Homework 7 — Numerical Analysis Fall 2024

Instructions:

- Due 12/18 at 6:00pm on Gradescope.
- You must follow the submission policy in the syllabus
- This homework is optional. If you submit it I will compute your homework grade from the all 7 homeworks. If you do not submit it, I will compute your homework grade from the homeworks 1-6.

Problem 1. Suppose $\mathbf{A}$ has SVD $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\mathsf{T}$ where

$$\mathbf{U} = \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \\ | & | & & | \end{bmatrix}, \qquad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}.$$

(a) Show that $\mathbf{v}_i$ is an eigenvector of $\mathbf{A}^\mathsf{T}\mathbf{A}$. What is the corresponding eigenvalue?

(b) Define the block matrix:

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^\mathsf{T} & \mathbf{0} \end{bmatrix}.$$

Show that

$$\mathbf{x} = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}$$

is an eigenvector of $\mathbf{B}$. What is the corresponding eigenvalue?

Problem 2. Suppose $\lambda_1 > \lambda_2 > \lambda_3 \geq \cdots \geq \lambda_n \geq 0$ and let

$$\mathbf{A} = \mathbf{V} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \mathbf{V}^{-1}, \qquad \mathbf{x} = \mathbf{V} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Find a vector $\mathbf{z}$ so that $\mathbf{A}^k\mathbf{x} = \mathbf{V}\mathbf{z}$.

(b) What vector does $\mathbf{z}/\|\mathbf{z}\|$ converge to as $k \to \infty$?

(c) What vector does $\mathbf{A}^k\mathbf{x}/\|\mathbf{A}^k\mathbf{x}\|$ converge to as $k \to \infty$?

(d) What if instead $\mathbf{x} = \mathbf{V} \begin{bmatrix} 10^{-100} \\ 1 \\ 1 \\ 1 \end{bmatrix}$?

Problem 3. (a) Let $f(x) = \exp(-x)$ and define $p_k(x)$ as the degree k polynomial interpolate to $f(x)$ at the $k+1$ Chebyshev nodes.

Make a plot of k vs
$$\max_{x\in[-1,1]} |f(x) - p_k(x)|$$
for $k = 0, 1, 2, \ldots 20$. Put the y-axis on a log scale and label the axes/plot/etc.

To approximate
$$\max_{x\in[-1,1]} |f(x) - p_k(x)|$$
you can instead take the maximum over 1000 equally spaced points in $[-1, 1]$ and use this instead.

(b) Repeat this for $f(x) = 1/(1 + 16x^2)$ and $k = 0, 1, \ldots, 100$.

(c) Repeat this for $f(x) = |\sin(5x)|^3 = (\sin(5x)^2)^{3/2}$ and $k = 0, 1, \ldots, 100$, but put both axes on log-scales.

Add a line k vs $k^{-\nu}$, where ν is the largest value so that the $(\nu-1)$-st derivative of $f(x)$ is continuous.

Problem 4. On $[-1, 1]$, the j-th Chebyshev polynomial is defined as
$$T_j(x) := \cos(j\cos^{-1}(x)).$$
Remarkably this is actually a polynomial of degree j.

(a) For $j = 0, 1, 2, 3, 4, 5$ plots $T_j(x)$ on $[-1, 1]$.

(b) What are the zeros of $T_k(x)$ in terms of k?

(c) Denote by $x_1, \ldots, x_k$ the zeros of $T_k(x)$. For $k = 10$ form the Vandermonde-like matrix
$$\mathbf{A} = \begin{bmatrix} T_0(x_1) & T_1(x_1) & \cdots & T_{k-1}(x_1) \\ T_0(x_2) & T_1(x_2) & \cdots & T_{k-1}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_k) & T_1(x_k) & \cdots & T_{k-1}(x_k) \end{bmatrix}.$$
Look at $\mathbf{A}^\mathsf{T}\mathbf{A}$. What do you notice?

What is the condition number of $\mathbf{A}$?

(d) How might this be relevant to polynomial interpolation?

Problem 5. Let $f(x) = 1/(1 + 16x^2)$.

For any non-negative integer k, set $n = k^2 + 1$ and let $x_1, \ldots, x_n$ be n equally spaced points from -1 to 1 and let $q_k(x)$ be the degree k polynomial minimizing
$$\min_{\deg(q)=k} \sum_{i=1}^{n} (f(x_i) - q(x_i))^2.$$

On a log-y plot, plot the error

$$\max_{x\in[-1,1]} |f(x) - q_k(x)|$$

for $k = 0, 1, \ldots, 100$. Add to this plot the error of the Chebyshev interpolant that you computed in 3(b).