Week 2 Section B

LU factorization

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Outline

- LU
- Cholesky
- SVD
- Python practice

LU Factorization

- Solving Ax = b becomes LUx = b:
  - 1. find A = LU
  - 2. solve Ly = b
  - 3. solve Ux = y

  - Tiem complexity? Ax = LUx? LUx = b?

LU Factorization

- matrix $Ax = b$. As a concrete example,
- $$\begin{pmatrix} 4 & -2 & -3 & 1 \\ 1 & 3 & 1 & 3 \\ 1 & 2 & -1 & -2 \\ 2 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 20 \\ 14 \\ 3 \\ 9 \end{pmatrix}$$
- LU decomposition facturs a matrix A into the product of two other matrices $A = LU$
- $$\begin{pmatrix} 4 & -2 & -3 & 1 \\ 1 & 3 & 1 & 3 \\ 1 & 2 & -1 & -2 \\ 2 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 \\ 0.25 & 0.71428571 & 1 & 0 \\ 0.5 & 0.57142857 & 0.33333333 & 1 \end{pmatrix} \begin{pmatrix} 4 & -2. & -3 & 1 \\ 0 & 3.5 & 1.75 & 2.75 \\ 0 & 0 & -1.5 & -4.21428571 \\ 0 & 0 & 0 & -1.66666667 \end{pmatrix}$$

LU Factorization

- Our equation is now,

- $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 \\ 0.25 & 0.71428571 & 1 & 0 \\ 0.5 & 0.57142857 & 0.33333333 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 4 & -2. & -3 & 1 \\ 0 & 3.5 & 1.75 & 2.75 \\ 0 & 0 & -1.5 & -4.21428571 \\ 0 & 0 & 0 & -1.66666667 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{\tilde{x}} = \begin{pmatrix} 20 \\ 14 \\ 3 \\ 9 \end{pmatrix}$$

- After making that substitution, we have

- $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 \\ 0.25 & 0.71428571 & 1 & 0 \\ 0.5 & 0.57142857 & 0.33333333 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix} = \begin{pmatrix} 20 \\ 14 \\ 3 \\ 9 \end{pmatrix}$$

- Which we can solve for $\tilde{x}$ by forward substitution.

LU Factorization

- we can solve for our unknown x using the definition of $\tilde{x}$ and using backward substitution.
- $$\begin{pmatrix} 4 & -2. & -3 & 1 \\ 0 & 3.5 & 1.75 & 2.75 \\ 0 & 0 & -1.5 & -4.21428571 \\ 0 & 0 & 0 & -1.66666667 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{pmatrix}$$

- **Demo**

Cholesky Factorization

- Cholesky decomposition facts a symmetric matrix into a lower triangular matrix L and its transpose such that
- $A = LL^{\dagger}$ (used only for square matrices)
- Example, $\begin{pmatrix} 4.31 & 0.094 & 1.599 & 0.634 & -0.16 \\ 0.094 & 6.961 & 1.984 & 3.656 & -3.814 \\ 1.599 & 1.984 & 7.412 & -1.473 & -2.258 \\ 0.634 & 3.656 & -1.473 & 4.386 & -1.518 \\ 0.16 & -3.814 & -2.258 & -1.518 & 2.553 \end{pmatrix}$
- decomposes into the product of
- $\begin{pmatrix} 2.076 & 0 & 0 & 0 & 0 \\ 0.045 & 2.6380 & 0 & 0 & \\ 0.77 & 0.739 & 2.505 & 0 & 0 \\ 0.305 & 1.381 & -1.089 & 1.095 & 0 \\ -0.077 & -1.445 & -0.452 & 0.007 & 0.506 \end{pmatrix}$ & $\begin{pmatrix} 2.076 & 0.045 & 0.77 & 0.305 & -0.077 \\ 0 & 2.638 & 0.739 & 1.381 & -1.445 \\ 0 & 0 & 2.505 & -1.089 & -0.452 \\ 0 & 0 & 0 & 1.095 & 0.007 \\ 0 & 0 & 0 & 0 & 0.506 \end{pmatrix}$
- **Demo**

Cholesky Factorization

- Let's solve the linear system

- $$\begin{pmatrix} 4.31 & 0.094 & 1.599 & 0.634 & -0.16 \\ 0.094 & 6.961 & 1.984 & 3.656 & -3.814 \\ 1.599 & 1.984 & 7.412 & -1.473 & -2.258 \\ 0.634 & 3.656 & -1.473 & 4.386 & -1.518 \\ 0.16 & -3.814 & -2.258 & -1.518 & 2.553 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -0.4693 \\ 0.6756 \\ -1.8170 \\ -0.1831 \\ 1.0590 \end{pmatrix}$$

- Cholesky decomposition

- $$\begin{pmatrix} 2.076 & 0 & 0 & 0 & 0 \\ 0.045 & 2.6380 & 0 & 0 & \\ 0.77 & 0.739 & 2.505 & 0 & 0 \\ 0.305 & 1.381 & -1.089 & 1.095 & 0 \\ -0.077 & -1.445 & -0.452 & 0.007 & 0.506 \end{pmatrix} \underbrace{\begin{pmatrix} 2.076 & 0.045 & 0.77 & 0.305 & -0.077 \\ 0 & 2.638 & 0.739 & 1.381 & -1.445 \\ 0 & 0 & 2.505 & -1.089 & -0.452 \\ 0 & 0 & 0 & 1.095 & 0.007 \\ 0 & 0 & 0 & 0 & 0.506 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}}_{\tilde{x}} = \begin{pmatrix} -0.4693 \\ 0.6756 \\ -1.8170 \\ -0.1831 \\ 1.0590 \end{pmatrix}$$

Cholesky Factorization

- Now we can write,

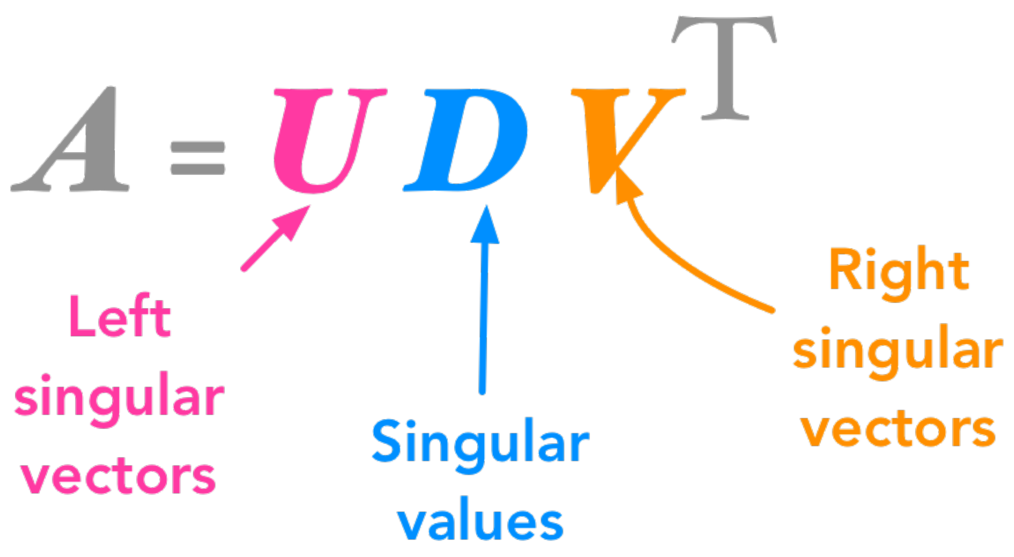
- $$\begin{pmatrix} 2.076 & 0 & 0 & 0 & 0 \\ 0.045 & 2.6380 & 0 & 0 & \\ 0.77 & 0.739 & 2.505 & 0 & 0 \\ 0.305 & 1.381 & -1.089 & 1.095 & 0 \\ -0.077 & -1.445 & -0.452 & 0.007 & 0.506 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \end{pmatrix} = \begin{pmatrix} -0.4693 \\ 0.6756 \\ -1.8170 \\ -0.1831 \\ 1.0590 \end{pmatrix}$$

- We can easily solve the above equation for $\tilde{x}$ by forward substitution. Then we can get our original unknown vector by backward substitution,

- $$\begin{pmatrix} 2.076 & 0.045 & 0.77 & 0.305 & -0.077 \\ 0 & 2.638 & 0.739 & 1.381 & -1.445 \\ 0 & 0 & 2.505 & -1.089 & -0.452 \\ 0 & 0 & 0 & 1.095 & 0.007 \\ 0 & 0 & 0 & 0 & 0.506 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \end{pmatrix}$$

- **Demo**

SVD - Singular Value Decomposition

- Is a way to go to decompose any type of matrices
- We will decompose A into 3 matrices
- 

> *A is a matrix that can be seen as a linear transformation. This transformation can be decomposed in three sub-transformations: 1. rotation, 2. re-scaling, 3. rotation. These three steps correspond to the three matrices U, D, and V.*

- U and V are orthogonal matrices ($U^{\mathsf{T}} = U^{-1}$ and $V^{\mathsf{T}} = V^{-1}$)
- D is a diagonal matrix (all 0 except the diagonal).
- However D is not necessarily square.
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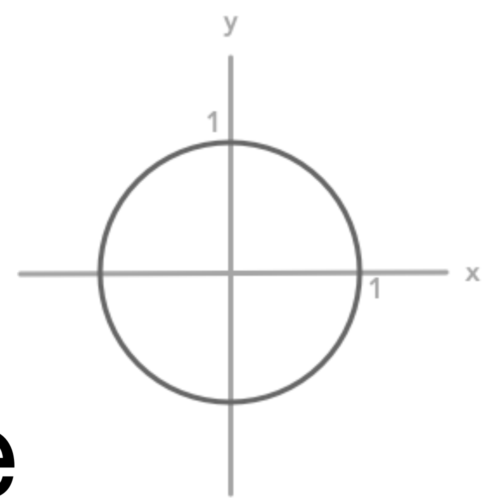
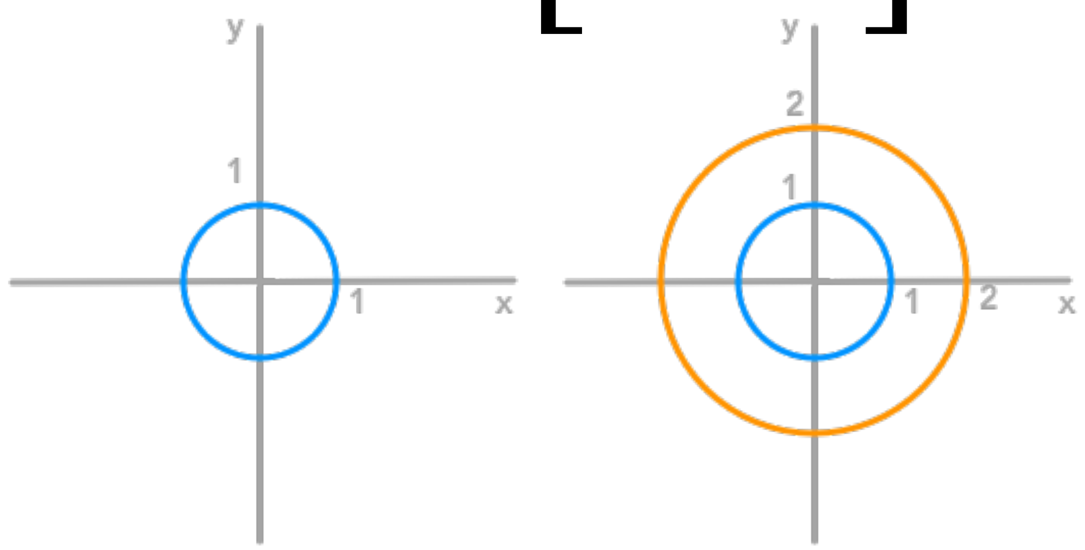
SVD

- Example
  - How the linear transformation associated with matrices?
  - The unit circle 

—> apply the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
  - $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ —> 

  - each coordinate of the unit circle was multiplied by two

SVD

- Apply SVD to an image processing problem
- Use the SVD to extract the more important features from the image