

Homework 3

Numerical Analysis Fall 2023

Instructions:

- Due 10/13 at 5:00pm on Gradescope.
- You must follow the submission policy in the syllabus

Problem 1. For each problem, write down a matrix or matrices which performs the stated operations to a 3×4 matrix. Make sure to specify whether you should be applied on the left or right.

For reference, we will also show what the operation does to the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 2 & -3 & 1 & 1 \\ 2 & -2 & 5 & 4 \end{bmatrix}$$

- (a) Extract the second column.

$$\mathbf{A} \rightarrow \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix}, \quad \mathbf{B} \rightarrow \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix}$$

- (b) Extract the second column and place it in the third column of a 3×3 matrix of zeros.

$$\mathbf{A} \rightarrow \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & 10 \end{bmatrix}, \quad \mathbf{B} \rightarrow \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \\ 0 & 0 & -2 \end{bmatrix}$$

- (c) Swap the second and third columns.

$$\mathbf{A} \rightarrow \begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 7 & 6 & 8 \\ 9 & 11 & 10 & 12 \end{bmatrix}, \quad \mathbf{B} \rightarrow \begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & 1 & -3 & 1 \\ 2 & 5 & -2 & 4 \end{bmatrix}$$

- (d) Sum up each column.

$$\mathbf{A} \rightarrow [15 \ 18 \ 21 \ 24], \quad \mathbf{B} \rightarrow [5 \ -2 \ 4 \ 5]$$

- (e) Swap the second and third columns, then sum up each column (this one requires using two matrices).

$$\mathbf{A} \rightarrow [15 \ 21 \ 18 \ 24], \quad \mathbf{B} \rightarrow [5 \ 4 \ -2 \ 5]$$

- (f) Sum all the entries (this one requires using two matrices).

$$\mathbf{A} \rightarrow [78], \quad \mathbf{B} \rightarrow [12]$$

- Problem 2.** (a) Prove that on a log-log plot, n vs n^k is a line. What is the slope?
(b) What will the plot of n vs $5n^2 + 3n + 1$ look like on a log-log plot when n is large?
(c) Prove that on a log-y plot, n vs ρ^n is a line. What is the slope?

Problem 3. In chapter 9, we see our implementation of a triangular solver was faster than numpy's `np.linalg.solve`.

Scipy has a solver `sp.linalg.solve_triangular` for triangular systems which uses LAPACK's triangular solver.

Add this solver to the runtime comparison of `np.linalg.solve` and our triangular solver in Chapter 9 of the notes (all the code from the notes is online on the Google Drive).

Include the new figure showing the runtime scaling of all three algorithms. How does scipy's triangular solver compare?

Remember to include your code as well as the figure.

Note also that the timings don't work that well in colab, so it is preferable to run locally if possible.

Problem 4. Consider the problem of adding n numbers. That is $f(x_1, \dots, x_n) = x_1 + x_2 + \dots + x_n$.

There are many ways we could implement an algorithm for this problem. Perhaps the simplest is

```
import numpy as np

def add_list(x, n):
    s = np.zeros(1, dtype=x.dtype)
    for i in range(n):
        s = s + x[i]
    return s
```

Consider $x = [10^8, 1, 1, \dots, 1]$ for $n = 34412$. We can set this problem up as

```
n = 34412
x = np.ones(n, dtype=np.float32)
x[0] = 1*10**8

add_list(x, n)
```

Note here we are using 32 bit floating point arithmetic throughout.

(a) What is the correct answer to this problem, and what does the algorithm output?

(b) Suppose instead we implement an algorithm

```
def add_list_sorted(x,n):  
    z = np.sort(x)  
    s = np.zeros(1,dtype=x.dtype)  
    for i in range(n):  
        s = s + z[i]  
    return s
```

What is the output of `add_list_sorted(x,n)`?

(c) Give a possible explication for why the two algorithms have different outputs.

Problem 5. Describe whether the following problems/tasks are well-conditioned or not. Justify each of your responses.

- (a) Problem/task: You are given a vector $[x_1, x_2]$ and must compute the solution $[z_1, z_2]$ to the linear system of equations

$$2.3z_1 - 1.01z_2 = x_1, \quad 2.31z_1 - 1.00z_2 = x_2.$$

Example inputs/outputs:

input x	solution $P(x)$
$[1, 2]$	$[30.816, 69.184]$
$[1, 1]$	$[0.30211, -0.302115]$





- (b) Problem/task: You are given function $h : [-1, 1] \rightarrow \mathbb{R}$ and must return $\int_{-1}^1 h(s)ds$. We will use the norm $\|\tilde{h} - h\|_\infty := \max_{s \in [-1, 1]} |\tilde{h}(s) - h(s)|$.

Example inputs/outputs:

input x	solution $P(x)$
$h(s) = 1$	2
$h(s) = s^2$	2/3
$h(s) = \sin(s)$	0

- (c) Problem/task: You're designing a self-driving car. As you approach an intersection, you are given a greyscale image of the stoplight. You must determine whether it is green or red; i.e. if the car should continue through the intersection or stop.

Example inputs/outputs:

input x	solution $P(x)$
	green
	red
	green
	red

- (d) Problem/task: You're designing a self-driving car. As you approach an intersection, you are given a color image of the stoplight. You must determine whether it is green or red; i.e. if the car should continue through the intersection or stop.

Example inputs/outputs:

input x	solution $P(x)$
	green
	red
	green
	red

Images from: Brooklyn 4K - Night Drive and Jazz New York City Drive