Homework 6 — Numerical Analysis Spring 2023

Instructions:

- Due 04/20 at 11:59pm on Gradescope.
- Write the names of anyone you work with on the top of your assignment. If you worked alone, write that you worked alone.
- Show your work.
- Include all code you use as copyable `monospaced` text in the PDF (i.e. not as a screenshot).
- Do not put the solutions to multiple problems on the same page.
- Tag your responses on gradescope. Each page should have a *single* problem tag. Improperly tagged responses will not receive credit.

Problem 1. Spend at least two hours working on your project prior to the 20th. Answer the following:

(a) What is the current status of your project?

(b) What are the big tasks you have left to do before your project is done?

(c) What is your plan for completing the project in a timely manner?

Problem 2. Suppose $\mathbf{A}$ has SVD $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathsf{T}}$ where

$$\mathbf{U} = \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \\ | & | & & | \end{bmatrix}, \qquad \mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}.$$

(a) Show that $\mathbf{v}_i$ is an eigenvector of $\mathbf{A}^{\mathsf{T}}\mathbf{A}$. What is the corresponding eigenvalue?

(b) Define the block matrix:

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^{\mathsf{T}} & \mathbf{0} \end{bmatrix}.$$

Show that

$$\mathbf{x} = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}$$

is an eigenvector of $\mathbf{B}$. What is the corresponding eigenvalue?

Problem 3. Suppose $\mathbf{A}$ has eigenvalue decomposition:

$$\mathbf{A} = \mathbf{V}\begin{bmatrix} -4 & & & \\ & -1 & & \\ & & 2 & \\ & & & 3 \end{bmatrix}\mathbf{V}^{-1}, \qquad \mathbf{V} = \begin{bmatrix} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ | & | & | & | \end{bmatrix}$$

where the $\mathbf{v}_i$ are all orthonormal.

Suppose we run inverse power method with shift c with $\mathbf{x} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$; that is, power method on $(\mathbf{A} - c\mathbf{I})^{-1}$.

If $c \in (0.5, 2.5)$, then we will converge to $\mathbf{v}_3$, the eigenvector corresponding to eigenvalue 2. The rate of converge is

$$\rho = \left|\frac{\lambda_2((\mathbf{A} - c\mathbf{I})^{-1})}{\lambda_1((\mathbf{A} - c\mathbf{I})^{-1})}\right|,$$

where $\lambda_1((\mathbf{A} - c\mathbf{I})^{-1})$ and $\lambda_2((\mathbf{A} - c\mathbf{I})^{-1})$ are the largest and second largest eigenvalues of $(\mathbf{A} - c\mathbf{I})^{-1}$ in magnitude respectively.

(a) Plot ρ as a function of c for c in the range $(0.5, 2.5)$.

(b) Let $\mathbf{y}_k$ be the output of k-steps of the power method, and assume $\|\mathbf{v}_3 - \mathbf{y}_k\|_2 \leq \rho^k$. For $\epsilon = 10^{-1}$, make a plot showing how large k has to be so that $\|\mathbf{v}_3 - \mathbf{y}_k\|_2 < \epsilon$ for the values of c in the range $(0.5, 2.5)$. Add more a new line to this plot for each $\epsilon = 10^{-2}, 10^{-5}, 10^{-10}$ plot. Label all the lines.

Problem 4. For the same matrix as in Problem 3, suppose we run power method with a starting vector:

$$\mathbf{x} = \mathbf{V}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Find a vector $\mathbf{z}$ so that $\mathbf{A}^k\mathbf{x} = \mathbf{V}\mathbf{z}$.

(b) What vector does $\mathbf{z}/\|\mathbf{z}\|$ converge to as $k \to \infty$?

(c) What vector does $\mathbf{A}^k\mathbf{v}/\|\mathbf{A}^k\mathbf{v}\|$ converge to as $k \to \infty$?

(d) Why did we get something different than on worksheet 8, where power method converged to a multiple of $\mathbf{v}_1$?