# STA4003: Time Series

Tutorial 4

The Chinese University of Hong Kong, Shenzhen

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# Overview

# AR Model: Definition

AR($p$) model:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + w_t$$

- $X_t$: stationary, $\mu = 0$
- $\phi_1 \ldots \phi_p$: constants, $\phi_p \neq 0$
- $w_t$: white noise

Use backshift operator,

$$(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) X_t = w_t$$

Define autoregressive operator $\phi(B) := 1 - \phi_1 B - \cdots - \phi_p B^p$,

$$\phi(B) X_t = w_t$$

# AR Model: Properties

**Property**: AR($p$) model can be written as a linear process.

Example: AR(1) model

$$\begin{aligned} X_t &= \phi X_{t-1} + w_t \\ &= \phi(\phi X_{t-2} + w_{t-1}) + w_t \\ &= \cdots = \phi^k X_{t-k} + \phi^{k-1} w_{t-(k-1)} + \cdots + \phi w_{t-1} + w_t \end{aligned}$$

If $|\phi| < 1$, $\phi^k X_{t-k} \to 0$ as $k \to \infty$. Thus

$$X_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}$$

# AR Model: Properties

**Property**: AR($p$) is causal stationary if and only if every root $r_i$ of $\phi(Z)$ satisfies $|r_i| > 1$, i.e., outside the unit circle.

$\phi(Z)$: autoregressive polynomial of the AR($p$) model

$$\begin{aligned}\phi(Z) &= 1 - \phi_1 Z - \phi_2 Z^2 - \cdots - \phi_p Z^p \\ &= \prod_{i=1}^{p}(1 - r_i^{-1} Z)\end{aligned}$$

The autoregressive operator $\phi(B)$ can also be written as $\prod_{i=1}^{p}(1 - r_i^{-1} B)$. So the AR($p$) model $\phi(B)X_t = w_t$ is equivalent to

$$(1 - r_1^{-1}B)(1 - r_2^{-1}B)\ldots(1 - r_p^{-1}B)X_t = w_t$$

If all $|r_i^{-1}| < 1$, i.e., $|r_i| > 1$, then $X_t$ is a causal stationary linear process.

# AR Model: Properties

Example (continued): AR(1) model $X_t = \phi X_{t-1} + w_t$
- if $|\phi| < 1$, $X_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}$, stationary
- if $|\phi| = 1$, $X_t = X_{t-1} + w_t$, nonstationary
- if $|\phi| > 1$,

$$\begin{aligned} X_t &= \phi^{-1} X_{t+1} - \phi^{-1} w_{t+1} \\ &= \phi^{-1}(\phi^{-1} X_{t+2} - \phi^{-1} w_{t+2}) - \phi^{-1} w_{t+1} \\ &= \cdots = (\phi^{-1})^k X_{t+k} - \sum_{j=1}^{k+1} \phi^{-j} w_{t+j} \end{aligned}$$

Since $|\phi^{-1}| < 1$, $(\phi^{-1})^k X_{t+k} \to 0$ as $k \to \infty$. Thus

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} w_{t+j}$$

not causal stationary!

# Matching Coefficients

- We know that an AR($p$) model (satisfying some conditions) can be written as a causal stationary linear process. (coefficient: $\phi$)
- We also know that a linear process has the form $X_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$. (coefficient: $\psi$)

**Aim**: find the relationship between $\phi$ and $\psi$.

Define $\psi(B) := \sum_{j=0}^{\infty} \psi_j B^j$, then the linear process has the form

$$X_t = \psi(B) w_t$$

Also, recall the expression of the AR($p$) model,

$$\phi(B) X_t = w_t$$

Therefore, we have

$$\phi(B)\psi(B) w_t = w_t \;\Rightarrow\; \phi(B)\psi(B) = 1$$

# Question 1

Find the autocovariance function $\gamma(h)$ and the autocorrelation function $\rho(h)$ of AR(1) model: $X_t = \phi X_{t-1} + w_t$, where $|\phi| < 1$.

# Question 2

Show that in AR(1) model: $X_t = \phi X_{t-1} + w_t$, we have $\psi_j = \phi^j$, where $\psi_j's$ are the coefficients in the corresponding linear process, and $|\phi| < 1$.

## Question 3

Consider a process that satisfies the zero-mean, "stationary" AR(1) equation $Y_t = \phi Y_{t-1} + e_t$ with $-1 < \phi < 1$. Let $c$ be any nonzero constant and define $W_t = Y_t + c\phi^t$.

1. Show that $E(W_t) = c\phi^t$.

2. Show that $\{W_t\}$ satisfies the "stationary" AR(1) equation $W_t = \phi W_{t-1} + e_t$.

3. Is $\{W_t\}$ stationary?

Thank you!