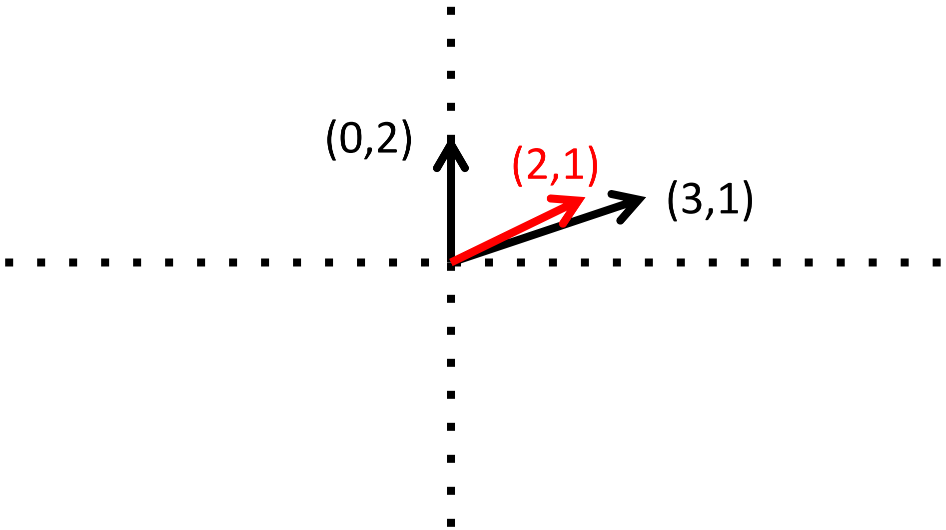
Eigenvectors & eigenvalues

Zichen Chen

What do matrices do to vectors?

$$\overleftrightarrow{M}$$

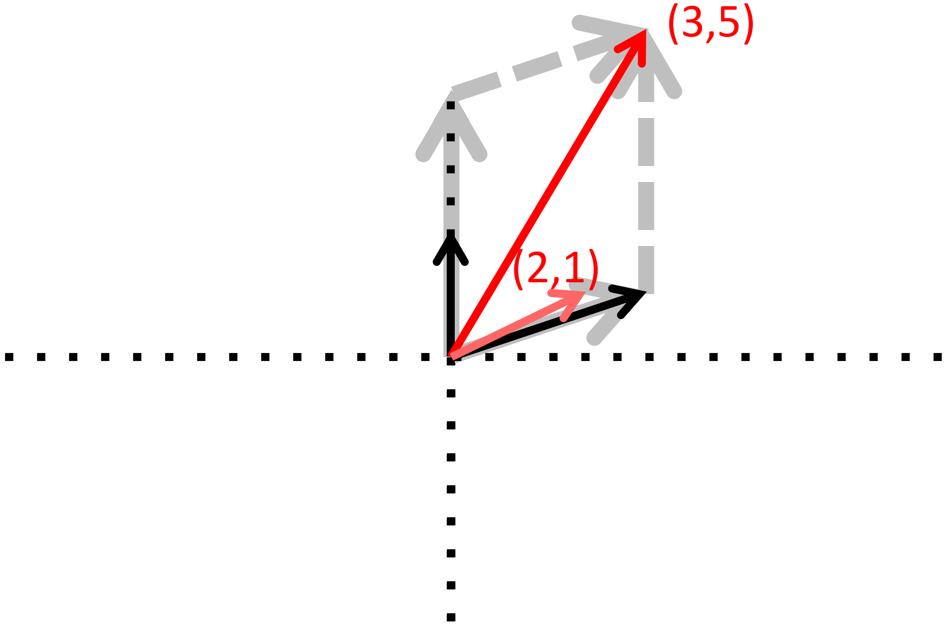
$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$



Recall

M

$$\begin{pmatrix}0 & 3\\ 2 & 1\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix} = 2\begin{pmatrix}0\\ 2\end{pmatrix} + 1\begin{pmatrix}3\\ 1\end{pmatrix} = \begin{pmatrix}0\\ 4\end{pmatrix} + \begin{pmatrix}3\\ 1\end{pmatrix} = \begin{pmatrix}3\\ 5\end{pmatrix}$$



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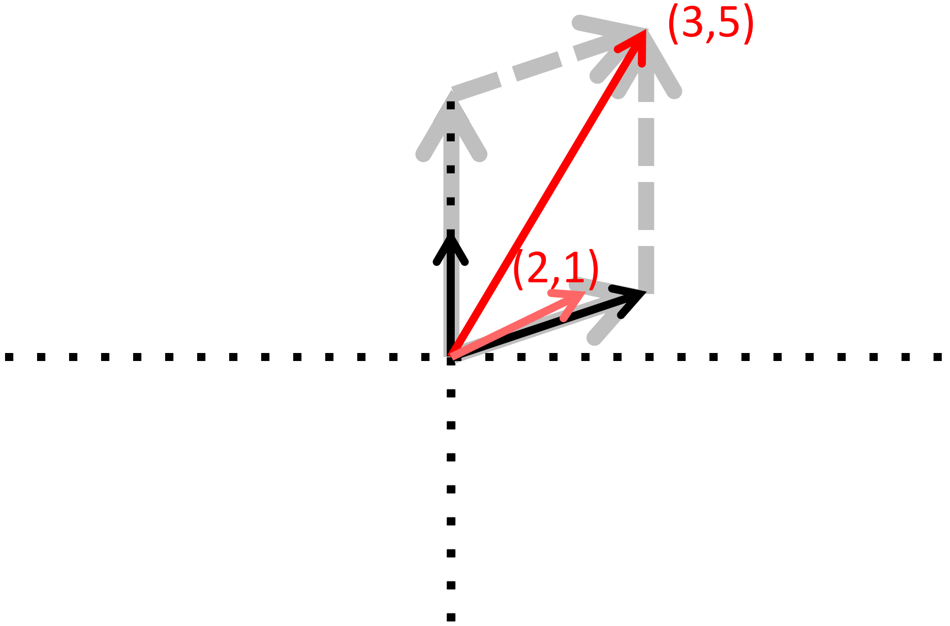

- The new vector is:
  1) **rotated**
  2) **scaled**

Are there any special vectors that **only get scaled**?

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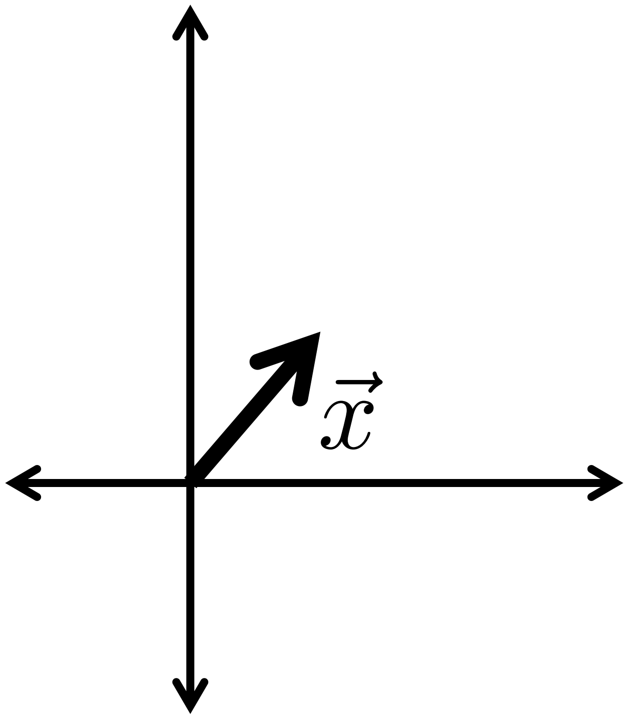
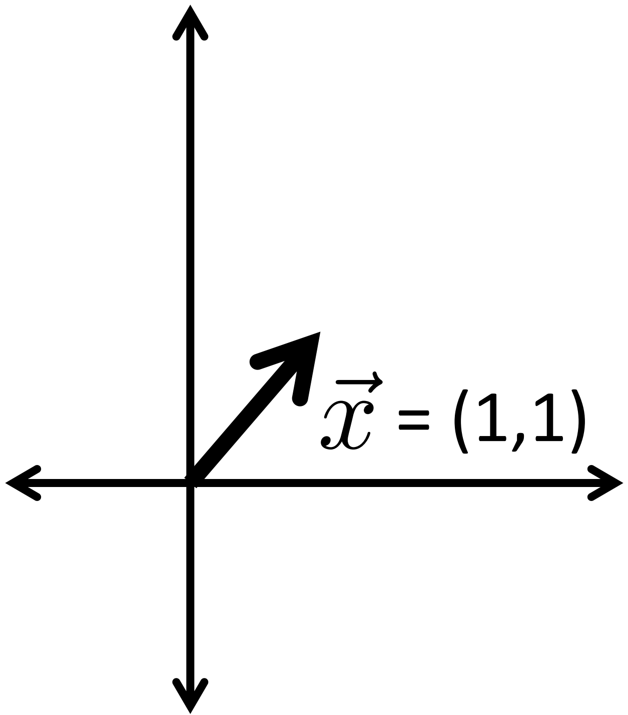
$$\overleftrightarrow{M} \quad \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



Try (1,1)

Are there any special vectors that **only get scaled**?

$$\underbrace{\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}}_{\overleftrightarrow{M}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$



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$\overleftrightarrow{M}$

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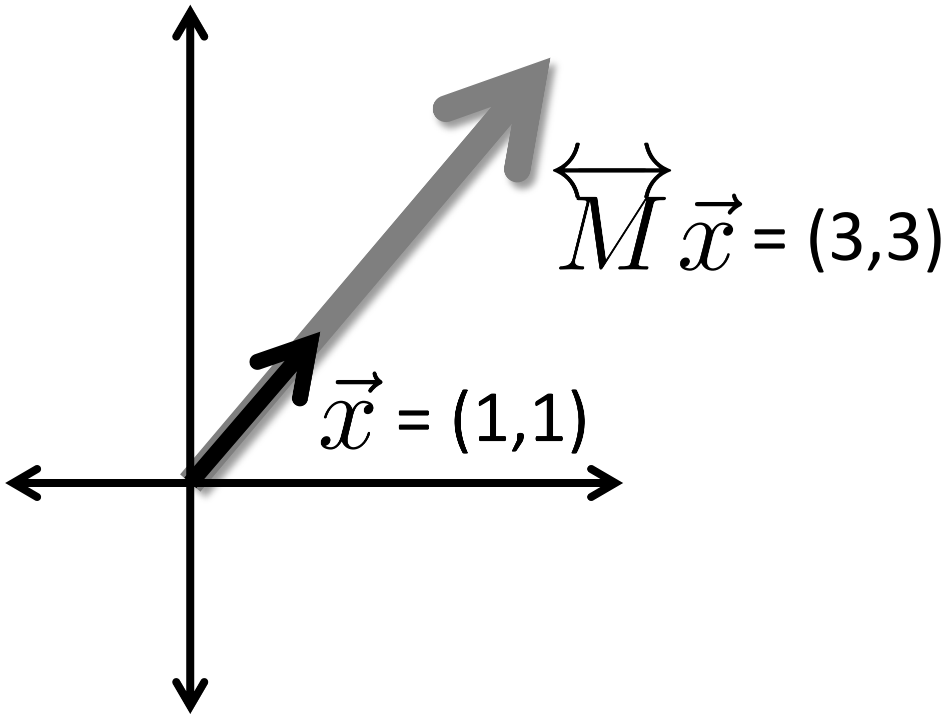

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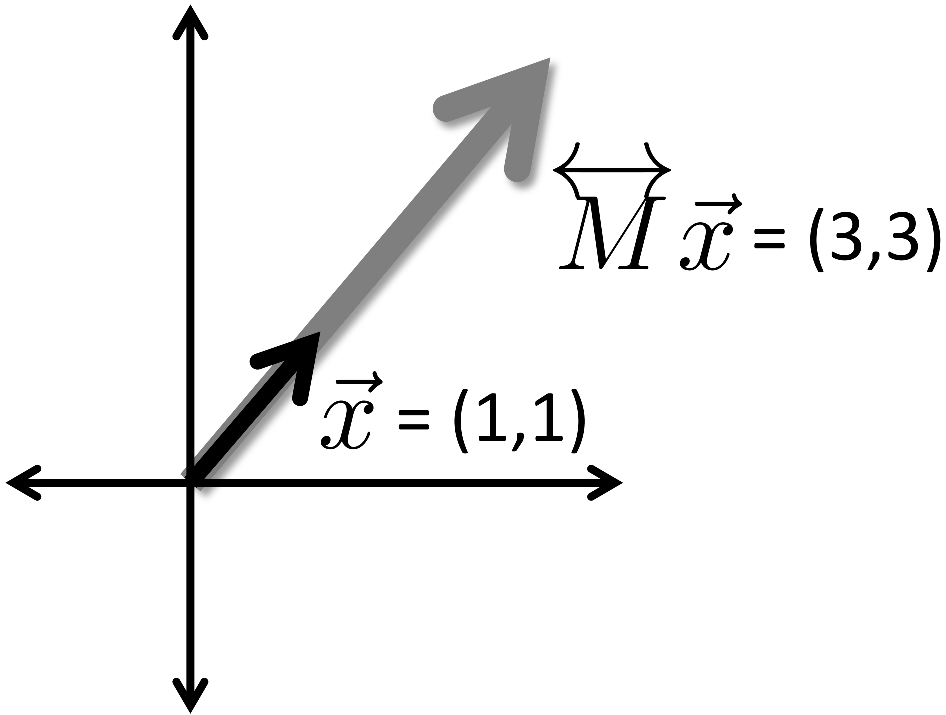

- For this special vector, multiplying by M is like multiplying by a scalar.
- (1,1) is called an **eigenvector** of M
- 3 (the scaling factor) is called the **eigenvalue** associated with this eigenvector

Are there any other eigenvectors?

- Yes! The easiest way to find is with python's eig command.

$$\vec{e}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \vec{e}^{(2)} = \begin{pmatrix} -1.5 \\ 1 \end{pmatrix}$$

- *Exercise*: verify that (-1.5, 1) is also an eigenvector of M.
- Note: eigenvectors are only defined up to a scale factor.
  – Conventions are either to make **e**'s unit vectors, or make one of the elements 1

Step back:
Eigenvectors obey this equation

$$\overleftrightarrow{M}\vec{e} = \lambda\vec{e}$$

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- This is called the characteristic equation for λ
- In general, for an N x N matrix, there are N eigenvectors