More problems will be added

Instructions:

- Due April 9 at 11:59pm on Gradescope.
- You must follow the submission policy in the syllabus

Problem 1 (Product and quotient space)**.**

(a) For a positive integer m, show that $V^m = \underbrace{V \times V \times \cdots \times V}_{m\text{times}}$ is isomorphic to $\mathcal{L}(\mathbb{F}^m, V)$. Do not assume V is finite-dimensional.

(b) Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subspaces U_1, U_2 of V. Prove that the intersection $A_1 \cap A_2$ is either a translate of some subspace of V or is the empty set.

(c) An equivalence relation is a binary relation that is reflexive, symmetric and transitive. Fix a subspace U of V. Show that $v \sim w$ if and only if $v - w \in U$ is an equivalence relation on V.

(d) Briefly explain how the previous problem relates to translates.

(e) Suppose U is a subspace of V such that V/U is finite-dimensional. Prove that V is isomorphic to $U \times (V/U)$.

Problem 2 (Duality)**.**

(a) Explain why each linear functional is surjective or is the zero map.

(b) Show that the dual map of the identity operator on V is the identity operator on V'.

(c) Suppose $m \geq 0$. What is the dual basis of $\{1, x-5, (x-5)^2, \ldots, (x-5)^m\}$ in $\mathcal{P}_m$?

(d) Suppose $T \in \mathcal{L}(V, W)$ and $w_1, \ldots, w_m$ is a basis of range T. Hence for each $v \in V$, there exist unique numbers $\varphi_1(v), \ldots, \varphi_m(v)$ such that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m,$$

thus defining functions $\varphi_1, \ldots, \varphi_m$ from V to $\mathbf{F}$. Show that each of the functions $\varphi_1, \ldots, \varphi_m$ is a linear functional on V.