Suppose X_t is stationary with 0 mean, define

$$\epsilon_t := X_t - \sum_{i=1}^{h-1} a_i X_{t-i}$$

$$\delta_{t-h} := X_{t-h} - \sum_{j=1}^{h-1} b_j X_{t-j}$$

where $\boldsymbol{a} = (a_{h-1}, a_{h-2}, \cdots, a_1)^\top$ and $\boldsymbol{b} = (b_{h-1}, b_{h-2}, \cdots, b_1)^\top$ are chosen such that $E[\epsilon_t^2]$ and $E[\delta_{t-h}^2]$ are minimized.
Note that

$$E[\epsilon_t^2] = E[(X_t - \hat{X}_t)^2]$$
$$E[\delta_{t-h}^2] = E[(X_{t-h} - \hat{X}_{t-h})^2]$$

So by definition,

$$\phi_{hh} = corr(\epsilon_t, \delta_{t-h}) = \frac{E[\epsilon_t \delta_{t-h}]}{\sqrt{E[\epsilon_t^2]E[\delta_{t-h}^2]}}$$

Consider the BLP of X_t given $\{X_{t-1}, X_{t-2}, \cdots, X_{t-h}\}$

$$X_t^h = \alpha_{h1} X_{t-1} + \alpha_{h2} X_{t-2} + \cdots + \alpha_{hh} X_{t-h}$$

By Property I (to be proved later),

$$\alpha_{hh} = \frac{\rho(h) - \tilde{\boldsymbol{\rho}}_{h-1}^\top \varrho_{h-1}^{-1} \boldsymbol{\rho}_{h-1}}{1 - \tilde{\boldsymbol{\rho}}_{h-1}^\top \varrho_{h-1}^{-1} \tilde{\boldsymbol{\rho}}_{h-1}}$$

where $\varrho_{h-1} = (\rho(i-j))_{1 \le i,j \le h-1}$, $\tilde{\boldsymbol{\rho}}_{h-1} = (\rho(h-1), \cdots, \rho(1))^\top$, $\boldsymbol{\rho}_{h-1} = (\rho(1), \cdots, \rho(h-1))^\top$.
Therefore, we would like to show that

$$\phi_{hh} = \frac{E[\epsilon_t \delta_{t-h}]}{\sqrt{E[\epsilon_t^2]E[\delta_{t-h}^2]}} = \frac{\rho(h) - \tilde{\boldsymbol{\rho}}_{h-1}^\top \varrho_{h-1}^{-1} \boldsymbol{\rho}_{h-1}}{1 - \tilde{\boldsymbol{\rho}}_{h-1}^\top \varrho_{h-1}^{-1} \tilde{\boldsymbol{\rho}}_{h-1}}$$

$\diamond$ $E[\epsilon_t \delta_{t-h}]$

$$\begin{aligned}
E[\epsilon_t \delta_{t-h}] &= Cov(\epsilon_t, \delta_{t-h}) \\
&= Cov(\epsilon_t, X_{t-h} - \sum_{j=1}^{h-1} b_j X_{t-j}) \\
&= Cov(\epsilon_t, X_{t-h}) \\
&= E[(X_t - \sum_{i=1}^{h-1} a_i X_{t-i}) X_{t-h}] \\
&= E[X_t X_{t-h}] - \sum_{i=1}^{h-1} a_i E[X_{t-i} X_{t-h}] \\
&= \gamma(h) - \sum_{i=1}^{h-1} a_i \gamma(h-i) \\
&= \gamma(0) \cdot (\rho(h) - \boldsymbol{a}^\top \boldsymbol{\rho}_{h-1})
\end{aligned}$$

$\diamond$ $E[\epsilon_t^2]$

$$\begin{aligned}
E[\epsilon_t^2] &= Cov(\epsilon_t, \epsilon_t) \\
&= Cov(\epsilon_t, X_t - \sum_{i=1}^{h-1} a_i X_{t-i}) \\
&= Cov(\epsilon_t, X_t) \\
&= E[\epsilon_t X_t] \\
&= E[(X_t - \sum_{i=1}^{h-1} a_i X_{t-i}) X_t] \\
&= \gamma(0) - \sum_{i=1}^{h-1} a_i \gamma(i) \\
&= \gamma(0) \cdot (1 - \boldsymbol{a}^\top \tilde{\boldsymbol{\rho}}_{h-1})
\end{aligned}$$

$\diamond$ $E[\delta_{t-h}^2]$

$$\begin{aligned}
E[\delta_{t-h}^2] &= Cov(\delta_{t-h}, \delta_{t-h}) \\
&= Cov(\delta_{t-h}, X_{t-h} - \sum_{j=1}^{h-1} b_j X_{t-j}) \\
&= Cov(\delta_{t-h}, X_{t-h}) \\
&= E[\delta_{t-h} X_{t-h}] \\
&= E[(X_{t-h} - \sum_{j=1}^{h-1} b_j X_{t-j}) X_{t-h}] \\
&= \gamma(0) - \sum_{j=1}^{h-1} b_j \gamma(h-j) \\
&= \gamma(0) \cdot (1 - \boldsymbol{b}^\top \boldsymbol{\rho}_{h-1})
\end{aligned}$$

Therefore,

$$\phi_{hh} = \frac{E[\epsilon_t \delta_{t-h}]}{\sqrt{E[\epsilon_t^2] E[\delta_{t-h}^2]}} = \frac{\rho(h) - \boldsymbol{a}^\top \boldsymbol{\rho}_{h-1}}{\sqrt{(1 - \boldsymbol{a}^\top \tilde{\boldsymbol{\rho}}_{h-1})(1 - \boldsymbol{b}^\top \boldsymbol{\rho}_{h-1})}} \tag{1}$$

Now, what are $\boldsymbol{a}$ and $\boldsymbol{b}$?

$$\begin{aligned}
E[\epsilon_t^2] &= E[(X_t - \sum_{i=1}^{h-1} a_i X_{t-i})^2] \\
&= E[X_t^2] - 2 \sum_{i=1}^{h-1} a_i E[X_t X_{t-i}] + E[(\sum_{i=1}^{h-1} a_i X_{t-i})^2]
\end{aligned}$$

Note that

$$\sum_{i=1}^{h-1} a_i X_{t-i} = \boldsymbol{a}^\top \cdot \begin{pmatrix} X_{t-(h-1)} \\ X_{t-(h-2)} \\ \vdots \\ X_{t-1} \end{pmatrix} := \boldsymbol{a}^\top \tilde{\boldsymbol{X}}_{h-1}$$

So

$$(\sum_{i=1}^{h-1} a_i X_{t-i})^2 = \boldsymbol{a}^\top \tilde{\boldsymbol{X}}_{h-1} \tilde{\boldsymbol{X}}_{h-1}^\top \boldsymbol{a}$$

Hence,

$$E[(\sum_{i=1}^{h-1} a_i X_{t-i})^2] = \boldsymbol{a}^\top E[\tilde{\boldsymbol{X}}_{h-1} \tilde{\boldsymbol{X}}_{h-1}^\top] \boldsymbol{a}$$

Now,

$$\begin{aligned} E[\tilde{\boldsymbol{X}}_{h-1} \tilde{\boldsymbol{X}}_{h-1}^\top] &= E\left[\begin{pmatrix} X_{t-h+1} \\ X_{t-h+2} \\ \vdots \\ X_{t-1} \end{pmatrix} (X_{t-h+1}, X_{t-h+2}, \cdots, X_{t-1}) \right] \\ &= \begin{pmatrix} \gamma(0) & \gamma(-1) & \cdots & \gamma(2-h) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(3-h) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(h-2) & \gamma(h-3) & \cdots & \gamma(0) \end{pmatrix} \\ &= \gamma(0) \cdot \varrho_{h-1} \end{aligned}$$

So

$$E[\epsilon_t^2] = \gamma(0) \cdot (1 - 2\boldsymbol{a}^\top \tilde{\boldsymbol{\rho}}_{h-1} + \boldsymbol{a}^\top \varrho_{h-1} \boldsymbol{a})$$

Set $\nabla \boldsymbol{a} = \boldsymbol{0}$,

$$\begin{aligned} -2\tilde{\boldsymbol{\rho}}_{h-1} + 2\varrho_{h-1}\boldsymbol{a} &= \boldsymbol{0} \\ \Rightarrow \ \boldsymbol{a} &= \varrho_{h-1}^{-1} \tilde{\boldsymbol{\rho}}_{h-1} \end{aligned}$$

Similarly,

$$\begin{aligned} E[\delta_{t-h}^2] &= E[(X_{t-h} - \sum_{j=1}^{h-1} b_j X_{t-j})^2] \\ &= E[X_{t-h}^2] - 2\sum_{j=1}^{h-1} b_j E[X_{t-h} X_{t-j}] + E[(\sum_{j=1}^{h-1} b_j X_{t-j})^2] \\ &= \gamma(0) \cdot (1 - 2\boldsymbol{b}^\top \boldsymbol{\rho}_{h-1} + \boldsymbol{b}^\top \varrho_{h-1} \boldsymbol{b}) \end{aligned}$$

Set $\nabla \boldsymbol{b} = \boldsymbol{0}$,

$$\begin{aligned} -2\boldsymbol{\rho}_{h-1} + 2\varrho_{h-1}\boldsymbol{b} &= \boldsymbol{0} \\ \Rightarrow \ \boldsymbol{b} &= \varrho_{h-1}^{-1} \boldsymbol{\rho}_{h-1} \end{aligned}$$

Plug $\boldsymbol{a}$ and $\boldsymbol{b}$ into (1),

$$\phi_{hh} = \frac{\rho(h) - \boldsymbol{a}^\top \boldsymbol{\rho}_{h-1}}{\sqrt{(1 - \boldsymbol{a}^\top \tilde{\boldsymbol{\rho}}_{h-1})(1 - \boldsymbol{b}^\top \boldsymbol{\rho}_{h-1})}} = \frac{\rho(h) - \tilde{\boldsymbol{\rho}}_{h-1}^\top \varrho_{h-1}^{-1} \boldsymbol{\rho}_{h-1}}{1 - \tilde{\boldsymbol{\rho}}_{h-1}^\top \varrho_{h-1}^{-1} \tilde{\boldsymbol{\rho}}_{h-1}}$$

Now we prove Property I.
We have our prediction equations

$$\sum_{j=1}^{h} \alpha_{hj}\gamma(k-j) = \gamma(k),\ k = 1, \cdots, h$$

Devide each side by $\gamma(0)$,

$$\sum_{j=1}^{h} \alpha_{hj}\rho(k-j) = \rho(k),\ k = 1, \cdots, h$$

Write into a matrix form,

$$\varrho_h \boldsymbol{\alpha}_h = \boldsymbol{\rho}_h$$

where

$$\varrho_h = \begin{pmatrix} \rho(0) & \rho(-1) & \cdots & \rho(1-h) \\ \rho(1) & \rho(0) & \cdots & \rho(2-h) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(h-1) & \rho(h-2) & \cdots & \rho(0) \end{pmatrix} \ \boldsymbol{\alpha}_h = \begin{pmatrix} \alpha_{h1} \\ \alpha_{h2} \\ \vdots \\ \alpha_{hh} \end{pmatrix} \ \boldsymbol{\rho}_h = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(h) \end{pmatrix}$$

Define $\boldsymbol{\alpha}_1 = (a_{h1}, a_{h2}, \cdots, a_{h,h-1})^\top$, and denote

$$\varrho_h = \begin{pmatrix} \rho(0) & \cdots & \rho(2-h) & \rho(1-h) \\ \vdots & \ddots & \vdots & \vdots \\ \rho(h-2) & \cdots & \rho(0) & \rho(1) \\ \rho(h-1) & \cdots & \rho(1) & \rho(0) \end{pmatrix} = \begin{pmatrix} \varrho_{h-1} & \tilde{\boldsymbol{\rho}}_{h-1} \\ \tilde{\boldsymbol{\rho}}_{h-1}^\top & 1 \end{pmatrix}$$

So

$$\varrho_h \boldsymbol{\alpha}_h = \begin{pmatrix} \varrho_{h-1} & \tilde{\boldsymbol{\rho}}_{h-1} \\ \tilde{\boldsymbol{\rho}}_{h-1}^\top & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \alpha_{hh} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\rho}_{h-1} \\ \rho(h) \end{pmatrix}$$

Hence,

$$\begin{cases} \tilde{\boldsymbol{\rho}}_{h-1}^\top \boldsymbol{\alpha}_1 + \alpha_{hh} = \rho(h) \\ \varrho_{h-1}\boldsymbol{\alpha}_1 + \tilde{\boldsymbol{\rho}}_{h-1}\alpha_{hh} = \boldsymbol{\rho}_{h-1} \end{cases}$$

And we finally obtain

$$\alpha_{hh} = \frac{\rho(h) - \tilde{\boldsymbol{\rho}}_{h-1}^\top \varrho_{h-1}^{-1} \boldsymbol{\rho}_{h-1}}{1 - \tilde{\boldsymbol{\rho}}_{h-1}^\top \varrho_{h-1}^{-1} \tilde{\boldsymbol{\rho}}_{h-1}}$$