

Homework 3: Mathematical Statistics (MATH-UA 234)

Due 10/06 at the beginning of class on Gradescope

Problem 1. Let

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$$

where $a_{i,j}$ are constants and $x_i \sim N(0, 1)$, $i = 1, 2, \dots, n$ are independent and identically distributed standard normal random variables.

- What is the covariance matrix $\vec{\Sigma}$ for \vec{x} ? (Recall the (i, j) -entry of $\vec{\Sigma}$ is $\text{CoV}[x_i, x_j]$.)
- Show that $\mathbb{E}[\vec{x}^T \vec{A} \vec{x}] = \text{tr}(\vec{A}) := a_{1,1} + a_{2,2} + \cdots + a_{n,n}$ (hint: write out the expression for $\vec{x}^T \vec{A} \vec{x}$ as a sum over the entries of \vec{A})
- Suppose \vec{A} is diagonal; i.e. $a_{i,j} = 0$ for all $i \neq j$. Compute the variance of $\vec{x}^T \vec{A} \vec{x}$. You may use the fact that x_i^2 is a Chi-square random variable with one degree of freedom so that $\mathbb{V}[x_i^2] = 2$.

This is an example of “stochastic trace estimation” which is an important algorithmic tool in a number of recent algorithms.

Problem 2 (Wasserstein 5.5). Let $X_1, X_2, \dots, X_n \sim \text{Ber}(p)$. Let $Z_n = (X_1^2 + X_2^2 + \cdots + X_n^2)/n$. Prove that

- Z_n converges in probability to the constant random variable p .
- Z_n converges in quadratic mean to the constant random variable p .

Problem 3 (Wasserstein 6.1). Let $X_1, X_2, \dots, X_n \sim \text{Poisson}(\lambda)$ and let $\hat{\lambda} = n^{-1} \sum_{i=1}^n X_i$ be our point estimator for λ . Find the bias $\text{Bias} = \mathbb{E}[\hat{\lambda}] - \lambda$, standard error $se_n = \sqrt{\mathbb{V}[\hat{\lambda}]}$, and mean squared error $\text{MSE}_n = \mathbb{E}[(\hat{\lambda} - \lambda)^2]$ of $\hat{\lambda}$.

Problem 4 (Wasserstein 6.2). Let $X_1, X_2, \dots, X_n \sim \text{Unif}(0, \theta)$ and let $\hat{\theta} = \max\{X_1, X_2, \dots, X_n\}$ be our point estimator for θ .

- Write down the distribution for $\hat{\theta}$ (hint: we’ve already don’t a similar problem)
- Find the bias $\text{Bias} = \mathbb{E}[\hat{\theta}] - \theta$, standard error $se_n = \sqrt{\mathbb{V}[\hat{\theta}]}$, and mean squared error $\text{MSE}_n = \mathbb{E}[(\hat{\theta} - \theta)^2]$ of $\hat{\theta}$.

Problem 5 (Wasserstein 6.3). Let $X_1, X_2, \dots, X_n \sim \text{Unif}(0, \theta)$ and let $\hat{\theta} = 2\bar{X}_n$ be our point estimator for θ . Find the bias $\text{Bias} = \mathbb{E}[\hat{\theta}] - \theta$, standard error $se_n = \sqrt{\mathbb{V}[\hat{\theta}]}$, and mean squared error $\text{MSE}_n = \mathbb{E}[(\hat{\theta} - \theta)^2]$ of $\hat{\theta}$.

Problem 6. Suppose $X_1, X_2, \dots, X_n \sim F_{\mu, \sigma^2}$ are independent and identically distributed samples from some distribution F_{μ, σ^2} with mean μ and variance σ^2 .

Recall that

$$\hat{S}_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2, \quad \hat{T}_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

are both point estimators for the parameter σ^2 .

problems with a textbook reference are based on, but not identical to, the given reference

- (a) Compute $\mathbb{E}[\hat{S}_n^2]$ and $\mathbb{E}[\hat{T}_n^2]$.
- (b) Which point estimator has smaller bias?
- (c) Which point estimator has smaller standard error?

Problem 7. Describe of point estimation of a parameter which you noticed in a different part of **your life** (e.g. in other classes, on the subway, at the park, in the dorm, etc.).