Week 7 Section Review of SVD

Zichen Chen

Content

- Review
- Demo

Singular value decomposition

- Eigen decomposition
- Rank of a Matrix
- Norm of a Matrix

Eigen decomposition

- Eigenvector, eigenvalue
- Eigenvector
  - a vector whose product when multiplied by the matrix is a scalar multiple of itself.
  - corresponding multiplier —> eigenvalue

- if *A* is a matrix, *v* is a eigenvector of *A*, and λ is the corresponding eigenvalue, then $Av=\lambda v$.

Eigen decomposition - Example

- $A = \begin{pmatrix} 4 & 0 & -1 \\ 2 & -2 & 3 \\ 7 & 5 & 0 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

- $Av = \begin{pmatrix} 4 & 0 & 1 \\ 2 & -2 & 3 \\ 5 & 7 & 0 \end{pmatrix} \underbrace{\boxed{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}}_{\text{Eigenvector}} = \begin{pmatrix} 4*1+0*1+1*2 \\ 2*1+-2*1+3*2 \\ 5*1+7*1+0*2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 12 \end{pmatrix} = \underbrace{\boxed{6}}_{\text{Eigenvalue}}v$

Find the eigenvalues/vectors

- To find the eigenvalues/vectors of a $n \times n$ square matrix
  - solve $det(A - \lambda I) = 0$
  - Where A is the matrix, λ is the eigenvalue, and I is an $n \times n$ identity matrix.

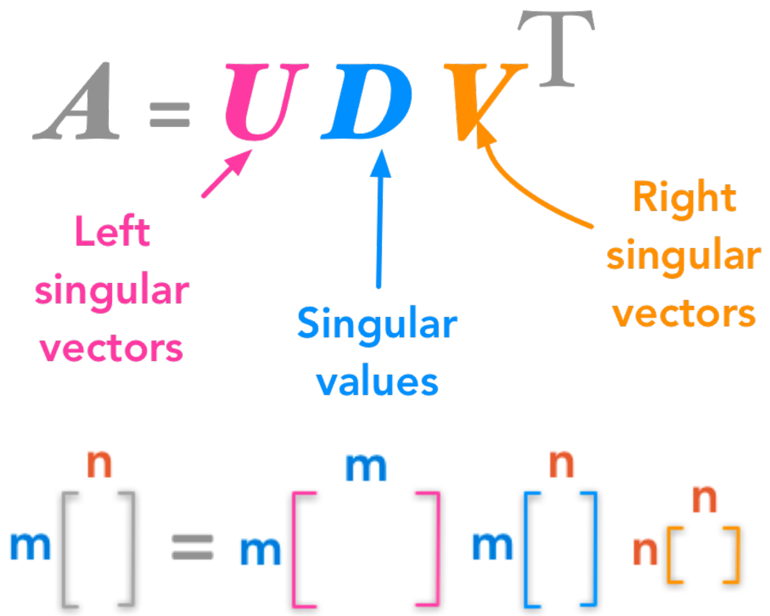
Find the eigenvalues/vectors - example

- $A = \begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}$
- $det(A - \lambda I) = det(\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = det \begin{pmatrix} 4 - \lambda & 3 \\ 2 & -1 - \lambda \end{pmatrix} = 0$
- $det(A - \lambda I) = (4 - \lambda)(-1 - \lambda) - 3 * 2 = \lambda^2 - 3\lambda - 10 = (\lambda + 2)(\lambda - 5) = 0$

- eigenvalues of A must be **-2 and 5** —> corresponding eigenvectors
- $\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix} * \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 5 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{—>} v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
- $\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix} * \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = -2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \text{—>} w = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

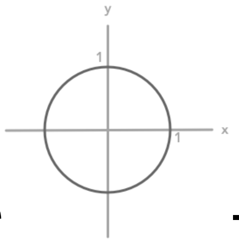
Singular Value Decomposition

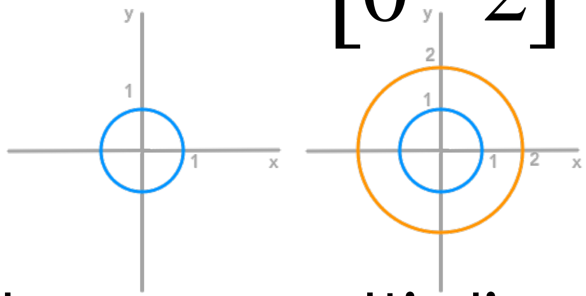
- SVD is a factorization of a matrix that **generalizes** the **eigen decomposition** of a square matrix to any mxn matrix
- We will decompose A into 3 matrices



- U and V are orthogonal matrices ($U^\top = U^{-1}$ and $V^\top = V^{-1}$)
- D is a diagonal matrix (all 0 except the diagonal).
  - Its elements σi are ordered: σ0 > σ1 > … $\sigma_{\min(m,n)-1} \geq 0$
- σ are called the **singular values** of M

SVD - example

- How the linear transformation associated with matrices?
- The unit circle 

—> apply the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
- $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ —> 

- each coordinate of the unit circle was multiplied by two

SVD - calculation

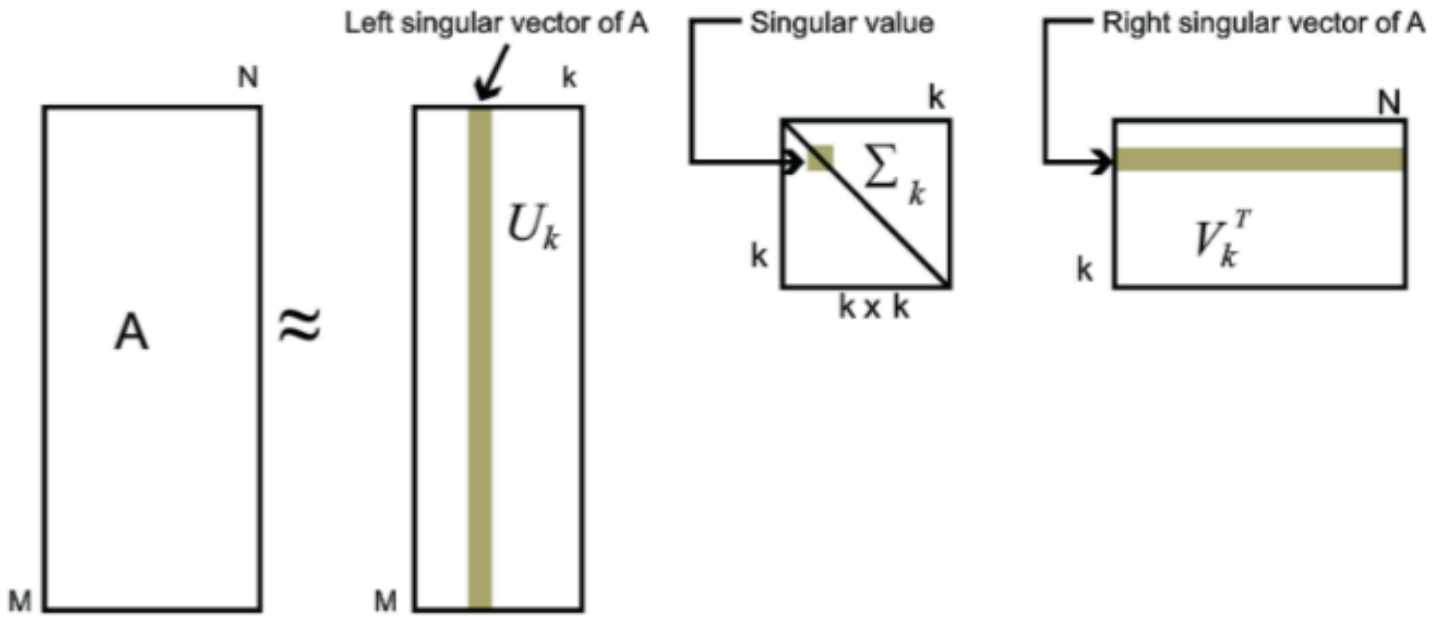
- $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$ —> $\mathbf{A^TA} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
  $\mathbf{AA^T} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$
- Get eigenvalue and eigenvector
- $A^TA\colon \lambda_1 = 3; v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}; \lambda_2 = 1; v_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$
- $AA^T\colon \lambda_1 = 3; u_1 = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}; \lambda_2 = 1; u_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}; \lambda_3 = 0; u_3 = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

SVD - calculation

- $Av_i = \sigma_i u_i, i = 1{,}2$ $((A^T A)v_i = \lambda_i v_i)$
- Singular value
- $\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \sigma_1 \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \Rightarrow \sigma_1 = \sqrt{3}$
- $\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \sigma_2 \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \Rightarrow \sigma_2 = 1$
- SVD:
- $A = U\Sigma V^T = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

SVD - importance

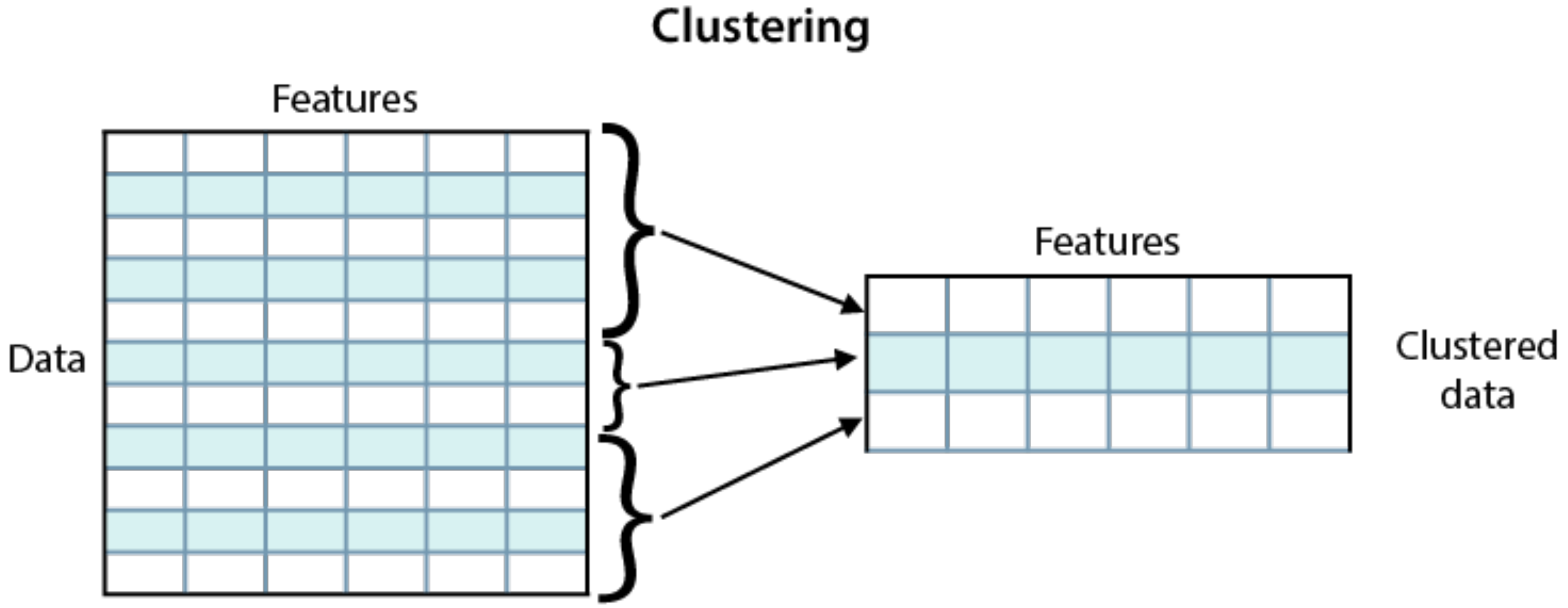
- $A_{m \times n} = U_{m \times m} \Sigma_{m \times n} V_{n \times n}^T \approx U_{m \times k} \Sigma_{k \times k} V_{k \times n}^T$
- To approach the matrix



Extend - from machine learning

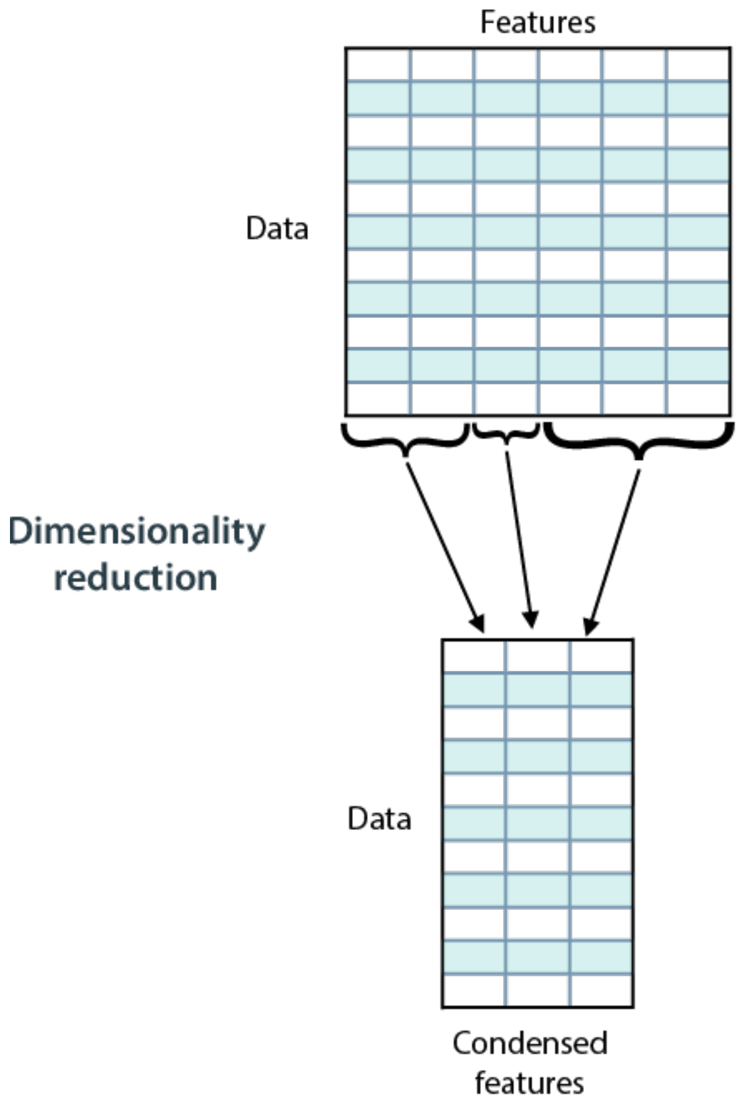
Clustering

- Clustering can be used to simplify our data by reducing the number of rows in our dataset by grouping several rows into one



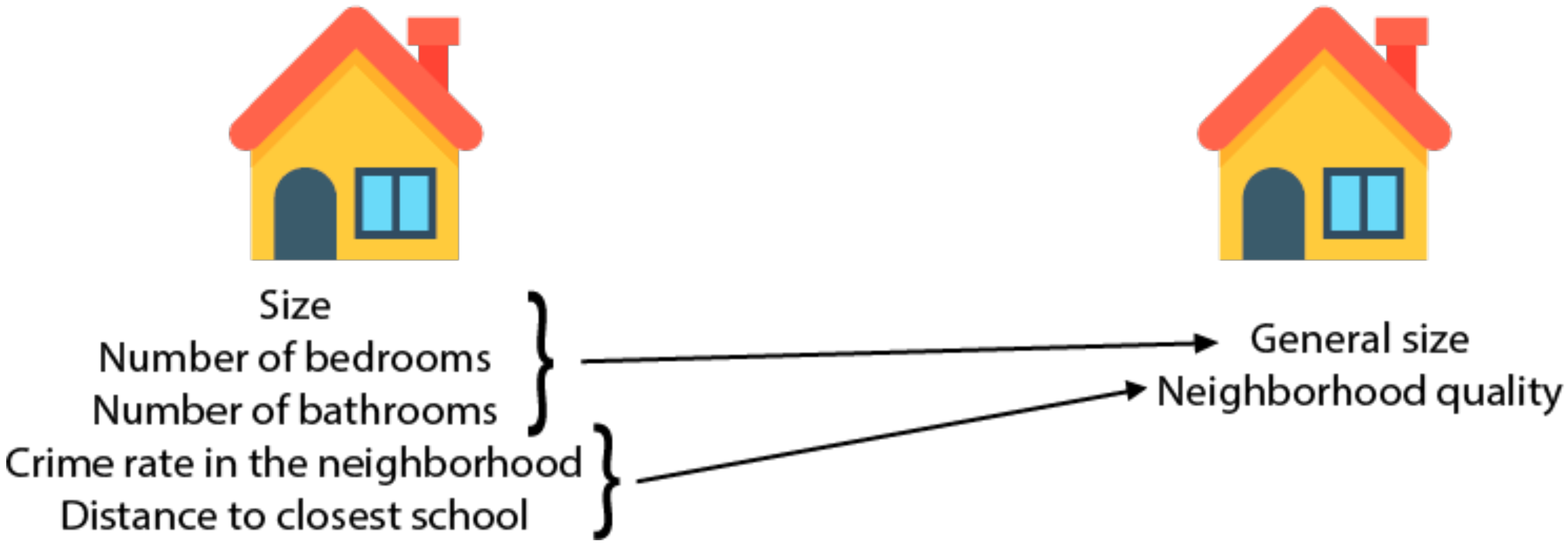
Dimensionality reduction

- Dimensionality reduction can be used to simplify our data by reducing the number of columns in our dataset.



Dimensionality reduction

- Dimensionality reduction algorithms help us simplify our data.
  - reduce the number of features in the dataset without losing much information



Summary

- A common type of unsupervised learning algorithms
  - clustering and dimensionality reduction
- Clustering is used to group data into similar clusters to extract information or make it easier to handle.
- Dimensionality reduction is a way to simplify our data, by joining certain similar features and losing as little information as possible.

- Singular value decomposition can simplify our data by reducing both the number of rows and columns.