The final will be in class April 30. The format and style will be similar to the quizzes, and you can use two pages of 8.5x11 notes. All content covered through April 16 may appear on the final.

The following are some selected practice problems which are broadly representative of the type of problems that could appear (but not exhaustive). You should also review the past quizzes, homeworks, and textbook problems.

Problem 1. Recall that a non-empty subset U of a vector space V is a *subspace* if it is closed under addition and scalar multiplication.

Let V be the vector space of continuous real-valued functions on $[-5, 5]$. For $b \in \mathbb{R}$, define a subset U of V by $U = \{f \in V : f'(0) + f''(3) = b\}$. Show that U is a subspace of V *if and only if* $b = 0$.

Problem 2. Suppose $\vec{v}_1, \dots, \vec{v}_n$ is a linearly independent. Prove $\vec{v}_1, \dots, \vec{v}_k$ is linearly independent if $k \le n$.

Problem 3. Suppose $\vec{v}_1, \dots, \vec{v}_n$ is a basis for a vector space X. Let $Y = \operatorname{span}(\vec{v}_1, \dots, \vec{v}_k)$ for some $k \le n$. Show that $\vec{v}_1, \dots, \vec{v}_k$ is a basis for Y.

Problem 4. Let $v_1, \dots, v_n$ be a linearly independent set of vectors. Suppose $v = c_1 v_1 + \cdots + c_n v_n$ and $v' = c_1' v_1 + \cdots + c_n' v_n$. Show that $v = v'$ if and only if $c_i = c_i'$ for all $i = 1, \dots, n$.

Problem 5. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \dots, u_m$ is a basis of U and $w_1, \dots, w_n$ is a basis of W. Prove that

$$u_1, \dots, u_m, w_1, \dots, w_n$$

is a basis of V.

Problem 6. Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that

$$U \cap \operatorname{null} T = \{0\} \quad \text{and} \quad \operatorname{range} T = \{Tu : u \in U\}.$$

Problem 7. Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \ \operatorname{null} ST \le \dim \ \operatorname{null} S + \dim \operatorname{null} T.$$

Problem 8. Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_m$ is a list of vectors in V such that $Tv_1, \dots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \dots, v_m$ is linearly independent.

Problem 9. Show that if $S, T \in \mathcal{L}(V, W)$, then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

Problem 10. **Fact 1.** If $q_0, \dots, q_k \in \mathcal{P}_k$ are such that $\deg(q_i) = i$, then they are a basis for $\mathcal{P}_k$.

Fact 2. Let $T : V \to W$ be a linear map. The fundamental theorem of linear maps says if $\dim(V) < \infty$, then

$$\dim(V) = \dim(\operatorname{null}(T)) + \dim(\operatorname{range}(T)).$$

Fact 3. Let V be a finite dimensional vector space and U a subspace of V. If $\dim(U) = \dim(v)$, then $U = V$.

Suppose T is a linear map from $\mathcal{P}_2$ to $\mathbb{R}^3$ with $\text{range}(T) = \{(a, b, 0) : a, b \in \mathbb{R}\}$. Let $q \in \mathcal{P}_2$ be defined by $q(x) = x + 1$ and suppose

$$Tq = (0, 0, 0).$$

Find vectors $p_1, p_2 \in \mathcal{P}_2$ so that $v_1 = Tp_1$ and $v_2 = Tp_2$ form a basis for range(T) (and prove it)

Hint: Start by extending q to a basis q, p_1, p_2 for $\mathcal{P}_2$. Then show v_1 and v_2 are linearly independent by considering $\alpha_1 v_1 + \alpha_2 v_2 = 0$ and using the definition of v_1 and v_2 and what you know about the null space of T. Finally, argue v_1 and v_2 are spanning.

Problem 11. Suppose T is invertible and S_1 and S_2 are inverses of T. Prove $S_1 = S_2$.

Problem 12. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$. Prove that $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

Problem 13. We are used to working with the monomial basis:

$$m_0(x) = 1, \quad m_1(x) = x, \quad m_2(x) = x^2, \quad m_3(x) = x^3, \quad m_4(x) = x^4$$

Physicists like the Legendre polynomials:

$$L_0(x) = 1, \quad L_1(x) = x, \quad L_2(x) = \frac{1}{2}(3x^2 - 1), \quad L_3(x) = \frac{1}{2}(5x^3 - 3x).$$

(a) Let $D : \mathcal{P}_4 \to \mathcal{P}_4$ be the differentiation operator: $Dp = p'$. What is $C = \mathcal{M}(D, \{m_i\}, \{m_i\})$?

(b) Let $p = 2L_3$. What is $v = \mathcal{M}(p, \{L_i\})$?

(c) The change of basis matrices for monomial to Legendre and Legendre to monomial are given by:

$$A = \mathcal{M}(I, \{L_i\}, \{m_i\}) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}, \qquad B = \mathcal{M}(I, \{m_i\}, \{L_i\}) = \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & \frac{3}{5} \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{2}{5} \end{bmatrix}.$$

What is the formula for $\mathcal{M}(Dp, \{L_i\})$ in terms of A,B,C and v? Also compute the numerical entries. You can do it by matrix products or other means.

Problem 14. Suppose $u_1, \ldots, u_n$ is a basis for U and $v_1, \ldots, v_m$ is a basis for V (both over the same field). Write a basis for $U \times V$.

Problem 15. Let V be a vector space and $U \subset V$. Recall $v + U = \{v + u : u \in U\}$.

Prove that $v + U = w + U$ if and only if $v - w \in U$.

Problem 16. Fact. Suppose R is upper triangular ($r_{i,j} = 0$ if $i > j$). If R is invertible, then the inverse R^{-1} is also upper triangular.

Suppose $v_1, \ldots, v_n$ and $u_1, \ldots, u_n$ are such that

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} R, \quad \text{where} \quad R = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ 0 & r_{2,2} & \cdots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{n,n} \end{bmatrix}.$$

(a) Show that $\text{span}\{v_1, \dots, v_k\}$ is a subset of $\text{span}\{u_1, \dots, u_k\}$ for each $k = 1, 2, \dots, n$.
(b) Prove that if R is invertible, $\text{span}\{v_1, \dots, v_k\} = \text{span}\{u_1, \dots, u_k\}$ for each $k = 1, 2, \dots, n$. This can be viewed as a converse to the homework problem 3(a).

Problem 17. (a) Find an orthonormal basis for

$$\text{span}\left(\begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \right)$$

(b) Solve the least squares problem

$$\min_{x,y,z} \left\| \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\|_2$$

Problem 18. Recall the Chebyshev polynomials

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_2(x) = 2x^2 - 1, \qquad T_3(x) = 4x^3 - 3x$$

satisfy the orthogonality condition:

$$\int_{-1}^{1} T_n(x)\, T_m(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \begin{cases} 0 & \text{if } n \neq m, \\ \pi & \text{if } n = m = 0, \\ \frac{\pi}{2} & \text{if } n = m \neq 0. \end{cases}$$

Let $f(x) = \sin(x)$. Find a degree 3 polynomial $p(x)$ and a function $g(x)$ so that $f(x) = p(x) + g(x)$ and

$$\int_{-1}^{1} p(x) g(x) \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = 0.$$

You can express your functions in terms of the Chebyshev polynomials and leave the coefficients as integrals.

Problem 19. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. (i) Prove that if $U \subseteq \text{null } T$, then U is invariant under T. (ii) Prove that if $\text{range } T \subseteq U$, then U is invariant under T.

Problem 20. Suppose $T \in \mathcal{L}(V)$. Then $\text{null}(T)$ and $\text{range}(T)$ are invariant under T.

Problem 21. Let e_1, e_2, e_3, e_4 be a basis for V. Suppose $T \in \mathcal{L}(V)$ is defined by

$$Te_1 = e_1, \quad Te_2 = e_2, \quad Te_3 = -2e_3, \quad Te_4 = 3e_4.$$

(a) Write the matrix of T with respect to $e_1, \dots, e_4$.
(b) Find the minimial polynomial of T; i.e. the monic polynomial of minimal degree for which $p(T) = 0$. Prove the polynomial you found is of minimal degree and unique.

Problem 22. Suppose T is invertible. Prove that $E(\lambda, T) = E(1/\lambda, T)$ for all $\lambda \neq 0$.

Problem 23. Let V be a vector space of $\mathbb{C}$ with an inner product $\langle \cdot, \cdot \rangle$. Recall an operator T is called self-adjoint if $\langle Tu, v \rangle = \overline{\langle u, Tv \rangle}$ for all $u, v \in V$.

Prove that T is self-adjoint if and only if $\langle Tv, v \rangle$ is real for all $v \in V$.