

# 2018 AIME II Problems

MAA

June 16, 2020

## 1 Acknowledgement

All the following problems are copyrighted by the Mathematical Association of America's American Mathematics Competitions.

## 2 Problems

- Problem 1** Points  $A$ ,  $B$ , and  $C$  lie in that order along a straight path where the distance from  $A$  to  $C$  is 1800 meters. Ina runs twice as fast as Eve, and Paul runs twice as fast as Ina. The three runners start running at the same time with Ina starting at  $A$  and running toward  $C$ , Paul starting at  $B$  and running toward  $C$ , and Eve starting at  $C$  and running toward  $A$ . When Paul meets Eve, he turns around and runs toward  $A$ . Paul and Ina both arrive at  $B$  at the same time. Find the number of meters from  $A$  to  $B$ .
- Problem 2** Let  $a_0 = 2$ ,  $a_1 = 5$ , and  $a_2 = 8$ , and for  $n > 2$  define  $a_n$  recursively to be the remainder when  $4(a_{n-1} + a_{n-2} + a_{n-3})$  is divided by 11. Find  $a_{2018} \cdot a_{2020} \cdot a_{2022}$ .
- Problem 3** Find the sum of all positive integers  $b < 1000$  such that the base- $b$  integer  $36_b$  is a perfect square and the base- $b$  integer  $27_b$  is a perfect cube.
- Problem 4** In equiangular octagon  $CAROLINE$ ,  $CA = RO = LI = NE = \sqrt{2}$  and  $AR = OL = IN = EC = 1$ . The self-intersecting octagon  $CORNELIA$  encloses six non-overlapping triangular regions. Let  $K$  be the area enclosed by  $CORNELIA$ , that is, the total area of the six triangular regions. Then  $K = \frac{a}{b}$ , where  $a$  and  $b$  are relatively prime positive integers. Find  $a + b$ .
- Problem 5** Suppose that  $x$ ,  $y$ , and  $z$  are complex numbers such that  $xy = -80 - 320i$ ,  $yz = 60$ , and  $zx = -96 + 24i$ , where  $i = \sqrt{-1}$ . Then there are real numbers  $a$  and  $b$  such that  $x + y + z = a + bi$ . Find  $a^2 + b^2$ .
- Problem 6** A real number  $a$  is chosen randomly and uniformly from the interval  $[-20, 18]$ . The probability that the roots of the polynomial

- Problem 7** Triangle  $ABC$  has side lengths  $AB = 9$ ,  $BC = 5\sqrt{3}$ , and  $AC = 12$ . Points  $A = P_0, P_1, P_2, \dots, P_{2450} = B$  are on segment  $\overline{AB}$  with  $P_k$  between  $P_{k-1}$  and  $P_{k+1}$  for  $k = 1, 2, \dots, 2449$ , and points  $A = Q_0, Q_1, Q_2, \dots, Q_{2450} = C$  are on segment  $\overline{AC}$  with  $Q_k$  between  $Q_{k-1}$  and  $Q_{k+1}$  for  $k = 1, 2, \dots, 2449$ . Furthermore, each segment  $\overline{P_k Q_k}$ ,  $k = 1, 2, \dots, 2449$ , is parallel to  $\overline{BC}$ . The segments cut the triangle into 2450 regions, consisting of 2449 trapezoids and 1 triangle. Each of the 2450 regions has the same area. Find the number of segments  $\overline{P_k Q_k}$ ,  $k = 1, 2, \dots, 2450$ , that have rational length.
- Problem 8** A frog is positioned at the origin of the coordinate plane. From the point  $(x, y)$ , the frog can jump to any of the points  $(x+1, y)$ ,  $(x+2, y)$ ,  $(x, y+1)$ , or  $(x, y+2)$ . Find the number of distinct sequences of jumps in which the frog begins at  $(0, 0)$  and ends at  $(4, 4)$ .
- Problem 9** Octagon  $ABCDEFGH$  with side lengths  $AB = CD = EF = GH = 10$  and  $BC = DE = FG = HA = 11$  is formed by removing 6-8-10 triangles from the corners of a  $23 \times 27$  rectangle with side  $\overline{AH}$  on a short side of the rectangle, as shown. Let  $J$  be the midpoint of  $\overline{AH}$ , and partition the octagon into 7 triangles by drawing segments  $\overline{JB}$ ,  $\overline{JC}$ ,  $\overline{JD}$ ,  $\overline{JE}$ ,  $\overline{JF}$ , and  $\overline{JG}$ . Find the area of the convex polygon whose vertices are the centroids of these 7 triangles.
- Problem 10** Find the number of functions  $f(x)$  from  $\{1, 2, 3, 4, 5\}$  to  $\{1, 2, 3, 4, 5\}$  that satisfy  $f(f(x)) = f(f(f(x)))$  for all  $x$  in  $\{1, 2, 3, 4, 5\}$ .
- Problem 11** Find the number of permutations of  $1, 2, 3, 4, 5, 6$  such that for each  $k$  with  $1 \leq k \leq 5$ , at least one of the first  $k$  terms of the permutation is greater than  $k$ .
- Problem 12** Let  $ABCD$  be a convex quadrilateral with  $AB = CD = 10$ ,  $BC = 14$ , and  $AD = 2\sqrt{65}$ . Assume that the diagonals of  $ABCD$  intersect at point  $P$ , and that the sum of the areas of triangles  $APB$  and  $CPD$  equals the sum of the areas of triangles  $BPC$  and  $APD$ . Find the area of quadrilateral  $ABCD$ .
- Problem 13** Misha rolls a standard, fair six-sided die until she rolls 1-2-3 in that order on three consecutive rolls. The probability that she will roll the die an odd number of times is  $\frac{m}{n}$  where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
- Problem 14** The incircle  $\omega$  of triangle  $ABC$  is tangent to  $\overline{BC}$  at  $X$ . Let  $Y \neq X$  be the other intersection of  $\overline{AX}$  with  $\omega$ . Points  $P$  and  $Q$  lie on  $\overline{AB}$  and  $\overline{AC}$ , respectively, so that  $\overline{PQ}$  is tangent to  $\omega$  at  $Y$ . Assume that  $AP = 3$ ,  $PB = 4$ ,  $AC = 8$ , and  $AQ = \frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
- Problem 15** Find the number of functions  $f$  from  $\{0, 1, 2, 3, 4, 5, 6\}$  to the integers such that  $f(0) = 0$ ,  $f(6) = 12$ , and