Quiz 2 (bonus) Numerical Analysis Spring 2023

Answer the following questions. You can discuss with others, but you must write up your solutions alone. Earning full points on this assignment will result in increasing your quiz 2 score by 7 points (with a maximum of 100%).

Problem 1. Let $\mathbb{F}$ denote some discrete set of numbers, and suppose that for some $\epsilon > 0$ the function $\mathrm{rd} : \mathbb{R} \to \mathbb{F}$ satisfies

$$|x - \mathrm{rd}(x)| < \epsilon |x|, \qquad \forall x \in \mathbb{R}.$$

Find the largest value of ϵ for which we can guarantee $\mathrm{rd}(10^5 + 1) \neq \mathrm{rd}(10^5)$.

Problem 2. Consider the following problem/task: You are given a differentiable function $h : [-1, 1] \to \mathbb{R}$ and must return the length 2 vector $[h'(0), \int_{-1}^{1} h(s)\mathrm{d}s]$.

Example inputs/outputs:

input	solution
$h(s) = 1$	$[0, 2]$
$h(s) = s^2 + 2s$	$[2, 2/3]$
$h(s) = \sin(s)$	$[1, 0]$

Define two inputs h and $\tilde{h}$ as near if $\mathrm{dist}(h, \tilde{h}) := \int_{-1}^{1} |h(s) - \tilde{h}(s)|\mathrm{d}s$ is small.

(a) Give a reasonable mathematical definition for the condition number of this problem at an input h. Your definition should be specific to this particular problem/task, not a generic answer.

(b) Decide whether this problem is well-conditioned or not. If it is, explain why. If it is not, provide an example showing that it is not (with justification).