Homework 3 — Linear Algbera I

Instructions:

- Due March 4 at 11:59pm on Gradescope.
- You must follow the submission policy in the syllabus

Problem 1 (Isomorphism)**.**

(a) Let $V = \mathbb{R}^n$ and $W = \mathcal{P}_{n-1}(\mathbb{R})$ (the set of polynomials of degree at most $n-1$). Write down an isomorphism ϕ from V to W (you must define $\phi(\vec{v})$ for each $\vec{v} \in V$), and prove it is an isomorphism.

(b) Suppose $\phi : U \to V$ is an isomorphism. Let $\phi^{-1} : V \to U$ be the inverse of ϕ (i.e. $\phi^{-1}(\phi(\vec{u})) = \vec{u}$ for all $u \in U$). Prove that ϕ^{-1} is an isomorphism from V to U.

(c) Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^4$. Prove that U and V are not isomorphic. (Hint: suppose ϕ is an isomorphism from U to V, and derive a contradiction.)

Problem 2 (Linear Maps)**.**

(a) Suppose $b, c \in \mathbf{R}$. Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

(b) Suppose $b, c \in \mathbf{R}$. Define $T : \mathcal{P}(\mathbf{R}) \to \mathbf{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)dx + c \sin p(0)\right).$$

Show that T is linear if and only if $b = c = 0$.

(c) Suppose $\vec{v}_1, \ldots, \vec{v}_n$ spans V. Prove that $T\vec{v}_1, \ldots, T\vec{v}_n$ spans range(T).

Problem 3 (FTLM Proof)**.** We will prove the FTLM in a slightly different way (so you can't use any consequences of the FTLM in this problem). While this is a bit more tedious at the moment, once things like (a) and (b) become second nature, I think this proof is more insightful.

Suppose V is finite dimensional and $T \in \mathcal{L}(V, W)$. Let $\vec{u}_1, \ldots, \vec{u}_m$ be a basis for null(T) and extend to a basis $\vec{u}_1, \ldots, \vec{u}_m, \vec{v}_1, \ldots, \vec{v}_n$ for V. Define a subspace $X =$ span$(v_1, \ldots, \vec{v}_n)$ of V.

(a) Define T_V as the restriction of T to X; i.e. $(T_V(\vec{x})) = T(\vec{x})$ for all $\vec{x} \in X$. Prove that $T_V \in \mathcal{L}(X, W)$.

(b) What is null(T_V)?

(c) What is $\text{range}(T_V)$?

(d) Prove that T_V is an isomorphism from X to $\text{range}(T)$; i.e. that it is injective and surjective.

(e) Conclude that $\dim(\text{range}(T)) = n$, and hence the FTLM.

Problem 4 (FTLM)**.**

(a) Give an example of a linear map with $\dim(\text{null}(T)) = 2$ and $\dim(\text{range}(T)) = 2$.

(b) Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \text{null}(ST) \leq \dim \text{null}(S) + \dim \text{null}(T).$$

(c) Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \text{range}\, ST \leq \min\{\dim \text{range}\, S, \dim \text{range}\, T\}.$$

(d) Prove there does not exist $T \in \mathcal{L}(\mathbb{R}^5)$ such that $\text{range}(T) = \text{null}(T)$.

(e) Find an example of $T \in \mathcal{L}(\mathbb{R}^4)$ such that $\text{range}(T) = \text{null}(T)$.

(f) For the operator from the previous problem, what is $\text{range}(T^2)$ and $\text{null}(T^2)$? (Here T^2 is notation for TT.)