Computational Finance Assignment Report

ABSTRACT

The theme of this report is to discuss different optimal portfolios derived by various methods. We start from Markowitz efficient portfolio (the Mean-Variance model) and compare the performance of random portfolios with the portfolios achieved by `Matlab` financial toolbox. The `CVX` toolbox and `linprog/quadprog` routines of `Matlab` have similar (identical) performances in deriving optimal portfolios. The naive 1/N portfolio outperforms other efficient portfolio models in the experiments. We apply greedy forward selection and sparse index tracking algorithm (with $l1$ norm) to find small subsets of the assets which have the smallest estimation errors. We also discuss the constraints on the transaction costs and the convex optimization problem in the last part of the report.

1 INTRODUCTION

In the first part of this report, we discuss Markowitz efficient portfolio and compare the performance of random portfolios with efficient frontier derived by `Matlab` financial toolbox. Then we apply `CVX` toolbox and `linprog/ quadprog` routines to achieve portfolios and find out that they have similar performances. We use FTSE100 data of the past 3 years to experiment two methods: naive 1/N portfolio and mean-variance portfolio. The result shows that the 1/N model is a better choice in the situation discussed in question. Then we apply both the greedy forward selection and sparse index tracking algorithm to find small subsets of the assets which have the lowest value of the tracking error. The sparse index tracking algorithm has a smaller error than that of greedy forward selection. In the final part, we discuss the objective functions and constraints on the transaction costs and analyse a specific case about the convex optimization problem.

2 QUESTIONS

2.1 Markowitz Efficient Frontier

We construct a portfolio of investment in the two assets with weights π_1, π_2 and get weight vector $\boldsymbol{\pi} = \binom{\pi_1}{\pi_2}$. Since π_1, π_2 are percentages, we have $\pi_1, \pi_2 \in [\,0, 1\,]$ and $\pi_1 + \pi_2 = 1$. The expected returns and covariances on returns are given by $\boldsymbol{m} = [\,0.10\ 0.10\,]^t$ and $\begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \end{bmatrix}$ respectively. The mean return M is calculated by **Eq.1** which equals to a constant: 0.1. The variance V calculated by **Eq.2** achieves V_{min}=0.0025 when $\pi_1 = 0.5$.

$$M = \boldsymbol{\pi}^T \boldsymbol{\mu} = \boldsymbol{\pi}^T \boldsymbol{m} = 0.1 \times \pi_1 + 0.1 \times \pi_2 = 0.1 \times 1 = 0.1 \quad \textbf{Eq.1}$$

$$V = \boldsymbol{\pi}^T \Sigma \boldsymbol{\pi} = \boldsymbol{\pi}^T C \boldsymbol{\pi} = 0.01 \times \pi_1^2 - 0.01 \times \pi_1 + 0.005 \quad \textbf{Eq.2}$$

The value of V is between 0.0025 and 0.005 while the Markowitz Efficient Frontier is at point $(0.0025, 0.1)$. We draw the figure of the $M - V$ model in Figure 1.

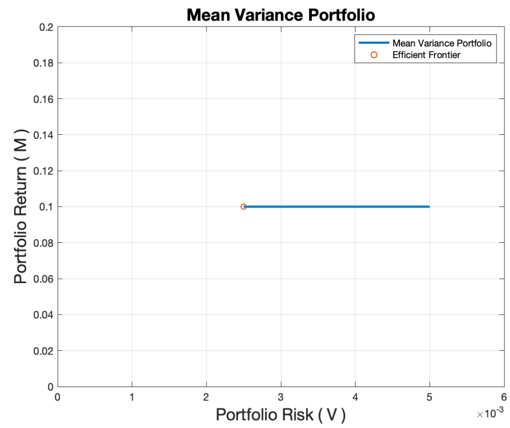

Figure 1: Mean Variance Portfolio

2.2 Efficient Portfolio Frontier vs Random Portfolios

In this section, we are given three securities which have the expected return $\boldsymbol{m} = [\,0.10\ 0.20\ 0.15]^t$, and corresponding covariances

$$C = \begin{bmatrix} 0.005 & -0.010 & 0.004 \\ -0.010 & 0.040 & -0.002 \\ 0.004 & -0.002 & 0.023 \end{bmatrix}$$

We generate 100 random portfolios of the combination of there three securities and compare them with the efficient portfolio frontier.

2.2.1 *Three-asset Model.*

Firstly, we plot scatter diagrams of three-assets portfolios and compare them with the efficient frontier which is shown in Figure2.

2.2.2 *Two-asset Model.*

Here we take the assets pair-wise and see the performance comparing to the efficient frontier.

The means and covariance of three combinations: asset 1 and 2, asset 1 and 3, asset 2 and 3 which are $\boldsymbol{m}_{12}, \boldsymbol{m}_{13}, \boldsymbol{m}_{23}$,C_{12}, C_{13}, C_{23} are given as below. The portfolios and their corresponding efficient frontiers are shown in Figure3, 4, 5 respectively.

$$\boldsymbol{m}_{12} = [\,0.10\ 0.20\,]^t, C_{12} = \begin{bmatrix} 0.005 & -0.010 \\ -0.010 & 0.040 \end{bmatrix}$$

$$\boldsymbol{m}_{13} = [\,0.10\ 0.15\,]^t, C_{13} = \begin{bmatrix} 0.005 & 0.004 \\ 0.004 & 0.023 \end{bmatrix}$$

$$\boldsymbol{m}_{23} = [\,0.20\ 0.15\,]^t, C_{23} = \begin{bmatrix} 0.040 & -0.002 \\ -0.002 & 0.023 \end{bmatrix}$$

Figure 2: Efficient Portfolio Frontier vs Three-asset Random Portfolios

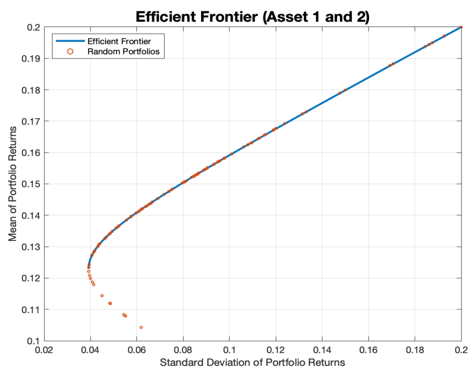

Figure 3: Efficient Portfolio Frontier vs Two-asset (1 and 2) Random Portfolios

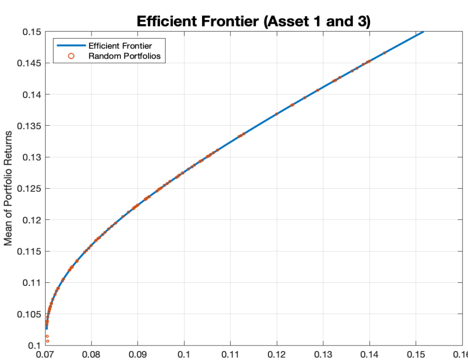

Figure 4: Efficient Portfolio Frontier vs Two-asset (1 and 3) Random Portfolios

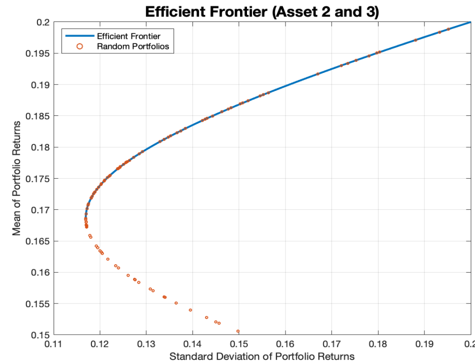

Figure 5: Efficient Portfolio Frontier vs Two-asset (2 and 3) Random Portfolios

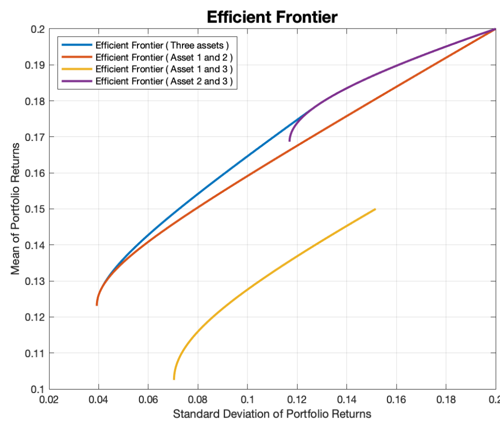

Figure 6: Efficient Portfolio Frontier (Different Assets Combinations)

The comparison of three assets and different combinations of two assets shown in Figure6 illustrates that the portfolio of three assests is better than any other combinations of two assets' portfolios, which corresponds to the core idea that investing in a portfolio of assets rather than in a single asset is an efficient way of eliminating risks of single assets. Besides, the performance of asset 1 and 3 is the worst of the four portfolios. The covariance of it is positive, which means that when two assets are positively correlated, the portfolio of them may not perform better than that with negative correlations.

2.3 NaiveMV function with CVX toolbox

In this part, we replace the `linprog` (Linear programming) and `quadprog` (Quadratic programming) in the NaiveMV function and use CVX toolbox instead to construct efficient portfolios. The `linprog` finds the maximum return constrained by:

$$\max \boldsymbol{\pi}^T \boldsymbol{\mu} \text{ subject to } \sum_{i=1}^{N} \pi_i = 1, \text{ and } \pi_i \geq 0$$

With CVX toolbox, we replace the function of `linprog` as below:

```
cvx_begin quiet
    variable MaxReturnWeights(NAssets,1)
    maximize (MaxReturnWeights.' * ERet)
    subject to
        V1*MaxReturnWeights== 1;
        MaxReturnWeights >= V0;
cvx_end
```

The quadprog finds the minimum variance constrained by:

$$\min \boldsymbol{\pi}^T \Sigma \boldsymbol{\pi} \text{ subject to } \sum_{i=1}^{N} \pi_i = 1$$

With CVX toolbox, we replace the function of quadprog as below:

```
cvx_begin quiet
    variable MinVarWeights(NAssets,1)
    minimize (MinVarWeights' * ECov * MinVarWeights)
    subject to
        V1*MinVarWeights== 1;
        MinVarWeights >= V0;
cvx_end
```

Figure7 shows the comparison of portfolios achieved by linprog/quadprog and CVX toolbox. From the figure we can find out that the portfolios by the two methods are overlapped and both of them are on the efficient frontier plotted by matlab financial toolbox.

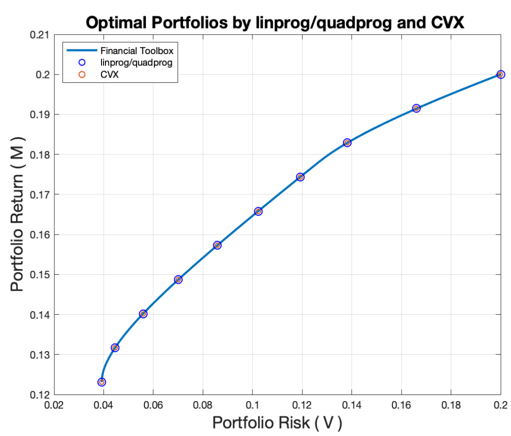

Figure 7: Optimal Portfolios constructed by linprog/quadprog and CVX

The mean squre error for the returns derived by `linprog/quadprog` and CVX toolbox is 1.8031e-14 which means the results can be regarded as identical.

2.4 Efficient Portfolio vs 1/N Portfolio

We obtain daily FTSE 100 data and data for the prices of 30 companies in the FTSE index from `Yahoo Finance`. The period of data is from 09/02/2016 to 08/02/2019. We connect the data in each file with the column of adjusted closing price, which is the closing price after adjustments for all applicable splits and dividend distributions [1]. The missing values in the historical data of the 30 components can be filled in by the function `fillmissing` with the adjusted closing price from the previous day so that the length of data in each file is equal. We use `tick2ret` function to convert price series to return series to achieve the daily return of each company. The shape of the matrix we construct for the next steps is 759× 30. Next, we choose three assets randomly by `randperm` and divide them into two parts: row 1 to 380 as the training set and row 381 to 759 as the test set.

The mean and covariance of the training set (asset 17, 13, 22) are computed as below ($T = 380$) :

$$\hat{\boldsymbol{\mu}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{r}(t) = \begin{pmatrix} 0.0003 & 0.0027 & 0.0015 \end{pmatrix}$$

$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T} (\boldsymbol{r}(t) - \hat{\boldsymbol{\mu}})^T (\boldsymbol{r}(t) - \hat{\boldsymbol{\mu}}) = 1\times 10^{-3} \begin{pmatrix} 0.3307 & 0.0918 & 0.0570 \\ 0.0918 & 0.3944 & 0.0595 \\ 0.0570 & 0.0595 & 0.1753 \end{pmatrix}$$

We use `NaiveMV` to derive Markowitz mean-variance portfolio with the highest return and lowest risk. The weight of the portfolio: $\boldsymbol{\pi}$ =[0.2307, 0.1752, 0.5941] is used to compute the daily return over the test set of all the three assests. The weights of 1/N portfolio is denoted as $\boldsymbol{\pi}_{1/N}$ =[1/3, 1/3, 1/3]. Figure 8 illustrates the daily return series obtained by two methods discussed above. With the function `cumsum` we construct the cumulative return series for the two methods shown as in Figure 9, which shows the 1/N portfolio returns is slightly higher than the Markowitz mean-variance portfolio on the test set within the second half of time period in this question.

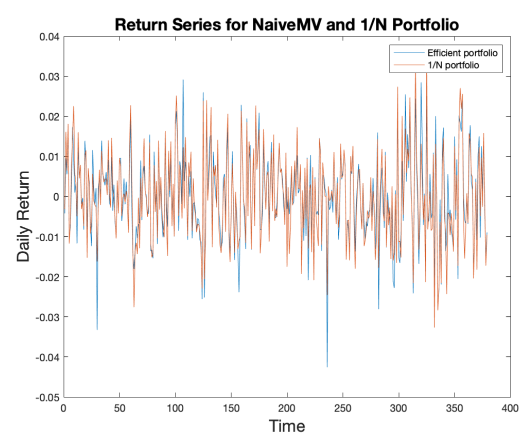

Figure 8: Return Series for NaiveMV and 1/N Portfolio

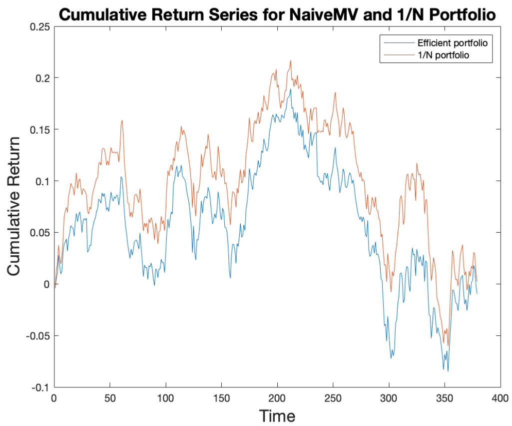

Figure 9: Cumulative Return Series for NaiveMV and 1/N Portfolio

In addition, we compare the Sharpe ratio of these two models by `Sharpe`. The ratio of the Markowitz mean-variance portfolio is -0.0023 which indicates the expected return is less than risk-free rate, i.e. the portfolio has a high risk. For the 1/N portfolio, the ratio is 5.4291e-04, which is higher than that of the M-V portfolio. We could conclude that the 1/N portfolio performs better in this particular case. As the assets are chosen randomly, we also find other portfolios by M-V model which performs better than that of the 1/N portfolio. In conclusion, there is no clear evidence to prove that the other efficient portfolio could perform better than 1/N portfolio consistently. The result in this question is consistent with the discussion by DeMiguel in [3].

2.5 Asset-pricing Models with Unobservable Factors

In this section, we aim to implement enhancements reviewed in [3] based on the method explained in [2], which is implying portfolios by asset-pricing models with unobservable factors. The core idea of the model is that with the unobservable factors of the returns, the mispricing result is contained in the covariance matrix of the residuals. Unlike traditional methods, the model constructed by the estimator of expected returns is more stable and reliable. The covariance matrix of the residuals (instead of returns) in the model is expressed as:

$$\Sigma = v\mu\mu^T + \sigma^2 I_N, \qquad \textbf{Eq.3}$$

where v, σ^2 are positive scalars achieved by maximum likelihood estimation, μ is used to estimate the mean and covariance matrix of the asset returns.

$$\max_{x_t} x_t^T \mu_t - \frac{\gamma}{2} x_t^T \Sigma_t x_t \qquad \textbf{Eq.4}$$

The optimal portfolio weights can be calculated by replacing the Σ in Eq.**Eq.4** with Eq.**Eq.3**. In the paper[2], it suggests to estimate the value of μ, v, σ^2 by maximizing the joint log-likelihood function **Eq.5** with the constraints that v, σ^2 are positive. The likelihood function used to estimate the scalars are shown as below

$$L(\mu, v, \sigma^2 | z_1, ...z_T) \propto \left| v\mu\mu' + \sigma^2 I \right|^{-T/2}$$
$$\times exp\Big\{ -\frac{1}{2} \sum_{t=1}^{T} (z_t - \mu)' (v\mu\mu' + \sigma^2 I)^{-1} (z_t - \mu) \Big\}. \qquad \textbf{Eq.5}$$

The expected returns μ A constrained quasi-Newton numerical procedure is applied to solve the maximum likelihood estimation by the sequential quadratic routine `constr`. In the experiment, we use `NaiveMV` to find the optimal weights as we did in the previous sections.

We assume that the asset returns are normally distributed. Hence, the MLE estimator of the mean for arbitrary estimator in the covariance matrix is the sample mean, which is irrelevant to the expected return μ. In this way, we could remove μ when we construct the MLE estimator of Σ.

In the experiment, we use NaiveMV and MP strategy to construct optimal portfolios respectively. When calculating the maximum likelihood of v, we use risk-free rate=3% based on the average risk-free rate of the UK during the past 20 years.

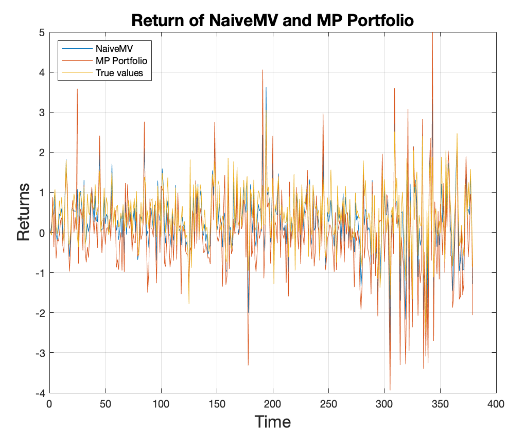

Figure 10: Return of NaiveMV and MP Portfolio

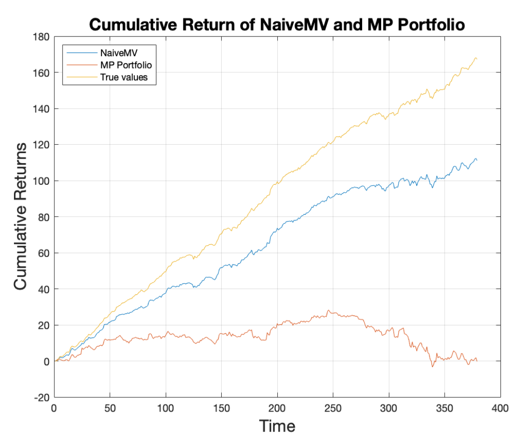

Figure 11: Cumulative Return of NaiveMV and MP Portfolio

Figure 10 and 11 show the returns and cumulative returns achieved by the two methods. The mp model performs worse than both the naive MV model and the 1/N portfolio. The Sharpe ratio of mp model is 0.9188 whereas the value of naive MV and 1/N portfolio is 0.9291 and 1.1754. We can conclude from the result that the mp model can not consistently outperform the naive MV and 1/N model.

2.6 Greedy Forward Selection vs Sparse Index Tracking

We use the same data set as discussed in section **2.4**. The historical data of the 30 companies for the past three years are used for estimating the total return whereas the FTSE 100 index is treated as our target. In this section, we use two different methods of index tracking: Greedy forward selection and Sparse portfolio to obtain two models. Our task is to find the minimum subset of the 30 companies which have the minimum estimation error and compare the number of the subsets derived by the two methods. In other words, we aim to find a vector of portfolio weights $\mathbf{w} = \arg\min_{w} \|\mathbf{y} - \mathbf{R}\mathbf{w}\|_2^2$. The $\mathbf{y}$ represents the FTSE 100 index returns and the $\mathbf{R}$ is the returns of each asset in the portfolio.

Before applying two methods, we use `tick2ret` function to obtain the daily return of the 30 companies and the FTSE 100 index separately.

2.6.1 Greedy Forward Selection.

First, we use the greedy forward selection to find the optimal portfolio. We start from choosing one company and compare the estimation errors $\|\mathbf{y} - \mathbf{R}\mathbf{w}\|_2^2$ for each asset and choose the one which has the minimum estimation error as the portfolio. By repeating the above process, we pick the portfolio which has the smallest error in each step. According to the limitation in the question, which is to select about a fifth of the available stocks, we will end up the search after finding the optimal portfolio of 6 assets.

The details of the steps are listed in Table 1 and shown in figure 12. The final 6-asset portfolio chosen by greedy forward selection are: 8, 3, 29, 25, 27, 1 and the estimation error of the final step is: 0.3032. It is noticable that the estimation error of the 6th step is actually higher than that in the step 5.

Step	Asset Added in Each Step	Weights of Each Step	Estimation Error
1	8	1.0	1.6930
2	3	0.7243, 0.2757	0.3465
3	29	0.4807, 0.3760, 0.1433	0.3216
4	25	0.3598, 0.2716, 0.2695, 0.0991	0.3082
5	27	0.2954, 0.2291, 0.2099, 0.1974, 0.0683	0.3016
6	1	0.3006, 0.2582, 0.2230, 0.1091, 0.1091, 1.8197e-06	0.3032

Table 1: Steps of Greedy Forward Selection

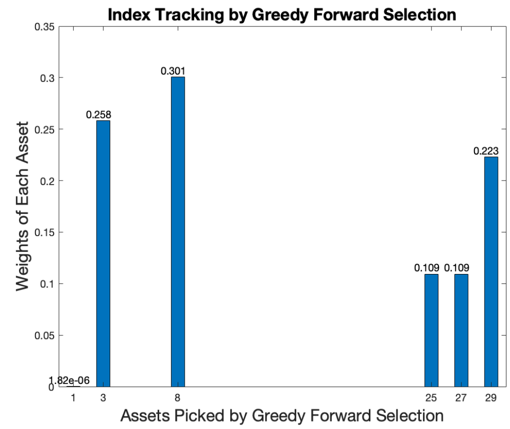

Figure 12: Index Tracking by Greedy Forward Selection

2.6.2 Sparse Index Tracking.

Now we try a different method of $l1$ regularization to construct a sparse portfolio. We use the CVX toolkit to solve the objective function:

$$\boldsymbol{w}^{[\tau]} = \arg\min_{w} \left[\|\mathbf{y} - \mathbf{R}\mathbf{w}\|_2^2 + \tau \|\boldsymbol{w}\|_1\right], \text{ subject to } \sum_{i=1}^{N} w_i = 1 \quad \textbf{Eq.6}$$

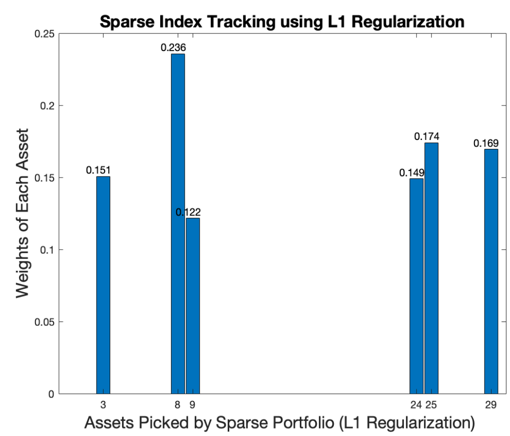

Figure 13: Sparse Index Tracking using L1 Regularization

The 6 assets chosen by sparse index tracking using $l1$ penalty are: 8, 3, 29, 24, 9, 25 (see in Figure13) and the estimation error is: 0.2952. We can notice that both two methods choose the assets: 8, 3, 25, 29 into the portfolios. In the experiment, we tuned τ for a wide range of values (from 0.0001 to over than 100) and found out that the weights of the portfolios returned by the `CVX` toolbox are similar to each other. To reduce the estimation error which is shown in **Eq.6**, we set τ=0.0001 and derive the smallest estimation error 0.2952, which is slightly smaller than that by the greedy forward selection algorithm. The comparison of the assets and their corresponding weights of these two methods are shown in Table2.

Content	Asset	Weight	Asset	Weight
Methods	Greedy		Sparse	
	3	0.258	3	0.151
	8	0.301	8	0.236
	25	0.109	25	0.174
	29	0.223	29	0.169
	1	1.82e-06	9	0.122
	27	0.109	24	0.149
Error	0.3032		0.2952	

Table 2: Comparison of Greedy and Sparse Algorithm

To compare the performance of the two methods with the target value: FTSE 100 index (time period: 09/02/2016 to 08/02/2019), we plot the cumulative return series for FTSE 100 index, greedy forward selection and sparse index tracking with $l1$ penalty shown in Figure14. The performance of greedy search is better than the true value for the most of the time within the period in question, while the return of sparse index tracking with $l1$ regularization outweigh both the target and performance by greedy algorithm all the time.

After comparing both the estimation error and the cumulative returns derived by the two methods, we can conclude that the sparse portfolio can maintain or even improve the performance of the assets. By adding only a small value of penalty, we can construct a powerful tool to reduce the estimation error and focus on a small subset of the assets. In this way, we could ensure a relatively stable or even higher performance as well as reduce the transaction cost of the whole assets.

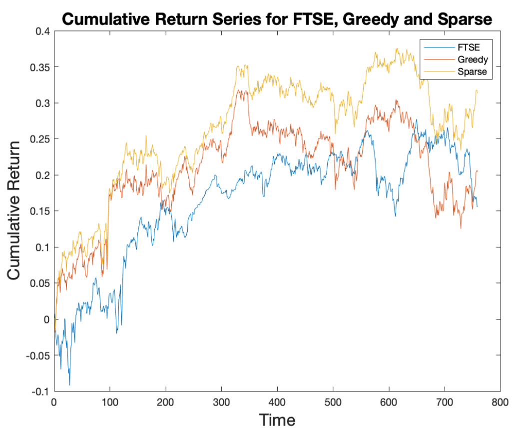

Figure 14: Cumulative Return Series for FTSE, Greedy and Sparse

2.7 Transaction Costs

2.7.1 *Constraints on Transaction Costs.*

In this section, we study the idea of Lobo et al. [4] about the effects of transaction costs in optimal portfolios. Firstly, we specify the notations of the variables as below:

The current holdings in each asset are $w = (w_1, \ldots w_n)^T$. The total current wealth is $\mathbf{1}^T w$, where $\mathbf{1}$ is a vector with all entries equal to one. The dollar amount transacted in each asset is $x = (x_1, \ldots x_n)^T$ ($x_i > 0$ for buying, $x_i < 0$ for selling). The adjusted portfolio after transaction is $w + x$, which is held for a fixed period of time. The return on asset i by the end of the period of time is denoted as a random variable a_i which the expectation is $\mathbf{E}$. With the knowledge of the first and second moments of the joint distribution being assumed as $a = (a_1, \ldots, a_n)$, we can denote the expected return $\bar{a}$ and the covariance matrix Σ as:

$$\bar{a} = \mathbf{E}a, \quad \Sigma = \mathbf{E}(a - \bar{a})(a - \bar{a})^T \qquad \textbf{Eq.7}$$

When we consider a riskless asset, the corresponding $\bar{a}_i$ equals to a_i, and $\Sigma_{ii} = 0$. The end of period wealth is $W = a^T(w + x)$, which the expected value and variance are:

$$\mathbf{E}W = \bar{a}^T(w + x), \quad \mathbf{E}(W - \mathbf{E}W)^2 = (w + x)^T \Sigma (w + x) \qquad \textbf{Eq.8}$$

The transaction costs in the paper are assumed to be separable, which means that the sum of the transaction costs can be calculated by each trade.

$$\phi(x) = \sum_{i=1}^{n} \phi_i(x_i) \qquad \textbf{Eq.9}$$

The realistic transaction cost in each trade is expressed as:

$$\phi_i(x_i) = \begin{cases} \alpha_i^+ x_i, & x_i \geqslant 0 \\ -\alpha_i^- x_i, & x_i \leqslant 0 \end{cases} \qquad \textbf{Eq.10}$$

where α_i^+ and α_i^- represent the cost rates associating with buying and selling asset i. The linear transaction cost functions ϕ_i can be handled by using variables $x^+, x^- \in \mathbf{R}^n$ to calculate the total transaction $x_i = x_i^+ - x_i^-$, where $x_i^+ \geq 0$ and $x_i^- \geq 0$. In this way, we can express cost functions ϕ_i as:

$$\phi_i(x_i) = \alpha_i^+ x_i^+ + \alpha_i^- x_i^- \qquad \textbf{Eq.11}$$

Various constraints are imposed in the paper, ranging from the budget constraint to the shortfall risk constraint, which will be discussed separately.

- Budget(self-financing) Constraint
  Given the amount transacted in each asset $x = (x_1, \ldots x_n)^T$, the sum of all corresponding transaction costs $\phi(x)$>0 and the assumption that $\bar{a}_i > 0$ we have

  $$\mathbf{1}^T x + \phi(x) \leq 0 \qquad \textbf{Eq.12}$$

  i.e. the sum of the transaction costs must not exceed the total transacted amount.
- Diversification Constraints

(1) Individual Diversification Constraints
The amount invested in each asset i is constrained to a maximum of pi shown in **Eq.13**, and its alternative way in **Eq.14**

$$w_i + xi \leq p_i, \quad i = 1, \ldots, n \qquad \textbf{Eq.13}$$

$$w_i + x_i \leq \gamma_i \mathbf{1}^T (w + x), \quad i = 1, \ldots, n \qquad \textbf{Eq.14}$$

The γ_i in the **Eq.14** denotes the proportion accounted by each asset of the total (post transaction) wealth.

(2) The Amount of Wealth invested in Small Subsets
The constraints limit the largest proportion γ of the total wealth which can be invested in no more than r assets (see

in **Eq.15**).

$$\sum_{i=1}^{r}(w+x)_{[i]} \leq \gamma_i \mathbf{1}^T(w+x), \quad i=1,\dots,n \qquad \textbf{Eq.15}$$

- Shortselling Constraints
  Shortselling constraints limit the maximum amount of shortselling allowed on asset i. The individual bound is denoted by s_i(which is set to zero if shortselling is forbidden):

$$w_i + x_i \geqslant -s_i, \quad i=1,\dots,n. \qquad \textbf{Eq.16}$$

  The bound on total shortselling is S:

$$\sum_{i=1}^{n}(w_i+x_i)_- \leq S, \qquad \textbf{Eq.17}$$

  where$(\xi)_- = \max\{-\xi, 0\}$.
- Variance
  The standard deviation of the wealth W by the end of period is limited to be no more than σ_{max}

$$(w+x)^T\Sigma(w+x) \leq \sigma_{max}^2, \qquad \textbf{Eq.18}$$

  The equation above is a convex quadratic inequality which can also be expressed as

$$\left\|\Sigma^{1/2}(w+x)\right\| \leq \sigma_{max}, \qquad \textbf{Eq.19}$$

  with $\|\cdot\|$ and $\Sigma^{1/2}$ denote the $l2$ norm and the symmetric matrix square root of Σ respectively.
  The standard deviation of the return R is defined as a ratio: $R = W/(1^T w)$, which is limited by a maximum value $\sigma_{R,max}$

$$\left\|\Sigma^{1/2}(w+x)\right\| \leq \sigma_{R,max} 1^T w. \qquad \textbf{Eq.20}$$

  The constraints on the wealth $\boldsymbol{W}$ (**Eq.20**) and the return $\boldsymbol{R}$ are all sencond-order cone constraints.
- Shortfall Risk Constraints
  Assuming the random vector of returns a have a jointly Gaussian distribution, $a \sim N(\hat{a}, \Sigma)$, we constrain the probablity of the end of period wealth W being greater than a lower level W^{low} exceeds η ($\eta \geqslant 0.5$):

$$\boldsymbol{Prob}(W \geq W^{low}) \geq \eta. \qquad \textbf{Eq.21}$$

  For $W^{low} < \mathbf{1}^T w$, the probablity constraint is equivalent to: Given a confidence level η, the value at risk per dollar invested of $VaR = 1 - W^{low}/\mathbf{1}^T w$. Let $\boldsymbol{\Phi}$ denotes the probability of a standard Gaussian distribution, $\boldsymbol{\mu} = \bar{a}^T(w+x)$ and $\sigma^2 = (w+x)^T\Sigma(w+x)$, the original constraint can be expressed as

$$\boldsymbol{\Phi}^{-1}(\eta)\left\|\Sigma^{1/2}(w+x)\right\| \leq \bar{a}^T(w+x) - \boldsymbol{W}^{low}, \qquad \textbf{Eq.22}$$

  which is also a second-order cone constraint when $\eta \geqslant 0.5$. With the assumption of Gaussian, we could balance the return and confidence by imposing shortfall risk constraints.

2.7.2 *Aspects of the Conclusions in Fig.2.*

We have concluded from the above section that imposing one or more shortfall risk constraints helps to preserve the convexity of the problem. As the constraint in **Eq.22** is a second-order cone constraint and the objective function is linear, we can solve it by convex optimisation. The extent of the confidence we impose depends on how bad the return is.

The Fig.2 of the paper shows the cumulative distribution function of the return the expected return by the dotted line. The two dashed lines show the shortfall probability. The higher limit limits the return below 0.9 to be less than 20% and the lower and the left line limits the probability of a return below 0.7 to be no more than 3%. It illustrates that the shortfall risk constraint ensures a relatively higher return by increasing the probability of the higher return to a higher level of confidence. In other words, the constraint helps to reduce the probability of returns if they are bad or even disastrous so that the lower the return, the smaller the probability.

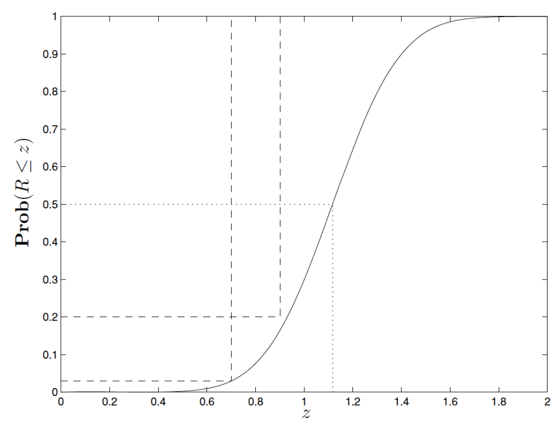

Figure 15: Fig. 2 in Paper[4]

2.7.3 *The Optimization Problem.*

The optimization problem discussed in paper[4] section 1.6 contains one riskless and 100 risky assets (n = 101). The constraints of the problem are as below, which specifies the objective function (i), the linear transaction costs constraints (ii), (iii), one limit (iv) on shortselling of s_i per asset, and shortfall risk constraints (v).

$$\begin{aligned}
&\text{maximize} \quad \bar{a}^T(w+x^+-x^-) \quad (i)\\
&\text{subject to}\\
&\mathbf{1}^T(x^+-x^-) + \textstyle\sum_{i=1}^{n}(\alpha_i^+ x_i^+ + \alpha_i^- x_i^-) \leqslant 0 \quad (ii)\\
&x_i^+ \geqslant 0, \quad x_i^- \geqslant 0, \quad i=1,\dots,n \quad (iii)\\
&w_i + x_i^+ - x_i^- \geqslant s_i, \quad x_i^- \geqslant 0, \quad i=1,\dots,n \quad (iv)\\
&\Phi^{-1}(\eta_j)\left\|\Sigma^{1/2}(w+x^+-x^-)\right\| \leqslant \bar{a}^T(w+x^x-x^-) - W_j^{low},\\
&j=1,2. \quad (v)
\end{aligned}$$

Eq.23

The parameters defined in this problem are:

$$\eta_1 = 80\%, \quad W_1^{low} = 0.9; \quad and \quad \eta_2 = 97\%, \quad W_2^{low} = 0.7. \qquad \textbf{Eq.24}$$

The Fig. 3 in the paper shows the efficient frontier which is the trade-off between the expected return*ER* and the standard deviation σ. With the shortfall constraints, the optimal solution to this problem is shown as the small circle.

In the previous sections, we have used the financial toolbox of matlab, the function `NaiveMV` and `CVX` toolbox to derive optimal portfolios. By tuning the parameters in the problem, we can constrain the variables η and W to get better performances (higher returns with lower variances). We may also use the sparse index tracking method to get a smaller subset so that we can reduce transaction costs.

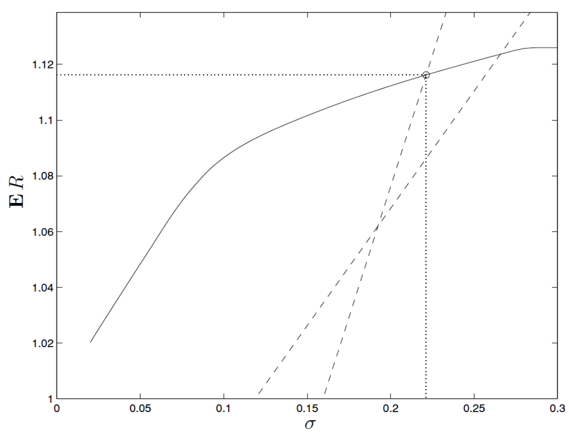

Figure 16: Fig. 3 in Paper[4]

3 CONCLUSION

The theme of this report is optimal portfolios. We discuss Markowitz efficient portfolio (Mean-Variance model) and derive the portfolios by `Matlab` financial toolbox with different routines. The `CVX` toolbox and `linprog/quadprog` routines have identical performances. We use FTSE100 data of the past 3 years, divide it into training set and test set to verify the results achieved by two methods: naive 1/N portfolio and mean-variance portfolio. The 1/N model is a better choice in comparison with other strategies. The greedy forward selection and sparse index tracking algorithm with $l1$ penalty are used to reduce components of the total assets so that we can reduce transaction costs as well as reducing the estimation error. The sparse index tracking algorithm performs better than the greedy forward selection. We discuss the objective functions and constraints on the transaction costs and analyse a specific case about the convex optimization problem in the last part. More study needs to be done in further research to evaluate different models performance and to solve the convex optimization problem.

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