]+\frac{1}{2\sigma^2}\left(\mu+\lambda\sigma^2\right)^2-\frac{\mu^2}{2\sigma^2}
\end{aligned}$$

So, the final integral becomes:

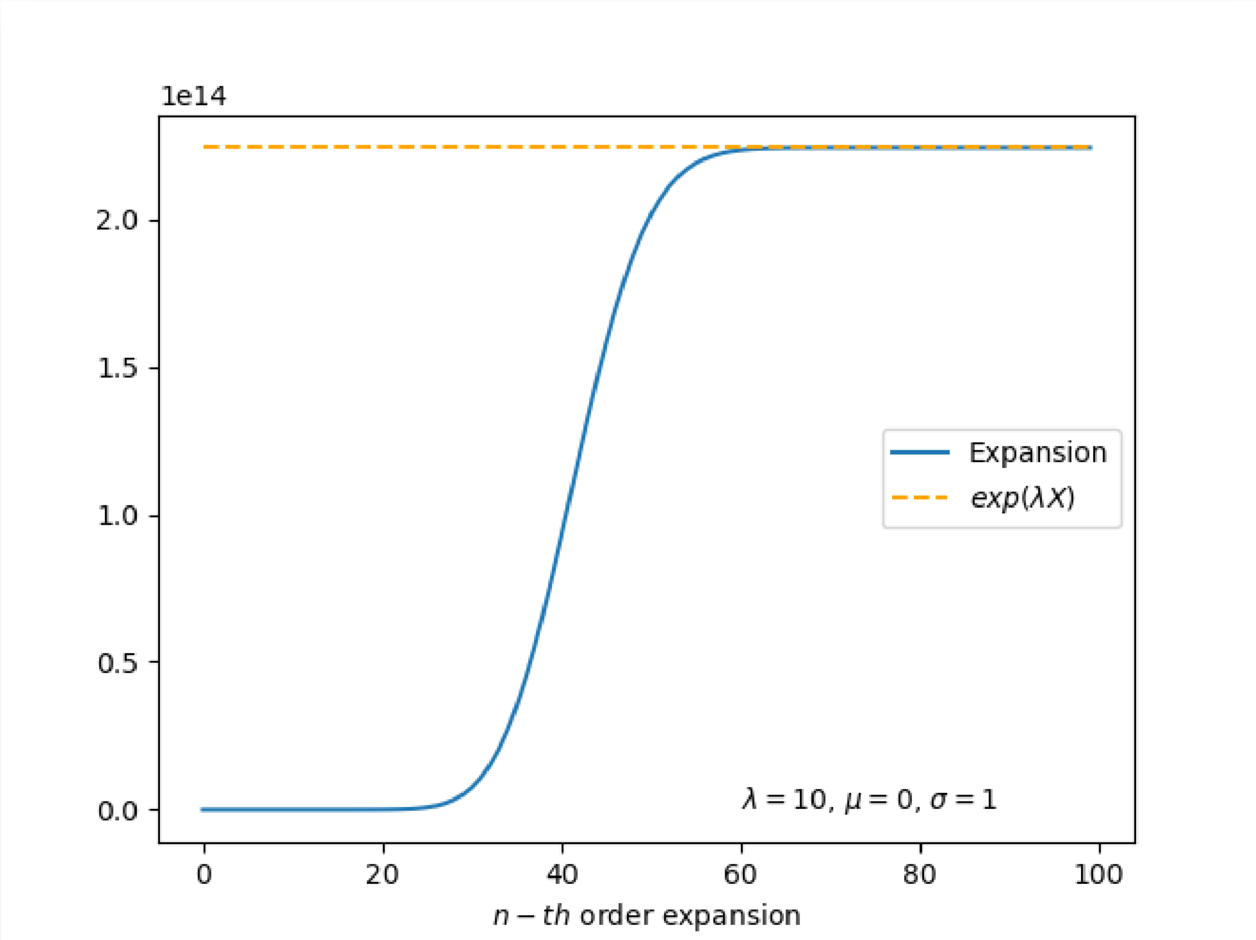
$$\begin{aligned}
\phi\left(\lambda|\mu,\sigma\right) &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{\lambda x}e^{-\frac{(x-\mu)}{2\sigma^2}}\,dx \\
&= e^{\frac{1}{2\sigma^2}\left(\mu+\lambda\sigma^2\right)^2-\frac{\mu^2}{2\sigma^2}} = \exp\left(\mu\lambda+\frac{1}{2}\sigma^2\lambda^2\right)
\end{aligned}$$

Problem 3





1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
$\lambda = 1, \mu = 0, \sigma = 1$
Expansion
$exp(\lambda X)$
0
20
40
60
80
100
$n - th$ order expansion



1e14
2.0
1.5
1.0
0.5
0.0
Expansion
$exp(\lambda X)$
$\lambda = 10, \mu = 0, \sigma = 1$
0
20
40
60
80
100
$n - th$ order expansion

Problem 4

Suppose $\left\langle \hat{X} \right\rangle = \mu$, $\left\langle \left(\hat{X} - \mu \right)^2 \right\rangle = \sigma^2$, and $f\left(\hat{X}\right)$ is any function. Suppose $\hat{X} \approx \mu$, then we have:

$$f\left(\hat{X}\right) = f(\mu) + f'(\mu)\left(\hat{X} - \mu\right) + \frac{f''(\mu)}{2!}\left(\hat{X} - \mu\right)^2 + \cdots + \frac{f^{(n)}(\mu)}{n!}\left(\hat{X} - \mu\right)^n$$

$$\left\langle f\left(\hat{X}\right) \right\rangle = f(\mu) + f'(\mu)\left\langle \left(\hat{X} - \mu\right) \right\rangle + \frac{f''(\mu)}{2!}\left\langle \left(\hat{X} - \mu\right)^2 \right\rangle + \cdots + \frac{f^{(n)}(\mu)}{n!}\left\langle \left(\hat{X} - \mu\right)^n \right\rangle$$

For a non-linear function, all the derivatives can not be zero simultaneously. For a specific example, let's say $\hat{Y} = \hat{X}^2$, then,

$$\left\langle \hat{Y} \right\rangle = \mu^2 + 0 + \sigma^2 = \mu^2 + \sigma^2$$