Homework 6: Mathematical Statistics (MATH-UA 234)

Due 11/17 at the beginning of class on Gradescope

Reminder. Don't forget the project proposals are due on 11/22. A good proposal is the key to doing well on the project, so if you have to choose between spending time on the homework or on the proposal, choose the proposal (although hopefully you can put sufficient time into both).

Problem 1. *Pick at least one of the following articles to read. Provide a one paragraph summary of what you think the most important points of the article were.*

- *An unhealthy obsession with p-values is ruining science*
- *Using Effect Size—or Why the P Value Is Not Enough*
- *The Extent and Consequences of P-Hacking in Science*

Problem 2. *Suppose that the size α test is of the form*

$$\text{reject } H_0 = \{\theta \in \Theta_0\} \text{ if and only if } T(X_1, \dots, X_n) > c_\alpha.$$

(a) *Given data $X_1, \dots, X_n$, prove that,*

$$p\text{-value} = \sup_{\theta \in \Theta_0} \mathbb{P}[T(X_1', \dots, X_n') > T(X_1, \dots, X_n) | X_1', \dots, X_n' \sim F_\theta].$$

(b) *Explain, in simple words, what this result says about the meaning of a p-value for this type of test.*

Problem 3 (Wasserman 10.5). *Let $X_1, \dots, X_n \sim \mathrm{Unif}(0, \theta)$ and let $Y = \max\{X_1, \dots, X_n\}$. We want to test $H_0 = \{\theta \leq 1/2\}$ vs $H_1 = \{\theta > 1/2\}$ and we will reject if $Y > c$ for some constant c.*

(a) *Find the power function $\beta(\theta) = \mathbb{P}[\max\{X_1, \dots, X_n\} > c | X_1, \dots, X_n \sim \mathrm{Unif}(0, \theta)]$.*

(b) *What choice of c will make the size of the test 0.05?*

(c) *In a sample of size $n = 20$ with $Y = 0.48$, what is the p-value? What is the conclusion about H_0 that you would make?*

(d) *In a sample of size $n = 5$ with $Y = 0.45$, what is the p-value? What is the conclusion about H_0 that you would make? Since $Y = 0.45$ is smaller than the previous case, we might expect it would be "harder to reject" H_0, yet the p-value is smaller than the previous case. Explain this observation.*

(e) *In a sample of size $n = 20$ with $Y = 0.52$, what is the p-value? What is the conclusion about H_0 that you would make?*

Solution.

(a) In homework 3 we computed the distribution of the maximum of n uniform random variables

problems with a textbook reference are based on, but not identical to, the given reference on $(0, \theta)$. Using this, we can compute

$$\begin{aligned}
\beta(\theta) &= \mathbb{P}[\max\{X_1, \dots, X_n\} > c | X_1, \dots, X_n \sim \text{Unif}(0, \theta)] \\
&= 1 - \mathbb{P}[\max\{X_1, \dots, X_n\} \le c | X_1, \dots, X_n \sim \text{Unif}(0, \theta)] \\
&= 1 - \begin{cases} c^n/\theta^n & c \in [0, \theta] \\ 1 & c > \theta \\ 0 & c < 0. \end{cases} \\
&= \begin{cases} 1 - c^n/\theta^n & c \in [0, \theta] \\ 0 & c > \theta \\ 1 & c < 0. \end{cases}
\end{aligned}$$

(b) The size is

$$\sup_{\theta \le 1/2} \beta(\theta) = \begin{cases} 1 - 2^n c^n & 0 \le c \le 1/2 \\ 0 & c > 1/2 \\ 1 & c < 0. \end{cases}$$

We can make the size equal to 0.05 if $0.05 = 1 - 2^n c^n$. I.e. if $c = (1 - .05)^{1/n}/2$.

(c) Here we will reject at the size α corresponding to $c < Y = 0.52$. This threshold is $\alpha = 1 - 2^{20}(0.48)^{20} \approx 0.558$, so $p \approx 0.558$.

(d) We now have $p = 1 - 2^5(0.45)^5 \approx 0.41$.

Since there is much less data, assuming the data was drawn from Unif$(0, 1/2)$, it is more probable that we see a maximum of 0.45 when we draw only 5 samples than we see a maximum of 0.48 if we draw 20 samples.

This is why p-values can be meaningless if the sample sizes aren't sufficiently large.

(e) In this case we reject even at the size zero test with $c = 1/2$. So $p = 0$.

Problem 4 (Wasserman 10.6). *There is a theory that people can postpone their death until after an important event. To test the theory, Phillips and King (1988) collected data on deaths around the Jewish holiday Passover. Of 1919 deaths, 922 died the week before the holiday and 997 died the week after.*

We can model this situation by thinking of whether each person died after or before the holiday as an iid Bernoulli random variable with parameter p. Then, the number of people who died after the holiday is a Binomial random variable with parameter (n, p), where $n = 1919$.

(a) for each $\alpha \in (0, 1)$, devise a size-α test for the null hypothesis that $p = 1/2$.

(b) Report and interpret the p-value.

(c) Construct a confidence interval for p.

Problem 5. *Suppose you work for a pharmaceutical company. You are trying to develop drugs to reduce the recovery time from infection by a particular virus.*

To model the effectiveness of a given drug, you could do the following: Let X_i the recovery time of a patient given your drug, and suppose $X_i \sim N(\mu, 1)$ some unknown μ. The mean recovery time for patients who did not receive any treatment is μ_0. Thus, you would like to test the null hypothesis $H_0 = \{\mu \leq \mu_0\}$ by analyzing the outcomes $X_1, \ldots, X_n$ of a drug trial. If you are able to reject the null hypothesis, then you will be able to market your drug.

In reality, you studied statistics in college instead of chemistry and every drug you make is ineffective ($\mu = \mu_0$). However, you are greedy and want to make money by selling drugs. Your boss doesn't know statistics, and in your area there are no governmental regulation agencies. Thus, if you perform an experiment where you reject the null hypothesis at a p-value of 0.05, then you will be able to sell the drug and get rich.

(a) *Explain how you could design an experiment (or series of experiments) in order to reject the null hypothesis at a p-value of 0.05 (or at least give you a decent shot at this). Your experiments should be legitimate (i.e. you cannot just lie about whether the drug helped someone or not).*

(b) *Now, suppose you are a governmental regulator. Explain what regulations you might introduce which would have prevented your alter-ego from getting away with such tricks.*