Homework 2: Mathematical Statistics (MATH-UA 234)

Due 09/22 at the beginning of class on Gradescope

Problem 1. *Solve each of the following:*

(a) *Let X be any random variable with $\mathbb{E}[X^4] < \infty$. Show that $\mathbb{E}[X^4] \geq \mathbb{E}[X^2]^2$.*

(b) *Suppose $X \sim \operatorname{Exp}(1)$. Then, as in Example 3.30, the moment generating function is $\psi_X(t) = 1/(1-t)$. Use the moment generating function to find $\mathbb{E}[X^k]$, for integer $k \geq 0$.*

(c) *Let $X \sim \operatorname{Exp}(1)$ and let $Y = \cos(X)$. Find $\mathbb{E}[Y]$.*

(d) *Let X, Y be random variables. Show that* $\operatorname{Co}\mathbb{V}[X,Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

Solution.

(a) Let $Y = X^2$. Then $\mathbb{V}[Y] = \mathbb{E}[(Y - \mathbb{E}[Y])^2] \geq 0$. But we also have $\mathbb{V}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$. Thus, $\mathbb{E}[X^4] = \mathbb{E}[Y^2] \geq \mathbb{E}[Y]^2 = \mathbb{E}[X^2]^2$.

(b) Recall that $\psi_X(t) = \mathbb{E}[X^0] + t\mathbb{E}[X] + \frac{t^2}{2!}\mathbb{E}[X^2] + \frac{t^3}{3!}\mathbb{E}[X^3] + \cdots$. Thus, we have that

$$\mathbb{E}[X^k] = \left[\frac{\mathrm{d}^k}{\mathrm{d}t^k}\psi_X(t)\right]_{t=0} = \left[\frac{k!}{(1-t)^{k+1}}\right]_{t=0} = k!$$

(c) We can use the "law of the lazy statistician" and compute,

$$\mathbb{E}[Y] = \mathbb{E}[\cos(X)] = \int \cos(x) f_X(x)\mathrm{d}x = \int_0^\infty \cos(x)\exp(-x)\mathrm{d}X = \left[\frac{1}{2}\exp(-x)(\sin(x) - \cos(x))\right]_{x=0}^\infty = \frac{1}{2}.$$

(d) By definition,

$$\operatorname{Co}\mathbb{V}[X,Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

Expanding and applying the linearity of expectation,

$$\begin{aligned}\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] &= \mathbb{E}[XY - \mathbb{E}[X]Y - X\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y]] \\ &= \mathbb{E}[XY] - \mathbb{E}[\mathbb{E}[X]Y] - \mathbb{E}[X\mathbb{E}[Y]] + \mathbb{E}[\mathbb{E}[X]\mathbb{E}[Y]].\end{aligned}$$

Now, using that expectations of random variables are constants,

$$\mathbb{E}[\mathbb{E}[X]Y] = \mathbb{E}[X\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[X]\mathbb{E}[Y]] = \mathbb{E}[X]\mathbb{E}[Y].$$

Combining everything above gives the result.

Problem 2. *Suppose X and Y are random variables with joint probability mass function,*

$$f_{X,Y}(x,y) = \mathbb{P}[X = x, Y = y] = \begin{cases} .1 & X = -1, Y = 1 \\ .3 & X = -1, Y = -1 \\ .2 & X = 1, Y = 1 \\ .4 & X = 1, Y = -1 \end{cases}$$

problems with a textbook reference are based on, but not identical to, the given reference

(a) *Compute the marginal probability mass function,* $f_X(x) = \mathbb{P}[X = x]$.

(b) *Compute* $f(y) = \mathbb{E}[X|Y = y]$.

(c) *Compute* $\mathbb{E}[X]$ *using the marginal pmf and then using the the law of iterated expectation. Do the results agree?*

Solution.

(a) Recall that

$$f_X(x) = \sum_y f_{X,Y}(x, y)$$

Thus, we find

$$f_X(x) = \begin{cases} .1 + .3 & x = -1 \\ .2 + .4 & x = 1 \end{cases} = \begin{cases} .4 & x = -1 \\ .6 & x = 1. \end{cases}$$

(b) Recall that

$$\mathbb{E}[X|Y = y] = \sum_x x\mathbb{P}[X = x|Y = y] = \sum_x x\frac{\mathbb{P}[X = x, Y = y]}{\mathbb{P}[Y = y]}.$$

Similar to above, we can compute

$$f_Y(y) = \mathbb{P}[Y = y] = \begin{cases} .7 & y = -1 \\ .3 & y = 1 \end{cases}$$

Thus, we find

$$f(y) = \begin{cases} -1(.3)/.7 + 1(.4)/.7 & y = -1 \\ -1(.1)/.3 + 1(.2)/.3 & y = 1 \end{cases} = \begin{cases} 1/7 & y = -1 \\ 1/3 & y = 1. \end{cases}$$

(c) Using the marginal pmf we have

$$\mathbb{E}[X] = \sum_x x\mathbb{P}[X = x] = -1(.4) + 1(.6) = .2.$$

Using the law of iterated expectation we have,

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \sum_y \mathbb{E}[X|Y = y]\mathbb{P}[Y = y] = (1/7)(.7) + (1/3)(.3) = .2.$$

Problem 3 (Wasserman 3.4 (statistics of a random walk))**.** *A particle starts at the origin of the real line and moves along the line in jumps of one unit. For each jump the probability is p that the particle will jump one unit to the left and the probability is* $1 - p$ *that the particle will jump one unit to the right. Let* X_n *be the position of the particle after n jumps. Find* $\mathbb{E}[X_n]$ *and* $\mathbb{V}[X_n]$*. (This is known as a random walk.)*

Solution.

We will use the standard convention that moving to the right one unit increases X_n by one.

At jump i, let $Y_i = -1$ if the particle moves to the left, and $Y_i = 1$ if the particle moves to the right. Then $X_n = Y_1 + Y_2 + \cdots + Y_n$. Since the probabilities that the particle moves left or right do not depend on the past history of the particle, the Y_i's are independent.

By direct computation,

$$\mathbb{E}[Y_i] = -1(p) + 1(1 - p) = 1 - 2p, \quad \mathbb{V}[Y_i] = \mathbb{E}[Y_i^2] - \mathbb{E}[Y_i]^2 = 1 - (1 - 2p)^2 = 4p(1 - p).$$

We then have

$$\mathbb{E}[X_n] = \mathbb{E}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = n\mathbb{E}[Y_i] = n(1-2p).$$

Since the Y_i are independent,

$$\mathbb{V}[X_n] = \mathbb{V}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \mathbb{V}[Y_i] = n\mathbb{V}[Y_i] = 4np(1-p).$$

Problem 4 (Wasserman 3.15 (variance of a mixture))**.** *Let*

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}(x+y) & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \textit{otherwise.} \end{cases}$$

Find $\mathbb{V}[2X - 3Y + 8]$.

Solution. One approach is to use the rules for sums of variances. In particular,

$$\mathbb{V}[2X - 3Y - 8] = 2^2\mathbb{V}[X] + (-3)^2\mathbb{V}[Y] + 2(2(-3))\,\mathrm{Co}\mathbb{V}[X,Y].$$

We can also just compute the variance by the definition. First, note that by Mathematica,

$$\mathbb{E}[2X - 3Y + 8] = \int_0^1\int_0^2 (2x - 3y + 8) f_{X,Y}(x,y)\mathrm{d}y\mathrm{d}x = \frac{49}{9}.$$

Next,

$$\mathbb{E}[(2X - 3Y + 8 - \mathbb{E}[2X - 3Y + 8])^2] = \int_0^1\int_0^2 (2x - 3y + 8 - 49/9)^2 f_{X,Y}(x,y)\mathrm{d}y\mathrm{d}x = \frac{245}{81}.$$

Problem 5 (Wasserman 5.4 (convergence))**.**

Let $X_1, X_2, \ldots$ be a sequence of random variables such that

$$\mathbb{P}[X_n = 1/n] = 1 - 1/n^2, \quad \mathbb{P}[X_n = n] = 1/n^2.$$

(a) Does X_n converge in probability to any random variable? If so, prove this. If no such variable exists, explain why not.

(b) Does X_n converge in quadratic mean? If so, prove this. If no such variable exists, explain why not.

Solution.

(a) The variable to which we are might be converging was not given. This means we'll have to find a candidate ourself. Note that X_n is usually near $1/n$ when n is large, which is also near 0. So we might expect X_n converges to the constant random variable 0 in some way.

In order to prove this, we must first fix an arbitrary $\epsilon > 0$. Now,

$$\mathbb{P}[|X_n - 0| > \epsilon] = \begin{cases} 0 & 1/n \le \epsilon, n \le \epsilon \\ 1/n^2 & 1/n \le \epsilon, n > \epsilon \\ 1 & 1/n > \epsilon. \end{cases}$$

Thus, when n is bigger than $1/\epsilon$, $\mathbb{P}[|X_n - 0| > \epsilon] = 0$. In particular,

$$\lim_{n\to\infty} \mathbb{P}[|X_n - 0| > \epsilon] = 0.$$

Thus, X_n converges to zero in probability.

Note that another approach would be to show that X_n converges in distribution to 0, and then use Theorem 5.4.

(b) If X_n converges to any random variable in quadratic mean, then it must also converge to that random variable in probability. In other words the random variable would have to be zero (or at least equal to zero with probability one).

By direct computation,

$$\mathbb{E}[(X_n - 0)^2] = \left(\frac{1}{n} - 0\right)^2 \left(1 - \frac{1}{n^2}\right) + (n-0)^2 \left(\frac{1}{n^2}\right) = \frac{1}{n^2} - \frac{1}{n^4} + 1.$$

But then,

$$\lim_{n\to\infty} \mathbb{E}[(X_n - 0)^2] = 1 \neq 0.$$

So X_n does not converge in quadratic mean to zero.

Problem 6. *Describe an instance of one of the probability concepts we've seen recently in the course which you noticed in a different part of your life (e.g. in other classes, on the subway, at the park, in the dorm, etc.).*